OT Methodology for non-parametric calibration (complements to Lecture 4)

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based on joint works with
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Transfer material from one site to another while minimising transportation costs.

- Monge (1781), Kantorovich (1948): Monge-Kantorovich problem
- Benamou & Brenier (2000): continuous-time formulation

Optimal transport, continuous-time formulation

Minimising the cost function F under given initial density ρ_0 and final density ρ_1

$$\inf_{\rho,v} \int_{\mathbb{R}^d} \int_0^1 \rho(t,x) F(v(t,x))\, dt dx,$$

subject to the continuity equation

$$\partial_t \rho(t,x) + \nabla \cdot (\rho(t,x) v(t,x)) = 0,$$

and the initial and final distributions

$$\rho(0,x) = \rho_0, \quad \rho(1,x) = \rho_1.$$

Stochastic optimal transport

Tan & Touzi (2013) (also Mikami & Thieullen (2006), Huesmann & Trevisan (2017), Backhoff et al. (2017)): Consider probability measures $\mathbb{P}$ such that X is a semimartingale,

$$dX_t = \beta_t^{\mathbb{P}}\, dt + (\alpha_t^{\mathbb{P}})^{1/2}\, dW_t^{\mathbb{P}}.$$

Stochastic optimal transport problem

We want to minimise

$$V(\mu_0, \mu_1) = \inf_{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)} \mathbb{E}^{\mathbb{P}} \int_0^1 F(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dt,$$

where $\mathcal{P}(\mu_0, \mu_1)$ contains probability measures satisfying

$$\mathbb{P} \circ X_0^{-1} = \mu_0, \quad \mathbb{P} \circ X_1^{-1} = \mu_1.$$

Note that the cost function F is convex and may depend on (t, X) as well.

Tan & Touzi (2013) established the following duality result

Dual formulation

The primal problem is equivalent to

$$V(\mu_0, \mu_1) = \sup_{\phi_1} \int \phi_1 \, d\mu_1 - \phi_0 \, d\mu_0,$$

where

$$\phi_0(x) := \sup_{\mathbb{P} \in \mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}} \left(\phi_1(X_1) - \int_0^1 F(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}) \, dt \right).$$

and for $F_t = F(t, X_t, \alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}})$ characterised ϕ_0 via PDEs.

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and for $F_t = F(t, X_t, \alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}})$ characterised ϕ_0 via PDEs.

Guo and Loeper (2018) extended this to path dependent constraints and cost.
Path-dependent PDEs & functional Itô used to describe the dual.

SOT induces a projection onto a subset of (semi)-martingales.

Use for calibration:

- Gather market data $\mathcal{G}$
- Fix a favourite reference model $\bar{\mathbb{P}}$
- Consider a cost F given by

$$F(\mathbb{P}) = \begin{cases} \mathrm{dist}(\mathbb{P}, \bar{\mathbb{P}}) & \text{if } \mathbb{P} \text{ is calibrated to } \mathcal{G}, \\ +\infty & \text{otherwise.} \end{cases}$$

- ensuring convexity to get duality
- Solve the dual via a non-linear (P)PDE
- $\mathbb{P}^*$ recovered via $\nabla F^*(\ldots)$.

Overview

SPX & VIX Calibration

SPX and VIX

- S&P 500 Index (SPX): a stock market index that measures the stock performance of 500 large companies listed in the US stock market.
- CBOE Volatility Index (VIX): a volatility index that measures the market's expectation of the volatility of SPX over the following 30 days.

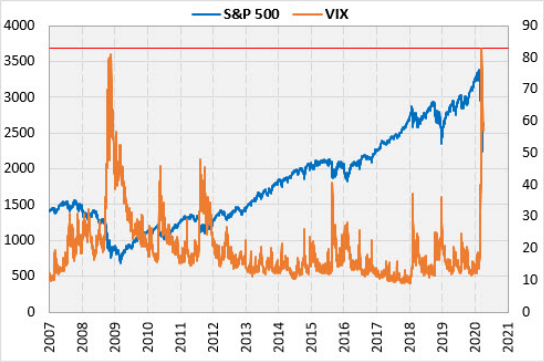


Figure: Historical SPX and VIX data. (Source: Schaeffer's Investment Research)

Why joint calibration?

- VIX futures and options are very popular hedging instruments.
  e.g., Szado (2009) shows that VIX call options are better than S&P 500 put options as a hedging instrument against the financial crisis in 2008.
- An arbitrage argument (Guyon 2020): existence of a liquid market
  $\Rightarrow$ need for models that jointly calibrate to the option prices of SPX and VIX
  $\Rightarrow$ avoid arbitrage between financial institutions (or even within the same institution)
- Joint calibration problem: build a (stochastic volatility) model that jointly calibrates to the prices of SPX options, VIX futures and VIX options.
- Very challenging problem, especially for short maturities.

Previous works:

- Continuous-time diffusion models (without jump):
  - Gatheral (2008): double CEV model
  - Goutte–Ismail–Pham (2017): Regime-switching Heston model
  - Fouque–Saporito (2018): Heston stochastic vol-of-vol
- Continuous-time jump-diffusion models: many works including
  - Cont–Kokholm (2013), Lian–Zhu (2013), Baldeaux–Badran (2014), Kokholm–Stisen (2015), Pacati–Pompa–Reno (2018), ...

However, even with jumps, these models have yet to achieve an exact fit.

Recent works:

- Guyon (2020): nonparametric discrete-time model calibrated by martingale optimal transport
- Gatheral–Jusselin–Rosenbaum (2020): (parametric) quadratic rough Heston model (no efficient calibration method yet)
- $\Rightarrow$ This work: nonparametric continuous-time model calibrated by semimartingale optimal transport

Modelling

Assumption: zero interest rates & dividends.

Let S_t be the SPX price:

$$S_t = S_0 + \int_0^t \sigma_s S_s \, dW_s.$$

Consider a time grid $0 < t_0 < t_1 < \cdots < t_n = T$ and an annualisation factor AF, e.g., if t_i corresponds to daily observations, then $AF = 100^2 \times 252/n$.

The *realised variance* of S_t during $[t_0, T]$:

$$AF \sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_i - 1}} \right)^2 \to \frac{100^2}{T - t_0} \int_{t_0}^{T} \sigma_t^2 \, dt, \quad a.s.$$

The VIX index at t_0:

$$VIX(t_0, T) = \sqrt{\mathbb{E}\left(\frac{100^2}{T - t_0} \int_{t_0}^{T} \sigma_t^2 \, dt \,\middle|\, \mathcal{F}_{t_0} \right)}$$

Modelling — Standard approach

Underlying assets:

$$S_t = S_0 + \int_0^t \sigma_s S_s \, dW_s$$

$$VIX(t_0, T) = \sqrt{\mathbb{E}\left(\frac{100^2}{T - t_0} \int_{t_0}^{T} \sigma_t^2 \, dt \,\middle|\, \mathcal{F}_{t_0} \right)}$$

Calibrating instruments:

SPX calls: $u^{SPX,c} = \mathbb{E}((S_T - K)^+)$
SPX puts: $u^{SPX,p} = \mathbb{E}((K - S_T)^+)$
VIX futures: $u^{VIX,f} = \mathbb{E}(VIX_{t_0})$
VIX calls: $u^{VIX,c} = \mathbb{E}((VIX_{t_0} - K)^+)$
VIX puts: $u^{VIX,p} = \mathbb{E}((K - VIX_{t_0})^+)$

Many previous works involve modelling (S_t, σ_t) or (S_t, σ_t^2)
$\Rightarrow$ the term VIX is a square root of conditional expectation
$\Rightarrow$ numerically difficult to compute the prices of VIX futures and VIX options.

Modelling — Our approach

Consider a two dimensional stochastic process $X = (X^1, X^2)$, let X^1 be the logarithm of S_t:

$$X_t^1 := \log S_t = X_0^1 - \frac{1}{2}\int_0^t \sigma_s^2\, ds + \int_0^t \sigma_s\, dW_s.$$

Let X^2 be a half of the expected forward quadratic variation of X^1 over $[t, T]$ observed at t:

$$X_t^2 = \mathbb{E}\left(\frac{1}{2}\int_t^T \sigma_s^2\, ds \,\middle|\, \mathcal{F}_t\right).$$

Calibrating instruments: for $\tau \le T$,

$$\begin{aligned}
\text{SPX calls:} \quad & u^{SPX,c} = \mathbb{E}((\exp(X_\tau^1) - K)^+) =: \mathbb{E}(G^{SPX,c}(X_\tau)) \\
\text{SPX puts:} \quad & u^{SPX,p} = \mathbb{E}((K - \exp(X_\tau^1))^+) =: \mathbb{E}(G^{SPX,p}(X_\tau)) \\
\text{VIX futures:} \quad & u^{VIX,f} = \mathbb{E}(100\sqrt{2X_{t_0}^2/(T-t_0)}) =: \mathbb{E}(G^{VIX,f}(X_{t_0})) \\
\text{VIX calls:} \quad & u^{VIX,c} = \mathbb{E}((100\sqrt{2X_{t_0}^2/(T-t_0)} - K)^+) =: \mathbb{E}(G^{VIX,c}(X_{t_0})) \\
\text{VIX puts:} \quad & u^{VIX,p} = \mathbb{E}((K - 100\sqrt{2X_{t_0}^2/(T-t_0)})^+) =: \mathbb{E}(G^{VIX,p}(X_{t_0}))
\end{aligned}$$

All payoffs depend on only the marginal distributions of X at fixed times
$\Rightarrow$ suitable for the calibration framework via optimal transport.

Modelling — a Heston model example

The Heston model:

$$\begin{aligned} dS_t &= \sqrt{\nu_t} S_t \, dW_t^1, \\ d\nu_t &= -\kappa(\nu_t - \theta)\, dt + \omega \sqrt{\nu_t}\, dW_t^2, \\ \langle dW^1, dW^2 \rangle_t &= \eta \, dt. \end{aligned}$$

We can derive that

$$X_t^2 = \mathbb{E}\left(\frac{1}{2} \int_t^T \nu_s \, ds \,\middle|\, \mathcal{F}_t \right) = \frac{1 - e^{-\kappa(T-t)}}{2\kappa}(\nu_t - \theta) + \frac{1}{2}\theta(T-t).$$

Define $A(t,\kappa) := (1 - e^{-\kappa(T-t)})/\kappa$ and $\nu(t, X_t^2, \kappa, \theta) := A(t,\kappa)^{-1}(2X_t^2 - \theta(T-t)) + \theta$, then the Heston model in terms of (X^1, X^2) is

$$\begin{aligned} dX_t^1 &= -\frac{1}{2}\nu(t, X_t^2, \kappa, \theta)\, dt + \sqrt{\nu(t, X_t^2, \kappa, \theta)}\, dW_t^1, \\ dX_t^2 &= -\frac{1}{2}\nu(t, X_t^2, \kappa, \theta)\, dt + \frac{1}{2}A(t,\kappa)\omega\sqrt{\nu(t, X_t^2, \kappa, \theta)}\, dW_t^2, \\ \langle dW_t^1, dW_t^2 \rangle &= \eta \, dt. \end{aligned}$$

Framework — Semimartingale optimal transport

Consider probability measures $\mathbb{P}$ under which X is a semimartingale:

$$dX_t = \alpha_t^{\mathbb{P}}\,dt + (\beta_t^{\mathbb{P}})^{\frac{1}{2}}\,dW_t^{\mathbb{P}}.$$

Semimartingale optimal transport with discrete constraints

Minimise

$$\inf_{\mathbb{P}\in\mathcal{P}(X_0,\tau,G,c)} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}},\beta_t^{\mathbb{P}})\,dt,$$

where $\mathcal{P}(X_0,\tau,G,c)$ contains probability measures $\mathbb{P}$ satisfying

$$\mathbb{P}\circ X_0^{-1} = \delta_{X_0} \quad \text{and} \quad \mathbb{E}^{\mathbb{P}} G_i(X_{\tau_i}) = c_i, \quad i=1,\ldots,m.$$

Note that the cost function F is convex in $(\alpha^{\mathbb{P}},\beta^{\mathbb{P}})$. It may depend on (t,X) as well.

Framework — Cost function

The cost function plays a regularisation role to ensure that X has the correct dynamics.

We want X to have the following dynamics:

$$X_t^1 = X_0^1 - \frac{1}{2}\int_0^t \sigma_s^2\,ds + \int_0^t \sigma_s\,dW_s, \quad X_t^2 = \mathbb{E}\left(\frac{1}{2}\int_t^T \sigma_s^2\,ds \,\middle|\, \mathcal{F}_t\right).$$

The above dynamics can be captured by $\mathbb{P}$ such that

$$(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) = \left(\begin{bmatrix} -\frac{1}{2}\sigma_t^2 \\ -\frac{1}{2}\sigma_t^2 \end{bmatrix}, \begin{bmatrix} \sigma_t^2 & (\beta_t)_{12} \\ (\beta_t)_{12} & (\beta_t)_{22} \end{bmatrix}\right), \quad 0 \le t \le T,$$

where $(\beta_t)_{12} = d\langle X^1, X^2\rangle_t \,/\, dt$ and $(\beta_t)_{22} = d\langle X^2\rangle_t \,/\, dt$ and with the additional property that $X_T^2 = 0$ $\mathbb{P}$-a.s.

Given $\bar{\beta}$, a reference for β, define the cost function:

$$F(\alpha,\beta) = \begin{cases} \displaystyle\sum_{i,j=1}^{2} (\beta_{ij} - \bar{\beta}_{ij})^2 & \text{if } \alpha_1 = \alpha_2 = -\frac{1}{2}\beta_{11}, \\ +\infty & \text{otherwise.} \end{cases}$$

Framework — Discrete constraints

The additional property $X_T^2 = 0$, $\mathbb{P}$-a.s. and the prices of calibrating instruments are imposed on X as discrete constraints $\Rightarrow$ exact calibration

We want to calibrate X to:

- m number of SPX options with payoffs $G = (G_1, \dots, G_m)$, maturities $\tau \in (0, T]^m$ and prices $u^{SPX} \in \mathbb{R}_+^m$, e.g.,
$$\mathbb{E}^{\mathbb{P}} G_i(X_{\tau_i}) = u_i^{SPX}, \quad i = 1, \dots, m,$$
- a VIX futures with payoff $J(x) = 100\sqrt{2x_2/(T - t_0)}$, maturity t_0 and price $u^{VIX,f} \in \mathbb{R}$, e.g.,
$$\mathbb{E}^{\mathbb{P}} J(X_{t_0}) = u^{VIX,f},$$
- n number of VIX options with payoffs $H = (H_1, \dots, H_n)$, maturity t_0 and prices $u^{VIX} \in \mathbb{R}_+^m$, e.g.,
$$\mathbb{E}^{\mathbb{P}} (H_i \circ J)(X_{t_0}) = u_i^{VIX}, \quad i = 1, \dots, n,$$
- a contract with payoff $\xi(x) = 1 - \exp(-(x_2)^2)$, maturity T and zero price, e.g.,
$$\mathbb{E}^{\mathbb{P}} \xi(X_T) = 0.$$

The last calibrating instrument ensures that $X_T^2 = 0$, $\mathbb{P}$-a.s. Since its price is always zero, we call it a *singular contract*.

Framework — Reformulation of the joint calibration problem

For simplicity, we represent all the discrete constraints by

$$\mathbb{E}^{\mathbb{P}} \mathcal{G}_i(X_{\mathcal{T}_i}) = c_i, \quad i = 1, \ldots, m+n+2,$$

where

$$\mathcal{G} = (\underbrace{G_1, \ldots, G_m}_{m \text{ SPX options}}, \underbrace{H_1 \circ J, \ldots, H_n \circ J}_{n \text{ VIX options}}, \underbrace{J}_{\text{VIX futures}}, \underbrace{\xi}_{\text{singular contract}}),$$

and $\mathcal{T}$ and c are defined in a similar manner.

Define a set of the probability measures $\mathcal{P}_{joint}$ such that

$$\mathcal{P}_{joint} := \{\mathbb{P} : \ \mathbb{P} \circ X_0^{-1} = \delta_{X_0} \text{ and } \mathbb{E}^{\mathbb{P}} \mathcal{G}_i(X_{\mathcal{T}_i}) = c_i, \ i = 1, \ldots, m+n+2\}$$

The joint calibration problem

$$\text{Minimise} \qquad V := \inf_{\mathbb{P} \in \mathcal{P}_{joint}} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) \, dt.$$

If we find an optimal solution $\tilde{\mathbb{P}}$ and $V < +\infty$, then we have a well-calibrated model

$$X_t = X_0 + \int_0^t \alpha_s^{\tilde{\mathbb{P}}} \, ds + \int_0^t (\beta_s^{\tilde{\mathbb{P}}})^{\frac{1}{2}} \, dW_s^{\mathbb{P}}.$$

Markovian projection: use a (Markovian) diffusion process mimic an Itô process by matching its marginals at fixed times. (Gyöngy (1986) and Brunick–Shreve (2013))

Lemma (Figalli (2008) and Trevisan (2016))

Let $\rho_t^{\mathbb{P}} = \mathbb{P} \circ X_t^{-1}$ be the marginal distribution of X_t under $\mathbb{P}$, $t \leq T$, then $\rho^{\mathbb{P}}$ is a weak solution to the Fokker–Planck equation:

$$\begin{cases} \partial_t \rho_t^{\mathbb{P}} + \nabla_x \cdot (\rho_t^{\mathbb{P}} \mathbb{E}_{t,x}^{\mathbb{P}} \alpha_t^{\mathbb{P}}) - \frac{1}{2} \sum_{i,j} \partial_{ij} (\rho_t^{\mathbb{P}} (\mathbb{E}_{t,x}^{\mathbb{P}} \beta_t^{\mathbb{P}})_{ij}) = 0 & \text{in } [0,T] \times \mathbb{R}^2, \\ \rho_0^{\mathbb{P}} = \delta_{X_0} & \text{in } \mathbb{R}^2. \end{cases}$$

Moreover, there exists another probability measure $\mathbb{P}'$ under which X has the same marginals, $\rho^{\mathbb{P}'} = \rho^{\mathbb{P}}$, and is a Markov process solving

$$dX_t = \alpha^{\mathbb{P}'}(t, X_t) dt + (\beta^{\mathbb{P}'}(t, X_t))^{\frac{1}{2}} \, dW_t^{\mathbb{P}'}, \quad 0 \leq t \leq T,$$

where $W^{\mathbb{P}'}$ is a $\mathbb{P}'$-Brownian motion, $\alpha^{\mathbb{P}'}(t,x) = \mathbb{E}_{t,x}^{\mathbb{P}} \alpha_t^{\mathbb{P}}$ and $\beta^{\mathbb{P}'}(t,x) = \mathbb{E}_{t,x}^{\mathbb{P}} \beta_t^{\mathbb{P}}$.

Notation: $\mathbb{E}_{t,x}^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}(\cdot \mid X_t = x)$.

Let $\mathcal{P}^{loc}_{joint}$ be a subset of $\mathcal{P}_{joint}$ such that, under any $\mathbb{P} \in \mathcal{P}^{loc}_{joint}$, X is a Markov process that solves

$$dX_t = \alpha^{\mathbb{P}}(t, X_t)dt + (\beta^{\mathbb{P}}(t, X_t))^{\frac{1}{2}}\, dW^{\mathbb{P}}_t, \quad 0 \le t \le T,$$

and X is fully calibrated to the calibrating instruments.

Proposition

$$V = \inf_{\mathbb{P} \in \mathcal{P}_{joint}} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha^{\mathbb{P}}_t, \beta^{\mathbb{P}}_t)\, dt = \inf_{\mathbb{P} \in \mathcal{P}^{loc}_{joint}} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha^{\mathbb{P}}_t(t, X_t), \beta^{\mathbb{P}}_t(t, X_t))\, dt$$

Proof: "$\geq$" follows by convexity of F via Jensen's inequality:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha^{\mathbb{P}}_t, \beta^{\mathbb{P}}_t)\, dt &= \mathbb{E}^{\mathbb{P}} \int_0^T \left(\mathbb{E}^{\mathbb{P}}_{t,x} F(\alpha^{\mathbb{P}}_t, \beta^{\mathbb{P}}_t) \right) dt \\ &\geq \mathbb{E}^{\mathbb{P}} \int_0^T F(\mathbb{E}^{\mathbb{P}}_{t,x} \alpha^{\mathbb{P}}_t, \mathbb{E}^{\mathbb{P}}_{t,x} \beta^{\mathbb{P}}_t)\, dt. \end{aligned}$$

"$\leq$" is clear since $\mathcal{P}^{loc}_{joint} \subset \mathcal{P}_{joint}$.

The problem can be made convex by introducing $A = \rho\alpha$ and $B = \rho\beta$, since

$$\rho F(\alpha, \beta) = \rho F\left(\frac{A}{\rho}, \frac{B}{\rho}\right) = \sup_{r + F^*(a,b) \leq 0} \{\rho r + A \cdot a + B : b\},$$

is convex in (ρ, A, B), where $F^*(a, b) = \sup_{\alpha,\beta}\{a \cdot \alpha + b : \beta - F(\alpha, \beta)\}$ is the convex conjugate of F, and $B : b = \mathrm{Tr}(Bb)$.

PDE formulation

Minimise

$$V = \inf_{\rho, A, B} \int_0^T \int_{\mathbb{R}^2} \rho F(A/\rho, B/\rho)\, dx dt,$$

subject to constraints

$$\partial_t \rho + \nabla_x \cdot A - \frac{1}{2} \sum_{i,j} \partial_{ij} B_{ij} = 0,$$

$$\int_{\mathbb{R}^2} \mathcal{G}_i \rho(t, \cdot)\, dx = c_i, \quad i = 1, \ldots, m + n + 2$$

$$\rho(0, \cdot) = \delta_{X_0}.$$

Introducing Lagrange multipliers $\phi \in C_c^\infty([0,T] \times \mathbb{R}^2)$ and $\lambda \in \mathbb{R}^{m+n+2}$, the problem can be formulated as:

$$V = \inf_{\rho,A,B} \sup_{\phi,\lambda} \left\{ \int_0^T \int_{\mathbb{R}^2} \left(\rho F\left(\frac{A}{\rho}, \frac{B}{\rho}\right) - (\partial_t \phi \rho + \nabla_x \phi \cdot A + \frac{1}{2}\nabla_x^2 \phi : B) - \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i)\rho \right) dx dt \right.$$
$$\left. + \lambda \cdot c - \phi(0, X_0) \right\}$$
$$= \sup_{\phi,\lambda} \inf_{\rho,A,B} \left\{ \underbrace{\int_0^T \int_{\mathbb{R}^2} \left(\rho F\left(\frac{A}{\rho}, \frac{B}{\rho}\right) \right.}_{\text{objective of the primal}} - (\partial_t \phi \rho + \nabla_x \phi \cdot A + \frac{1}{2}\nabla_x^2 \phi : B) - \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i)\rho \Bigg) dx dt$$
$$+ \underbrace{\lambda \cdot c - \phi(0, X_0)}_{\text{objective of the dual}} \Bigg\}$$

The interchange of $\inf$ and $\sup$ can be formally established by the Fenchel–Rockafellar duality theorem.

By applying the Fenchel–Rockafellar duality theorem and a smoothing technique:

Dual formulation

Maximise

$$V = \sup_{\lambda \in \mathbb{R}^{m+n+2}} \lambda \cdot c - \phi(0, X_0),$$

where ϕ is the viscosity solution to the HJB equation:

$$\partial_t \phi + F^*(\nabla_x \phi, \frac{1}{2}\nabla_x^2 \phi) = -\sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i),$$

with the terminal condition $\phi(T, \cdot) = 0$. If the supremum is attained and the associated solution to the HJB equation is $\tilde{\phi} \in BV([0,T], C_b^2(\mathbb{R}^2))$, then an optimal (α, β) of the PDE formulation can be found by

$$(\alpha, \beta) = \nabla F^*(\nabla_x \tilde{\phi}, \frac{1}{2}\nabla_x^2 \tilde{\phi}).$$

Note: $F^*(a,b) = \sup_{\alpha,\beta}\{a \cdot \alpha + b : \beta - F(\alpha, \beta)\}$ is the convex conjugate of F.

Framework — Gradient

Given $\lambda \in \mathbb{R}^{m+n+2}$ with the associated solution ϕ^λ, let $\mathbb{P}(\lambda)$ be the probability measure under which X has $(\alpha, \beta) = (\alpha^\lambda, \beta^\lambda) := \nabla F^*(\nabla_x \phi^\lambda, \frac{1}{2}\nabla_x^2 \phi^\lambda)$.

Define

$$L(\lambda) := \lambda \cdot c - \phi^\lambda(0, X_0).$$

The gradients of the objective can be formulated as the difference between the market prices and the model prices:

$$\partial_{\lambda_i} L(\lambda) = \underbrace{c_i}_{\text{market price}} - \underbrace{\mathbb{E}^{\mathbb{P}(\lambda)} \mathcal{G}_i(X_{\mathcal{T}_i})}_{\text{model price}}, \quad i = 1, \ldots, m.$$

The model price $\mathbb{E}^{\mathbb{P}(\lambda)} \mathcal{G}_i(X_{\mathcal{T}_i}) = \phi'(0, X_0)$ where ϕ' satisfies

$$\begin{cases} \partial_t \phi' + \alpha^\lambda \cdot \nabla_x \phi' + \dfrac{1}{2}\beta^\lambda : \nabla_x^2 \phi' = 0, & \text{in } [0, \mathcal{T}_i) \times \mathbb{R}^2, \\ \phi'(\mathcal{T}_i, \cdot) = \mathcal{G}_i. \end{cases}$$

Note: For the calculation of different gradients, the PDEs are the same but with different terminal conditions. The inversion of the linear operator is only required once for all gradients.

Numerical method — Solving the dual formulation

Dual formulation:

$$\text{maximise} \quad V = \sup_{\lambda \in \mathbb{R}^{m+n+2}} \lambda \cdot c - \phi^{\lambda}(0, X_0),$$

$$\text{subject to} \quad \partial_t \phi^{\lambda} + F^*(\nabla_x \phi^{\lambda}, \frac{1}{2} \nabla_x^2 \phi^{\lambda}) = - \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i), \quad \phi(T, \cdot) = 0.$$

Numerical solution:

1. Set an initial λ (e.g., $\lambda = \mathbf{0}$),
2. Solve the HJB equation backward to get $\phi^{\lambda}(0, X_0)$ (see next slide),
3. Solve the linear PDEs and calculate all gradients,
4. Update λ by gradient descent.

Numerical method — Solving HJB equations

HJB: $$\partial_t \phi + \sup_{\alpha,\beta}\{\alpha\cdot\nabla_x\phi + \frac{1}{2}\beta : \nabla_x^2\phi - F(\alpha,\beta)\} = -\sum_{i=1}^{m+n+2}\lambda_i\mathcal{G}_i\delta(t-\mathcal{T}_i), \quad \phi(T,\cdot)=0$$

Algorithm 1: Solving the HJB equation

for $k = N-1,\dots,0$ **do**

```
/* Handling the source term */
```

$\phi_{t_{k+1}} \leftarrow \phi_{t_{k+1}} + \sum_{i=1}^{m+n+2}\lambda_i\mathcal{G}_i\mathbb{1}(t_{k+1}=\mathcal{T}_i)$

```
/* Policy iteration */
```

$\phi_{t_k}^{new} \leftarrow \phi_{t_{k+1}}$

do

$\phi_{t_k}^{old} \leftarrow \phi_{t_k}^{new}$

Approximate the optimal $(\alpha_{t_k},\beta_{t_k})$ by solving the supremum with $\phi_{t_k}^{old}$

Solve the linearised HJB equation with $(\alpha_{t_k},\beta_{t_k})$ by a fully implicit finite difference method, and set the solution to $\phi_{t_k}^{new}$

while $\|\phi_{t_k}^{new} - \phi_{t_k}^{old}\|_\infty > \epsilon$

$\phi_{t_k} \leftarrow \phi_{t_k}^{new}$

end

Numerical method — Scaling

Scaling the discrete constraints with proper scales might improve the stability and convergence.

$$\mathbb{E}^{\mathbb{P}}\hat{\mathcal{G}}(X_T) := \mathbb{E}^{\mathbb{P}}\frac{1}{\Gamma}\mathcal{G}(X_T) = \frac{c}{\Gamma} =: \hat{c}$$

Recommended values of Γ:

- for SPX and VIX options, set Γ to their Black–Scholes Vega
  $\Rightarrow$ 1e-4 error of $\hat{c} \approx 1$ bp error in implied vol,
- for VIX futures, set $\Gamma = 100$
  $\Rightarrow$ 1e-4 error of $\hat{c} \approx 1$ cent error in price.

Numerical method — Smoothing

So far we have ignored the significance of the reference model $\bar{\beta}$.
When the gaps between strikes are too large or $\bar{\beta}$ is too far away from the β that describes the actual market dynamics, there might be spikes in the volatility surfaces, which might cause hump-shaped model volatility skews.

Smoothing technique:

1. Set an initial reference $\bar{\beta}$
2. Solve the dual formulation to get an optimal $\beta = \beta^*$
3. Smooth β^* by a smoothing method and set the result to $\bar{\beta}$
4. Repeat steps 2-4 with the new $\bar{\beta}$

In the numerical example, we smooth β^* by the simple moving average method over (X^1, X^2) with bandwidths of $(3, 3)$.

Simulated data example — Generating model

Simulated calibrating instruments:

- SPX call options maturing at 44 days and 79 days
- VIX futures maturing at 49 days
- VIX call options maturing at 49 days

Prices of the above instruments are *generated* using *Heston dynamics* and parameters $(\kappa, \theta, \omega, \eta) = (0.6, 0.09, 0.4, -0.5)$, i.e., X satisfies

$$X_t = X_0 + \int_0^t \alpha_s^{\mathbb{P}} \, ds + \int_0^t (\beta_s^{\mathbb{P}})^{\frac{1}{2}} \, dW_s^{\mathbb{P}},$$

and

$$(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) = \left(\begin{bmatrix} -\frac{1}{2}\nu(t, X_t^2, \kappa, \theta) \\ -\frac{1}{2}\nu(t, X_t^2, \kappa, \theta) \end{bmatrix}, \begin{bmatrix} \nu(t, X_t^2, \kappa, \theta) & \frac{1}{2}\eta\omega A(t,\kappa)\nu(t, X_t^2, \kappa, \theta) \\ \frac{1}{2}\eta\omega A(t,\kappa)\nu(t, X_t^2, \kappa, \theta) & \frac{1}{4}\omega^2 A(t,\kappa)^2 \nu(t, X_t^2, \kappa, \theta) \end{bmatrix} \right),$$

where $A(t, \kappa) := (1 - e^{-\kappa(T-t)})/\kappa$ and $\nu(t, X_t^2, \kappa, \theta) := A(t, \kappa)^{-1}(2X_t^2 - \theta(T-t)) + \theta$.

$\Rightarrow$ Solution exists!

Simulated data example — OT-Model

Recall our joint calibration problem is

$$\inf_{\mathbb{P}\in\mathcal{P}^{loc}_{joint}} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}})\, dt, \quad \text{where } F(\alpha,\beta) = \begin{cases} \sum_{i,j=1}^{2} (\beta_{ij} - \bar{\beta}_{ij})^2 & \text{if } \alpha_1 = \alpha_2 = -\frac{1}{2}\beta_{11}, \\ +\infty & \text{otherwise.} \end{cases}$$

We consider two references:

(a) a Heston reference with parameters $(\bar{\kappa}, \bar{\theta}, \bar{\omega}, \bar{\eta}) = (0.9, 0.04, 0.6, -0.3)$:

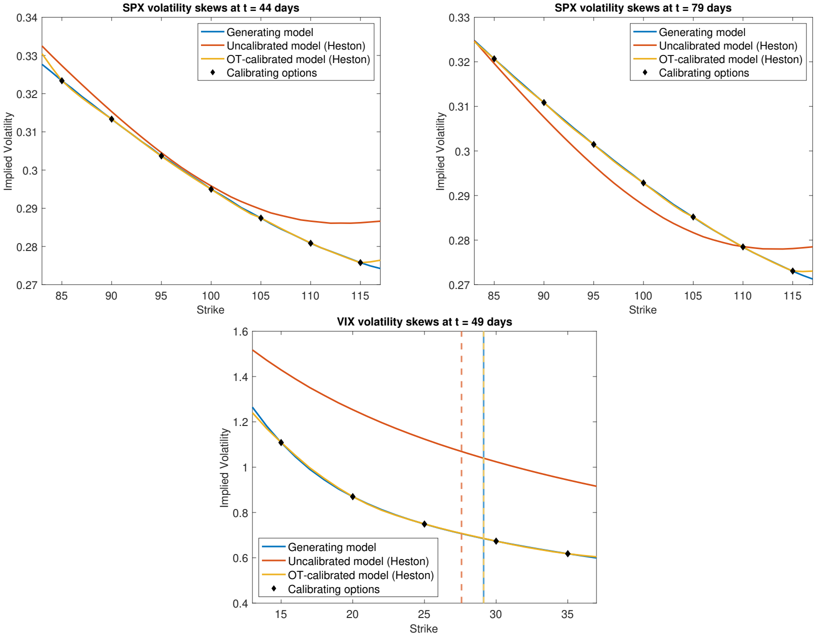
$$\bar{\beta}(t, X_t^1, X_t^2) = \begin{bmatrix} \nu(t, X_t^2, \bar{\kappa}, \bar{\theta}) & \frac{1}{2}\bar{\eta}\bar{\omega}A(t,\bar{\kappa})\nu(t, X_t^2, \bar{\kappa}, \bar{\theta}) \\ \frac{1}{2}\bar{\eta}\bar{\omega}A(t,\bar{\kappa})\nu(t, X_t^2, \bar{\kappa}, \bar{\theta}) & \frac{1}{4}\bar{\omega}^2 A(t,\bar{\kappa})^2\nu(t, X_t^2, \bar{\kappa}, \bar{\theta}) \end{bmatrix};$$

(b) a constant reference:

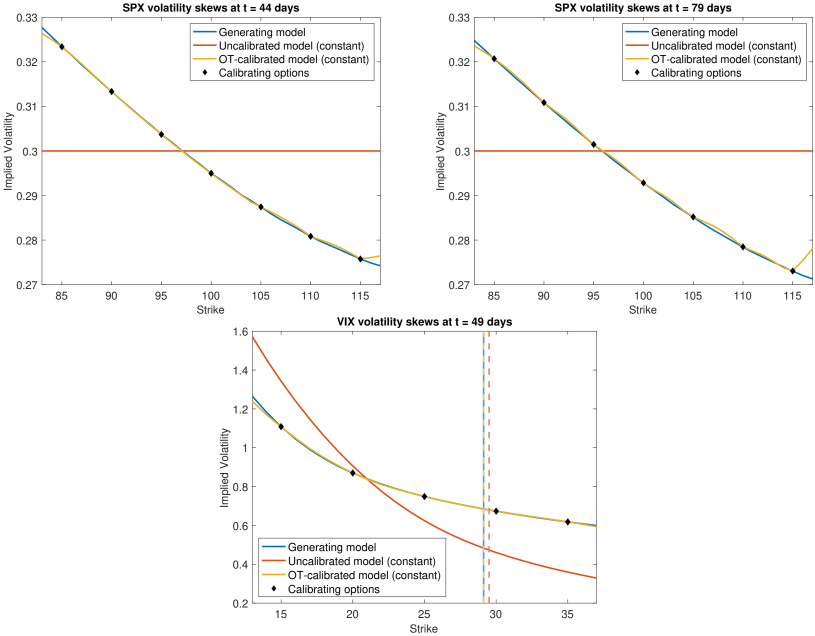
$$\bar{\beta}(t, X_t^1, X_t^2) = \begin{bmatrix} 0.09 & -0.01 \\ -0.01 & 0.04 \end{bmatrix}.$$

Rk: if in (a) we took the reference to be the generating model, $(\bar{\kappa}, \bar{\theta}, \bar{\omega}, \bar{\eta}) = (\kappa, \theta, \omega, \eta)$, then the algorithm quickly recovers OT-model = generating model by $\lambda = \mathbf{0}$, and $V = 0$.

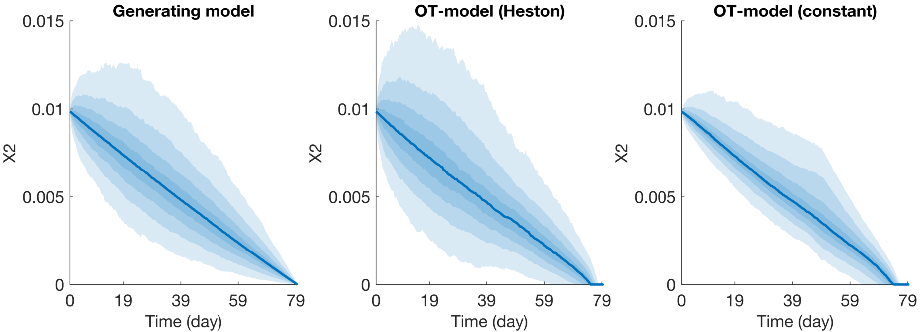
Simulated data example — Calibration results for Heston reference



Simulated data example — Calibrating results for constant reference



Simulated data example — Simulation of X^2



Market data example

Market data as of 1st September 2020:

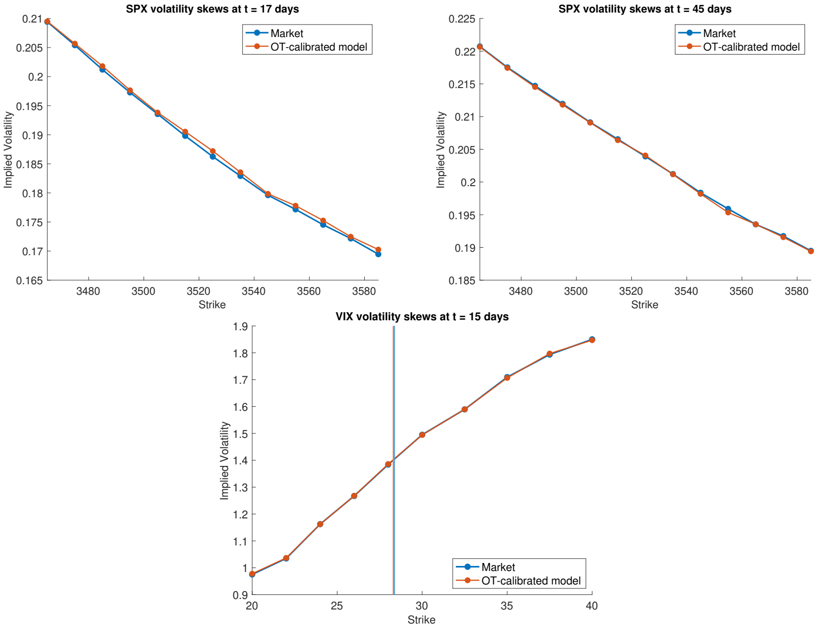
- SPX call options maturing at 17 days and 45 days
- VIX futures maturing at 15 days
- VIX call option maturing at 15 days

These are the shortest maturities, which is known as the most challenging case!

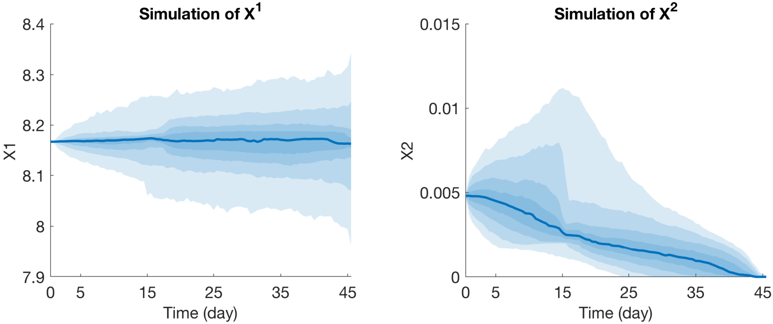
We calibrate the OT-model with a Heston reference $\bar{\beta}$. The parameters $(\bar{\kappa}, \bar{\theta}, \bar{\omega}, \bar{\eta}) = (4.99, 0.038, 0.52, -0.99)$ are obtained by (roughly) calibrating a standard Heston model to the SPX option prices.

Remark. Interest rates and dividends are NOT zero
$\Rightarrow$ model X^1 as the log of T-forward SPX price (instead of the spot price)
$\Rightarrow$ $\mathbb{P}$ are T-forward measures under which $\exp(X^1)$ is still a martingale.

Market data example — Calibration results



Market data example — Simulation of X^1 and X^2



Fixed Income Modelling — Our Approach

Assumption: A pre-calibrated short rate model fitting the term structure, zero dividends

Take a two dimensional stochastic process $X = (X^1, X^2)$, let X^1 log-stock price of some underlying asset and X^2 represent the short rate

$$X_t^1 = X_0^1 + X_t^2 - \frac{1}{2}\int_0^t \sigma_s^2 \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}W_s^1,$$

we assume that X^2 is a Hull-White short rate process given by

$$X_t^2 = X_0^2 + \int_0^t (\theta(s) - a(s)X_s^2)\,\mathrm{d}s + \int_0^t \sigma_r(s)\,\mathrm{d}W_s^2.$$

We assume that W_t^1 and W_t^2 are correlated standard Brownian motions such that

$$\langle W_\cdot^1, W_\cdot^2 \rangle_t = \int_0^t \xi_s \,\mathrm{d}s.$$

Note that since r_t is assumed to be pre-calibrated, the parameters θ, a, and σ_r are all assumed to be known. We calibrate σ and ξ using Call options on the underlying at 60 and 120 days.

Given n Call options observed in the market with prices u_i, strikes K_i and maturities τ_i, our calibration constraints become

$$\mathbb{E}\left[e^{-\int_0^{\tau_i} X_s^2\,\mathrm{d}s}\left(e^{X^1_{\tau_i}} - K_i\right)\right] = u_i, \qquad i = 1, \ldots, n.$$

We therefore consider the set $\mathcal{P}(X_0, \tau, K, u)$ containing measures $\mathbb{P}$ such that X is a semimartingale and satisfies the calibration constraints.
Moreover, we may localise using Markovian projection and consider the subset $\mathcal{P}_{\mathrm{loc}}(X_0, \tau, K, u) \subset \mathcal{P}(X_0, \tau, K, u)$ such that under the mimicking measure $\mathbb{P}' \in \mathcal{P}_{\mathrm{loc}}(X_0, \tau, K, u)$, X is a Markov process satisfying

$$\mathrm{d}X_t = \alpha(t, X_t)\mathrm{d}t + (\beta(t, X_t))^{\frac{1}{2}}\mathrm{d}W_t,$$

where W is a $\mathbb{P}'$ Brownian motion.

The discount term $e^{-\int_0^{\tau_i} X_s^2 \, \mathrm{d}s}$ is path dependent and thus incompatible with our PDE formulation framework.
We could add an extra state variable, but that would increase the computational complexity when solving the HJB equation, so we provide a conditioning argument.

Discounted Density Transformation

Let $\bar{\rho}$ be the joint law of X_t and $\int_0^t X_s^2 \, \mathrm{d}s$ and $\eta_{t,x}(y)$ the law of $\int_0^t X_s^2 \, \mathrm{d}s$ conditional on $X_t = [x^1, x^2]^\intercal$.
Define the 'discounted density' $\tilde{\rho}(t,x) = \left(\int_{\mathbb{R}} e^{-y} \eta_{t,x}(\mathrm{d}y)\right) \rho(t,x) = F(t,x)\rho(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^2$. Then $\tilde{\rho}$ satisfies for $(t,x) \in [0,T] \times \mathbb{R}^2$:

$$\partial_t \tilde{\rho}(t,x) + \nabla_x \cdot (\alpha(t,x)\tilde{\rho}(t,x)) - \frac{1}{2}\nabla_x^2 : (\beta(t,x)\tilde{\rho}(t,x)) + x_2\tilde{\rho}(t,x) = 0.$$

Primal Problem

Minimise

$$V = \inf_{\rho, A, B} \int_0^T \int_{\mathbb{R}^2} \rho F\left(\frac{A}{\rho}, \frac{B}{\rho}\right) \mathrm{d}x\mathrm{d}t,$$

subject to the constraints

$$\partial_t \rho + \nabla_x \cdot A - \frac{1}{2}\nabla^2 : B + x_2 \rho = 0$$

$$\int_{\mathbb{R}^2} \left(e^{x_1} - K_i\right)^+ \rho(\tau_i, \mathrm{d}x) = u_i, \qquad i = 1, \dots, n$$

$$\rho(0, \cdot) = \delta_{X_0}$$

Framework — Dual Problem

Introduce the Lagrange multipliers $\phi \in C_c^\infty([0,T] \times \mathbb{R}^2)$ and $\lambda \in \mathbb{R}^n$, then

$$V = \inf_{\rho,A,B} \sup_{\phi,\lambda} \Big\{ \int_0^T \int_{\mathbb{R}^2} \Big(\rho F\Big(\frac{A}{\rho}, \frac{B}{\rho}\Big) - \Big(\partial_t \phi \rho + \nabla_x \phi \cdot A + \frac{1}{2} \nabla_x^2 \phi : B - x_2 \phi \rho \Big) - \sum_{i=1}^n \lambda_i (e^{x_1} - K_i)^+ \delta_{\tau_i} \rho \Big) \, \mathrm{d}x\mathrm{d}t + \lambda \cdot u - \phi(0, X_0) \Big\}$$

Dual Problem

Maximise

$$V = \sup_{\lambda \in \mathbb{R}^n} \lambda \cdot u - \phi(0, X_0),$$

where ϕ is the viscosity solution to the HJB equation:

$$\partial_t \phi - x_2 \phi + F^*(\nabla_x \phi, \frac{1}{2} \nabla_x^2 \phi) + \sum_{i=1}^n \lambda_i (e_1^x - K_i)^+ \delta_{\tau_i} = 0$$

with the terminal condition $\phi(T, \cdot) = 0$. If the supremum is attained and the associated solution to the HJB equation is $\tilde{\phi} \in \mathrm{BV}([0,T], C_b^2(\mathbb{R}^2))$, then an optimal (α, β) of the PDE formulation can be found by

$$(\alpha, \beta) = \nabla F^*(\nabla_x \tilde{\phi}, \frac{1}{2} \nabla_x^2 \tilde{\phi}).$$

First choose a reference correlation $\xi(t, Z_t, r_t) = \frac{\sigma_r(t)}{\sigma(t,Z_t,r_t)}\bar{\xi}(t), \quad$ for $t \in [0, T]$. Then define for $p > 1$

$$H(x, \bar{x}, s) = \begin{cases} (p-1)\left(\frac{x-s}{\bar{x}-s}\right)^{1+p} + (p+1)\left(\frac{x-s}{\bar{x}-s}\right)^{1-p} - 2p, & \text{if } x, \bar{x} > s, \\ +\infty, & \text{otherwise.} \end{cases}$$

Notice that the coefficients are chosen such that H is minimised over x at $x = \bar{x}$ with $\min H = 0$. Also define the convex set

$$\Gamma(t, X_t) = \Big\{(\alpha, \beta) \in \mathbb{R}^2 \times \mathbb{S}^2 \; : \; \alpha_1 = X_t^2 - \frac{1}{2}\beta_{11}, \; \alpha_2 = (b(t) - aX_t^2), \\ \beta_{12} = \beta_{21} = \bar{\xi}\sigma_r(t), \; \beta_{22} = \sigma_r^2\Big\}$$

Define the cost function $F(\alpha, \beta) = \begin{cases} H(\beta_{11}, \bar{\sigma}^2, \bar{\xi}^2\sigma_r^2), & \text{if } (\alpha, \beta) \in \Gamma(t, X_t), \\ +\infty, & \text{otherwise.} \end{cases}$

$\bar{\sigma}^2 = \bar{\sigma}^2(t, X_t)$ is some reference value for the volatility

HJB Equation and Model Prices

HJB Equation

$$\sum_{i=1}^{n} \lambda_i(\exp(x_1) - K_i)^+ \delta_{\tau_i} + \partial_t \phi + \sup_{\beta_{11}} \Bigg(\left(x_2 - \frac{1}{2}\beta_{11} \right) \partial_{x_1}\phi$$
$$+ (b(t) - a x_2)\partial_{x_2}\phi + \frac{1}{2}\beta_{11}\partial^2_{x_1 x_1}\phi + \bar{\xi}\sigma_r \partial^2_{x_1 x_2}\phi + \frac{1}{2}\sigma_r^2 \partial^2_{x_2 x_2}\phi - x_2\phi$$
$$- H(\beta_{11}, \bar{\sigma}^2, \bar{\xi}^2\sigma_r^2) \Bigg) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^2.$$

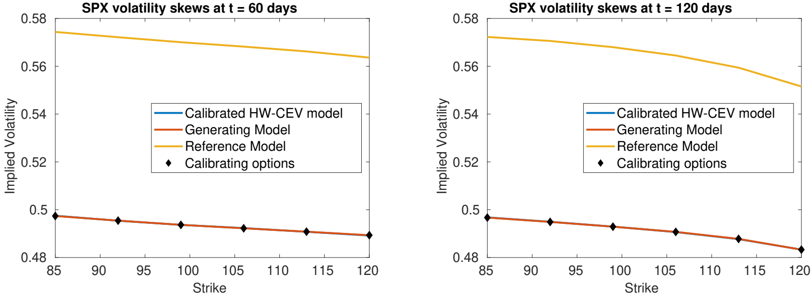
Given λ with associated solution $\mathbb{P}^\lambda$ of the dual problem, let $\mathbb{P}(\lambda)$ be the probability measure under which X has the characteristics $(\alpha^\lambda, \beta^\lambda) = \nabla F^*(\nabla_x \phi^\lambda, \frac{1}{2}\nabla_x^2 \phi^\lambda)$. Then the model price of an instrument with payoff $\mathcal{G}$ and maturity $\mathcal{T}$ is given by $\mathbb{E}^{\mathbb{P}(\lambda)}\left[e^{-\int_0^{\mathcal{T}} X_s^2 \, \mathrm{d}s} \mathcal{G}(X_{\mathcal{T}}) \right] = \phi'(0, X_0)$, where ϕ' solves

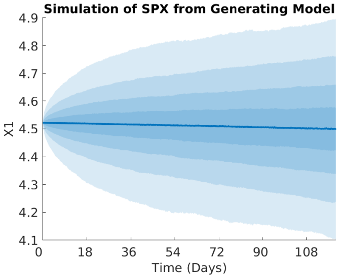
$$\begin{cases} \partial_t \phi + \alpha^\lambda \cdot \nabla_x \phi' + \frac{1}{2}\beta^\lambda : \nabla_x^2 \phi' - x_2 \phi' = 0, & (t, x) \in [0, \mathcal{T}) \times \mathbb{R}^2 \\ \phi'(\mathcal{T}, \cdot) = \mathcal{G}(\cdot) \end{cases}$$

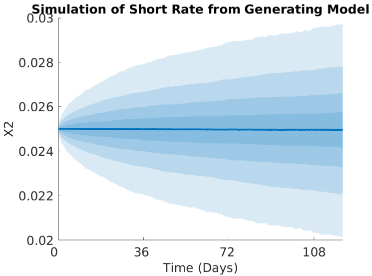
The numerical method is analogous in this case, and we may analytically compute the optimal β_{11} in the HJB equation with our chosen cost function.

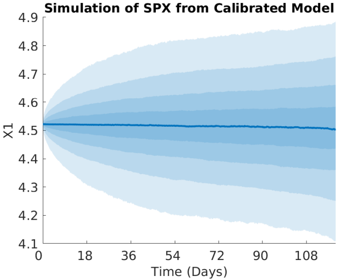
Simulated Data Example

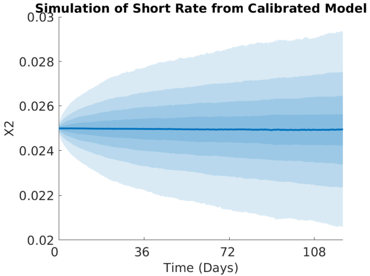
We used a CEV-Hull-White reference and generating model with the interest rate parameters the same in both. This gave us that $\bar{\sigma}(t, x) = \sigma \exp(x_1)^{\gamma-1}$. The generating model had parameters $(\sigma, \gamma, a, \sigma_r, \xi) = (0.78, 0.9, 0.4, 0.005, -0.6)$, and the "good" reference had $(\bar{\sigma}, \bar{\gamma}, \bar{\xi}) = (0.9, 0.9, -0.4)$, whereas the "bad" reference had $(\bar{\sigma}, \bar{\gamma}, \bar{\xi}) = (1.2, 0.78, 0.4)$




Simulation of SPX from Generating Model
X1
4.9
4.8
4.7
4.6
4.5
4.4
4.3
4.2
4.1
0
18
36
54
72
90
108
Time (Days)


Simulation of Short Rate from Generating Model
X2
0.03
0.028
0.026
0.024
0.022
0.02
0
36
72
108
Time (Days)


Simulation of SPX from Calibrated Model
X1
4.9
4.8
4.7
4.6
4.5
4.4
4.3
4.2
4.1
0
18
36
54
72
90
108
Time (Days)


Simulation of Short Rate from Calibrated Model
X2
0.03
0.028
0.026
0.024
0.022
0.02
0
36
72
108
Time (Days)

Simulated Data Example — Plots of Characteristics

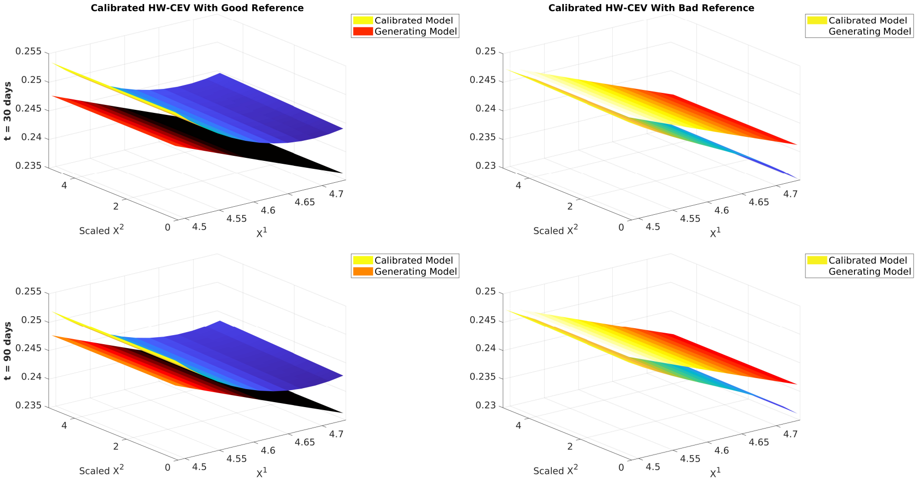


Figure: Comparison of β_{11} with the generating vol surface for a ‘good’ and a ‘bad’ reference model

Simulated Data Example — Plots of Characteristics

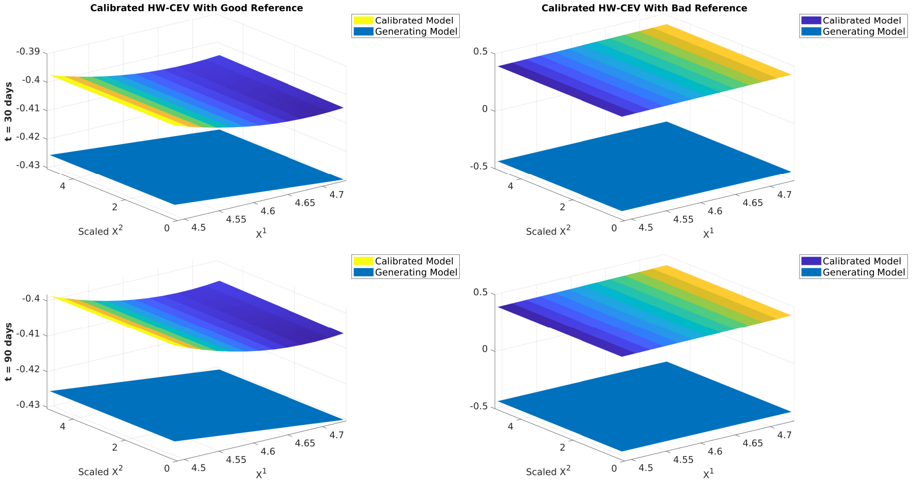


Figure: Comparison of ξ with the generating vol surface for a ‘good’ and a ‘bad’ reference model

Cost Function for Joint Calibration

Now assume we have no prior knowledge of the interest rate, our characteristics for the log-stock and short rate are therefore given by:

$$\alpha_t = \begin{bmatrix} X_t^2 - \frac{1}{2}(\beta_t)_{11} \\ (\alpha_t)_2 \end{bmatrix}, \qquad \beta_t = \begin{bmatrix} (\beta_t)_{11} & (\beta_t)_{12} \\ (\beta_t)_{12} & (\beta_t)_{22} \end{bmatrix}.$$

Define the convex set

$$\Gamma(t,x) = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \times \mathbb{S}_+^2 \ : \ \alpha_1 = x_2 - \frac{1}{2}\beta_{11} \right\}.$$

Define the cost function

$$F(\alpha, \beta) = \begin{cases} ||\alpha - \overline{\alpha}||_2^2 + ||\beta - \overline{\beta}||_{\mathrm{Fro}}^2, & \text{if } (\alpha, \beta) \in \Gamma(t,x), \\ +\infty, & \text{otherwise.} \end{cases}$$

Where $\bar{\alpha}$ and $\bar{\beta}$ correspond to some reference model. We remark that we will calibrate with interest rate derivatives as well.

Joint Calibration HJB Equation

The dual formulation is similar to the sequential calibration, but with a different cost function. Let $G_i(x)$ denote the payoffs of instruments with maturity τ_i and market value u_i.

Joint Calibration Dual Formulation

Maximise

$$V = \sup_{\lambda \in \mathbb{R}^n} \lambda \cdot u - \phi(0, X_0)$$

Subject to

$$\begin{aligned}
\partial_t \phi + \sup_{\alpha_2 \in \mathbb{R}, \beta \in \mathbb{S}^2_+} \Bigg\{ & \left(x_2 - \frac{1}{2}\beta_{11} \right) \partial_{x_1}\phi + \alpha_2 \partial_{x_2}\phi + \frac{1}{2}\beta_{11}\partial^2_{x_1 x_1}\phi \\
& + \frac{1}{2}\beta_{22}\partial^2_{x_2 x_2}\phi + \beta_{12}\partial^2_{x_1 x_2}\phi - ||\alpha - \overline{\alpha}||_2^2 - ||\beta - \overline{\beta}||^2_{\text{Fro}} \Bigg\} \\
& - x_2\phi + \sum_{i=1}^{n} \lambda_i G_i(x)\delta_{\tau_i} = 0, \qquad \text{for } (t,x) \in [0,T] \times \mathbb{R}^2
\end{aligned}$$

Joint Calibration Numerical Method

The numerical method is identical to the sequential calibration method, and we can analytically compute the supremum in the HJB equation.

We calibrate using Call options on the stock and Caplets on the interest rate with a fixed notional of $1,000 at 60 and 120 days.

The reference models are the CEV local volatility model with a Hull-White interest rate and a CIR interest rate. In both cases, the generating model was the same with shifted parameters. The parameters were given as follows:

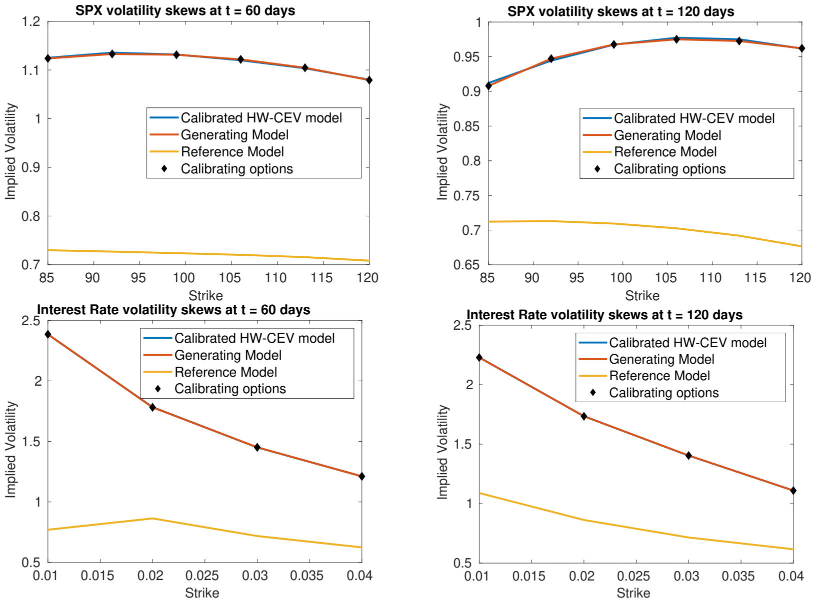
Generating		Reference	
σ	1.50	$\overline{\sigma}$	1.2
γ	0.95	$\overline{\gamma}$	0.89
a	0.05	$\overline{a}$	0.03
σ_r	0.04	$\overline{\sigma}_r$	0.02
ρ	-0.05	$\overline{\rho}$	-0.2

Table: CEV-Hull-White Parameters

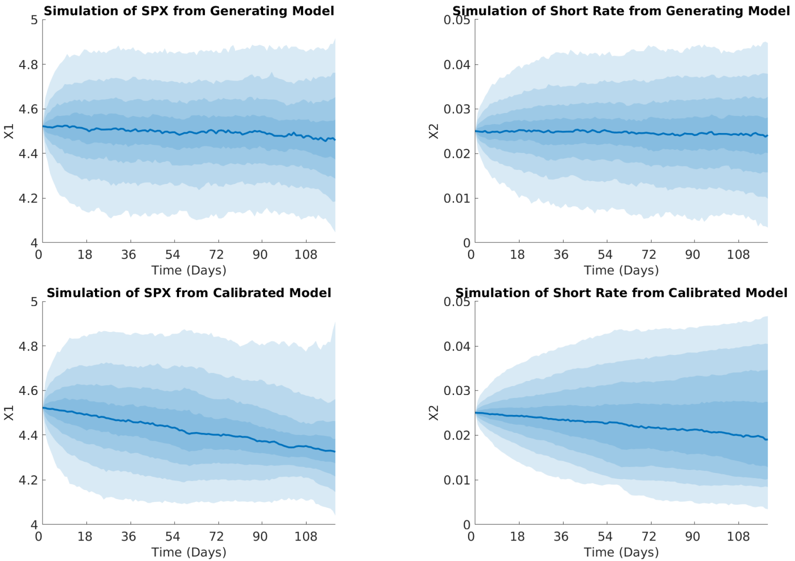
Generating		Reference	
σ	1.5	$\overline{\sigma}$	1.2
γ	0.95	$\overline{\gamma}$	0.89
b	0.03	$\overline{b}$	0.03
a	0.5	$\overline{a}$	0.4
σ_r	0.5	$\overline{\sigma}_r$	0.3
ρ	-0.4	$\overline{\rho}$	-0.2

Table: CEV-CIR Parameters

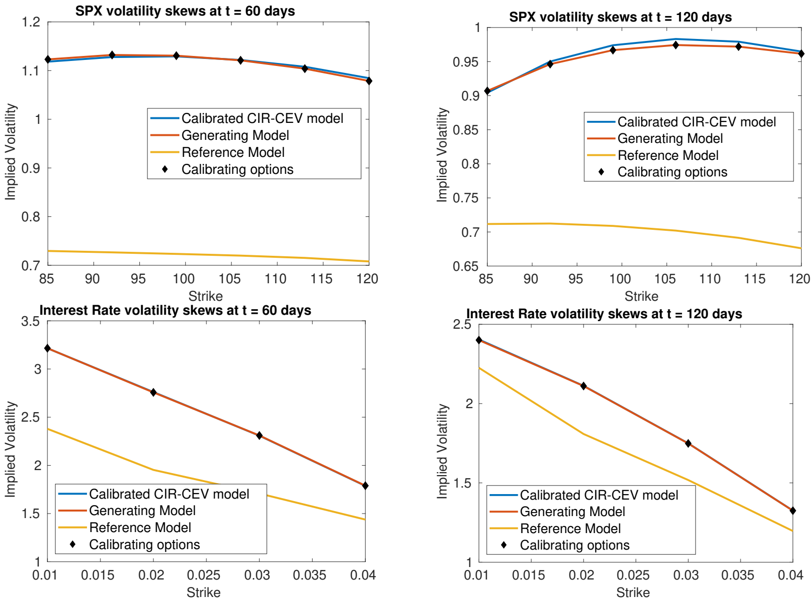
Simulated Data Example — CEV-HW



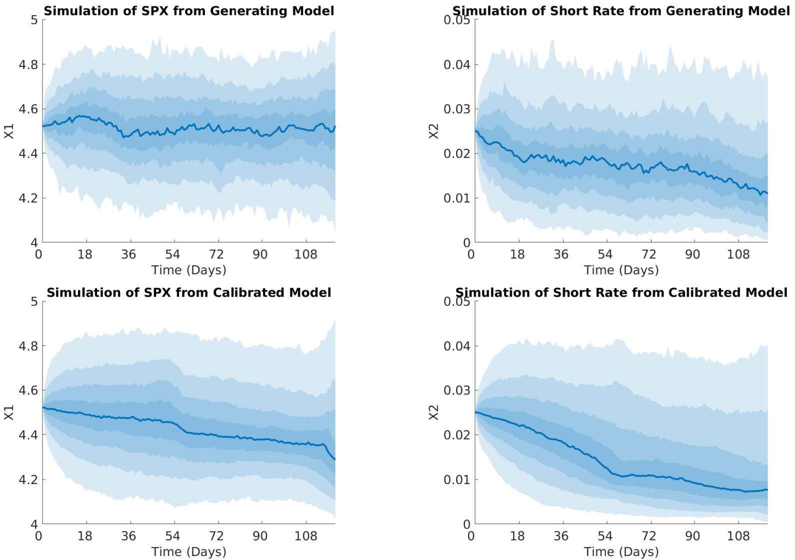
Simulated Data Example — Simulation of Calibrated Models



Simulated Data Example — CEV-CIR



Simulated Data Example — Simulation of Calibrated Models



Simulated Data Example — Plots of Characteristics

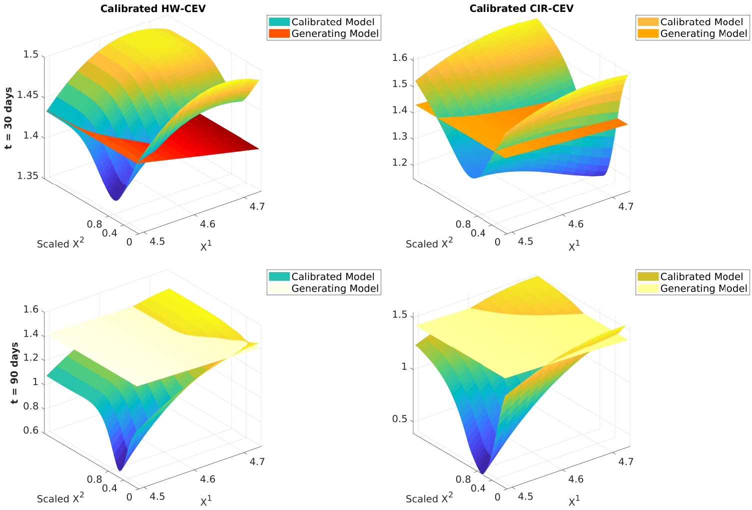


Figure: Compatison of β_{11} for the calibrated and generating model

Simulated Data Example — Plots of Characteristics

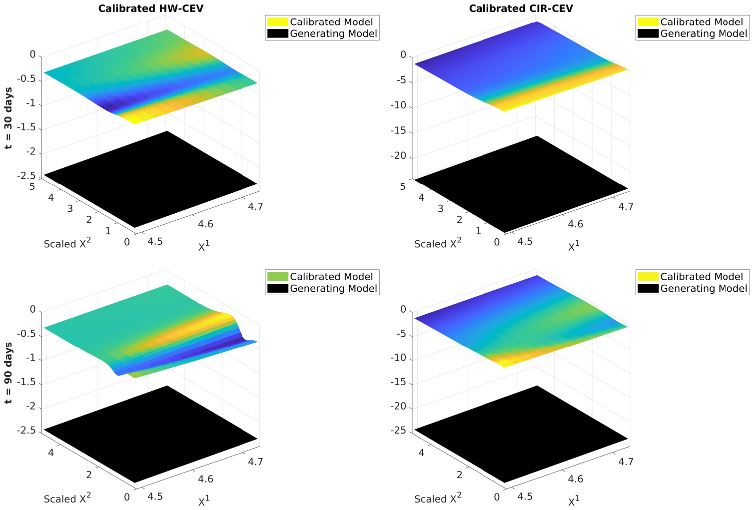


Figure: Comparison of β_{12} for the calibrated and generating model

Simulated Data Example — Plots of Characteristics

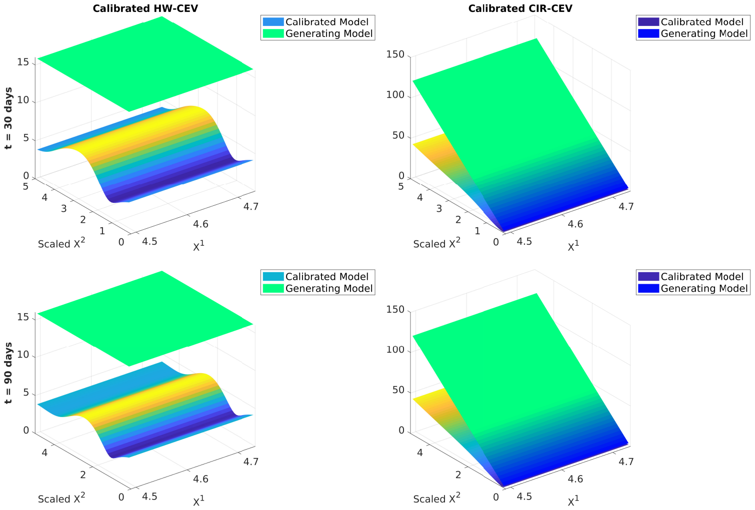


Figure: Comparison of β_{22} for the calibrated and generating model

Simulated Data Example — Plots of Characteristics

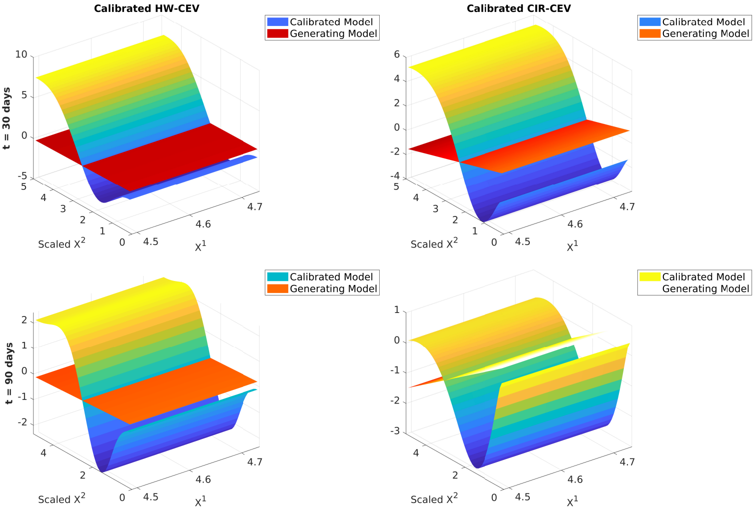


Figure: Comparison of α_2 for the calibrated and generating model

Investigating the Compatibility of Both Methods

Both methods calibrate the market data in a non-parametric way and recover a volatility surface that is different from the generating model while still pricing the options data. We can use a simulated data approach to check if these are compatible via the following procedure:

1. Specify a generating model, in this case CEV-Hull-White.
2. Assume that the Hull-White component perfectly matches the market data, so select a reference with the same Hull-White parameters, but different CEV parameters.
3. Perform sequential calibration and joint calibration and compare.

We used the following parameters:

Generating		Reference	
σ	1.20	$\overline{\sigma}$	1.50
γ	0.95	$\overline{\gamma}$	0.95
a	0.4	$\overline{a}$	0.4
σ_r	0.03	$\overline{\sigma}_r$	0.03
ρ	-0.3	$\overline{\rho}$	-0.1

Table: Generating and Reference model parameters - note that the interest rate parameters are the same as it is assumed to be pre-calibrated.

Investigating Compatibility — Plots of Characteristics

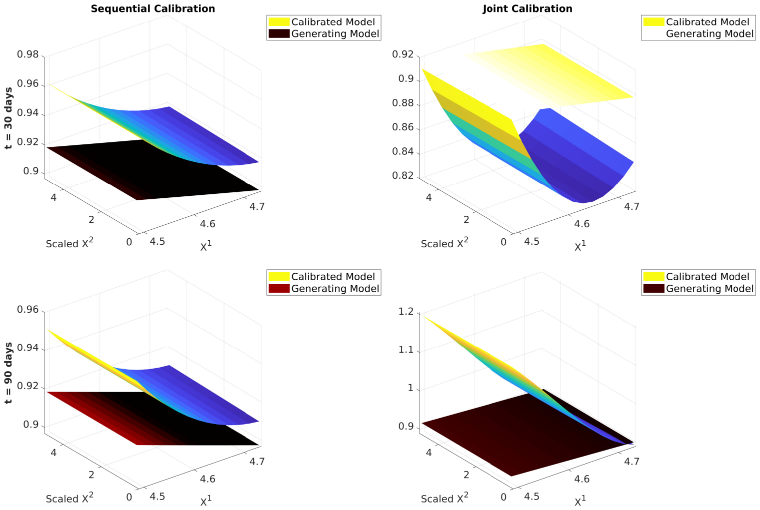


Figure: Compatison of β_{11} for the calibrated and generating model

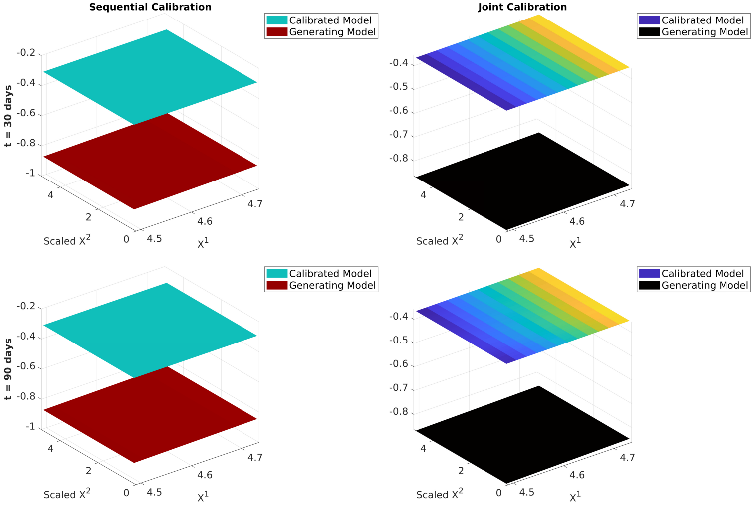


Figure: Comparison of β_{12} for the calibrated and generating model

Investigating Compatibility — Plots of Characteristics

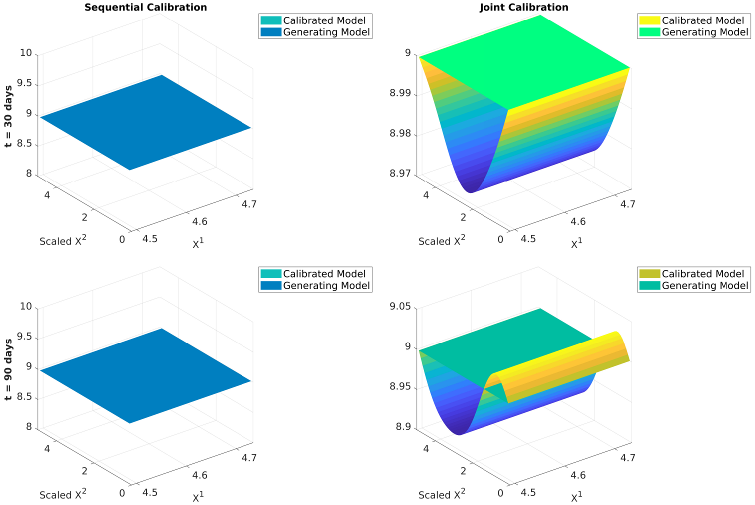


Figure: Comparison of β_{22} for the calibrated and generating model

Investigating Compatibility — Plots of Characteristics

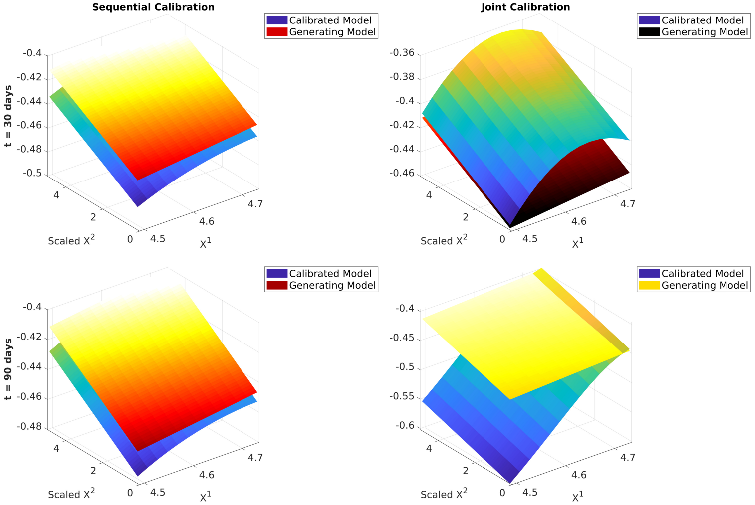


Figure: Comparison of α_1 for the calibrated and generating model

Investigating Compatibility — Plots of Characteristics

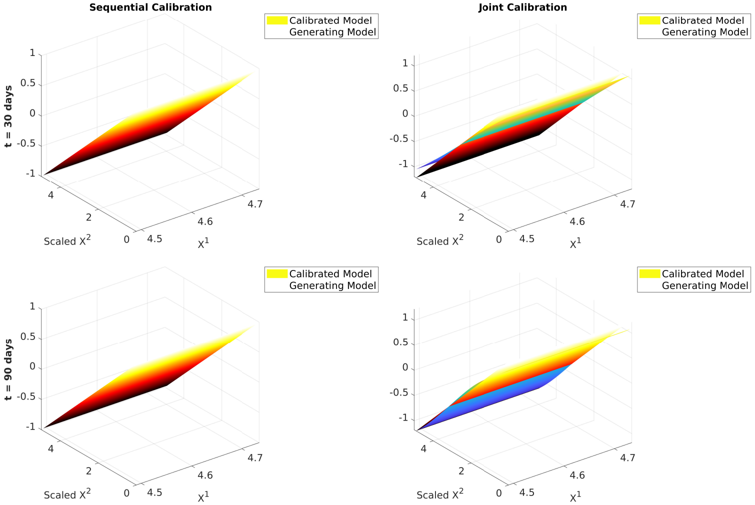


Figure: Comparison of α_2 for the calibrated and generating model

Joint Calibration with Market Data

An ongoing element of the research is a joint calibration with market data. We took the SPX as the underlying and the 1M US LIBOR for a proxy of the short rate. We obtained the following data on 23/05/2022 from a Bloomberg terminal:

- Calls on the SPX with expiry 19/08/2022,
- Caps on the one month LIBOR with notional $10,000,000 and expiry 23/08/2022,
- Calls on the SPX with expiry 18/11/2022,
- Caps on the one month LIBOR with notional $10,000,000 and expiry 23/11/2022.

The choice of a reference model is tricky here as it must be reasonably close to the market data for convergence. In addition, the existence of a measure under which our model is calibrated to the market data is not guaranteed.

We observed a good fit for the interest rate volatility, but a very poor fit for the SPX volatility and this is still ongoing.

Conclusion and future research

Conclusion:

- We build a nonparametric continuous-time stochastic volatility model that is accurately calibrated to the SPX options, VIX futures and VIX options prices
- The model selection/calibration is achieved using an optimal transport perspective

Future research:

- Improving computational efficiency and exploring applications in higher dimensions
  - Deep PDE solvers (see, e.g., Han et al. (2020))
  - Neural SDE (see, e.g., Cuchiero et al. (2020))

Thank you !

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SPX & Interest Rates Calibration

Case I: A pre-calibrated short rate model fitting the term structure, zero dividends

Take a two dimensional stochastic process $X = (X^1, X^2)$, let X^1 log-stock price of some underlying asset and X^2 represent the short rate

$$X_t^1 = X_0^1 + X_t^2 - \frac{1}{2}\int_0^t \sigma_s^2 \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}W_s^1,$$

we assume that X^2 is a Hull-White short rate process given by

$$X_t^2 = X_0^2 + \int_0^t (\theta(s) - a(s)X_s^2)\,\mathrm{d}s + \int_0^t \sigma_r(s)\,\mathrm{d}W_s^2.$$

We assume that W_t^1 and W_t^2 are correlated standard Brownian motions such that

$$\langle W_\cdot^1, W_\cdot^2 \rangle_t = \int_0^t \xi_s \,\mathrm{d}s.$$

Note that since r_t is assumed to be pre-calibrated, the parameters θ, a, and σ_r are all assumed to be known. We calibrate σ and ξ using Call options on the underlying at 60 and 120 days.

Given n Call options observed in the market with prices u_i, strikes K_i and maturities τ_i, our calibration constraints become

$$\mathbb{E}\left[e^{-\int_0^{\tau_i} X_s^2 \, \mathrm{d}s}\left(e^{X^1_{\tau_i}} - K_i\right)\right] = u_i, \qquad i = 1, \dots, n.$$

We therefore consider the set $\mathcal{P}(X_0, \tau, K, u)$ containing measures $\mathbb{P}$ such that X is a semimartingale and satisfies the calibration constraints.
Moreover, we may localise using Markovian projection and consider the subset $\mathcal{P}_{\mathrm{loc}}(X_0, \tau, K, u) \subset \mathcal{P}(X_0, \tau, K, u)$ such that under the mimicking measure $\mathbb{P}' \in \mathcal{P}_{\mathrm{loc}}(X_0, \tau, K, u)$, X is a Markov process satisfying

$$\mathrm{d}X_t = \alpha(t, X_t)\mathrm{d}t + (\beta(t, X_t))^{\frac{1}{2}}\mathrm{d}W_t,$$

where W is a $\mathbb{P}'$ Brownian motion.

The discount term $e^{-\int_0^{\tau_i} X_s^2 \,\mathrm{d}s}$ is path dependent and thus incompatible with our PDE formulation framework.
We could add an extra state variable, but that would increase the computational complexity when solving the HJB equation, so we provide a conditioning argument.

Discounted Density Transformation

Let $\bar{\rho}$ be the joint law of X_t^1, X_t^2 and $\int_0^t X_s^2 \,\mathrm{d}s$ and $\eta_{t,x}(y)$ the law of $\int_0^t X_s^2 \,\mathrm{d}s$ conditional on $X_t = [x^1, x^2]^\intercal$.
Define the 'discounted density' $\tilde{\rho}(t,x) = \left(\int_{\mathbb{R}} e^{-y} \eta_{t,x}(\mathrm{d}y)\right) \rho(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^2$.
Then $\tilde{\rho}$ satisfies for $(t,x) \in [0,T] \times \mathbb{R}^2$:

$$\partial_t \tilde{\rho}(t,x) + \nabla_x \cdot (\alpha(t,x)\tilde{\rho}(t,x)) - \frac{1}{2}\nabla_x^2 : (\beta(t,x)\tilde{\rho}(t,x)) + x_2 \tilde{\rho}(t,x) = 0.$$

Primal Problem

Minimise

$$V = \inf_{\rho, A, B} \int_0^T \int_{\mathbb{R}^2} \rho F\left(\frac{A}{\rho}, \frac{B}{\rho}\right) \mathrm{d}x\mathrm{d}t,$$

subject to the constraints

$$\partial_t \rho + \nabla_x \cdot A - \frac{1}{2}\nabla^2 : B + x_2 \rho = 0$$

$$\int_{\mathbb{R}^2} \left(e^{x_1} - K_i\right)^+ \rho(\tau_i, \mathrm{d}x) = u_i, \qquad i = 1, \ldots, n$$

$$\rho(0, \cdot) = \delta_{X_0}$$

Framework — Dual Problem

Introduce the Lagrange multipliers $\phi \in C_c^\infty([0,T]\times\mathbb{R}^2)$ and $\lambda \in \mathbb{R}^n$, then

$$V = \inf_{\rho,A,B}\sup_{\phi,\lambda}\left\{\int_0^T\int_{\mathbb{R}^2}\left(\rho F\left(\frac{A}{\rho},\frac{B}{\rho}\right) - \left(\partial_t\phi\rho + \nabla_x\phi\cdot A + \frac{1}{2}\nabla_x^2\phi : B - x_2\phi\rho\right)\right.\right.$$
$$\left.\left. - \sum_{i=1}^n \lambda_i(e^{x_1}-K_i)^+\delta_{\tau_i}\rho\right)\mathrm{d}x\mathrm{d}t + \lambda\cdot u - \phi(0,X_0)\right\}$$

Dual Problem

Maximise

$$V = \sup_{\lambda\in\mathbb{R}^n} \lambda\cdot u - \phi(0,X_0),$$

where ϕ is the viscosity solution to the HJB equation:

$$\partial_t\phi - x_2\phi + F^*(\nabla_x\phi, \frac{1}{2}\nabla_x^2\phi) + \sum_{i=1}^n \lambda_i(e_1^x - K_i)^+\delta_{\tau_i} = 0$$

with the terminal condition $\phi(T,\cdot) = 0$. If the supremum is attained and the associated solution to the HJB equation is $\tilde{\phi}\in \mathrm{BV}([0,T], C_b^2(\mathbb{R}^2))$, then an optimal (α,β) of the PDE formulation can be found by

$$(\alpha,\beta) = \nabla F^*(\nabla_x\tilde{\phi}, \frac{1}{2}\nabla_x^2\tilde{\phi}).$$

Cost function for Sequential Calibration

Choose a reference correlation $\bar{\xi}(t)$ and require $\xi(t, Z_t, r_t) = \frac{\sigma_r(t)}{\sigma(t, Z_t, r_t)}\bar{\xi}(t)$, for $t \in [0, T]$.
Define for $p > 1$

$$H(x, \bar{x}, s) = \begin{cases} (p-1)\left(\frac{x-s}{\bar{x}-s}\right)^{1+p} + (p+1)\left(\frac{x-s}{\bar{x}-s}\right)^{1-p} - 2p, & \text{if } x, \bar{x} > s, \\ +\infty, & \text{otherwise.} \end{cases}$$

Notice that the coefficients are chosen such that H is minimised over x at $x = \bar{x}$ with $\min H = 0$. Also define the convex set

$$\Gamma(t, X_t) = \Big\{(\alpha, \beta) \in \mathbb{R}^2 \times \mathbb{S}^2 \ : \ \alpha_1 = X_t^2 - \frac{1}{2}\beta_{11}, \ \alpha_2 = (b(t) - aX_t^2),$$
$$\beta_{12} = \beta_{21} = \bar{\xi}\sigma_r(t), \ \beta_{22} = \sigma_r^2\Big\}$$

Define the cost function $F(\alpha, \beta) = \begin{cases} H(\beta_{11}, \bar{\sigma}^2, \bar{\xi}^2\sigma_r^2), & \text{if } (\alpha, \beta) \in \Gamma(t, X_t), \\ +\infty, & \text{otherwise.} \end{cases}$

$\bar{\sigma}^2 = \bar{\sigma}^2(t, X_t)$ is some reference value for the volatility

HJB Equation and Model Prices

HJB Equation

$$\sum_{i=1}^{n} \lambda_i (\exp(x_1) - K_i)^+ \delta_{\tau_i} + \partial_t \phi + \sup_{\beta_{11}} \Bigg(\left(x_2 - \frac{1}{2}\beta_{11} \right) \partial_{x_1} \phi$$
$$+ (b(t) - a x_2) \partial_{x_2} \phi + \frac{1}{2} \beta_{11} \partial^2_{x_1 x_1} \phi + \bar{\xi} \sigma_r \partial^2_{x_1 x_2} \phi + \frac{1}{2} \sigma_r^2 \partial^2_{x_2 x_2} \phi - x_2 \phi$$
$$- H(\beta_{11}, \bar{\sigma}^2, \bar{\xi}^2 \sigma_r^2) \Bigg) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^2.$$

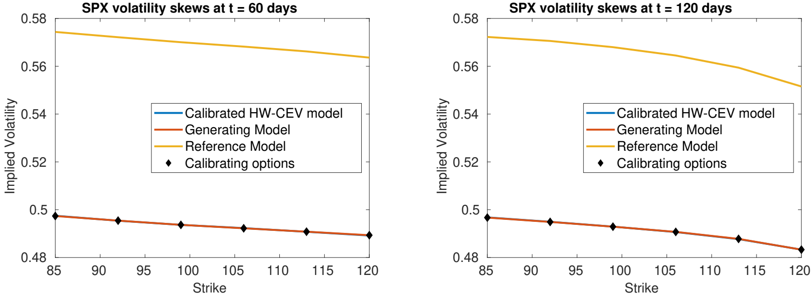
Given λ with associated solution $\mathbb{P}^\lambda$ of the dual problem, let $\mathbb{P}(\lambda)$ be the probability measure under which X has the characteristics $(\alpha^\lambda, \beta^\lambda) = \nabla F^*(\nabla_x \phi^\lambda, \frac{1}{2}\nabla_x^2 \phi^\lambda)$. Then the model price of an instrument with payoff $\mathcal{G}$ and maturity $\mathcal{T}$ is given by $\mathbb{E}^{\mathbb{P}(\lambda)} \left[e^{-\int_0^{\mathcal{T}} X_s^2 \, \mathrm{d}s} \mathcal{G}(X_{\mathcal{T}}) \right] = \phi'(0, X_0)$, where ϕ' solves

$$\begin{cases} \partial_t \phi + \alpha^\lambda \cdot \nabla_x \phi' + \frac{1}{2} \beta^\lambda : \nabla_x^2 \phi' - x_2 \phi' = 0, & (t,x) \in [0, \mathcal{T}) \times \mathbb{R}^2 \\ \phi'(\mathcal{T}, \cdot) = \mathcal{G}(\cdot) \end{cases}$$

The numerical method is analogous in this case, and we may analytically compute the optimal β_{11} in the HJB equation with our chosen cost function.

Simulated Data Example

We used a CEV-Hull-White reference and generating model with the interest rate parameters the same in both. This gave us that $\bar{\sigma}(t,x) = \sigma\exp(x_1)^{\gamma-1}$. The generating model had parameters $(\sigma, \gamma, a, \sigma_r, \xi) = (0.78, 0.9, 0.4, 0.005, -0.6)$, and the "good" reference had $(\bar{\sigma}, \bar{\gamma}, \bar{\xi}) = (0.9, 0.9, -0.4)$, whereas the "bad" reference had $(\bar{\sigma}, \bar{\gamma}, \bar{\xi}) = (1.2, 0.78, 0.4)$



Simulated Data Example — Plots of Characteristics

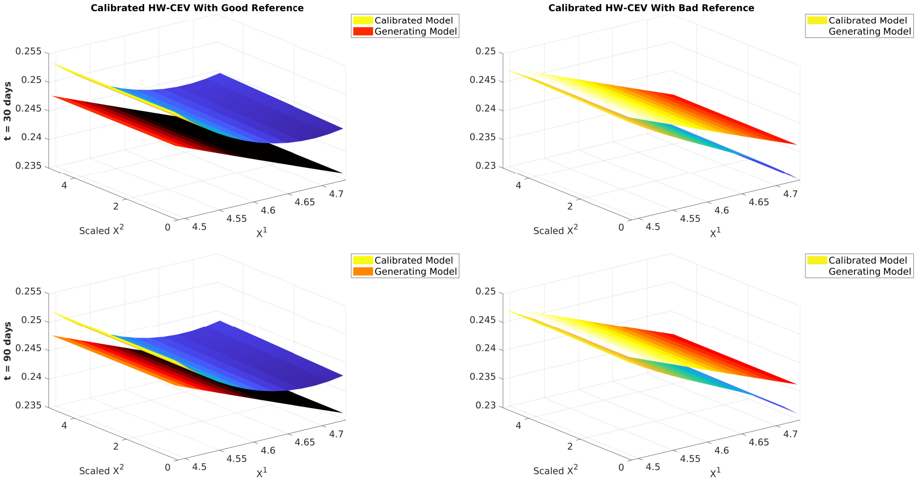


Figure: Comparison of β_{11} with the generating vol surface for a ‘good’ and a ‘bad’ reference model

Simulated Data Example — Plots of Characteristics

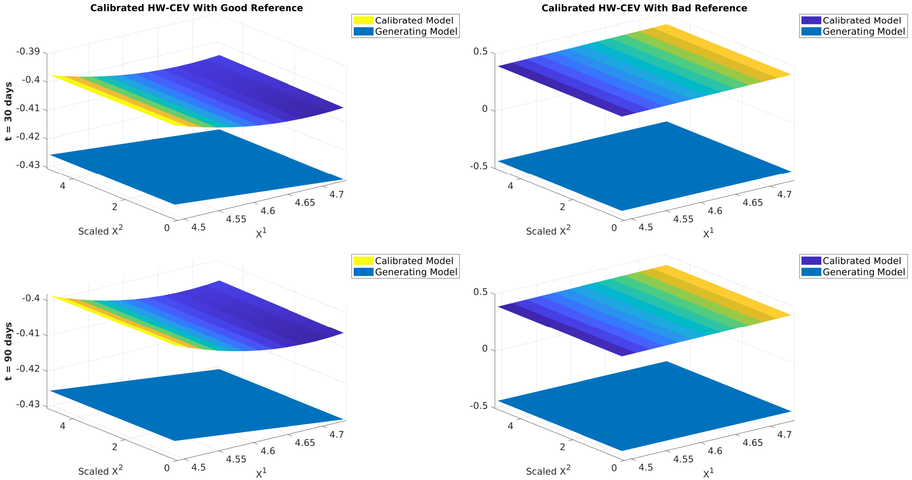


Figure: Comparison of ξ with the generating vol surface for a ‘good’ and a ‘bad’ reference model

Cost Function for Joint Calibration

Case II: Joint & simultaneous calibration exercise, zero dividends

Now assume we have no prior knowledge of the interest rate, our characteristics for the log-stock and short rate are therefore given by:

$$\alpha_t = \begin{bmatrix} X_t^2 - \frac{1}{2}(\beta_t)_{11} \\ (\alpha_t)_2 \end{bmatrix}, \qquad \beta_t = \begin{bmatrix} (\beta_t)_{11} & (\beta_t)_{12} \\ (\beta_t)_{12} & (\beta_t)_{22} \end{bmatrix}.$$

Define the convex set

$$\Gamma(t,x) = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \times \mathbb{S}_+^2 \;:\; \alpha_1 = x_2 - \frac{1}{2}\beta_{11} \right\}.$$

Define the cost function

$$F(\alpha,\beta) = \begin{cases} ||\alpha - \overline{\alpha}||_2^2 + ||\beta - \overline{\beta}||_{\text{Fro}}^2, & \text{if } (\alpha,\beta) \in \Gamma(t,x), \\ +\infty, & \text{otherwise.} \end{cases}$$

Where $\bar{\alpha}$ and $\bar{\beta}$ correspond to some reference model. We remark that we will calibrate with interest rate derivatives as well.

Joint Calibration HJB Equation

The dual formulation is similar to the sequential calibration, but with a different cost function. Let $G_i(x)$ denote the payoffs of instruments with maturity τ_i and market value u_i.

Joint Calibration Dual Formulation

Maximise

$$V = \sup_{\lambda \in \mathbb{R}^n} \lambda \cdot u - \phi(0, X_0)$$

Subject to

$$\begin{aligned}\partial_t \phi + \sup_{\alpha_2 \in \mathbb{R}, \beta \in \mathbb{S}^2_+} \Bigg\{ & \left(x_2 - \frac{1}{2}\beta_{11}\right) \partial_{x_1}\phi + \alpha_2 \partial_{x_2}\phi + \frac{1}{2}\beta_{11}\partial^2_{x_1 x_1}\phi \\ & + \frac{1}{2}\beta_{22}\partial^2_{x_2 x_2}\phi + \beta_{12}\partial^2_{x_1 x_2}\phi - ||\alpha - \overline{\alpha}||_2^2 - ||\beta - \overline{\beta}||^2_{\mathrm{Fro}} \Bigg\} \\ & - x_2\phi + \sum_{i=1}^n \lambda_i G_i(x)\delta_{\tau_i} = 0, \qquad \text{for } (t, x) \in [0, T] \times \mathbb{R}^2\end{aligned}$$

Joint Calibration Numerical Method

The numerical method is identical to the sequential calibration method, and we can analytically compute the supremum in the HJB equation.

We calibrate using Call options on the stock and Caplets on the interest rate with a fixed notional of $1,000 at 60 and 120 days.

The reference models are the CEV local volatility model with a Hull-White interest rate and a CIR interest rate. In both cases, the generating model was the same with shifted parameters. The parameters were given as follows:

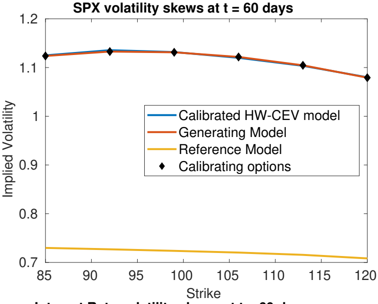
Generating		Reference	
σ	1.50	$\overline{\sigma}$	1.2
γ	0.95	$\overline{\gamma}$	0.89
a	0.05	$\overline{a}$	0.03
σ_r	0.04	$\overline{\sigma}_r$	0.02
ρ	-0.05	$\overline{\rho}$	-0.2

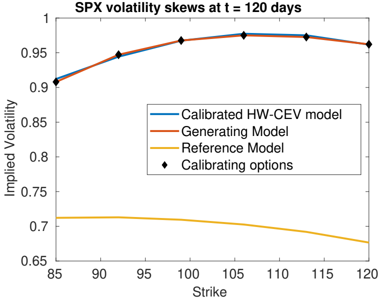
Table: CEV-Hull-White Parameters

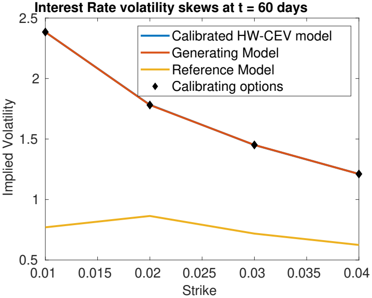
Generating		Reference	
σ	1.5	$\overline{\sigma}$	1.2
γ	0.95	$\overline{\gamma}$	0.89
b	0.03	$\overline{b}$	0.03
a	0.5	$\overline{a}$	0.4
σ_r	0.5	$\overline{\sigma}_r$	0.3
ρ	-0.4	$\overline{\rho}$	-0.2

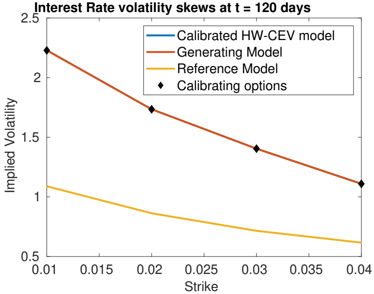
Table: CEV-CIR Parameters

Simulated Data Example — CEV-HW

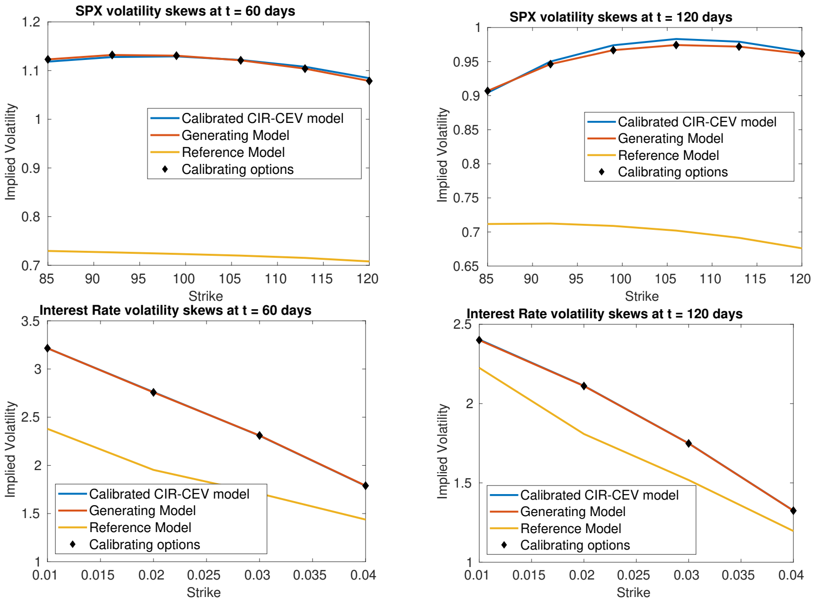








Simulated Data Example — CEV-CIR



Simulated Data Example — Plots of Characteristics

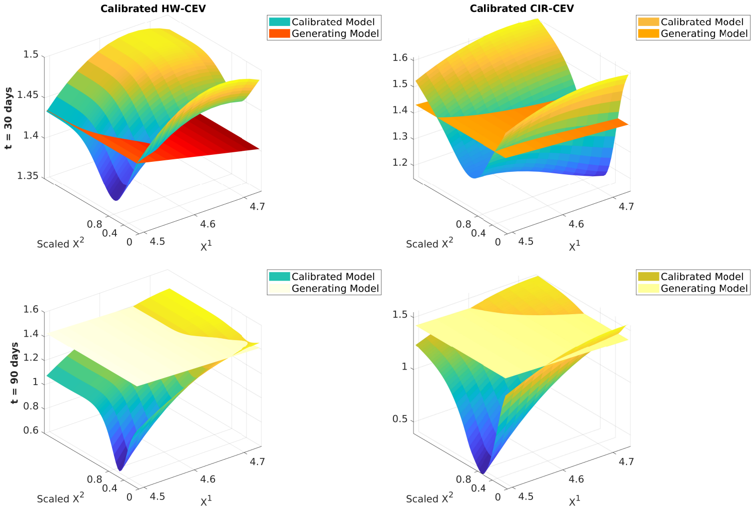


Figure: Compatison of β_{11} for the calibrated and generating model

Simulated Data Example — Plots of Characteristics

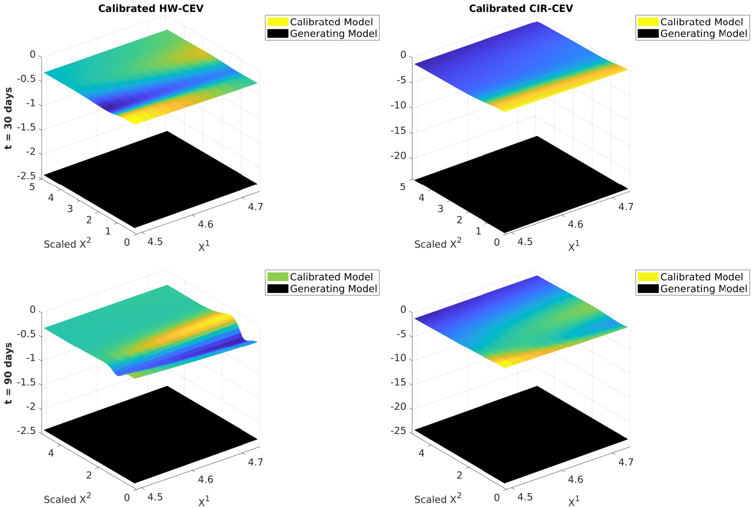


Figure: Comparison of β_{12} for the calibrated and generating model

Simulated Data Example — Plots of Characteristics

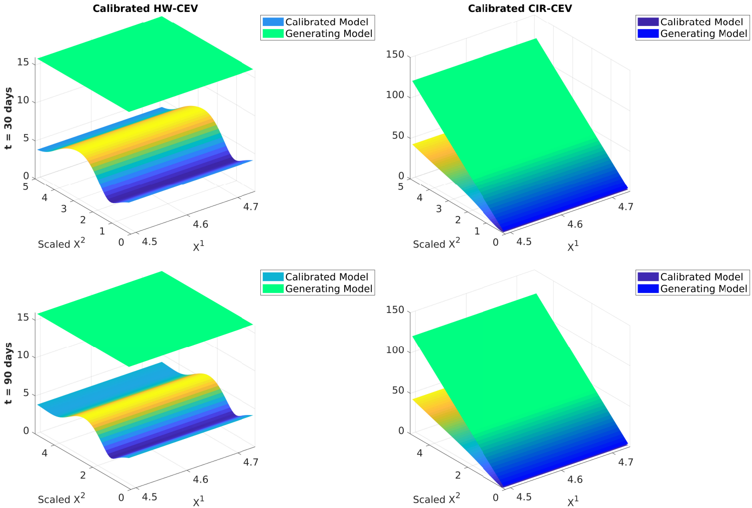


Figure: Comparison of β_{22} for the calibrated and generating model

Simulated Data Example — Plots of Characteristics

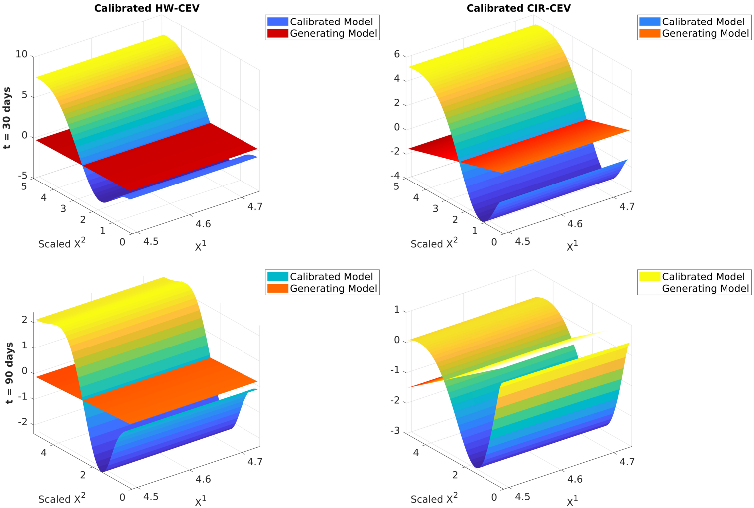


Figure: Comparison of α_2 for the calibrated and generating model

Robust pricing and hedging of American options via OT

Consider a market with stocks X and some European claims g which WLOG have initial prices of 0. We are allowed to trade X dynamically and g statically.
Let $\mathcal{Q} \subset \mathcal{P}$ be the set of possible "models", i.e., X is martingale, g has zero expectation, etc.

Robust hedging: model uncertainty

Consider a market with stocks X and some European claims g which WLOG have initial prices of 0. We are allowed to trade X dynamically and g statically.
Let $\mathcal{Q} \subset \mathcal{P}$ be the set of possible "models", i.e., X is martingale, g has zero expectation, etc.
Consider a European claim Z. Worst case model price:

$$\sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z.$$

Super-hedging price:

$$\pi(Z) := \inf\{x : \exists (q,h), \text{ s.t. } x + \int_0^1 q \cdot dX_t + h \cdot g \geq Z, \ \mathcal{Q}\text{-q.s.}\}.$$

Robust hedging: model uncertainty

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Super-hedging price:

$$\pi(Z) := \inf\{x : \exists (q,h),\ \text{s.t. } x + \int_0^1 q \cdot dX_t + h \cdot g \geq Z,\ \mathcal{Q}\text{-q.s.}\}.$$

It is easy to check that

$$\pi(Z) \geq \sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z.$$

Duality (equality) results are obtained in various settings by Denis & Martini (2006); Soner, Touzi & Zhang (2013); Neufeld & Nutz (2013); and Possamaï, Royer & Touzi (2013); Hou & O. (2018) and many more.

Hedging duality via Path-dependent OT

Theorem

Let $F : \Lambda \times \mathbb{S}^d \to \mathbb{R} \cup \{+\infty\}$ *satisfy some assumptions and* $F^*(t, \omega, \cdot)$ *be the convex conjugate of* $F(t, \omega, \cdot)$*. Define*

$$V := \sup_{\mathbb{P}} \inf_{h \in \mathbb{R}^m} \mathbb{E}^{\mathbb{P}}(-h \cdot g + Z) - \mathbb{E}^{\mathbb{P}} \int_0^1 F(\beta_t^{\mathbb{P}})\, dt,$$

$$\mathcal{V} := \inf_{h \in \mathbb{R}^m, \phi \in C^{1,2}(\Lambda)} \phi(0, X_0),$$

subject to $\phi(1, \cdot) \geq Z - h \cdot g$ *and* $\mathcal{D}_t \phi + F^* \left(\frac{1}{2} \nabla_x^2 \phi \right) \leq 0.$

Then $V = \mathcal{V}$*. Moreover, if* V *is finite, then the supremum is attained.*

Let $F(\beta)$ be 0 if $\beta \in D$ (volatility constraint), or ∞ otherwise. Then the dual is

$$\mathcal{V} = \inf_{h\in\mathbb{R}^m,\phi\in C^{1,2}(\Lambda)} \phi(0, X_0),$$

$$\text{subject to} \quad \phi(1,\cdot) \geq Z - h\cdot g \quad \text{and} \quad \mathcal{D}_t\phi + \sup_{\beta\in D} \frac{1}{2}\nabla_x^2\phi : \beta \leq 0.$$

Each ϕ is actually a super-hedge. For every $\mathbb{P} \in \mathcal{Q}$

$$\begin{aligned} Z - h\cdot g - \phi(0, X_0) &\leq \phi(1, X) - \phi(0, X_0) \\ &= \int_0^1 (\mathcal{D}_t\phi + \frac{1}{2}\beta^{\mathbb{P}} : \nabla_x^2\phi)dt + \nabla_x\phi\cdot dX_t, \quad \mathbb{P}\text{-a.s.} \\ &\leq \int_0^1 \nabla_x\phi\cdot dX_t. \end{aligned}$$

Hence $\phi(0, X_0) \geq \pi(Z)$. Since this works for all ϕ satisfying (92), it implies

$$\mathcal{V} = \inf_{\phi\in C_0^{1,2}(\Lambda),(92)} \phi(0, X_0) \geq \pi(Z) \geq \sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z = V = \mathcal{V}.$$

Robust hedging American options

Let Z be an American-style claim. Worst case model price:

$$\sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot).$$

Super-hedging price:

$$\pi^A(Z) := \inf\{x : \exists (p, q, h) \text{ s.t. } x + \int_0^\tau p \cdot dX_t + \int_\tau^1 q^\tau \cdot dX_t + hg \geq Z_\tau, \mathcal{Q}^D\text{-q.s.}, \forall \tau \in \mathcal{T}\}.$$

Again, it is easy to check $\pi^A(Z) \geq \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot)$.

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When the set of statically traded European options is non-empty, there may be a duality gap, which can be eliminated by enlarging the probability space.
In discrete time, various duality results for American options are obtained by Dolinsky (2014); Hobson & Neuberger (2017); Bayraktar & Zhou (2017); Aksamit, Deng, O. & Tan (2019); and more. Some relevant works in continuous time include Herrmann & Stebegg (2017); Tiplea (2019); Grigorova, Quenez & Sulem (2021) etc.

Overview

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In the case where there is no statically traded European options g.

$$\bar{\pi}(Z) = \pi^A(Z) \geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \bar{\pi}(Z),$$

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot).$$

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When g does exist, then we have to introduce a second enlarged space $\hat{\Omega}$ which includes the price process of g as another martingale.

$$\bar{\pi}_g(Z) = \pi_g^A(Z) \geq \widehat{\pi}^A(Z) = \bar{\widehat{\pi}}(Z) \geq \sup_{\bar{\widehat{\mathbb{P}}} \in \bar{\widehat{\mathcal{Q}}}} \mathbb{E}^{\bar{\widehat{\mathbb{P}}}} Z \geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}_g} \mathbb{E}^{\bar{\mathbb{P}}} Z = \bar{\pi}_g(Z),$$

$$\sup_{\bar{\widehat{\mathbb{P}}} \in \bar{\widehat{\mathcal{Q}}}} \mathbb{E}^{\bar{\widehat{\mathbb{P}}}} Z = \sup_{\widehat{\tau} \in \widehat{\mathcal{T}}, \widehat{\mathbb{P}} \in \widehat{\mathcal{Q}}} \mathbb{E}^{\widehat{\mathbb{P}}} Z(\widehat{\tau}, \cdot).$$

Continuous time issues

We mostly focus on the case where there is no g.
Pricing hedging duality for European options is known in continuous time, and naturally extends to the enlarged space.
The equality $\bar{\pi}(Z) = \pi^A(Z)$ can also be argued in mostly the same way.
However, the equality

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot),$$

creates difficulties in continuous time. Possible approaches include approximating with discrete time, Doob-Meyer type decomposition of non-linear Snell envelopes, reflected 2BSDEs, etc.

Enlarged space

The original space for our model is $\Omega := C([0,1];\mathbb{R}^d)$ with canonical process X. We enlarge it to $\bar{\Omega} := \Theta \times \Omega$ where

$$\Theta := \{\vartheta \in C([0,1],\mathbb{R}) : \vartheta_t = \theta \wedge t, \text{ for some } \theta \in [0,1]\}.$$

Θ is isometric to $[0,1]$.

Most aspects of Ω can be naturally extended to $\bar{\Omega}$, include semimartingale measures (since ϑ semimartingale with characteristics $(\mathbb{1}(t \leq \theta), 0))$. E.g., we define $\bar{\mathcal{Q}}$ to be the set of measures under which X is a martingale.

Also define the "stopped paths" of $\bar{\Omega}$, by

$$\bar{\Lambda} := \{(t, \bar{\omega}_{\cdot\wedge t}) : t \in [0,1], \bar{\omega} \in \bar{\Omega}\}.$$

So elements of $\bar{\Lambda}$ are $(t, \bar{\omega}_{\cdot\wedge t}) = (t, \vartheta_{\cdot\wedge t}, \omega_{\cdot\wedge t}) = (t, \theta \wedge t, \omega_{\cdot\wedge t})$.
Functional Itô calculus and PPDEs can be extended in the same way.
The path-dependent optimal transport duality results (Guo and Loeper (2021)) can be applied here.

Theorem

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}^D} \mathbb{E}^{\bar{\mathbb{P}}} f = \inf_{\phi \in C_0^{1,1,2}(\bar{\Lambda})} \phi(0,0,X_0),$$

subject to $\phi(1,\cdot,\cdot) \geq f$ *and* $\mathcal{D}_t\phi + \mathbb{1}(t \leq \theta)\nabla_\theta \phi + \sup_{\beta \in D} \frac{1}{2}\beta : \nabla_x^2 \phi \leq 0.$

European duality on the enlarged space

Theorem

$$\sup_{\bar{\mathbb{P}}\in\bar{\mathcal{Q}}^D} \mathbb{E}^{\bar{\mathbb{P}}} f = \inf_{\phi\in C_0^{1,1,2}(\bar{\Lambda})} \phi(0,0,X_0),$$

subject to $\phi(1,\cdot,\cdot) \geq f$ *and* $\mathcal{D}_t\phi + \mathbb{1}(t\leq\theta)\nabla_\theta\phi + \sup_{\beta\in D}\frac{1}{2}\beta : \nabla_x^2\phi \leq 0.$

By the functional Itô formula, for each ϕ and $\bar{\mathbb{P}} \in \bar{\mathcal{Q}}^D$, the following holds $\bar{\mathbb{P}}$-a.s.

$$\begin{aligned} f - \phi(0,0,X_0) &\leq \phi(1,\cdot,\cdot) - \phi(0,0,X_0) \\ &= \int_0^1 (\mathcal{D}_t\phi + \mathbb{1}(t\leq\theta)\nabla_\theta\phi + \frac{1}{2}\beta^{\mathbb{P}} : \nabla_x^2\phi)dt + \nabla_x\phi\cdot dX_t \\ &\leq \int_0^1 \nabla_x\phi\cdot dX_t. \end{aligned}$$

Hence $\phi(0,0,X_0) \geq \bar{\pi}(f)$. Since this holds for all ϕ, it implies

$$\sup_{\bar{\mathbb{P}}\in\bar{\mathcal{Q}}^D} \mathbb{E}^{\bar{\mathbb{P}}} f \geq \bar{\pi}(f).$$

BUT we still need to address $\sup_{\bar{\mathbb{P}}\in\bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau\in\mathcal{T},\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau,\cdot)$.

From random times to stopping times

Lemma

(a) For every $\mu \in \mathcal{P}(\bar{\Omega})$*, there exists an increasing and adapted* A *with* $A_0 = 0$ *and* $A_1 = 1$*, and* $\mathbb{P} \in \mathcal{P}(\Omega)$ *with* $\mathbb{P} \ll \mu_\Omega$*, such that for every (non-anticipative)* $\psi \in L^\infty(\Lambda)$*,*

$$\mu(\psi(\theta, \omega_{\cdot\wedge\theta})) = \mathbb{E}^{\mathbb{P}} \int_0^1 \psi(t, \omega_{\cdot\wedge t})\, dA_t.$$

(b) For every $\mu \in \mathcal{P}(\bar{\Omega})$*, there exists a family of true stopping times* τ_r *and probability measures* $\mathbb{P}^r$*, indexed by* $r \in [0,1]$*, such that for every* $\eta \in L^\infty(\bar{\Omega})$*,*

$$\mu(\eta(\theta, \omega)) = \int_0^1 \mathbb{E}^{\mathbb{P}^r} \eta(\theta = \tau_r, \omega)\, dr.$$

(c) For any $a \in [0,1]$ *and any bounded and* $\mathcal{F}_{\tau_a}$*-measurable function* γ*,*

$$\int_a^1 \mathbb{E}^{\mathbb{P}^r} \gamma\, dr = (1-a)\mathbb{E}^{\mathbb{P}} \gamma.$$

Roughly speaking, we obtain r by disintegrating μ according to the value of A.

Disintegrating martingale measures

Lemma

Suppose that X *is a martingale under* $\mu \in \mathcal{P}(\bar{\Omega})$ *with characteristic* $(0, \beta)$*. Then there exists a family of true stopping times* τ_r *and probability measures* $\mathbb{P}^r \in \mathcal{P}(\Omega)$*, indexed by* $r \in [0, 1]$*, such that for every* $\eta \in L^\infty(\bar{\Omega})$*,*

$$\mu(\eta(\theta, \omega)) = \int_0^1 \mathbb{E}^{\mathbb{P}^r} \eta(\theta = \tau_r, \cdot)\, dr.$$

Moreover, each $\mathbb{P}^r \in \mathcal{P}(\Omega)$ *is a martingale measure with characteristic* $(0, \beta(t, t \wedge \tau_r(\omega), \omega_{\cdot \wedge t}))$*.*

Corollary

For any $Z \in L^\infty(\bar{\Omega})$*, and any* $E \subseteq \Omega$*,*

$$\sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}, \mathbb{P}(E)=1} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot) = \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}, \bar{\mathbb{P}}(\Theta \times E)=1} \mathbb{E}^{\bar{\mathbb{P}}} Z.$$

Conclusion and future research

Conclusion:

- We consider Semimartingale OT perspective on pricing and hedging
- This includes European options, path-dependent options and now also American options
- We develop generic approach to Calibration via OT
- We use it to tackle difficult joint calibration problems: SPX options + VIX futures + VIX options prices; interest rates and SPX options
- Numerical proof-of-concept results

Future research:

- Improving computational efficiency and exploring applications in higher dimensions
  - Deep PDE solvers (see, e.g., Han et al. (2020))
  - Neural SDE (see, e.g., Cuchiero et al. (2020))
- OT Calibration to American options

Thank you!