

# OT Methodology for non-parametric calibration (complements to Lecture 6)

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based on joint works with  
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Transfer material from one site to another while minimising transportation costs.

- Monge (1781), Kantorovich (1948): Monge-Kantorovich problem
- Benamou & Brenier (2000): continuous-time formulation

## Optimal transport, continuous-time formulation

Minimising the cost function  $F$  under given initial density  $\rho_0$  and final density  $\rho_1$

$$\inf_{\rho, v} \int_{\mathbb{R}^d} \int_0^1 \rho(t, x) F(v(t, x)) dt dx,$$

subject to the continuity equation

$$\partial_t \rho(t, x) + \nabla \cdot (\rho(t, x) v(t, x)) = 0,$$

and the initial and final distributions

$$\rho(0, x) = \rho_0, \quad \rho(1, x) = \rho_1.$$

Tan & Touzi (2013) (also Mikami & Thieullen (2006), Huesmann & Trevisan (2017), Backhoff et al. (2017)): Consider probability measures  $\mathbb{P}$  such that  $X$  is a semimartingale,

$$dX_t = \beta_t^{\mathbb{P}} dt + (\alpha_t^{\mathbb{P}})^{1/2} dW_t^{\mathbb{P}}.$$

## Stochastic optimal transport problem

We want to minimise

$$V(\mu_0, \mu_1) = \inf_{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)} \mathbb{E}^{\mathbb{P}} \int_0^1 F(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}) dt,$$

where  $\mathcal{P}(\mu_0, \mu_1)$  contains probability measures satisfying

$$\mathbb{P} \circ X_0^{-1} = \mu_0, \quad \mathbb{P} \circ X_1^{-1} = \mu_1.$$

Note that the cost function  $F$  is convex and may depend on  $(t, X)$  as well.

Tan & Touzi (2013) established the following duality result

### Dual formulation

The primal problem is equivalent to

$$V(\mu_0, \mu_1) = \sup_{\phi_1} \int \phi_1 d\mu_1 - \phi_0 d\mu_0,$$

where

$$\phi_0(x) := \sup_{\mathbb{P} \in \mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}} \left( \phi_1(X_1) - \int_0^1 F(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}) dt \right).$$

and for  $F_t = F(t, X_t, \alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}})$  characterised  $\phi_0$  via PDEs.

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and for  $F_t = F(t, X_t, \alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}})$  characterised  $\phi_0$  via PDEs.

Guo and Loeper (2018) extended this to **path dependent constraints and cost**.

Path-dependent PDEs & functional Itô used to describe the dual.

SOT induces a **projection** onto a subset of (semi)-martingales.

Use for **calibration**:

- Gather market data  $\mathcal{G}$
- Fix a favourite reference model  $\bar{\mathbb{P}}$
- Consider a cost  $F$  given by

$$F(\mathbb{P}) = \begin{cases} \text{dist}(\mathbb{P}, \bar{\mathbb{P}}) & \text{if } \mathbb{P} \text{ is calibrated to } \mathcal{G}, \\ +\infty & \text{otherwise.} \end{cases}$$

- ensuring **convexity** to get **duality**
- Solve the dual via a non-linear (P)PDE
- $\mathbb{P}^*$  recovered via  $\nabla F^*(\dots)$ .

- 1 Introduction
- 2 SPX & VIX Modelling
  - Calibration framework
- 3 Numerical method
- 4 Numerical results
- 5 Calibrating Fixed Income Models
- 6 Robust pricing and hedging of Americans via OT
- 7 Conclusion

# SPX & VIX CALIBRATION



- S&P 500 Index (SPX): a stock market index that measures the stock performance of 500 large companies listed in the US stock market.
- CBOE Volatility Index (VIX): a volatility index that measures the market's expectation of the volatility of SPX over the following 30 days.

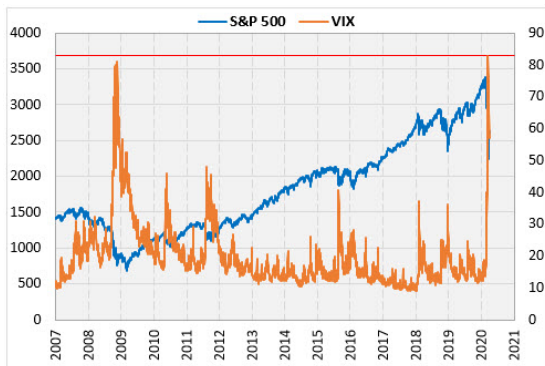


Figure: Historical SPX and VIX data. (Source: Schaeffer's Investment Research)

- VIX futures and options are very popular hedging instruments.  
e.g., Szado (2009) shows that VIX call options are better than S&P 500 put options as a hedging instrument against the financial crisis in 2008.
- An arbitrage argument (Guyon 2020): existence of a liquid market  
⇒ need for models that jointly calibrate to the option prices of SPX and VIX  
⇒ avoid arbitrage between financial institutions (or even within the same institution)
- Joint calibration problem: build a (stochastic volatility) model that jointly calibrates to the prices of SPX options, VIX futures and VIX options.
- Very challenging problem, especially for short maturities.

### Previous works:

- Continuous-time diffusion models (without jump):
  - Gatheral (2008): double CEV model
  - Goutte–Ismail–Pham (2017): Regime-switching Heston model
  - Fouque–Saporito (2018): Heston stochastic vol-of-vol
- Continuous-time jump-diffusion models: many works including
  - Cont–Kokholm (2013), Lian–Zhu (2013), Baldeaux–Badran (2014), Kokholm–Stisen (2015), Pacati–Pompa–Reno (2018), ...

However, even with jumps, these models have yet to achieve an exact fit.

### Recent works:

- Guyon (2020): nonparametric discrete-time model calibrated by martingale optimal transport
- Gatheral–Jusselin–Rosenbaum (2020): (parametric) quadratic rough Heston model (no efficient calibration method yet)
- $\Rightarrow$  This work: nonparametric continuous-time model calibrated by semimartingale optimal transport

*Assumption: zero interest rates & dividends.*

Let  $S_t$  be the SPX price:

$$S_t = S_0 + \int_0^t \sigma_s S_s dW_s.$$

Consider a time grid  $0 < t_0 < t_1 < \dots < t_n = T$  and an annualisation factor  $AF$ , e.g., if  $t_i$  corresponds to daily observations, then  $AF = 100^2 \times 252/n$ .

The *realised variance* of  $S_t$  during  $[t_0, T]$ :

$$AF \sum_{i=1}^n \left( \log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \rightarrow \frac{100^2}{T - t_0} \int_{t_0}^T \sigma_t^2 dt, \quad a.s.$$

The VIX index at  $t_0$ :

$$VIX(t_0, T) = \sqrt{\mathbb{E} \left( \frac{100^2}{T - t_0} \int_{t_0}^T \sigma_t^2 dt \mid \mathcal{F}_{t_0} \right)}$$

Underlying assets:

$$S_t = S_0 + \int_0^t \sigma_s S_s dW_s$$
$$VIX(t_0, T) = \sqrt{\mathbb{E} \left( \frac{100^2}{T - t_0} \int_{t_0}^T \sigma_t^2 dt \mid \mathcal{F}_{t_0} \right)}$$

Calibrating instruments:

$$\begin{aligned} \text{SPX calls:} \quad & u^{SPX,c} = \mathbb{E}((S_T - K)^+) \\ \text{SPX puts:} \quad & u^{SPX,p} = \mathbb{E}((K - S_T)^+) \\ \text{VIX futures:} \quad & u^{VIX,f} = \mathbb{E}(VIX_{t_0}) \\ \text{VIX calls:} \quad & u^{VIX,c} = \mathbb{E}((VIX_{t_0} - K)^+) \\ \text{VIX puts:} \quad & u^{VIX,p} = \mathbb{E}((K - VIX_{t_0})^+) \end{aligned}$$

Many previous works involve modelling  $(S_t, \sigma_t)$  or  $(S_t, \sigma_t^2)$

⇒ the term  $VIX$  is a square root of conditional expectation

⇒ numerically difficult to compute the prices of VIX futures and VIX options.

Consider a two dimensional stochastic process  $X = (X^1, X^2)$ , let  $X^1$  be the logarithm of  $S_t$ :

$$X_t^1 := \log S_t = X_0^1 - \frac{1}{2} \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s.$$

Let  $X^2$  be a half of the expected forward quadratic variation of  $X^1$  over  $[t, T]$  observed at  $t$ :

$$X_t^2 = \mathbb{E} \left( \frac{1}{2} \int_t^T \sigma_s^2 ds \mid \mathcal{F}_t \right).$$

Calibrating instruments: for  $\tau \leq T$ ,

SPX calls:  $u^{SPX,c} = \mathbb{E}((\exp(X_\tau^1) - K)^+) =: \mathbb{E}(G^{SPX,c}(X_\tau))$

SPX puts:  $u^{SPX,p} = \mathbb{E}((K - \exp(X_\tau^1))^+) =: \mathbb{E}(G^{SPX,p}(X_\tau))$

VIX futures:  $u^{VIX,f} = \mathbb{E}(100\sqrt{2X_{t_0}^2/(T-t_0)}) =: \mathbb{E}(G^{VIX,f}(X_{t_0}))$

VIX calls:  $u^{VIX,c} = \mathbb{E}((100\sqrt{2X_{t_0}^2/(T-t_0)} - K)^+) =: \mathbb{E}(G^{VIX,c}(X_{t_0}))$

VIX puts:  $u^{VIX,p} = \mathbb{E}((K - 100\sqrt{2X_{t_0}^2/(T-t_0)})^+) =: \mathbb{E}(G^{VIX,p}(X_{t_0}))$

All payoffs depend on only the marginal distributions of  $X$  at fixed times  
 $\Rightarrow$  suitable for the calibration framework via optimal transport.

The Heston model:

$$\begin{aligned} dS_t &= \sqrt{\nu_t} S_t dW_t^1, \\ d\nu_t &= -\kappa(\nu_t - \theta) dt + \omega\sqrt{\nu_t} dW_t^2, \\ \langle dW^1, dW^2 \rangle_t &= \eta dt. \end{aligned}$$

We can derive that

$$X_t^2 = \mathbb{E} \left( \frac{1}{2} \int_t^T \nu_s ds \mid \mathcal{F}_t \right) = \frac{1 - e^{-\kappa(T-t)}}{2\kappa} (\nu_t - \theta) + \frac{1}{2} \theta (T - t).$$

Define  $A(t, \kappa) := (1 - e^{-\kappa(T-t)})/\kappa$  and  $\nu(t, X_t^2, \kappa, \theta) := A(t, \kappa)^{-1} (2X_t^2 - \theta(T - t)) + \theta$ , then the Heston model in terms of  $(X^1, X^2)$  is

$$\begin{aligned} dX_t^1 &= -\frac{1}{2} \nu(t, X_t^2, \kappa, \theta) dt + \sqrt{\nu(t, X_t^2, \kappa, \theta)} dW_t^1, \\ dX_t^2 &= -\frac{1}{2} \nu(t, X_t^2, \kappa, \theta) dt + \frac{1}{2} A(t, \kappa) \omega \sqrt{\nu(t, X_t^2, \kappa, \theta)} dW_t^2, \\ \langle dW_t^1, dW_t^2 \rangle &= \eta dt. \end{aligned}$$

Consider probability measures  $\mathbb{P}$  under which  $X$  is a semimartingale:

$$dX_t = \alpha_t^{\mathbb{P}} dt + (\beta_t^{\mathbb{P}})^{\frac{1}{2}} dW_t^{\mathbb{P}}.$$

## Semimartingale optimal transport with discrete constraints

Minimise

$$\inf_{\mathbb{P} \in \mathcal{P}(X_0, \tau, G, c)} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) dt,$$

where  $\mathcal{P}(X_0, \tau, G, c)$  contains probability measures  $\mathbb{P}$  satisfying

$$\mathbb{P} \circ X_0^{-1} = \delta_{X_0} \quad \text{and} \quad \mathbb{E}^{\mathbb{P}} G_i(X_{\tau_i}) = c_i, \quad i = 1, \dots, m.$$

Note that the cost function  $F$  is convex in  $(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ . It may depend on  $(t, X)$  as well.



The cost function plays a regularisation role to ensure that  $X$  has the correct dynamics.

We want  $X$  to have the following dynamics:

$$X_t^1 = X_0^1 - \frac{1}{2} \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s, \quad X_t^2 = \mathbb{E} \left( \frac{1}{2} \int_t^T \sigma_s^2 ds \mid \mathcal{F}_t \right).$$

The above dynamics can be captured by  $\mathbb{P}$  such that

$$(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) = \left( \left[ \begin{array}{c} -\frac{1}{2}\sigma_t^2 \\ -\frac{1}{2}\sigma_t^2 \end{array} \right], \left[ \begin{array}{cc} \sigma_t^2 & (\beta_t)_{12} \\ (\beta_t)_{12} & (\beta_t)_{22} \end{array} \right] \right), \quad 0 \leq t \leq T,$$

where  $(\beta_t)_{12} = d\langle X^1, X^2 \rangle_t / dt$  and  $(\beta_t)_{22} = d\langle X^2 \rangle_t / dt$  and with the additional property that  $X_T^2 = 0$   $\mathbb{P}$ -a.s.

Given  $\bar{\beta}$ , a reference for  $\beta$ , define the cost function:

$$F(\alpha, \beta) = \begin{cases} \sum_{i,j=1}^2 (\beta_{ij} - \bar{\beta}_{ij})^2 & \text{if } \alpha_1 = \alpha_2 = -\frac{1}{2}\beta_{11}, \\ +\infty & \text{otherwise.} \end{cases}$$

The additional property  $X_T^2 = 0$ ,  $\mathbb{P}$ -a.s. and the prices of calibrating instruments are imposed on  $X$  as discrete constraints  $\Rightarrow$  exact calibration

We want to calibrate  $X$  to:

- $m$  number of SPX options with payoffs  $G = (G_1, \dots, G_m)$ , maturities  $\tau \in (0, T]^m$  and prices  $u^{SPX} \in \mathbb{R}_+^m$ , e.g.,

$$\mathbb{E}^{\mathbb{P}} G_i(X_{\tau_i}) = u_i^{SPX}, \quad i = 1, \dots, m,$$

- a VIX futures with payoff  $J(x) = 100\sqrt{2x_2/(T - t_0)}$ , maturity  $t_0$  and price  $u^{VIX,f} \in \mathbb{R}$ , e.g.,

$$\mathbb{E}^{\mathbb{P}} J(X_{t_0}) = u^{VIX,f},$$

- $n$  number of VIX options with payoffs  $H = (H_1, \dots, H_n)$ , maturity  $t_0$  and prices  $u^{VIX} \in \mathbb{R}_+^m$ , e.g.,

$$\mathbb{E}^{\mathbb{P}} (H_i \circ J)(X_{t_0}) = u_i^{VIX}, \quad i = 1, \dots, n,$$

- a contract with payoff  $\xi(x) = 1 - \exp(-(x_2)^2)$ , maturity  $T$  and zero price, e.g.,

$$\mathbb{E}^{\mathbb{P}} \xi(X_T) = 0.$$

The last calibrating instrument ensures that  $X_T^2 = 0$ ,  $\mathbb{P}$ -a.s. Since its price is always zero, we call it a *singular contract*.

# Framework — Reformulation of the joint calibration problem

For simplicity, we represent all the discrete constraints by

$$\mathbb{E}^{\mathbb{P}} \mathcal{G}_i(X_{\mathcal{T}_i}) = c_i, \quad i = 1, \dots, m + n + 2,$$

where

$$\mathcal{G} = \left( \underbrace{(G_1, \dots, G_m)}_{m \text{ SPX options}}, \underbrace{(H_1 \circ J, \dots, H_n \circ J)}_{n \text{ VIX options}}, \underbrace{J}_{\text{VIX futures}}, \underbrace{\xi}_{\text{singular contract}} \right),$$

and  $\mathcal{T}$  and  $c$  are defined in a similar manner.

Define a set of the probability measures  $\mathcal{P}_{joint}$  such that

$$\mathcal{P}_{joint} := \{ \mathbb{P} : \mathbb{P} \circ X_0^{-1} = \delta_{X_0} \text{ and } \mathbb{E}^{\mathbb{P}} \mathcal{G}_i(X_{\mathcal{T}_i}) = c_i, \quad i = 1, \dots, m + n + 2 \}$$

## The joint calibration problem

$$\text{Minimise} \quad V := \inf_{\mathbb{P} \in \mathcal{P}_{joint}} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) dt.$$

If we find an optimal solution  $\tilde{\mathbb{P}}$  and  $V < +\infty$ , then we have a well-calibrated model

$$X_t = X_0 + \int_0^t \alpha_s^{\tilde{\mathbb{P}}} ds + \int_0^t (\beta_s^{\tilde{\mathbb{P}}})^{\frac{1}{2}} dW_s^{\tilde{\mathbb{P}}}.$$

Markovian projection: use a (Markovian) diffusion process mimic an Itô process by matching its marginals at fixed times. (Gyöngy (1986) and Brunick–Shreve (2013))

**Lemma (Figalli (2008) and Trevisan (2016))**

Let  $\rho_t^{\mathbb{P}} = \mathbb{P} \circ X_t^{-1}$  be the marginal distribution of  $X_t$  under  $\mathbb{P}$ ,  $t \leq T$ , then  $\rho^{\mathbb{P}}$  is a weak solution to the Fokker–Planck equation:

$$\begin{cases} \partial_t \rho_t^{\mathbb{P}} + \nabla_x \cdot (\rho_t^{\mathbb{P}} \mathbb{E}_{t,x}^{\mathbb{P}} \alpha_t^{\mathbb{P}}) - \frac{1}{2} \sum_{i,j} \partial_{ij} (\rho_t^{\mathbb{P}} (\mathbb{E}_{t,x}^{\mathbb{P}} \beta_t^{\mathbb{P}})_{ij}) = 0 & \text{in } [0, T] \times \mathbb{R}^2, \\ \rho_0^{\mathbb{P}} = \delta_{X_0} & \text{in } \mathbb{R}^2. \end{cases}$$

Moreover, there exists another probability measure  $\mathbb{P}'$  under which  $X$  has the same marginals,  $\rho^{\mathbb{P}'} = \rho^{\mathbb{P}}$ , and is a Markov process solving

$$dX_t = \alpha^{\mathbb{P}'}(t, X_t) dt + (\beta^{\mathbb{P}'}(t, X_t))^{\frac{1}{2}} dW_t^{\mathbb{P}'}, \quad 0 \leq t \leq T,$$

where  $W^{\mathbb{P}'}$  is a  $\mathbb{P}'$ -Brownian motion,  $\alpha^{\mathbb{P}'}(t, x) = \mathbb{E}_{t,x}^{\mathbb{P}} \alpha_t^{\mathbb{P}}$  and  $\beta^{\mathbb{P}'}(t, x) = \mathbb{E}_{t,x}^{\mathbb{P}} \beta_t^{\mathbb{P}}$ .

Notation:  $\mathbb{E}_{t,x}^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}(\cdot \mid X_t = x)$ .

Let  $\mathcal{P}_{joint}^{loc}$  be a subset of  $\mathcal{P}_{joint}$  such that, under any  $\mathbb{P} \in \mathcal{P}_{joint}^{loc}$ ,  $X$  is a Markov process that solves

$$dX_t = \alpha^{\mathbb{P}}(t, X_t)dt + (\beta^{\mathbb{P}}(t, X_t))^{\frac{1}{2}} dW_t^{\mathbb{P}}, \quad 0 \leq t \leq T,$$

and  $X$  is fully calibrated to the calibrating instruments.

## Proposition

$$V = \inf_{\mathbb{P} \in \mathcal{P}_{joint}^{loc}} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) dt = \inf_{\mathbb{P} \in \mathcal{P}_{joint}^{loc}} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}}(t, X_t), \beta_t^{\mathbb{P}}(t, X_t)) dt$$

Proof: “ $\geq$ ” follows by convexity of  $F$  via Jensen’s inequality:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) dt &= \mathbb{E}^{\mathbb{P}} \int_0^T \left( \mathbb{E}_{t,x}^{\mathbb{P}} F(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) \right) dt \\ &\geq \mathbb{E}^{\mathbb{P}} \int_0^T F(\mathbb{E}_{t,x}^{\mathbb{P}} \alpha_t^{\mathbb{P}}, \mathbb{E}_{t,x}^{\mathbb{P}} \beta_t^{\mathbb{P}}) dt. \end{aligned}$$

“ $\leq$ ” is clear since  $\mathcal{P}_{joint}^{loc} \subset \mathcal{P}_{joint}$ .

The problem can be made convex by introducing  $A = \rho\alpha$  and  $B = \rho\beta$ , since

$$\rho F(\alpha, \beta) = \rho F\left(\frac{A}{\rho}, \frac{B}{\rho}\right) = \sup_{r + F^*(a, b) \leq 0} \{\rho r + A \cdot a + B : b\},$$

is convex in  $(\rho, A, B)$ , where  $F^*(a, b) = \sup_{\alpha, \beta} \{a \cdot \alpha + b : \beta - F(\alpha, \beta)\}$  is the convex conjugate of  $F$ , and  $B : b = \text{Tr}(Bb)$ .

## PDE formulation

Minimise

$$V = \inf_{\rho, A, B} \int_0^T \int_{\mathbb{R}^2} \rho F(A/\rho, B/\rho) dx dt,$$

subject to constraints

$$\partial_t \rho + \nabla_x \cdot A - \frac{1}{2} \sum_{i, j} \partial_{ij} B_{ij} = 0,$$

$$\int_{\mathbb{R}^2} \mathcal{G}_i \rho(t, \cdot) dx = c_i, \quad i = 1, \dots, m + n + 2$$

$$\rho(0, \cdot) = \delta_{X_0}.$$

Introducing Lagrange multipliers  $\phi \in C_c^\infty([0, T] \times \mathbb{R}^2)$  and  $\lambda \in \mathbb{R}^{m+n+2}$ , the problem can be formulated as:

$$\begin{aligned}
 V &= \inf_{\rho, A, B} \sup_{\phi, \lambda} \left\{ \int_0^T \int_{\mathbb{R}^2} \left( \rho F \left( \frac{A}{\rho}, \frac{B}{\rho} \right) - (\partial_t \phi \rho + \nabla_x \phi \cdot A + \frac{1}{2} \nabla_x^2 \phi : B) - \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i) \rho \right) dx dt \right. \\
 &\qquad \qquad \qquad \left. + \lambda \cdot c - \phi(0, X_0) \right\} \\
 &= \sup_{\phi, \lambda} \inf_{\rho, A, B} \left\{ \underbrace{\int_0^T \int_{\mathbb{R}^2} \left( \rho F \left( \frac{A}{\rho}, \frac{B}{\rho} \right) - (\partial_t \phi \rho + \nabla_x \phi \cdot A + \frac{1}{2} \nabla_x^2 \phi : B) - \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i) \rho \right) dx dt}_{\text{objective of the primal}} \right. \\
 &\qquad \qquad \qquad \left. + \underbrace{\lambda \cdot c - \phi(0, X_0)}_{\text{objective of the dual}} \right\}
 \end{aligned}$$

The interchange of inf and sup can be formally established by the Fenchel–Rockafellar duality theorem.

By applying the Fenchel–Rockafellar duality theorem and a smoothing technique:

## Dual formulation

Maximise

$$V = \sup_{\lambda \in \mathbb{R}^{m+n+2}} \lambda \cdot c - \phi(0, X_0),$$

where  $\phi$  is the viscosity solution to the HJB equation:

$$\partial_t \phi + F^*(\nabla_x \phi, \frac{1}{2} \nabla_x^2 \phi) = - \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i),$$

with the terminal condition  $\phi(T, \cdot) = 0$ . If the supremum is attained and the associated solution to the HJB equation is  $\tilde{\phi} \in BV([0, T], C_b^2(\mathbb{R}^2))$ , then an optimal  $(\alpha, \beta)$  of the PDE formulation can be found by

$$(\alpha, \beta) = \nabla F^*(\nabla_x \tilde{\phi}, \frac{1}{2} \nabla_x^2 \tilde{\phi}).$$

Note:  $F^*(a, b) = \sup_{\alpha, \beta} \{a \cdot \alpha + b : \beta - F(\alpha, \beta)\}$  is the convex conjugate of  $F$ .



Given  $\lambda \in \mathbb{R}^{m+n+2}$  with the associated solution  $\phi^\lambda$ , let  $\mathbb{P}(\lambda)$  be the probability measure under which  $X$  has  $(\alpha, \beta) = (\alpha^\lambda, \beta^\lambda) := \nabla F^*(\nabla_x \phi^\lambda, \frac{1}{2} \nabla_x^2 \phi^\lambda)$ .

Define

$$L(\lambda) := \lambda \cdot c - \phi^\lambda(0, X_0).$$

The gradients of the objective can be formulated as the difference between the market prices and the model prices:

$$\partial_{\lambda_i} L(\lambda) = \underbrace{c_i}_{\text{market price}} - \underbrace{\mathbb{E}^{\mathbb{P}(\lambda)} \mathcal{G}_i(X_{\mathcal{T}_i})}_{\text{model price}}, \quad i = 1, \dots, m.$$

The model price  $\mathbb{E}^{\mathbb{P}(\lambda)} \mathcal{G}_i(X_{\mathcal{T}_i}) = \phi'(0, X_0)$  where  $\phi'$  satisfies

$$\begin{cases} \partial_t \phi' + \alpha^\lambda \cdot \nabla_x \phi' + \frac{1}{2} \beta^\lambda : \nabla_x^2 \phi' = 0, & \text{in } [0, \mathcal{T}_i) \times \mathbb{R}^2, \\ \phi'(\mathcal{T}_i, \cdot) = \mathcal{G}_i. \end{cases}$$

Note: For the calculation of different gradients, the PDEs are the same but with different terminal conditions. The inversion of the linear operator is only required once for all gradients.

## Dual formulation:

$$\text{maximise } V = \sup_{\lambda \in \mathbb{R}^{m+n+2}} \lambda \cdot c - \phi^\lambda(0, X_0),$$

$$\text{subject to } \partial_t \phi^\lambda + F^*(\nabla_x \phi^\lambda, \frac{1}{2} \nabla_x^2 \phi^\lambda) = - \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i), \quad \phi(T, \cdot) = 0.$$

## Numerical solution:

- 1 Set an initial  $\lambda$  (e.g.,  $\lambda = \mathbf{0}$ ),
- 2 Solve the HJB equation backward to get  $\phi^\lambda(0, X_0)$  (see next slide),
- 3 Solve the linear PDEs and calculate all gradients,
- 4 Update  $\lambda$  by gradient descent.

$$\text{HJB: } \partial_t \phi + \sup_{\alpha, \beta} \left\{ \alpha \cdot \nabla_x \phi + \frac{1}{2} \beta : \nabla_x^2 \phi - F(\alpha, \beta) \right\} = - \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \delta(t - \mathcal{T}_i), \quad \phi(T, \cdot) = 0$$

---

## Algorithm 1: Solving the HJB equation

---

**for**  $k = N - 1, \dots, 0$  **do**

    /\* Handling the source term \*/

$\phi_{t_{k+1}} \leftarrow \phi_{t_{k+1}} + \sum_{i=1}^{m+n+2} \lambda_i \mathcal{G}_i \mathbb{1}(t_{k+1} = \mathcal{T}_i)$

    /\* Policy iteration \*/

$\phi_{t_k}^{new} \leftarrow \phi_{t_{k+1}}$

**do**

$\phi_{t_k}^{old} \leftarrow \phi_{t_k}^{new}$

        Approximate the optimal  $(\alpha_{t_k}, \beta_{t_k})$  by solving the supremum with  $\phi_{t_k}^{old}$

        Solve the linearised HJB equation with  $(\alpha_{t_k}, \beta_{t_k})$  by a fully implicit finite difference method, and set the solution to  $\phi_{t_k}^{new}$

**while**  $\|\phi_{t_k}^{new} - \phi_{t_k}^{old}\|_{\infty} > \epsilon$

$\phi_{t_k} \leftarrow \phi_{t_k}^{new}$

**end**

---

Scaling the discrete constraints with proper scales might improve the stability and convergence.

$$\mathbb{E}^{\mathbb{P}} \hat{\mathcal{G}}(X_T) := \mathbb{E}^{\mathbb{P}} \frac{1}{\Gamma} \mathcal{G}(X_T) = \frac{c}{\Gamma} =: \hat{c}$$

Recommended values of  $\Gamma$ :

- for SPX and VIX options, set  $\Gamma$  to their Black–Scholes Vega  
⇒ 1e-4 error of  $\hat{c} \approx 1$  bp error in implied vol,
- for VIX futures, set  $\Gamma = 100$   
⇒ 1e-4 error of  $\hat{c} \approx 1$  cent error in price.

So far we have ignored the significance of the reference model  $\bar{\beta}$ .

When the gaps between strikes are too large or  $\bar{\beta}$  is too far away from the  $\beta$  that describes the actual market dynamics, there might be spikes in the volatility surfaces, which might cause hump-shaped model volatility skews.

### Smoothing technique:

- 1 Set an initial reference  $\bar{\beta}$
- 2 Solve the dual formulation to get an optimal  $\beta = \beta^*$
- 3 Smooth  $\beta^*$  by a smoothing method and set the result to  $\bar{\beta}$
- 4 Repeat steps 2-4 with the new  $\bar{\beta}$

In the numerical example, we smooth  $\beta^*$  by the simple moving average method over  $(X^1, X^2)$  with bandwidths of  $(3, 3)$ .

Simulated calibrating instruments:

- SPX call options maturing at 44 days and 79 days
- VIX futures maturing at 49 days
- VIX call options maturing at 49 days

Prices of the above instruments are *generated* using *Heston dynamics* and parameters  $(\kappa, \theta, \omega, \eta) = (0.6, 0.09, 0.4, -0.5)$ , i.e.,  $X$  satisfies

$$X_t = X_0 + \int_0^t \alpha_s^{\mathbb{P}} ds + \int_0^t (\beta_s^{\mathbb{P}})^{\frac{1}{2}} dW_s^{\mathbb{P}},$$

and

$$(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) = \left( \left[ \begin{array}{c} -\frac{1}{2}\nu(t, X_t^2, \kappa, \theta) \\ -\frac{1}{2}\nu(t, X_t^2, \kappa, \theta) \end{array} \right], \left[ \begin{array}{cc} \nu(t, X_t^2, \kappa, \theta) & \frac{1}{2}\eta\omega A(t, \kappa)\nu(t, X_t^2, \kappa, \theta) \\ \frac{1}{2}\eta\omega A(t, \kappa)\nu(t, X_t^2, \kappa, \theta) & \frac{1}{4}\omega^2 A(t, \kappa)^2 \nu(t, X_t^2, \kappa, \theta) \end{array} \right] \right),$$

where  $A(t, \kappa) := (1 - e^{-\kappa(T-t)})/\kappa$  and  $\nu(t, X_t^2, \kappa, \theta) := A(t, \kappa)^{-1}(2X_t^2 - \theta(T-t)) + \theta$ .

⇒ Solution exists!

Recall our joint calibration problem is

$$\inf_{\mathbb{P} \in \mathcal{P}_{joint}^{loc}} \mathbb{E}^{\mathbb{P}} \int_0^T F(\alpha_t^{\mathbb{P}}, \beta_t^{\mathbb{P}}) dt, \quad \text{where } F(\alpha, \beta) = \begin{cases} \sum_{i,j=1}^2 (\beta_{ij} - \bar{\beta}_{ij})^2 & \text{if } \alpha_1 = \alpha_2 = -\frac{1}{2}\beta_{11}, \\ +\infty & \text{otherwise.} \end{cases}$$

We consider two references:

(a) a Heston reference with parameters  $(\bar{\kappa}, \bar{\theta}, \bar{\omega}, \bar{\eta}) = (0.9, 0.04, 0.6, -0.3)$ :

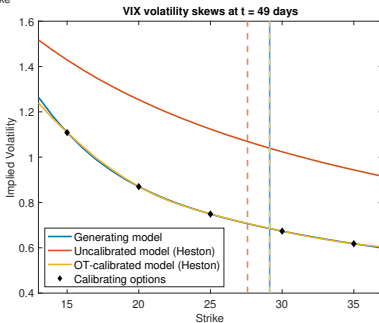
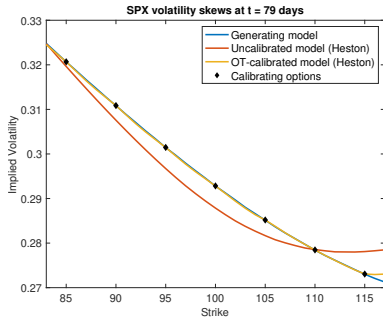
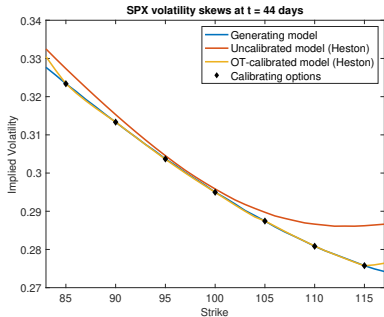
$$\bar{\beta}(t, X_t^1, X_t^2) = \begin{bmatrix} \nu(t, X_t^2, \bar{\kappa}, \bar{\theta}) & \frac{1}{2}\bar{\eta}\bar{\omega}A(t, \bar{\kappa})\nu(t, X_t^2, \bar{\kappa}, \bar{\theta}) \\ \frac{1}{2}\bar{\eta}\bar{\omega}A(t, \bar{\kappa})\nu(t, X_t^2, \bar{\kappa}, \bar{\theta}) & \frac{1}{4}\bar{\omega}^2A(t, \bar{\kappa})^2\nu(t, X_t^2, \bar{\kappa}, \bar{\theta}) \end{bmatrix};$$

(b) a constant reference:

$$\bar{\beta}(t, X_t^1, X_t^2) = \begin{bmatrix} 0.09 & -0.01 \\ -0.01 & 0.04 \end{bmatrix}.$$

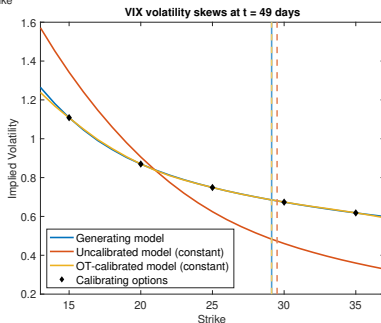
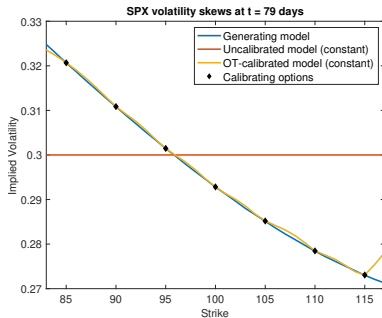
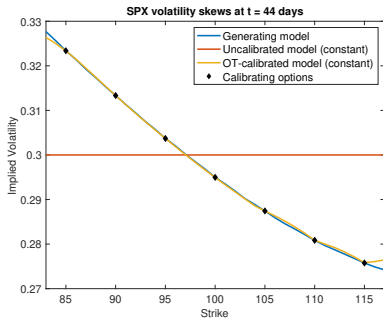
*Rk: if in (a) we took the reference to be the generating model,  $(\bar{\kappa}, \bar{\theta}, \bar{\omega}, \bar{\eta}) = (\kappa, \theta, \omega, \eta)$ , then the algorithm quickly recovers OT-model = generating model by  $\lambda = \mathbf{0}$ , and  $V = 0$ .*

# Simulated data example — Calibration results for Heston reference

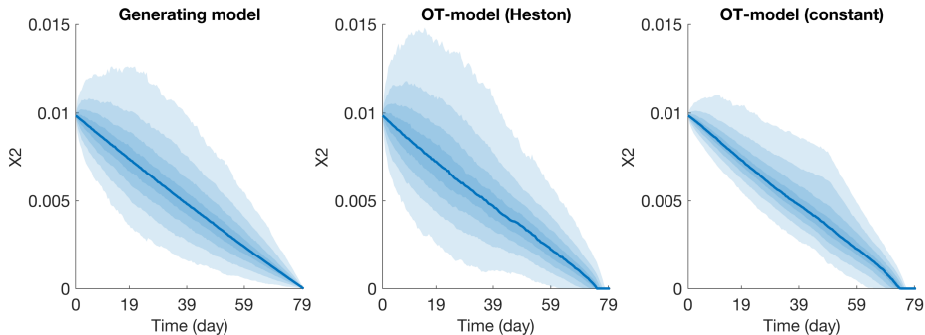




# Simulated data example — Calibrating results for constant reference



# Simulated data example — Simulation of $X^2$



Market data as of 1st September 2020:

- SPX call options maturing at 17 days and 45 days
- VIX futures maturing at 15 days
- VIX call option maturing at 15 days

These are the shortest maturities, which is known as the most challenging case!

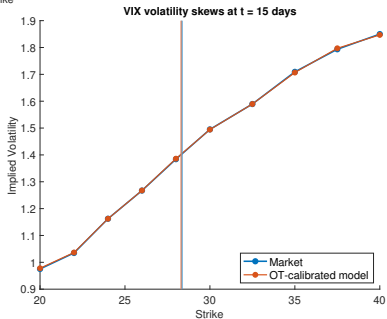
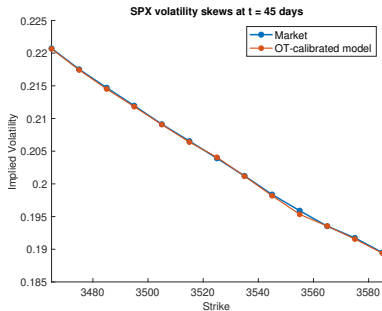
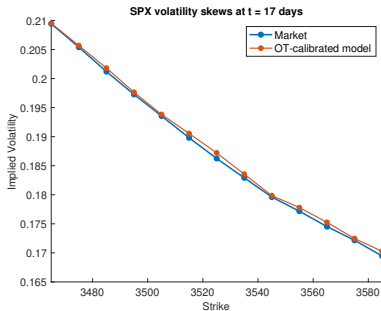
We calibrate the OT-model with a Heston reference  $\bar{\beta}$ . The parameters  $(\bar{\kappa}, \bar{\theta}, \bar{\omega}, \bar{\eta}) = (4.99, 0.038, 0.52, -0.99)$  are obtained by (roughly) calibrating a standard Heston model to the SPX option prices.

*Remark.* Interest rates and dividends are NOT zero

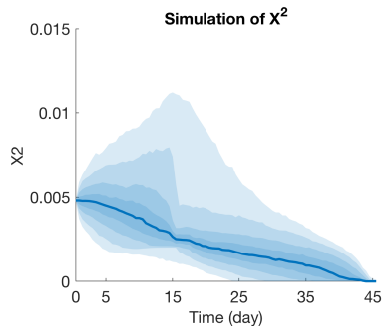
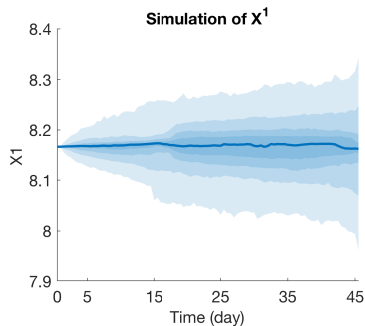
⇒ model  $X^1$  as the log of T-forward SPX price (instead of the spot price)

⇒  $\mathbb{P}$  are T-forward measures under which  $\exp(X^1)$  is still a martingale.

# Market data example — Calibration results



# Market data example — Simulation of $X^1$ and $X^2$





Benamou, J.-D., and Brenier, Y.

A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem.  
*Numerische Mathematik* 84, 3 (2000), 375–393.



Brunick, G., and Shreve, S.

Mimicking an Itô process by a solution of a stochastic differential equation.  
*Annals of Applied Probability* 23, 4 (2013), 1584–1628.



Cuchiero, C., Khosrawi, W., and Teichmann, J.

A generative adversarial network approach to calibration of local stochastic volatility models.  
*arXiv preprint arXiv:2005.02505* (2020).



Figalli, A.

Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients.  
*Journal of Functional Analysis* 254, 1 (2008), 109–153.



Fouque, J.-P., and Saporito, Y. F.

Heston stochastic vol-of-vol model for joint calibration of VIX and S&P 500 options.  
*Quantitative Finance* 18, 6 (2018), 1003–1016.



Gatheral, J.

Consistent modeling of SPX and VIX options.  
*In Bachelier congress (2008)*, vol. 37, pp. 39–51.



Gatheral, J., Jusselin, P., and Rosenbaum, M.

The quadratic rough Heston model and the joint S&P 500/VIX smile calibration problem.  
*arXiv preprint arXiv:2001.01789* (2020).



Goutte, S., Ismail, A., and Pham, H.

Regime-switching stochastic volatility model: estimation and calibration to VIX options.  
*Applied Mathematical Finance* 24, 1 (2017), 38–75.



Guo, I., and Loeper, G.

Path dependent optimal transport and model calibration on exotic derivatives.  
*arXiv preprint arXiv:1812.03526 (2018)*.



Guo, I., Loeper, G., and Wang, S.

Local volatility calibration by optimal transport.  
*2017 MATRIX Annals (2019)*.



Guo, I., Loeper, G., and Wang, S.

Calibration of local-stochastic volatility models by optimal transport.  
*Submitted to Mathematical Finance*



Gyöngy, I.

Mimicking the one-dimensional marginal distributions of processes having an Ito differential.  
*Probability theory and related fields 71, 4 (1986), 501–516*



Guyon, J.

The joint S&P 500/VIX smile calibration puzzle solved.  
*Risk, April (2020)*.



Han, J., Jentzen, A., et al.

Algorithms for solving high dimensional pdes: From nonlinear monte carlo to machine learning.  
*arXiv preprint arXiv:2008.13333 (2020)*.



Tan, X., Touzi, N.

Optimal transportation under controlled stochastic dynamics.  
*The Annals of Probability 41, 5 (2013), 3201–3240*.



Trevisan, D.

Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients.  
*Electronic Journal of Probability 21 (2016), Paper No. 22, 41*.

# SPX & INTEREST RATES CALIBRATION



*Case I: A pre-calibrated short rate model fitting the term structure, zero dividends*

Take a two dimensional stochastic process  $X = (X^1, X^2)$ , let  $X^1$  **log-stock price** of some underlying asset and  $X^2$  represent the short rate

$$X_t^1 = X_0^1 + X_t^2 - \frac{1}{2} \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^1,$$

we assume that  $X^2$  is a **Hull-White short rate process** given by

$$X_t^2 = X_0^2 + \int_0^t (\theta(s) - a(s)X_s^2) ds + \int_0^t \sigma_r(s) dW_s^2.$$

We assume that  $W_t^1$  and  $W_t^2$  are correlated standard Brownian motions such that

$$\langle W^1, W^2 \rangle_t = \int_0^t \xi_s ds.$$

Note that since  $r_t$  is assumed to be pre-calibrated, the parameters  $\theta$ ,  $a$ , and  $\sigma_r$  are all assumed to be known. We calibrate  $\sigma$  and  $\xi$  using Call options on the underlying at 60 and 120 days.

Given  $n$  Call options observed in the market with prices  $u_i$ , strikes  $K_i$  and maturities  $\tau_i$ , our calibration constraints become

$$\mathbb{E} \left[ e^{-\int_0^{\tau_i} X_s^2 ds} \left( e^{X_{\tau_i}^1} - K_i \right) \right] = u_i, \quad i = 1, \dots, n.$$

We therefore consider the set  $\mathcal{P}(X_0, \tau, K, u)$  containing measures  $\mathbb{P}$  such that  $X$  is a semimartingale and satisfies the calibration constraints.

Moreover, we may localise using Markovian projection and consider the subset  $\mathcal{P}_{\text{loc}}(X_0, \tau, K, u) \subset \mathcal{P}(X_0, \tau, K, u)$  such that under the mimicking measure  $\mathbb{P}' \in \mathcal{P}_{\text{loc}}(X_0, \tau, K, u)$ ,  $X$  is a Markov process satisfying

$$dX_t = \alpha(t, X_t)dt + (\beta(t, X_t))^{\frac{1}{2}} dW_t,$$

where  $W$  is a  $\mathbb{P}'$  Brownian motion.

The discount term  $e^{-\int_0^{\tau_i} X_s^2 ds}$  is path dependent and thus incompatible with our PDE formulation framework.

We could add an extra state variable, but that would increase the computational complexity when solving the HJB equation, so we provide a conditioning argument.

### Discounted Density Transformation

Let  $\bar{\rho}$  be the joint law of  $X_t^1, X_t^2$  and  $\int_0^t X_s^2 ds$  and  $\eta_{t,x}(y)$  the law of  $\int_0^t X_s^2 ds$  conditional on  $X_t = [x^1, x^2]^\top$ .

Define the 'discounted density'  $\tilde{\rho}(t, x) = \left( \int_{\mathbb{R}} e^{-y} \eta_{t,x}(dy) \right) \rho(t, x)$ ,  $(t, x) \in [0, T] \times \mathbb{R}^2$ . Then  $\tilde{\rho}$  satisfies for  $(t, x) \in [0, T] \times \mathbb{R}^2$ :

$$\partial_t \tilde{\rho}(t, x) + \nabla_x \cdot (\alpha(t, x) \tilde{\rho}(t, x)) - \frac{1}{2} \nabla_x^2 : (\beta(t, x) \tilde{\rho}(t, x)) + x_2 \tilde{\rho}(t, x) = 0.$$

## Primal Problem

Minimise

$$V = \inf_{\rho, A, B} \int_0^T \int_{\mathbb{R}^2} \rho F \left( \frac{A}{\rho}, \frac{B}{\rho} \right) dx dt,$$

subject to the constraints

$$\partial_t \rho + \nabla_x \cdot A - \frac{1}{2} \nabla^2 : B + x_2 \rho = 0$$

$$\int_{\mathbb{R}^2} (e^{x_1} - K_i)^+ \rho(\tau_i, dx) = u_i, \quad i = 1, \dots, n$$

$$\rho(0, \cdot) = \delta_{X_0}$$

Introduce the Lagrange multipliers  $\phi \in C_c^\infty([0, T] \times \mathbb{R}^2)$  and  $\lambda \in \mathbb{R}^n$ , then

$$V = \inf_{\rho, A, B} \sup_{\phi, \lambda} \left\{ \int_0^T \int_{\mathbb{R}^2} \left( \rho F \left( \frac{A}{\rho}, \frac{B}{\rho} \right) - \left( \partial_t \phi \rho + \nabla_x \phi \cdot A + \frac{1}{2} \nabla_x^2 \phi : B - x_2 \phi \rho \right) - \sum_{i=1}^n \lambda_i (e^{x_1} - K_i)^+ \delta_{\tau_i} \rho \right) dx dt + \lambda \cdot u - \phi(0, X_0) \right\}$$

## Dual Problem

Maximise

$$V = \sup_{\lambda \in \mathbb{R}^n} \lambda \cdot u - \phi(0, X_0),$$

where  $\phi$  is the viscosity solution to the HJB equation:

$$\partial_t \phi - x_2 \phi + F^*(\nabla_x \phi, \frac{1}{2} \nabla_x^2 \phi) + \sum_{i=1}^n \lambda_i (e^{x_1} - K_i)^+ \delta_{\tau_i} = 0$$

with the terminal condition  $\phi(T, \cdot) = 0$ . If the supremum is attained and the associated solution to the HJB equation is  $\tilde{\phi} \in \text{BV}([0, T], C_b^2(\mathbb{R}^2))$ , then an optimal  $(\alpha, \beta)$  of the PDE formulation can be found by

$$(\alpha, \beta) = \nabla F^*(\nabla_x \tilde{\phi}, \frac{1}{2} \nabla_x^2 \tilde{\phi}).$$

# Cost function for Sequential Calibration

Choose a **reference correlation**  $\bar{\xi}(t)$  and require  $\xi(t, Z_t, r_t) = \frac{\sigma_r(t)}{\sigma(t, Z_t, r_t)} \bar{\xi}(t)$ , for  $t \in [0, T]$ . Define for  $p > 1$

$$H(x, \bar{x}, s) = \begin{cases} (p-1) \left(\frac{x-s}{\bar{x}-s}\right)^{1+p} + (p+1) \left(\frac{x-s}{\bar{x}-s}\right)^{1-p} - 2p, & \text{if } x, \bar{x} > s, \\ +\infty, & \text{otherwise.} \end{cases}$$

Notice that the coefficients are chosen such that  $H$  is minimised over  $x$  at  $x = \bar{x}$  with  $\min H = 0$ . Also define the convex set

$$\Gamma(t, X_t) = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \times \mathbb{S}^2 : \alpha_1 = X_t^2 - \frac{1}{2}\beta_{11}, \alpha_2 = (b(t) - aX_t^2), \right. \\ \left. \beta_{12} = \beta_{21} = \bar{\xi}\sigma_r(t), \beta_{22} = \sigma_r^2 \right\}$$

Define the cost function  $F(\alpha, \beta) = \begin{cases} H(\beta_{11}, \bar{\sigma}^2, \bar{\xi}^2\sigma_r^2), & \text{if } (\alpha, \beta) \in \Gamma(t, X_t), \\ +\infty, & \text{otherwise.} \end{cases}$

$\bar{\sigma}^2 = \bar{\sigma}^2(t, X_t)$  is some reference value for the volatility

## HJB Equation

$$\begin{aligned} & \sum_{i=1}^n \lambda_i (\exp(x_1) - K_i)^+ \delta_{\tau_i} + \partial_t \phi + \sup_{\beta_{11}} \left( \left( x_2 - \frac{1}{2} \beta_{11} \right) \partial_{x_1} \phi \right. \\ & + (b(t) - ax_2) \partial_{x_2} \phi + \frac{1}{2} \beta_{11} \partial_{x_1 x_1}^2 \phi + \bar{\xi} \sigma_r \partial_{x_1 x_2}^2 \phi + \frac{1}{2} \sigma_r^2 \partial_{x_2 x_2}^2 \phi - x_2 \phi \\ & \left. - H(\beta_{11}, \bar{\sigma}^2, \bar{\xi}^2 \sigma_r^2) \right) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^2. \end{aligned}$$

Given  $\lambda$  with associated solution  $\mathbb{P}^\lambda$  of the dual problem, let  $\mathbb{P}(\lambda)$  be the probability measure under which  $X$  has the characteristics  $(\alpha^\lambda, \beta^\lambda) = \nabla F^*(\nabla_x \phi^\lambda, \frac{1}{2} \nabla_x^2 \phi^\lambda)$ . Then the model price of an instrument with payoff  $\mathcal{G}$  and maturity  $\mathcal{T}$  is given by

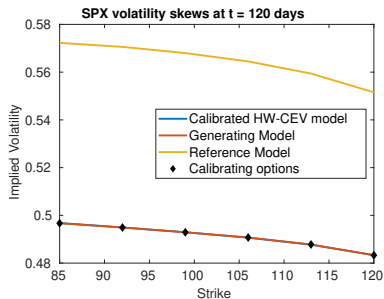
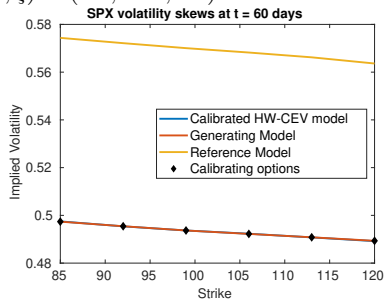
$\mathbb{E}^{\mathbb{P}(\lambda)} \left[ e^{-\int_0^T X_s^2 ds} \mathcal{G}(X_{\mathcal{T}}) \right] = \phi'(0, X_0)$ , where  $\phi'$  solves

$$\begin{cases} \partial_t \phi' + \alpha^\lambda \cdot \nabla_x \phi' + \frac{1}{2} \beta^\lambda : \nabla_x^2 \phi' - x_2 \phi' = 0, & (t, x) \in [0, \mathcal{T}] \times \mathbb{R}^2 \\ \phi'(\mathcal{T}, \cdot) = \mathcal{G}(\cdot) \end{cases}$$

The numerical method is analogous in this case, and we may analytically compute the optimal  $\beta_{11}$  in the HJB equation with our chosen cost function.

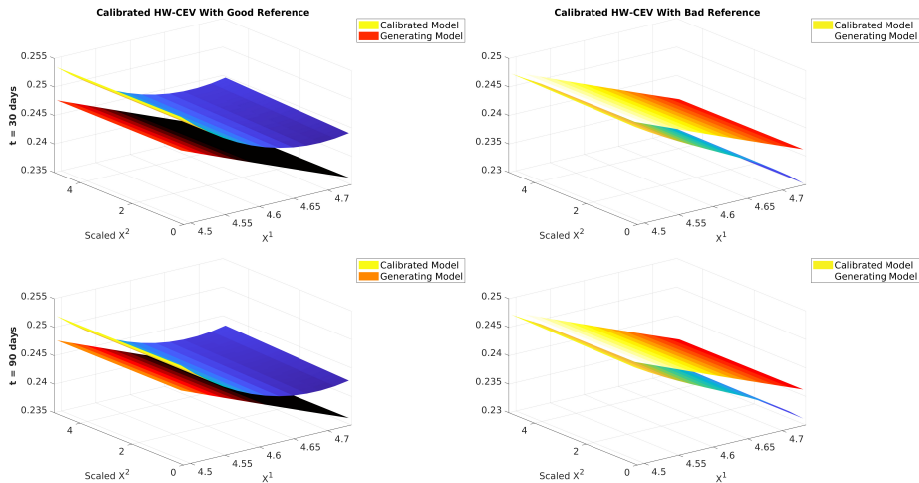
# Simulated Data Example

We used a CEV-Hull-White reference and generating model with the interest rate parameters the same in both. This gave us that  $\bar{\sigma}(t, x) = \sigma \exp(x_1)^{\gamma-1}$ . The generating model had parameters  $(\sigma, \gamma, a, \sigma_r, \xi) = (0.78, 0.9, 0.4, 0.005, -0.6)$ , and the “good” reference had  $(\bar{\sigma}, \bar{\gamma}, \bar{\xi}) = (0.9, 0.9, -0.4)$ , whereas the “bad” reference had  $(\bar{\sigma}, \bar{\gamma}, \bar{\xi}) = (1.2, 0.78, 0.4)$





# Simulated Data Example — Plots of Characteristics



**Figure:** Comparison of  $\beta_{11}$  with the generating vol surface for a 'good' and a 'bad' reference model

# Simulated Data Example — Plots of Characteristics

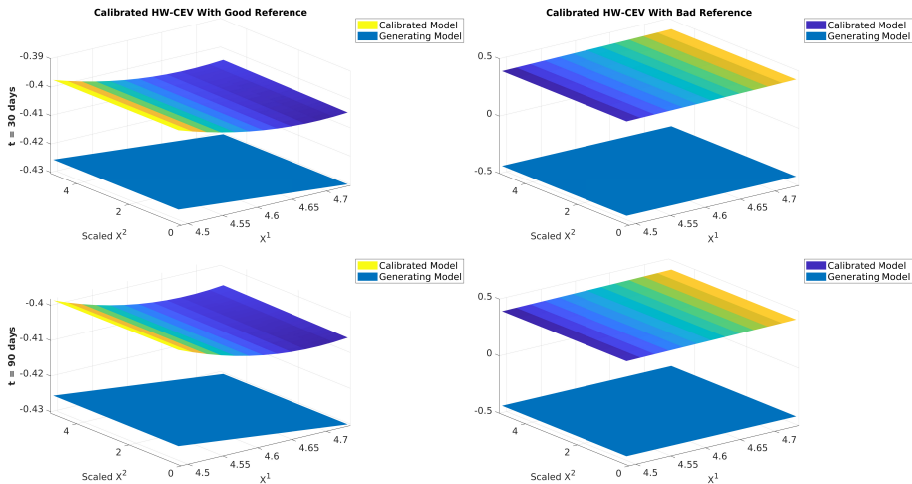


Figure: Comparison of  $\xi$  with the generating vol surface for a 'good' and a 'bad' reference model

*Case II: Joint & simultaneous calibration exercise, zero dividends*

Now assume we have no prior knowledge of the interest rate, our characteristics for the log-stock and short rate are therefore given by:

$$\alpha_t = \begin{bmatrix} X_t^2 - \frac{1}{2}(\beta_t)_{11} \\ (\alpha_t)_2 \end{bmatrix}, \quad \beta_t = \begin{bmatrix} (\beta_t)_{11} & (\beta_t)_{12} \\ (\beta_t)_{12} & (\beta_t)_{22} \end{bmatrix}.$$

Define the convex set

$$\Gamma(t, x) = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \times \mathbb{S}_+^2 : \alpha_1 = x_2 - \frac{1}{2}\beta_{11} \right\}.$$

Define the cost function

$$F(\alpha, \beta) = \begin{cases} \|\alpha - \bar{\alpha}\|_2^2 + \|\beta - \bar{\beta}\|_{\text{Fro}}^2, & \text{if } (\alpha, \beta) \in \Gamma(t, x), \\ +\infty, & \text{otherwise.} \end{cases}$$

Where  $\bar{\alpha}$  and  $\bar{\beta}$  correspond to some reference model. We remark that we will calibrate with interest rate derivatives as well.

The dual formulation is similar to the sequential calibration, but with a different cost function. Let  $G_i(x)$  denote the payoffs of instruments with maturity  $\tau_i$  and market value  $u_i$ .

## Joint Calibration Dual Formulation

Maximise

$$V = \sup_{\lambda \in \mathbb{R}^n} \lambda \cdot u - \phi(0, X_0)$$

Subject to

$$\begin{aligned} \partial_t \phi + \sup_{\alpha_2 \in \mathbb{R}, \beta \in \mathbb{S}_+^2} \left\{ \left( x_2 - \frac{1}{2} \beta_{11} \right) \partial_{x_1} \phi + \alpha_2 \partial_{x_2} \phi + \frac{1}{2} \beta_{11} \partial_{x_1 x_1}^2 \phi \right. \\ \left. + \frac{1}{2} \beta_{22} \partial_{x_2 x_2}^2 \phi + \beta_{12} \partial_{x_1 x_2}^2 \phi - \|\alpha - \bar{\alpha}\|_2^2 - \|\beta - \bar{\beta}\|_{\text{Fro}}^2 \right\} \\ - x_2 \phi + \sum_{i=1}^n \lambda_i G_i(x) \delta_{\tau_i} = 0, \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^2 \end{aligned}$$

The numerical method is identical to the sequential calibration method, and we can analytically compute the supremum in the HJB equation.

We calibrate using Call options on the stock and Caplets on the interest rate with a fixed notional of \$1,000 at 60 and 120 days.

The reference models are the CEV local volatility model with a Hull-White interest rate and a CIR interest rate. In both cases, the generating model was the same with shifted parameters. The parameters were given as follows:

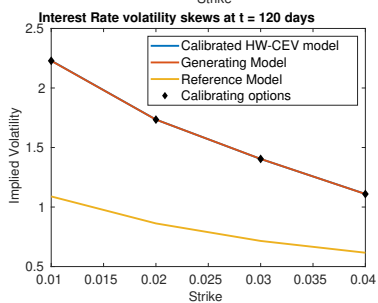
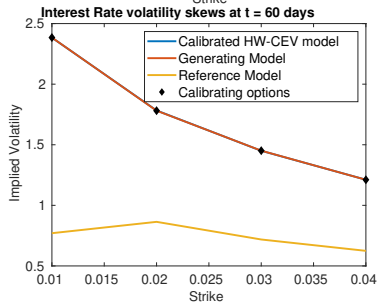
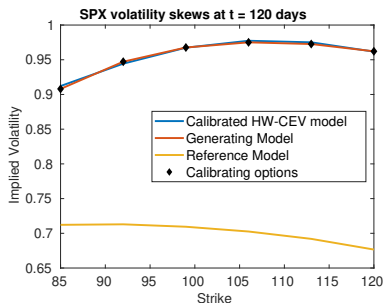
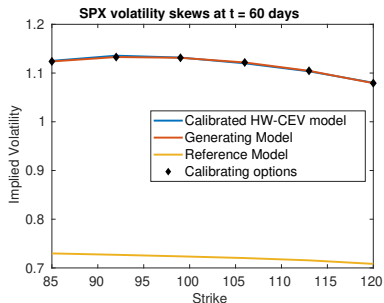
Generating		Reference	
$\sigma$	1.50	$\bar{\sigma}$	1.2
$\gamma$	0.95	$\bar{\gamma}$	0.89
$a$	0.05	$\bar{a}$	0.03
$\sigma_r$	0.04	$\bar{\sigma}_r$	0.02
$\rho$	-0.05	$\bar{\rho}$	-0.2

Table: CEV-Hull-White Parameters

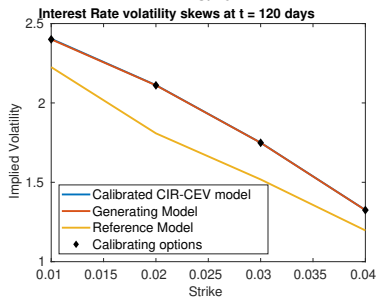
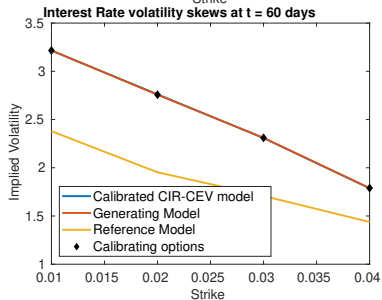
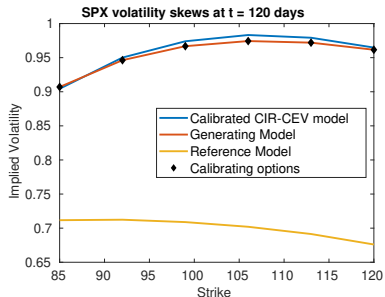
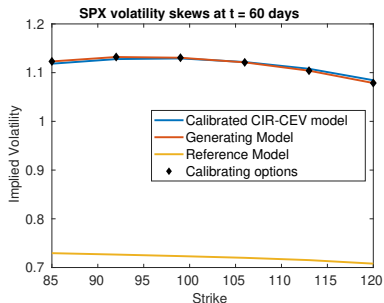
Generating		Reference	
$\sigma$	1.5	$\bar{\sigma}$	1.2
$\gamma$	0.95	$\bar{\gamma}$	0.89
$b$	0.03	$\bar{b}$	0.03
$a$	0.5	$\bar{a}$	0.4
$\sigma_r$	0.5	$\bar{\sigma}_r$	0.3
$\rho$	-0.4	$\bar{\rho}$	-0.2

Table: CEV-CIR Parameters

# Simulated Data Example — CEV-HW



# Simulated Data Example — CEV-CIR



# Simulated Data Example — Plots of Characteristics

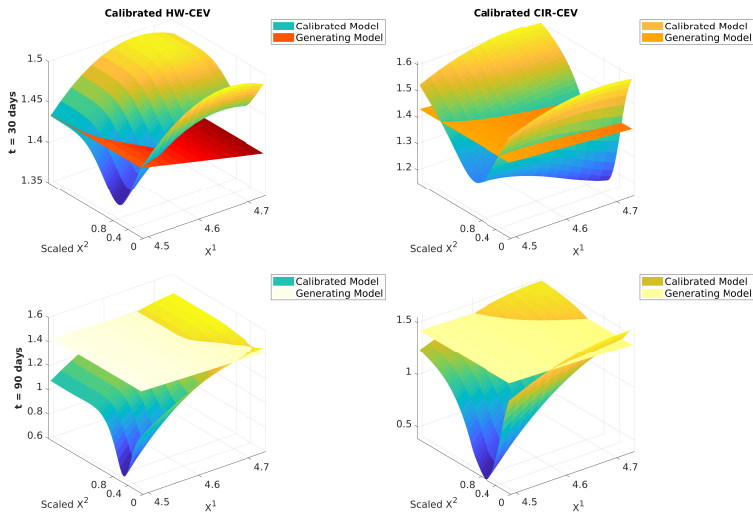


Figure: Comparison of  $\beta_{11}$  for the calibrated and generating model



# Simulated Data Example — Plots of Characteristics

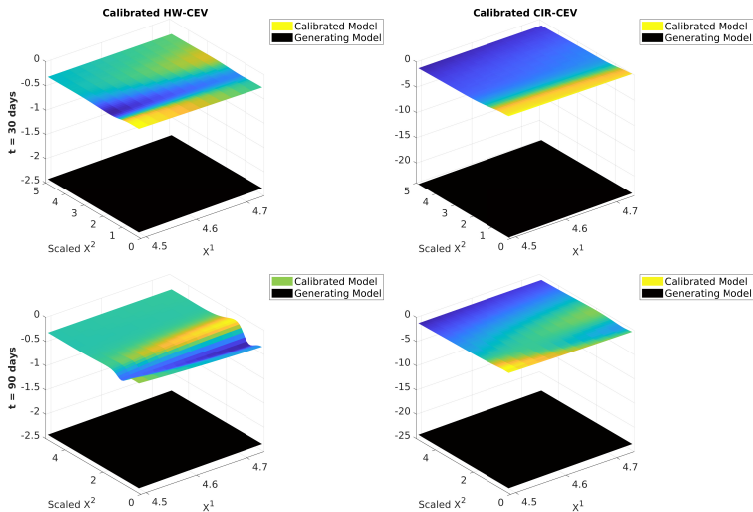


Figure: Comparison of  $\beta_{12}$  for the calibrated and generating model

# Simulated Data Example — Plots of Characteristics

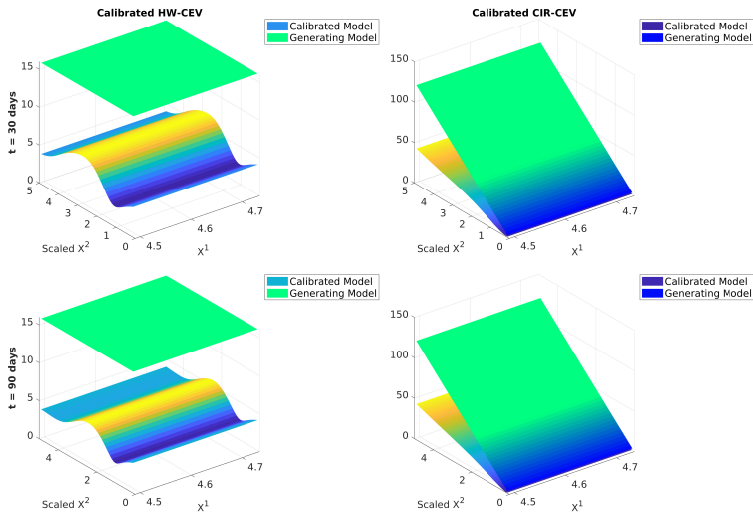


Figure: Comparison of  $\beta_{22}$  for the calibrated and generating model

# Simulated Data Example — Plots of Characteristics

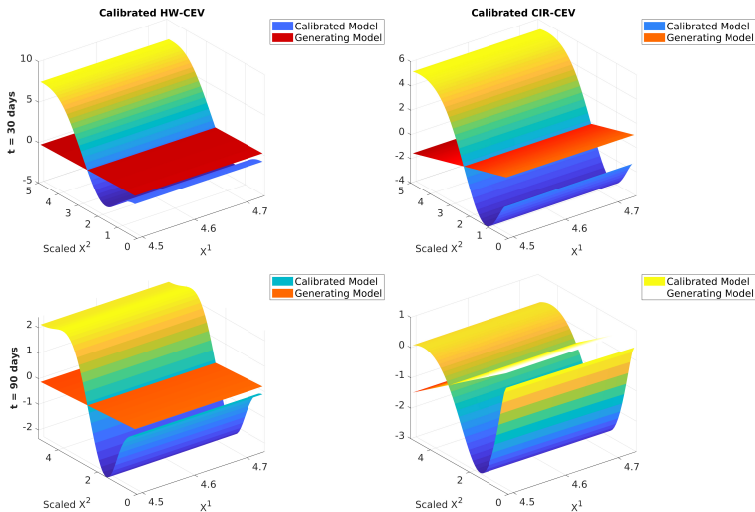


Figure: Comparison of  $\alpha_2$  for the calibrated and generating model

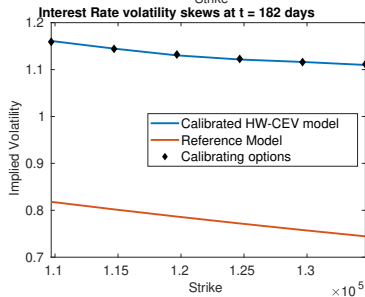
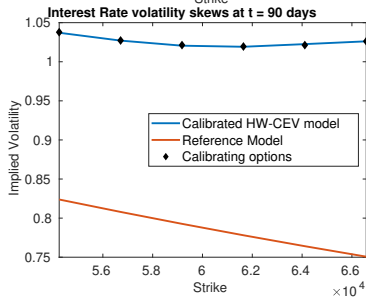
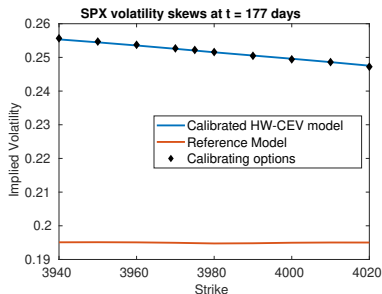
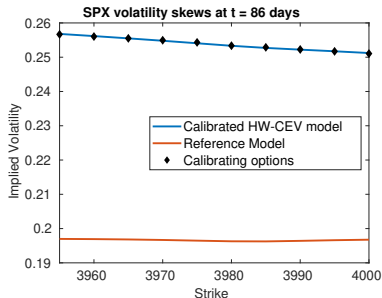
The numerical method is identical to the sequential calibration method, and we can analytically compute the supremum in the HJB equation. We took the SPX as the underlying and the 1M US LIBOR for a proxy of the short rate. We obtained the following data on 23/05/2022 from a Bloomberg terminal:

- Calls on the SPX with expiry 19/08/2022,
- Caps on the one month LIBOR with notional \$10,000,000 and expiry 23/08/2022,
- Calls on the SPX with expiry 18/11/2022,
- Caps on the one month LIBOR with notional \$10,000,000 and expiry 23/11/2022.

We additionally took a CEV-Hull-White reference model with parameters

$$(\bar{\sigma}, \gamma, \bar{a}, \bar{\sigma}_r, \bar{\rho}) = (0.3, 0.95, 0.01, 0.02, -0.7)$$

# Market Data Example — CEV-HW



# Market Data Example — Plots of Characteristics

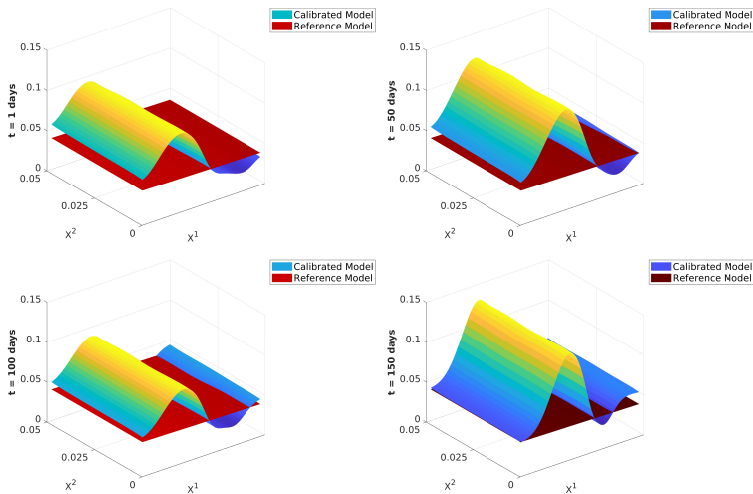


Figure: Comparison of  $\beta_{11} = \sigma_X^2$  for the calibrated and generating model

# Market Data Example — Plots of Characteristics

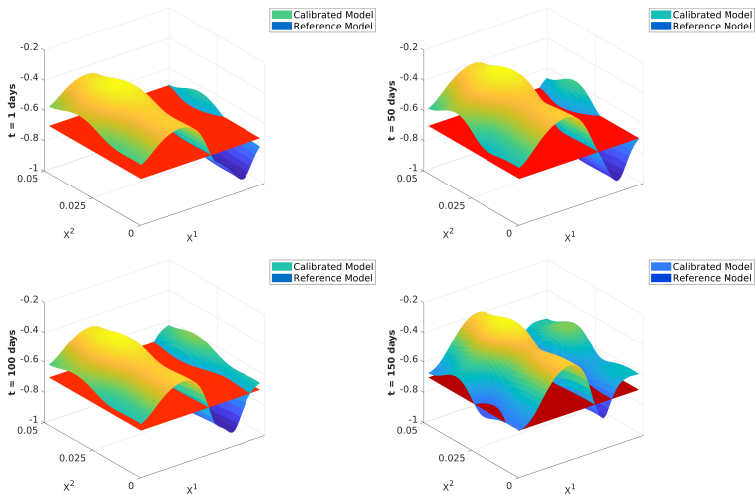


Figure: Comparison of  $\beta_{12} = \rho$  for the calibrated and generating model

# Market Data Example — Plots of Characteristics

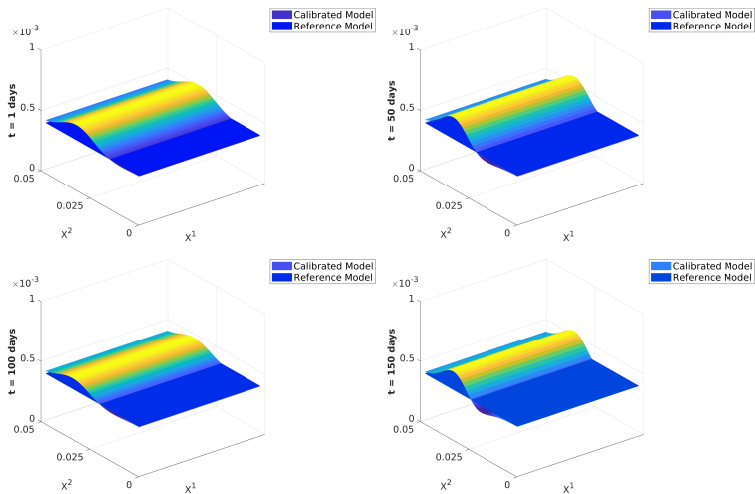


Figure: Comparison of  $\beta_{22} = \sigma_r^2$  for the calibrated and generating model



# Market Data Example — Plots of Characteristics

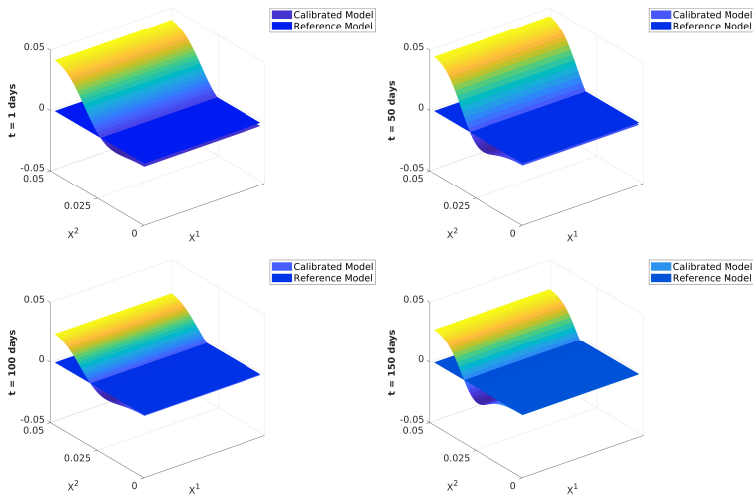


Figure: Comparison of  $\alpha_2 = \mu_r$  for the calibrated and generating model

ROBUST PRICING AND HEDGING OF AMERICAN OPTIONS  
VIA OT

Consider a market with stocks  $X$  and some European claims  $g$  which WLOG have initial prices of 0. We are allowed to trade  $X$  dynamically and  $g$  statically. Let  $\mathcal{Q} \subset \mathcal{P}$  be the set of possible “models”, i.e.,  $X$  is martingale,  $g$  has zero expectation, etc.

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Consider a European claim  $Z$ . Worst case model price:

$$\sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z.$$

Super-hedging price:

$$\pi(Z) := \inf\{x : \exists(q, h), \text{ s.t. } x + \int_0^1 q \cdot dX_t + h \cdot g \geq Z, \mathcal{Q}\text{-q.s.}\}.$$

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It is easy to check that

$$\pi(Z) \geq \sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z.$$

Duality (equality) results are obtained in various settings by Denis & Martini (2006); Soner, Touzi & Zhang (2013); Neufeld & Nutz (2013); and Possamaï, Royer & Touzi (2013); Hou & O. (2018) and many more.

## Theorem

Let  $F : \Lambda \times \mathbb{S}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  satisfy some assumptions and  $F^*(t, \omega, \cdot)$  be the convex conjugate of  $F(t, \omega, \cdot)$ . Define

$$V := \sup_{\mathbb{P}} \inf_{h \in \mathbb{R}^m} \mathbb{E}^{\mathbb{P}}(-h \cdot g + Z) - \mathbb{E}^{\mathbb{P}} \int_0^1 F(\beta_t^{\mathbb{P}}) dt,$$

$$\mathcal{V} := \inf_{h \in \mathbb{R}^m, \phi \in C^{1,2}(\Lambda)} \phi(0, X_0),$$

$$\text{subject to } \phi(1, \cdot) \geq Z - h \cdot g \quad \text{and} \quad \mathcal{D}_t \phi + F^* \left( \frac{1}{2} \nabla_x^2 \phi \right) \leq 0.$$

Then  $V = \mathcal{V}$ . Moreover, if  $V$  is finite, then the supremum is attained.

Let  $F(\beta)$  be 0 if  $\beta \in D$  (volatility constraint), or  $\infty$  otherwise. Then the dual is

$$\mathcal{V} = \inf_{h \in \mathbb{R}^m, \phi \in C^{1,2}(\Lambda)} \phi(0, X_0),$$

subject to  $\phi(1, \cdot) \geq Z - h \cdot g$  and  $\mathcal{D}_t \phi + \sup_{\beta \in D} \frac{1}{2} \nabla_x^2 \phi : \beta \leq 0$ .

Each  $\phi$  is actually a super-hedge. For every  $\mathbb{P} \in \mathcal{Q}$

$$\begin{aligned} Z - h \cdot g - \phi(0, X_0) &\leq \phi(1, X) - \phi(0, X_0) \\ &= \int_0^1 (\mathcal{D}_t \phi + \frac{1}{2} \beta^{\mathbb{P}} : \nabla_x^2 \phi) dt + \nabla_x \phi \cdot dX_t, \quad \mathbb{P}\text{-a.s.} \\ &\leq \int_0^1 \nabla_x \phi \cdot dX_t. \end{aligned}$$

Hence  $\phi(0, X_0) \geq \pi(Z)$ . Since this works for all  $\phi$  satisfying (68), it implies

$$\mathcal{V} = \inf_{\phi \in C_0^{1,2}(\Lambda), (68)} \phi(0, X_0) \geq \pi(Z) \geq \sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z = V = \mathcal{V}.$$

Let  $Z$  be an American-style claim. Worst case model price:

$$\sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot).$$

Super-hedging price:

$$\pi^A(Z) := \inf \{x : \exists (p, q, h) \text{ s.t.}$$

$$x + \int_0^\tau p \cdot dX_t + \int_\tau^1 q^\tau \cdot dX_t + hg \geq Z_\tau, \mathcal{Q}^D\text{-q.s.}, \forall \tau \in \mathcal{T}\}.$$

Again, it is easy to check  $\pi^A(Z) \geq \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot)$ .



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Again, it is easy to check  $\pi^A(Z) \geq \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot)$ .

When the set of statically traded European options is non-empty, there may be a duality gap, which can be eliminated by enlarging the probability space.

In discrete time, various duality results for American options are obtained by Dolinsky (2014); Hobson & Neuberger (2017); Bayraktar & Zhou (2017); Aksamit, Deng, O. & Tan (2019); and more. Some relevant works in continuous time include Herrmann & Stebegg (2017); Tiplea (2019); Grigорова, Quenez & Sulem (2021) etc.

The main idea of Aksamit et al. (2019) is to enlarge the space  $\Omega$  with the stopping decisions to obtain  $\bar{\Omega}$ . Then the American option can be seen as a European option under the enlarged space.

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In the case where there is no statically traded European options  $g$ .

$$\bar{\pi}(Z) = \pi^A(Z) \geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \bar{\pi}(Z),$$

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot).$$

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In the case where there is no statically traded European options  $g$ .

$$\begin{aligned}\bar{\pi}(Z) = \pi^A(Z) &\geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \bar{\pi}(Z), \\ \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z &= \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot).\end{aligned}$$

When  $g$  does exist, then we have to introduce a second enlarged space  $\hat{\Omega}$  which includes the price process of  $g$  as another martingale.

$$\begin{aligned}\bar{\pi}_g(Z) = \pi_g^A(Z) &\geq \hat{\pi}^A(Z) = \hat{\pi}(Z) \geq \sup_{\hat{\mathbb{P}} \in \hat{\mathcal{Q}}} \mathbb{E}^{\hat{\mathbb{P}}} Z \geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}_g} \mathbb{E}^{\bar{\mathbb{P}}} Z = \bar{\pi}_g(Z), \\ \sup_{\hat{\mathbb{P}} \in \hat{\mathcal{Q}}} \mathbb{E}^{\hat{\mathbb{P}}} Z &= \sup_{\hat{\tau} \in \hat{\mathcal{T}}, \hat{\mathbb{P}} \in \hat{\mathcal{Q}}} \mathbb{E}^{\hat{\mathbb{P}}} Z(\hat{\tau}, \cdot).\end{aligned}$$

We mostly focus on the case where there is no  $g$ .

Pricing hedging duality for European options is known in continuous time, and naturally extends to the enlarged space.

The equality  $\bar{\pi}(Z) = \pi^A(Z)$  can also be argued in mostly the same way.

However, the equality

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot),$$

creates difficulties in continuous time. Possible approaches include approximating with discrete time, Doob-Meyer type decomposition of non-linear Snell envelopes, reflected 2BSDEs, etc.

The original space for our model is  $\Omega := C([0, 1]; \mathbb{R}^d)$  with canonical process  $X$ . We enlarge it to  $\bar{\Omega} := \Theta \times \Omega$  where

$$\Theta := \{\vartheta \in C([0, 1], \mathbb{R}) : \vartheta_t = \theta \wedge t, \text{ for some } \theta \in [0, 1]\}.$$

$\Theta$  is isometric to  $[0, 1]$ .

Most aspects of  $\Omega$  can be naturally extended to  $\bar{\Omega}$ , include semimartingale measures (since  $\vartheta$  semimartingale with characteristics  $(\mathbb{1}(t \leq \theta), 0)$ ). E.g., we define  $\bar{\mathcal{Q}}$  to be the set of measures under which  $X$  is a martingale.

Also define the “stopped paths” of  $\bar{\Omega}$ , by

$$\bar{\Lambda} := \{(t, \bar{\omega}_{\cdot \wedge t}) : t \in [0, 1], \bar{\omega} \in \bar{\Omega}\}.$$

So elements of  $\bar{\Lambda}$  are  $(t, \bar{\omega}_{\cdot \wedge t}) = (t, \vartheta_{\cdot \wedge t}, \omega_{\cdot \wedge t}) = (t, \theta \wedge t, \omega_{\cdot \wedge t})$ .

Functional Itô calculus and PPDEs can be extended in the same way.

The path-dependent optimal transport duality results (Guo and Loeper (2021)) can be applied here.

## Theorem

$$\sup_{\mathbb{P} \in \bar{\mathcal{Q}}^D} \mathbb{E}^{\mathbb{P}} f = \inf_{\phi \in C_0^{1,1,2}(\bar{\Lambda})} \phi(0, 0, X_0),$$

$$\text{subject to } \phi(1, \cdot, \cdot) \geq f \quad \text{and} \quad \mathcal{D}_t \phi + \mathbf{1}(t \leq \theta) \nabla_{\theta} \phi + \sup_{\beta \in D} \frac{1}{2} \beta : \nabla_x^2 \phi \leq 0.$$

## Theorem

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}^D} \mathbb{E}^{\bar{\mathbb{P}}} f = \inf_{\phi \in C_0^{1,1,2}(\bar{\Lambda})} \phi(0, 0, X_0),$$

subject to  $\phi(1, \cdot, \cdot) \geq f$  and  $\mathcal{D}_t \phi + \mathbf{1}(t \leq \theta) \nabla_\theta \phi + \sup_{\beta \in D} \frac{1}{2} \beta : \nabla_x^2 \phi \leq 0.$

By the functional Itô formula, for each  $\phi$  and  $\bar{\mathbb{P}} \in \bar{\mathcal{Q}}^D$ , the following holds  $\bar{\mathbb{P}}$ -a.s.

$$\begin{aligned} f - \phi(0, 0, X_0) &\leq \phi(1, \cdot, \cdot) - \phi(0, 0, X_0) \\ &= \int_0^1 (\mathcal{D}_t \phi + \mathbf{1}(t \leq \theta) \nabla_\theta \phi + \frac{1}{2} \beta^{\mathbb{P}} : \nabla_x^2 \phi) dt + \nabla_x \phi \cdot dX_t \\ &\leq \int_0^1 \nabla_x \phi \cdot dX_t. \end{aligned}$$

Hence  $\phi(0, 0, X_0) \geq \bar{\pi}(f)$ . Since this holds for all  $\phi$ , it implies

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}^D} \mathbb{E}^{\bar{\mathbb{P}}} f \geq \bar{\pi}(f).$$

**BUT** we still need to address  $\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot).$



## Lemma

(a) For every  $\mu \in \mathcal{P}(\bar{\Omega})$ , there exists an increasing and adapted  $A$  with  $A_0 = 0$  and  $A_1 = 1$ , and  $\mathbb{P} \in \mathcal{P}(\Omega)$  with  $\mathbb{P} \ll \mu_\Omega$ , such that for every (non-anticipative)  $\psi \in L^\infty(\Lambda)$ ,

$$\mu(\psi(\theta, \omega \cdot \wedge \theta)) = \mathbb{E}^{\mathbb{P}} \int_0^1 \psi(t, \omega \cdot \wedge t) dA_t.$$

(b) For every  $\mu \in \mathcal{P}(\bar{\Omega})$ , there exists a family of true stopping times  $\tau_r$  and probability measures  $\mathbb{P}^r$ , indexed by  $r \in [0, 1]$ , such that for every  $\eta \in L^\infty(\bar{\Omega})$ ,

$$\mu(\eta(\theta, \omega)) = \int_0^1 \mathbb{E}^{\mathbb{P}^r} \eta(\theta = \tau_r, \omega) dr.$$

(c) For any  $a \in [0, 1]$  and any bounded and  $\mathcal{F}_{\tau_a}$ -measurable function  $\gamma$ ,

$$\int_a^1 \mathbb{E}^{\mathbb{P}^r} \gamma dr = (1 - a) \mathbb{E}^{\mathbb{P}} \gamma.$$

Roughly speaking, we obtain  $r$  by disintegrating  $\mu$  according to the value of  $A$ .

## Lemma

Suppose that  $X$  is a martingale under  $\mu \in \mathcal{P}(\bar{\Omega})$  with characteristic  $(0, \beta)$ . Then there exists a family of true stopping times  $\tau_r$  and probability measures  $\mathbb{P}^r \in \mathcal{P}(\Omega)$ , indexed by  $r \in [0, 1]$ , such that for every  $\eta \in L^\infty(\bar{\Omega})$ ,

$$\mu(\eta(\theta, \omega)) = \int_0^1 \mathbb{E}^{\mathbb{P}^r} \eta(\theta = \tau_r, \cdot) dr.$$

Moreover, each  $\mathbb{P}^r \in \mathcal{P}(\Omega)$  is a martingale measure with characteristic  $(0, \beta(t, t \wedge \tau_r(\omega), \omega_{\cdot \wedge t}))$ .

## Corollary

For any  $Z \in L^\infty(\bar{\Omega})$ , and any  $E \subseteq \Omega$ ,

$$\sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}, \mathbb{P}(E)=1} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot) = \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}, \bar{\mathbb{P}}(\Theta \times E)=1} \mathbb{E}^{\bar{\mathbb{P}}} Z.$$

### Conclusion:

- We consider Semimartingale OT perspective on pricing and hedging
- This includes European options, path-dependent options and now also American options
- We develop generic approach to Calibration via OT
- We use it to tackle difficult joint calibration problems: SPX options + VIX futures + VIX options prices; interest rates and SPX options
- Numerical proof-of-concept results

### Future research:

- Improving computational efficiency and exploring applications in higher dimensions
  - Deep PDE solvers (see, e.g., Han et al. (2020))
  - Neural SDE (see, e.g., Cuchiero et al. (2020))
- OT Calibration to American options

Thank you!