Functional convex ordering of stochastic processes : a constructive approach with applications to Finance

Gilles Pagès

(including joint works with Benjamin Jourdain & Yating Liu)

LPSM-Sorbonne-Université (Paris)

20th Winter School on Mathematical Finance

24th January 2023

Outline

1. Convex ordering: definitions and fist (static) examples
   - Convex ordering
   - Convexity (without order. . .)
2. Characterization of convex orderings
3. Functional convex ordering
4. Martingale (and scaled) Brownian diffusions
   - Discrete time: ARCH model. . .
   - Functional limit theorem: . . . to continuous time
   - Back to $1D$ (Jourdain-P. '22)
5. Jump diffusions
6. Bermuda and American options
   - Brownian diffusions
   - Jump martingale diffusions
7. McKean-Vlasov diffusions (with Y. Liu, to appear in *AAP*)
   - McKean-Vlasov equations
   - Convex order for McKean-Vlasov diffusions
8. Volterra equations (with B. Jourdain'22)

Definitions

Definition (Convex orderings)

Let $U,\ V \in L^1_{\mathbb{R}^d}(\mathbb{P})$ be two $\mathbb{R}^d$-valued random vectors with distributions μ and ν.
(*a*) *Convex ordering*. We say that U is dominated for the convex ordering by V, denoted

$$U \preceq_{cvx} V$$

if, for every convex function $f:\mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}\, f(U) \le \mathbb{E}\, f(V) \in (-\infty, +\infty] \tag{1}$$

or, equivalently, that μ is dominated by ν for the convex ordering if, for every convex function $f:\mathbb{R}^d \to \mathbb{R}$, $\int_{\mathbb{R}^d} f\, d\mu \le \int_{\mathbb{R}^d} f\, d\nu$.
(*b*) *Monotone convex ordering* $(d=1)$. When (1) only holds for non-decreasing/non-increasing convex functions f, the convex ordering is called increasing/decreasing convex order respectively denoted

$$U \preceq_{icv} V \quad \text{and} \quad U \preceq_{dcv} V.$$

Consistency

- For every $x \in \mathbb{R}^d$, by convexity of $f : \mathbb{R}^d \to \mathbb{R}$,

$$f(x) \geq f(0) + \langle \nabla_s f(0) \,|\, x \rangle.$$

where $\nabla_s f(0)$ denotes a subgradient of f at 0.

- Hence

$$f^-(x) \leq \Big(f(0) + \langle \nabla_s f(0) \,|\, x \rangle \Big)^- \leq |f(0)| + |\nabla_s f(0)||x|$$

so that

$$\mathbb{E}\, f^-(U) \leq |f(0)| + |\nabla_s f(0)| \mathbb{E}\,|U| < +\infty$$

and

$$\mathbb{E}\, f(U) = \underbrace{\mathbb{E}\, f^+(U)}_{\in[0,+\infty]} - \underbrace{\mathbb{E}\, f^-(U)}_{\in[0,+\infty)} \in (-\infty, +\infty] \text{ is well-defined.}$$

First properties (of $\preceq_{cvx}$)

- **P1**. As $f(x) = \pm x$ are both convex, $U \preceq_{cvx} V$ implies
$$\mathbb{E}\, U = \mathbb{E}\, V.$$
- **P2**. If, U, $V \in L^2(\mathbb{P})$, $U \preceq_{cvx} V$, then
$$\mathrm{Var}(U) \leq \mathrm{Var}(V).$$
[Set $f(x) = x^2$].
- **P3**. If $U \preceq_{icv} V$, then $\mathbb{E}\, U \leq \mathbb{E}\, V$.
- **P4**.
$$U \preceq_{dcv} V \Longleftrightarrow -V \preceq_{icv} -U$$
since $f(x) = f\big(-(-x)\big)$.

Convex ordering is a kind of generalization of the measure of risk

through the variance.

Examples I

- If $U = \mathbb{E}\,(V\,|\,U)$ then, for every convex function such that $f(V) \in L^1(\mathbb{P})$,
$$\mathbb{E}\,f(U) = \mathbb{E}\,f\big(\mathbb{E}\,(V\,|\,U)\big) \le \mathbb{E}\,\big[\mathbb{E}\,(f(V)\,|\,U)\big] = \mathbb{E}\,f(V).$$
owing to Jensen's inequality. Obvious if $\mathbb{E}\,f(V) = +\infty$.
- If $U \perp\!\!\!\perp W$, $W \in L^1(\mathbb{P})$, $\mathbb{E}\,W = 0$, then $U \preceq_{cvx} V = U + W$. $[\mu \preceq_{cvx} \mu * \nu]$
- $\forall\, u \in \mathbb{R}^d$, $\delta_u \preceq_{cvx} V$. $[\delta_u \preceq_{cvx} \mu]$
- Gaussian distributions (centered): Let $Z \sim \mathcal{N}(0, I_q)$ on $\mathbb{R}^q$ and let A, $B \in \mathbb{M}_{d,q}$ be $d \times q$ matrices
$$AA^* \le BB^* \;\text{ in }\; \mathcal{S}^+(d, \mathbb{R}) \Longrightarrow AZ \preceq_{cvx} BZ$$
or equivalently $\mathcal{N}(0, AA^*) \preceq_{cvx} \mathcal{N}(0, BB^*)$.
In particular if $d = q = 1$, $|\sigma| \le |\vartheta| \Rightarrow \mathcal{N}(0, \sigma^2) \preceq_{cvx} \mathcal{N}(0, \vartheta^2)$.
- Proof. Let $Z_1, Z_2 \sim \mathcal{N}(0; I_q)$ be independent. Set
$$U = AZ_1, \quad V = U + (BB^* - AA^*)^{1/2} Z_2.$$
Then $U = \mathbb{E}\,(V\,|\,U)$ and $V \sim \mathcal{N}\Big(0, AA^* + \big((BB^* - AA^*)^{1/2}\big)^2\Big) = \mathcal{N}(0, BB^*)$.

- Radial distributions (generalization): Let $Z : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^Q$ having a radial distribution in the sense

$$\forall\, O \in \mathcal{O}(q), \quad OZ \sim Z.$$

Let A, $B \in \mathbb{M}_{d,q}$. Then

$$AA^* \leq BB^* \ \text{ in } \ \mathcal{S}^+(d, \mathbb{R}) \Longrightarrow \ AZ \preceq_{cvx} BZ$$

We skip the proof (exercise with solution in ([1])).

[1] B. Jourdain, G. Pagès, Convex order, quantization and monotone approximations of ARCH models, *Journal of Theoretical Probability*, 35, (4), 2480–2517,2022

- If $U \preceq_{cvx} V$ and $U' \preceq_{cvx} V'$, $U \perp\!\!\!\perp U'$, $V \perp\!\!\!\perp V'$ then

$$U + U' \preceq_{cvx} V + V'.$$

$[\mu \preceq_{cvx} \nu$ and $\mu' \preceq_{cvx} \nu' \Rightarrow \mu * \mu' \preceq_{cvx} \nu * \nu']$. By Fubini's Theorem

$$\mathbb{E}\, f(U+U') = \int_{\mathbb{R}^d} \mathbb{E}\, f(u+U')\mathbb{P}_U(du) \le \int_{\mathbb{R}^d} \mathbb{E}\, f(u+V')\mathbb{P}_U(du)$$
$$\le \int_{\mathbb{R}^d} \mathbb{E}\, f(u+V')\mathbb{P}_{U'}(du) = \mathbb{E}\, f(U'+V').$$

- If $(U_n)_{n\ge1}$ i.i.d.$\sim U$ and $(V_n)_{n\ge1}$ i.i.d.$\sim V$, centered, $\perp\!\!\!\perp N, M$, $N \le M$, having values in $\mathbb{N}_0$, integrable

$$\sum_{k=1}^{N} U_k \preceq_{cvx} \sum_{k=1}^{N} V_k \preceq_{cvx} \sum_{k=1}^{M} V_k.$$

Obvious by induction.

Example II: martingales, peacocks

- If $(X_t)_{t\geq 0}$ is a martingale, then

$$t \longmapsto X_t \text{ is non-decreasing for the convex ordering}$$

i.e. $0 \leq s \leq t \Rightarrow X_s \preceq_{cvx} X_t$ since

$$\forall\, 0 \leq s \leq t, \quad X_s = \mathbb{E}\left(X_t \,|\, X_s\right).$$

- More generally, a process such that

$$t \longmapsto X_t \text{ is non-decreasing for the convex ordering}$$

is called p.c.o.c (for "Processus Croissant pour l'Ordre Convexe" in French) or even "peacock"...).

- Thus, any martingale is a peacock !
- More generally, if $X_t \sim M_t$, $t \geq 0$, where $(M_t)_{t\geq 0}$ is a martingale, then $(X_t)_{t\geq 0}$ is a peacock

About converses of "$U = \mathbb{E}(V \mid U) \Rightarrow U \preceq_{cvx} V$" and "1-martingale $\Rightarrow$ p.c.o.c."

- Strassen's Theorem (1965): $\mu \preceq_{cvx} \nu \iff \exists$ transition $P(x, dy)$ s.t.

$$\nu = \mu P \quad \text{and} \quad \forall\, x \in \mathbb{R}^d, \quad \int y\, P(x, dy) = x.$$

- Kellerer's Theorem (1972): X is a p.c.o.c $\iff$

  There exists a martingale $(M_t)_{t\geq 0}$ such that $X_t \stackrel{d}{=} M_t,\ t \geq 0$,

  (X is sometimes called a "1-martingale").

- Both proofs are unfortunately non-constructive.
- In Hirsch, Roynette, Profeta & Yor's monography ([2]), many (many...) explicit "representations" of p.c.o.c. by true martingales. Also, investigations on 2-martingales, n-martingales...

[2] *Peacocks and Associated Martingales, with Explicit Constructions*, Springer, 2011.

A revival motivated by Finance...

- A starter! t being fixed, $\sigma \mapsto e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ is a p.c.o.c. since

$$\forall\, \sigma > 0, \quad e^{\sigma W_t - \frac{\sigma^2 t}{2}} \stackrel{d}{=} e^{W_{\sigma^2 t} - \frac{\sigma^2 t}{2}} \ (\to \sigma\text{-martingale}).$$

- Application to Black-Scholes model $S^\sigma_t = s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}}$. For every convex payoff function $f : \mathbb{R}_+ \to \mathbb{R}_+$

$$\sigma \le \sigma' \Longrightarrow \mathbb{E}\, f(S^\sigma_t) \le \mathbb{E}\, f(S^{\sigma'}_t)$$

- Vanilla options: *Call* and *Put* options: $f(S_T) = (S_T - K)^+$, $f(S_T) = (K - S_T)^+$, etc.

Path-dependent payoffs

- E.g. what about path-dependent options like Asian payoffs. Let $f:\mathbb{R}_+\to\mathbb{R}_+$ convex

$$\sigma \longmapsto \mathrm{Premium}(\sigma) = \mathbb{E}\Big[f\Big(\frac{1}{T}\int_0^T \underbrace{s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}}}_{=S_t^\sigma} dt\Big)\Big] \;?$$

- P. Carr et al. (2008): Non-decreasing in σ when $f(x)=(x-K)^+$ (Asian Call).
- M. Yor (2010): $\sigma \mapsto \frac{1}{T}\int_0^T s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}} dt$ is a p.c.o.c. though not a martingale).

  (Hint: Representation using a Brownian sheet so that it has the 1-marginals of a martingale).
- Yields bounds on the option prices of vanilla options: $\sigma_{\min}\le\sigma\le\sigma_{\max}\Longrightarrow$ etc.
- This is a functional convex ordering of the first kind based on path-dependence. (see e.g. (for discrete time) path-dependent payoff functions [Brown, Rogers, Hobson 2001, Rüschendorf, 2008]).

▷ This suggests many other (new or not so new) questions !

- Switch from *BS* to local volatility models *i.e.* from scalar (or vector) parameter to a functional parameter.

  $\sigma \rightsquigarrow \sigma(x)$ "functional" convex ordering of the second kind

  (see [El Karoui-Jeanblanc-Schreve, 1998]), etc) i.e.

  $dX_t = \sigma(X_t)dW_t,\ X_0 \perp\!\!\!\perp W$ versus $dY_t = \theta(Y_t)dW_t,\ Y_0 \perp\!\!\!\perp W,\ X_0 \preceq_{cvx} Y_0$?

- Non-decreasing convex ordering: $\exists$ drift b! (see [Hajek, 1985] ([3]).
- "Fully" path-dependent convex ordering (twice functional. . .) (see [P.2016]).
- Bermuda and American options (see [Pham 2005, Rüschendorf 2008], [P. 2016]).
- Jumpy risky asset dynamics for (X_t^σ)? (see [Rüschendorf-Bergenthum, 2007], [P. 2016]).
- P.c.o.c. trough Martingale Optimal Transport. [Beïgelbock, Henry-Labordère et al, 2013, Tan, Touzi,Henry-Labordère 2015, Jourdain-P. 2020].

[3] Hajek, B., Mean stochastic comparison of diffusions. Z. Wahrsch. Verw. Gebiete 68 (1985), no. 3, 315–329.

More questions about convexity

- A side (?) question of interest : propagation of convexity in the sense

$$f : \mathbb{R} \to \mathbb{R} \text{ convex } \Longrightarrow x \longmapsto \mathbb{E}\, f(X^x_T) \text{ convex ?}$$

e.g. in a1D- local volatility model like

$$X^x_t = x + \int_0^t r\, X^x_s ds + \int_0^t X^x_s \vartheta(s, X^x_s) dW_s.$$

- More generally, when do we have such propagation of convexity if

$$X^x_t = x + \int_0^t \alpha(X^x_s + \beta) ds + \int_0^t \sigma(s, X^x_s) dW_s \quad ?$$

- Extensions to convex functionals $F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}$ and to higher dimensional processes ($d \geq 2$) ?
- Similar questions for monotonic convexity with a more general drift

$$X^x_t = x + \int_0^t b(s, X^x_s) ds + \int_0^t \sigma(s, X^x_s) dW_s.$$

Direct approach: first reduction

- Assume $\sigma(t,y)$ Lipschitz in y uniformly in $t \in [0,T]$ and $\sigma(\cdot,0)$ bounded.
- Let $f:\mathbb{R}\to\mathbb{R}$ be convex

$$X_t^x = x + \int_0^t \alpha(X_s^x + \beta)ds + \int_0^t \sigma(s, X_s^x)dW_s.$$

- Setting

$$\widetilde{X}_t^x = e^{\alpha t}X_t - \beta(1-e^{\alpha t})$$

and

$$\widetilde{\sigma}(t,y) = e^{\alpha t}\sigma\big(t, e^{-\alpha t}y - \beta(1-e^{-\alpha t})\big)$$

yields

$$\widetilde{X}^x = x + \int_0^t \widetilde{\sigma}(s, \widetilde{X}_s^x)dW_s$$

where $\widetilde{\sigma}(t,y)$ Lipschitz in y uniformly in $t \in [0,T]$.

- Hence, we may assume w.l.g. $\alpha = \beta = 0$.

Direct approach: Tangent flow ($d = 1$)

- If ([4]) f is smooth then

$$\partial_x \mathbb{E}\, f(X_T^x) = \mathbb{E}\, f'(X_T^x)\, Y_T^{(x)}$$

where

$$Y_t^{(x)} = \mathcal{E}\Big(\int_0^{\cdot} \sigma'_x(s, X_s^x)dW_s\Big)_t = \exp\Big(\int_0^t \sigma'_x(s, X_s^x)dW_s - \tfrac{1}{2}\int_0^t \sigma'_x(s, X_s^x)^2 ds\Big).$$

- Let $\mathbb{Q} = Y_T^{(x)} \cdot \mathbb{P}$, the probability on $(\Omega, \mathcal{A}, \mathbb{P})$ under which (Girsanov)

$$B_t = W_t - \int_0^t \sigma'_x(s, X_s^x)ds \quad \text{is a standard } \mathbb{Q} \text{ Brownian motion.}$$

- Then

$$X_t^x = x + \int_0^t \sigma\sigma'_x(s, X_s^x)ds + \int_0^t \sigma(s, X_s)dB_s$$

and

$$\partial_x \mathbb{E}\, f(X_T^x) = \mathbb{E}_{\mathbb{Q}}\, f'(X_T^x).$$

[4] see El Karoui et al. 1998, Robustness of the Black and Scholes formula, *Math. Fin.*

Direct approach: conclusion ($d = 1$)

- If $\sigma\sigma'_x$ is Lipschitz in space uniformly in time, then ([5]).

$$\mathbb{Q}\text{-}a.s. \quad x \mapsto X_t^x \quad \text{is non-decreasing} \ldots$$

- Hence

$$\mathbb{Q}\text{-}a.s. \quad x \mapsto f'(X_t^x) \quad \text{is non-decreasing} \ldots$$

- and so is

$$\partial_x \mathbb{E}\, f(X_T^x) = \mathbb{E}_{\mathbb{Q}}\, f'(X_T^x).$$

- Which ensures that $x \mapsto \mathbb{E}\, f(X_T^x)$ is convex. □

- Few comments:

  ▷ Extension for free to any convex function using the right derivative f'_r.

  ▷ Note that there is no convexity assumption required on σ.

  ▷ But beyond: the present proof is one-dimensional. What about $d \geq 2$ or switching from $f(X_T^x) \rightsquigarrow F\big((X_t^x)_{t\in[0,T]}\big)$?

[5] see Thm 3.7, chap. IX, Revuz-Yor, *Continuous martingales and Brownian motion*, Springer,3rd ed. 1998

Monotone convexity ?

- If f is smooth then

$$\partial_x \mathbb{E}\, f(X_T^x) = \mathbb{E}\Big[f'(X_T^x)\underbrace{e^{\int_0^T b'_x(s,X_s^x)ds}\, Y_T^{(x)}}_{\text{“new” tangent flow}}\Big] = \mathbb{E}_{\mathbb{Q}}\Big[f'(X_T^x)e^{\int_0^T b'_x(s,X_s^x)ds}\Big]$$

with

$$X_t^x = x + \int_0^t \big(b + \sigma\sigma'_x\big)(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s.$$

- If f is convex non-decreasing and $b(t,\cdot)$ is convex in x then f' is non-negative and non-decreasing and $b'_x(t, cdot)$ is non-decreasing. Hence

$$\partial_x \mathbb{E}\, f(X_T^x) \qquad \text{is non-negative non-decreasing}$$

i.e. $x \mapsto \mathbb{E}\, f(X_T^x)$ is is convex non-decreasing.

Aims and methods

1. Unify and generalize existing results with of focus on both functional aspects of functional convex ordering.
   - with a focus on both functional aspects of functional convex ordering.
   - As a by-product establish the convexity of $x \mapsto \mathbb{E}\, f(X_T^x)$ and/or $x \mapsto \mathbb{E}\, F(x^x)$.
2. Constraint: provide a constructive method of proof.
   - based on time discretization of continuous time martingale dynamics (risky assets in Finance) .
   - using numerical schemes that preserve the functional convex order satisfied by the process under consideration...
   - to avoid arbitrages.
3. Apply the paradigm to various frameworks:
   - American style options,
   - jump diffusions,
   - stochastic integrals,
   - McKean-Vlasov diffusions,
   - Volterra equations,
   - etc?

Example III: risk measure

- Let $X \in L^1\mathbb{P}$ be representative of a loss (with no atom for convenience) with c.d.f F_x.
- Let $\alpha \in (0,1]$, $\alpha \simeq 1$ be a risk level. Then

$$\mathrm{VaR}_\alpha(X) := (F_x)^{-1}(\alpha) \quad \text{and} \quad \mathrm{CVaR}_\alpha(X) := \mathbb{E}(X \,|\, X \geq \mathrm{Var}_\alpha(X))$$

- Rockafeller-Uryasev's representation of these two risk measures

$$L_{\alpha,X}(\xi) = \xi + \frac{1}{1-\alpha}\mathbb{E}\,(X-\xi)^+$$

satisfies

$$\mathrm{Var}_\alpha(X) = \mathrm{argmin}_{\mathbb{R}} L_{\alpha,X} \quad \text{and} \quad \mathrm{CVaR}_\alpha(X) = \min_{\mathbb{R}} L_{\alpha,X}.$$

- As a consequence

$$X \preceq_{icv} Y \Longrightarrow L_{\alpha,X} \leq L_{\alpha,Y}$$

so that

$$\mathrm{CVaR}_\alpha(X) \leq \mathrm{CVaR}_\alpha(Y).$$

- WARNING! Not true for the value-at-risk.

Characterization of convex ordering

Proposition

(a) Let $U,\ V \in L^1_{\mathbb{R}^d}(\mathbb{P})$. There is equivalence between

$$U \preceq_{cvx} V$$

and

$$\forall\, f : \mathbb{R}^d \to \mathbb{R} \text{ convex and Lipschitz continuous } \mathbb{E}\, f(U) \le \mathbb{E}\, f(V)$$

(b) Similar equivalence for $\le_{icv}$ and $\le_{dcv}$ (when $d = q = 1$).

The proof relies on the following lemma based on inf-convolution.

Lemma

Any convex function $f : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$f = \lim_n {}^{\uparrow} f_n, \quad f_n \text{ convex and Lipschitz continuous}, \ n \ge 1.$$

The functions f_n have the same monotonicity as f, if any.

Proof (lemma)

- We introduce the functions f_n defined through inf-convolution on $\mathbb{R}^d$ by

$$f_n(x) := \inf_{y\in\mathbb{R}^d}\big(f(y)+n|x-y|\big),\ n\geq 1.$$

One has by construction

$$\forall\, n\geq 1,\quad f_n\leq f_{n+1}\leq f.$$

- $f_n \uparrow f$ in a stationary way: let denote by $\nabla_s f(x)$ any subgradient of f at x.

$$\begin{aligned}\forall\, y\in\mathbb{R}^d,\ f(y)+n|y-x| &\geq f(x)+\langle\nabla_s f(x)\,|\,y-x\rangle+n|y-x| \text{ by convexity of } f\\ &\geq f(x)+(n-|\nabla_s f(x)|)|y-x|\\ &\geq f(x)\end{aligned}$$

Hence, $\forall\, n\geq|\nabla_s f(x)|,\quad f_n(x)\geq f(x)$ so that $f_n(x)=f(x)$.

- f_n is convex since, for $x,x'\in\mathbb{R}^d$, $\lambda\in[0,1]$,

$$\begin{aligned}f_n(\lambda x+(1-\lambda)x') &= \inf_{y,y'} f(\lambda y+(1-\lambda)y'))+n|\lambda(x-y)+(1-\lambda)(x'-y')|\\ &\leq \lambda\inf_y\big(f(y)+n|x-y|\big)+(1-\lambda)\inf_{y'}\big(f(y')+n|x'-y'|\big)\\ &=\lambda f_n(x)+(1-\lambda)f_n(x').\end{aligned}$$

Proof ($\Leftarrow$ of proposition)

- f_n are n-Lipschitz continuous since

$$|f_n(x) - f_n(x')| \le \sup_{y\in\mathbb{R}^d} \big| n|x-y| - n|x'-y| \big| \le n|x-x'|.$$

- $f_n(x) = \inf_y \big(f(x+y) + n|y|\big)$ has the same monotonicity as f ... if any. $\square$

Proof of the proposition.

- Assume f convex, then for every $n \ge 1$, $\mathbb{E}\, f_n(U) \le \mathbb{E}\, f_n(V)$.
- The functions f_n^-, $n \ge |\nabla_s f(0)|$, are dominated since

$$\begin{aligned}\forall\, x,\ y \in \mathbb{R}^d,\ f_n(x) &\ge f(0) + \langle \nabla_s f(0)\,|\,y\rangle + n|y-x|.\\ &\ge f(0) + |y|(n - |\nabla_s f(0)|) - n|x| \ge f(0) - n|x|.\end{aligned}$$

- As $U,\ V \in L^1(\mathbb{P})$, one has by the monotone convergence theorem

$$-\infty < \mathbb{E}\, f(U) \le \mathbb{E}\, f(V) \le +\infty.$$

Functional convex ordering: Definition

Assume $\mathcal{C}_T = \mathcal{C}([0,T],\mathbb{R}^d)$ is equipped with sup-norm $\|f\|_{\sup} = \sup_{u\in[0,T]} |f(u)|$.

Definition

Let $X, Y : (\Omega,\mathcal{F},\mathbb{P}) \to \mathcal{C}([0,T],\mathbb{R}^d)$ be two integrable continuous processes such that $\mathbb{E}[\|X\|_{\sup} + \|Y\|_{\sup}] < +\infty$.

(a) *Convex ordering.* We say that X is dominated by Y for the *convex ordering* – denoted by $X \preceq_{cvx} Y$ – if, for every l.s.c. (for the $\|\cdot\|_{\sup}$-norm topology) convex functional $F : \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbb{R}$,

$$\mathbb{E}\, F(X) \le \mathbb{E}\, F(Y). \tag{2}$$

(b) *Monotone convex ordering ($d = 1$).* We say that X is dominated by Y for the *increasing/decreasing convex ordering* if (2) holds for every non-increasing/non-decreasing for the pointwise partial order on $\mathcal{C}$ l.s.c. convex functional $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$. These orderings are denoted by

$$X \preceq_{icv} Y \quad \text{and} \quad X \preceq_{dcv} Y \quad \text{respectively.}$$

Characterization of functional convex ordering

- Do we have the same characterization for Lipschitz functionals ? Yesss!

Proposition

Let X, Y be two $\mathcal{C}([0,T],\mathbb{R}^d)$-valued r.v. (i.e. pathwise continuous stochastic processes) such that $\mathbb{E}[\|X\|_{\sup}+\|Y\|_{\sup}]<+\infty$.

(a) *Convex order*. Both statements are equivalent:

$$X \preceq_{cvx} Y$$

and

$$\forall\, F\in\mathcal{C}([0,T],\mathbb{R}^d)\to\mathbb{R},\ \|\cdot\|_\infty\text{-Lipschitz continuous},\ \mathbb{E}\,F(X)\le\mathbb{E}\,F(Y). \tag{3}$$

(b) *Pointwise monotonic convex ordering $(d=1)$*. Similar equivalence for $X\preceq_{icv} Y$ and $X\preceq_{dcv} Y$ with respect to pointwise non-decreasing (resp. non-increasing) Lipschitz convex functionals $F:\mathcal{C}([0,T],\mathbb{R}^d)\to\mathbb{R}$.

- The key is the following miracle-lemma!

Miracle lemma

Lemma (Quasi-subgradient)

([a]) Let $(E, \|\cdot\|)$ be a normed vector space and let $F : E \to \mathbb{R}$ be an l.s.c. convex functional (for the norm topology).

For every $x \in E$ and every $a \in (-\infty, F(x))$; there exists $G = G_{x,a} \in E'$ and $g = g_{x,a} \in \mathbb{R}$ such that

$$\begin{array}{ll} (i) & \forall\, u \in E, \quad G(u) + g \leq F(u), \\ (ii) & G(x) + g = a. \end{array}$$

[a] See Lemma 7.5 in Aliprantis, Charalambos D. and Border, Kim C., *Infinite dimensional Analysis*, Springer, 2006.

- The linear forms $G_{x,a}$, $-\infty < a < F(x)$ play the role of the sub gradient and the characterization in $\mathbb{R}^d$ can be extended to this framework with $E = \mathcal{C}([0, T], \mathbb{R}^d)$.
- One shows likewise that $\mathbb{E}\, F(X) \in (-\infty, +\infty]$ and the characterization by Lipschitz continuous functionals.

Paradigm of convex ordering by Wasserstein approximation

- Let $(E, |\cdot|_E)$ be a Banach space and

$$\mathcal{P}_1(E) = \Big\{\mu \text{ distribution on } \big(E, \mathcal{B}or(E)\big) : \int_E |\xi|_E \mu(d\xi) < +\infty\Big\}$$

be the convex set of integrable probability measures equipped with the (metric) topology of $\mathcal{W}_1$ the Wasserstein/Monge-Kantorovich distance.

$$\mathcal{W}_1(\mu,\nu) = \inf\Big\{\int |x-y| m(dx,dy),\ m(dx,E)=\mu,\ m(E,dy)=\nu\Big\} = \sup\Big\{\int f d\mu - \int f d\nu, [f]_{\rm Lip} \le 1\Big\}.$$

- Let X and Y be two E-valued random variables and let $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ two sequences of E-valued random variables such that

$$\begin{aligned} (i) &\quad \forall\, n \geq 1, \quad X_n \preceq_{cvx} Y_n \\ (ii) &\quad \mathcal{W}_1([X_n],[X]) + \mathcal{W}_1([Y_n],[Y]) \to 0 \text{ as } n \to +\infty \end{aligned}$$

where $[X] \in \mathcal{P}_1(E)$ denotes the distribution of X. Then

$$X \preceq_{cvx} Y.$$

Proof of the paradigm

- Let $F : E \to \mathbb{R}$ be a Lipschitz continuous function. Assumption (i) implies that

$$\mathbb{E}\, F(X_n) \le \mathbb{E}\, F(Y_n), \quad n \ge 1.$$

- Then, by (ii) and the Monge-Kantorovich characterization of $\mathcal{W}_1$-distance

$$\big|\mathbb{E}\, F(X_n) - \mathbb{E}\, F(X)\big| \le [F]_{\rm Lip}\mathcal{W}_1\big([X_n],[X]\big) \to 0 \text{ as } n \to +\infty,$$

- Idem for Y_n and Y.
- Letting $n \to +\infty$ in the first inequality yields the conclusion. □

▷ Application to $E = \mathcal{C}([0,T],\mathbb{R}^d), \|\cdot\|_{\sup})$.

▷ Adaptation to partially-ordered Banach space is straightforward.

▷ Other extensions e.g. to metric vector spaces (think to Skorokhod topology on $\mathbb{D}([0,T],\mathbb{R}^d)$.)

Martingale (and scaled) Brownian diffusions

- If we want to compare on (l.s.c.) convex functionals $F : \mathcal{C}([0,T], \mathbb{R}^d) \to \mathbb{R}$,

$$\mathbb{E}\, F(X) \quad ? \quad \mathbb{E}\, F(Y)$$

where

$$dX_t = \sigma(t, X_t)dW_t,\ X_0 \perp\!\!\!\perp W \ \text{ versus } \ dY_t = \theta(t, Y_t)dW_t,\ Y_0 \perp\!\!\!\perp W,\ X_0 \preceq_{cvx} Y_0?$$

in a higher dimensional setting:

– W q-dimensional B.M.,

– $\sigma(t, \cdot) : \mathbb{R}^d \to \mathbb{M}_{d,q}(\mathbb{R})$

we need:

- a pre-order on matrices,
- the resulting notion of convexity for matrix-valued vector fields.

Martingale (and scaled) Brownian diffusions

- Pre-order $\preceq$ on $\mathbb{M}_{d,q}(\mathbb{R})$: let $A,\ B \in \mathbb{M}_{d,q}(\mathbb{R})$.

$$A \preceq B \quad \text{if} \quad BB^* - AA^* \in \mathcal{S}^+(d,\mathbb{R}).$$

  [If $d = q = 1$, $a \preceq b$ iff $|a| \leq |b|$]

- $\preceq$-Convexity: $\sigma : \mathbb{R}^d \to \mathbb{M}_{d,q}$ is $\preceq$-convex if

  $\forall\, x,\ y \in \mathbb{R}^d,\ \lambda \in [0,1]$, there exists $O_{\lambda,x},\ O_{\lambda,y} \in \mathcal{O}(q,\mathbb{R})$ such that

$$\sigma\big(\lambda x + (1-\lambda)y\big) \preceq \lambda\sigma(x)O_{\lambda,x} + (1-\lambda)\sigma(y)O_{\lambda,y}$$

  i.e.

$$\sigma\sigma^*\big(\lambda x+(1-\lambda)y\big) \leq \big(\lambda\sigma(x)O_{\lambda,x}+(1-\lambda)\sigma(y)O_{\lambda,y}\big)\big(\lambda\sigma(x)O_{\lambda,x}+(1-\lambda)\sigma(y)O_{\lambda,y}\big)^*$$

- $d = q = 1$ with $O_{\lambda,x} = \mathrm{sign}\big(\sigma(x)\big)$ this simply reads

$$|\sigma| convex.$$

- $\Longrightarrow$ WARNING! Then, f $d = q = 1$, σ $\preceq$-convex means $|\sigma|$ convex !!

Examples

- Let $\lambda_k : \mathbb{R} \to \mathbb{R}$, $k = 1 : q$ be Lipschitz functions such that $|\lambda_k|$ are all convex. Set

$$\sigma(x) := A\,\mathrm{Diag}(\lambda_1(x), \ldots, \lambda_q(x))O,\; A \in \mathbb{M}_{d,q}(\mathbb{R}),\; O \in \mathcal{O}(q, \mathbb{R})$$

then σ is $\preceq$-convex.

- When $q = d$, σ $\preceq$-convex is equivalent to

$$\sigma\sigma^*(\alpha x + (1-\alpha)y) \le \Big(\alpha\sqrt{\sigma\sigma^*}(x) + (1-\alpha)\sqrt{\sigma\sigma^*}(y)\Big)\Big(\alpha\sqrt{\sigma\sigma^*}(x) + (1-\alpha)\sqrt{\sigma\sigma^*}(y)\Big)^*.$$

Theorem (Strong martingale diffusion, P. 2016, Fadili-P. 2017, Jourdain-P. 2021)

Let $\sigma, \theta \in \mathrm{Lip}\big([0,T]\times\mathbb{R}^d, \mathbb{M}_{d,q}\big)$, *$W$ q-S.B.M.. Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the unique strong solutions to*

$$\begin{aligned} dX_t^{(\sigma)} &= \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},\ X_0^{(\sigma)} \in L^1 \\ dX_t^{(\theta)} &= \theta(t, X_t^{(\theta)})dW_t^{(\theta)},\ X_0^{(\theta)} \in L^1, \quad (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.} \end{aligned}$$

(a) If $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ *and*

$$\begin{cases} (i)_\sigma & \sigma(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ \text{or} & \\ (i)_\theta & \theta(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ \text{and} & \\ (ii) & \sigma(t,\cdot) \preceq \theta(t,\cdot) \text{ for every } t \in [0,T], \end{cases}$$

then:

– for every l.s.c. convex $F : \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbb{R}$, $\mathbb{E}\,F(X^{(\sigma)}) \le \mathbb{E}\,F(X^{(\theta)})$

– if $(i)_\sigma$ *holds true, then one also have*

$$x \mapsto \mathbb{E}\,F(X^{(\sigma),x}) \text{ is convex.}$$

Theorem (Weak Martingale diffusions, P. 2016, Fadili-P. 2017)

Let $\sigma, \theta \in \mathcal{C}_{lin_x, unif}\big([0,T]\times\mathbb{R}^d, \mathbb{M}_{d,q}\big)$, $W^{(\sigma)}$, $W^{(\theta)}$ *q-S.B.M.. Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the unique weak solutions to*

$$\begin{aligned} dX_t^{(\sigma)} &= \sigma(t, X_t^{(\sigma)})dW_t,\ X_0^{(\sigma)} \in L^{1+\eta} \\ dX_t^{(\theta)} &= \theta(t, X_t^{(\theta)})dW_t,\ X_0^{(\theta)} \in L^{1+\eta}, \quad (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.} \end{aligned}$$

(a) *If* $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ *and*

$$\begin{cases} (i)_\sigma & \sigma(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ \text{or} & \\ (i)_\theta & \theta(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ \text{and} & \\ (ii) & \sigma(t,\cdot) \preceq \theta(t,\cdot) \text{ for every } t \in [0,T], \end{cases}$$

then:

– *for every convex* $F : \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbb{R}$, $\mathbb{E}\,F(X^{(\sigma)}) \le \mathbb{E}\,F(X^{(\theta)})$

– *if* $(i)_\sigma$ *holds true and* F *has* $\|\,.\,\|_{\sup}$*-polynomial growth*

$$x \mapsto \mathbb{E}\,F(X^{(\sigma),x}) \text{ is convex.}$$

The 1D case (martingale case)

Theorem (P. 2016)

Let $\sigma, \theta \in \mathcal{C}_{lin_x,unif}([0,T]\times\mathbb{R},\mathbb{R})$. *Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the* unique *weak solutions to*

$$dX_t^{(\sigma)} = \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},\ X_0^{(\sigma)} \in L^1$$
$$dX_t^{(\theta)} = \theta(t, X_t^{(\theta)})dW_t^{(\theta)},\ \ X_0^{(\theta)} \in L^1,\ (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.}$$

(a) *If* $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ *and*

$$\begin{cases} (i)_\sigma & |\sigma(t,.)| : \mathbb{R}\to\mathbb{R}_+ \text{ is convex for every } t\in[0,T],\\ \text{or} & \\ (i)_\theta & |\theta(t,.)| : \mathbb{R}\to\mathbb{R}_+ \text{ is convex for every } t\in[0,T],\\ \text{and} & \\ (ii) & |\sigma(t,\cdot)| \le |\theta(t,\cdot)| \text{ for every } t\in[0,T] \end{cases}$$

then:

– *for every l.s.c. convex* $F : \mathcal{C}([0,T],\mathbb{R}^d)\to\mathbb{R}$, $\mathbb{E}\,F(X^{(\sigma)}) \le \mathbb{E}\,F(X^{(\theta)})$

– *if* $(i)_\sigma$ *holds true and* F *has* $\|\,.\,\|_{\sup}$*-polynomial growth*

$$x \longmapsto \mathbb{E}\,F(X^{(\sigma),x}) \text{ is convex}$$

Scaled/drifted martingale diffusions (extension to)

- The former theorems still hold true for

$$X_t^{(\sigma)} = X_0^{(\sigma)} + \int_0^t \alpha(t)\big(X_t^{(\sigma)} + \beta(t)\big)dt + \int_0^t \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},$$

$$X_t^{(\theta)} = X_0^{(\theta)} + \int_0^t \alpha(t)\big(X_t^{(\theta)} + \beta(t)\big)dt + \int_0^t \theta(t, X_t^{(\theta)})dW_t^{(\theta)},$$

  where $\alpha(t) \in \mathbb{M}_{d,d}$ and $\beta(t) \in \mathbb{R}^d$ are Hölder continuous.

- Change of variable:

$$\widetilde{X}_t^{(\sigma)} = e^{-\int_0^t \alpha(s)ds}\big(X_t^{(\sigma)} + \beta(t)\big), \;\; \text{etc.}$$

- Finance: spot interest rate $\alpha(t) = r(t)\mathbf{1}$ and $\beta(t) = 0$ since typical (risk-neutral) dynamics of traded assets read

$$dS_t = r(t)S_t dt + S_t\sigma(S_t,)dW_t.$$

Functional Hajek's Theorem on Monotone convex ordering ($d = q = 1$)

Let

$$X_t^{(\sigma)} = X_0^{(\sigma)} + \int_0^t b_1(t, X_t^{(\sigma)})dt + \int_0^t \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},$$
$$X_t^{(\theta)} = X_0^{(\theta)} + \int_0^t b_2(t, X_t^{(\theta)})dt + \int_0^t \theta(t, X_t^{(\theta)})dW_t^{(\theta)}.$$

where all coefficients $b_i(t,\cdot)$, $\sigma(t,\cdot)$, $\theta(t,\cdot)$ are Lipchitz, uniformly in $t \in [0,T]$.

Theorem (Strong solution version)

Assume furthermore

$$(*)_1 \equiv b_1(t,\cdot) \textit{ and } |\sigma(t,\cdot)| \textit{ convex } \forall\, t \in [0,T])$$
or
$$(*)_2 \equiv b_2(t,\cdot) \textit{ and } |\theta(t,\cdot)| \textit{ convex } \forall\, t \in [0,T],$$

and $\quad b_1(t,\cdot) \le b_2(t,\cdot),\ |\sigma(t,\cdot)| \le |\theta(t,\cdot)|$ *and* $X_0^{(\sigma)} \le_{icv} X_0^{(\theta)}$

Theorem (continued)

Then:

– *for every l.s.c. convex, pointwise non-decreasing* $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$,

$$\mathbb{E}\, F(X^{(\sigma)}) \le \mathbb{E}\, F(X^{(\theta)}).$$

– *if* $(i)_\sigma$ *holds true*

$$x \mapsto \mathbb{E}\, F(X^{(\sigma),x}) \text{ is non-decreasing and convex.}$$

- Hajek's original theorem dealt with marginal convex ordering.
- Assume $(*)_1$. One defines for f non-decreasing and convex and $0 < h < 1/[b_1]_{\mathrm{Lip}}$. Then

$$Q_\gamma f(x,u) = \mathbb{E}\, f\big(x + hb_1(x) + \sqrt{h}\sigma(x)Z\big)$$

  is convex and nondecreasing in both x and u.
- Mimick the former proof.

Strategy (constructive)

- Time discretization (preferably) accessible to simulation: typically the Euler scheme.
- Propagate convexity (marginal or pathwise)
- Propagate comparison (marginal or pathwise)
- Transfer by Wasserstein distance or by functional limit theorems "à la Jacod-Shiryaev".

Step 1: discrete time ARCH models

- ARCH dynamics: Let $(Z_k)_{1\le k\le n}$ be a sequence of independent, radial r.v. on $(\Omega, \mathcal{A}, \mathbb{P})$. Two ARCH models: $X_0, Y_0 \in L^1(\mathbb{P})$,

$$\begin{aligned} X_{k+1} &= X_k + \sigma_k(X_k)\, Z_{k+1}, \\ Y_{k+1} &= Y_k + \theta_k(Y_k)\, Z_{k+1}, \quad k = 0 : n-1, \end{aligned}$$

where $\sigma_k,\ \theta_k : \mathbb{R} \to \mathbb{R},\ k = 0 : n-1$ have linear growth.

Proposition (Propagation result)

If $\sigma_k,\ k = 0 = n-1$ are $\preceq$-convex with linear growth,

$$X_0 = x \quad \text{and} \quad \forall\, k \in \{0, \ldots, n-1\}, \quad \sigma_k \preceq \theta_k,$$

then, for every convex function $F : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ convex with linear growth

$$x \longmapsto \mathbb{E}\, F(x, X_1^x \ldots, X_n^x) \quad \textit{is convex}.$$

Partial proof (marginal) with radial white noise

- $Z_k \sim \mathcal{N}(0, I_q)$, $1 \le k \le n$ or, more generally, $Z_k \sim OZ_k$, $\forall\, O \in \mathcal{O}(q, \mathbb{R})$.
- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function (with linear growth). Let
$$P_k^\sigma f(x) := \mathbb{E} f(x + \sigma_{k-1}(x) Z_k) = \big[\mathbb{E}\, f(x + AZ_k)\big]_{|A=\sigma_{k-1}(x)}.$$
- Set $A \in \mathbb{M}_{d,q} \mapsto Q_k f(x, A) := \mathbb{E}\, f(x + AZ_k)$, $k = 1 : n$, is right $O(q, \mathbb{R})$-invariant, convex and $\preceq$-non-decreasing in A by the starting example.
  - $Q_k f(x, AO) = \mathbb{E}\, f(x + AOZ_k) = \mathbb{E}\, f(x + AZ_k)$,
  - $Q_k f(\lambda(x, A) + (1-\lambda)(y, B)) = \mathbb{E}\, f\big(\lambda(x + AZ_k) + (1-\lambda)(y + BZ_k)\big)$
    $\le \lambda Q_k f(x, A) + (1-\lambda) Q_k f(y, B)$ by convexity of f.
  - If $A \preceq B$, then $AZ_k \preceq_{cvx} BZ_k$ and $f(x + \cdot)$ is convex.
- Hence if $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$

$$\begin{aligned}
P_k^\sigma f\big(\lambda x + (1-\lambda)y\big) &= Q_k f\big(\lambda x + (1-\lambda)y, \sigma_{k-1}(\lambda x + (1-\lambda)y)\big)\\
&\le Q_k f\big(\lambda x + (1-\lambda)y, \lambda\sigma_{k-1}(x) + (1-\lambda)\sigma_{k-1}(y)\big)\\
&\le \lambda Q_k f\big(x, \sigma_{k-1}(x)\big) + (1-\lambda) Q_k f\big(y, \sigma_{k-1}(y)\big)\\
&= \lambda P_k^\sigma f(x) + (1-\lambda) P_k^\sigma f(y).
\end{aligned}$$

- Hence the transition kernels P_k^σ propagate convexity:

$$f \;\text{ convex} \Longrightarrow P_k^\sigma(f) \;\text{ convex.}$$

- by a either forward or backward induction on k, one finally gets.

$$x \longmapsto \mathbb{E}\, f(X_n^x) = P_{1:n}^\sigma f(x) := P_1^\sigma \circ \cdots P_n^\sigma f(x) \quad \text{is convex.}$$

Proposition (Discrete time convex ordering result)

If all σ_k, $k = 0 = n-1$ or all θ_k, $k = 0 : n-1$ are $\preceq$-convex with linear growth,

$$X_0 \preceq_{cvx} Y_0 \quad \textit{and} \quad \forall\, k \in \{0, \ldots, n-1\}, \quad \sigma_k \preceq \theta_k,$$

then

$$(X_0, \ldots, X_n) \preceq_{cvx} (Y_0, \ldots, Y_n).$$

Partial proof (marginal) with radial white noise

- Assume e.g. that all σ_k are convex.
- Backward induction on k.
- For $k = n$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function with linear growth.

$$P_n^\sigma f(x) = Q_n f\big(x, \sigma_{n-1}(x)\big) \le Q_n f\big(x, \theta_{n-1}(x)\big) = P_n^\theta f(x)$$

by non-decreasing $\preceq$-monotony of Q_n.

- Assume $\underbrace{P^\sigma_{k+1:n} f}_{\text{convex}} \le P^\theta_{k+1:n} f$. Then

$$\forall\, x \in \mathbb{R}^d, \quad A \in \mathbb{M}_{d,q} \longmapsto Q_k\big(P^\sigma_{k+1:n} f\big)(x, A) \quad \text{is } \preceq\text{-non-decreasing}$$

so that

$$\begin{aligned} P^\sigma_{k:n} f(x) = Q_k\big(P^\sigma_{k+1:n} f\big)\big(x, \sigma_{k-1}(x)\big) &\overset{\downarrow}{\le} Q_k\big(P^\sigma_{k+1:n} f\big)\big(x, \theta_{k-1}(x)\big) \\ &\le Q_k\big(P^\theta_{k+1:n} f\big)\big(x, \theta_{k-1}(x)\big) \\ &= P^{x,\theta}_{k:n} f(x). \end{aligned}$$

- Hence, in particular for $f : \mathbb{R}^d \to \mathbb{R}$ Lipschitz and convex

$$\mathbb{E}\, f(X_n^\sigma) = \mathbb{E}\, P^\sigma_{1:n} f(X_0) \le \mathbb{E}\, P^\sigma_{1:n} f(Y_0) \le \mathbb{E}\, P^\theta_{1:n} f(Y_0) = \mathbb{E}\, f(X_n^\theta). \qquad \square$$

Global convex ordering

- Same strategy
- But entirely backward.
- $q = d = 1$ for simplicity.

▷ Dynamic programming: We introduce two martingales

$$M_k = \mathbb{E}\big(F(X_{0:n})\,|\,\mathcal{F}_k^Z\big) \;\text{ and }\; N_k = \mathbb{E}\big(F(Y_{0:n})\,|\,\mathcal{F}_k^Z\big),\; k = 0:n$$

and again the sequence of operators

$$Q_k(f)(x,u) = \mathbb{E}\, f(x + uZ_k),\; u \in \mathbb{R},\; k = 1:n.$$

Warning (for the mini-course)

- For convenience we will make the proof in a one-dimensional setting.
- Then a slightly revisited version of Jensen's inequality simplifies the communication.
- It follows ([6])

[6] G. Pagès, Convex order for path-dependent derivatives: a dynamic programing approach, Séminaire de Probabilités, XLVIII, LNM 2168, Springer, Berlin, 33-96, 2016.

Jensen's Inequality (a bit) revisited = Key Lemma

Lemma (Jensen's Inequality revisited)

Let $Z : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ be an centered integrable r.v.: $Z \in L^1$, $\mathbb{E}\, Z = 0$.

▷ *Let $f : \mathbb{R} \to \mathbb{R}$, convex, such that*

$$\forall\, x, u \in \mathbb{R},\ Qf(x,u) := \mathbb{E}\, f\big(x + u\, Z\big) \text{ is well-defined in } \mathbb{R}.$$

Then $Qf(x + \cdot)$ is convex, attains its minimum at 0 so that

$Qf(x + .)$ is non-decreasing on $\mathbb{R}_+$, non-increasing on $\mathbb{R}_-$.

▷ *If $Z \sim -Z$ (symmetric distribution), then $Qf(x + \cdot)$ is an even function and*

$$\forall\, x \in \mathbb{R},\ \forall\, a \in \mathbb{R}_+,\quad \sup_{|u| \le a} Qf(x,u) = Qf(x,a).$$

Proof. The function Qf is clearly convex and by Jensen's Inequality

$$Qf(x,u) \ge f\big(\mathbb{E}(x + u\, Z)\big) = f\big(x + u\, \mathbb{E}\, Z\big) = f(x) = Qf(x,0).$$

Hence $Q\, f$ is convex, $Qf(x + \cdot)$ attains its minimum at $u = 0$ hence is non-increasing on $\mathbb{R}_-$ and non-decreasing on $\mathbb{R}_+$. □

- A (first) backward induction and the definition of the kernels Q_k imply

$$M_k = \Phi_k(X_{0:k}) \quad \text{and} \quad N_k = \Psi_k(Y_{0:k}), \quad k = 0, \ldots, n.$$

where $\Phi_k, \Psi_k : \mathbb{R}^{k+1} \to \mathbb{R}$, $k = 0, \ldots, n$ are recursively defined by

$$\begin{aligned} \Phi_n &:= F, \\ \Phi_k(x_{0:k}) &= \left[\mathbb{E}\, \Phi_{k+1}(x_{0:k}, x_k + uZ_{k+1})\right]_{|u=\sigma_k(x_k)} \\ &:= \big(Q_{k+1}\Phi_{k+1}(x_{0:k}, \cdot)\big)(x_k, \sigma_k(x_k)), \quad k = 0 : n-1. \end{aligned}$$

Likewise

$$\Psi_n := F, \;\; \Psi_k(y_{0:k}) := \big(Q_{k+1}\Psi_{k+1}(y_{0:k}, \cdot)\big)(y_k, \theta_k(y_k)),\; k = 0 : n-1.$$

▷ Assume now that all functions σ_k are ≥ 0 and convex:

Lemma

$$\Big(G:\mathbb{R}^{k+2}\to\mathbb{R} \text{ convex}\Big)$$
$$\Downarrow$$
$$\Big((x_{0:k},u)\mapsto \mathbb{E}G(x_{0:k},x_k+uZ_{k+1}) = Q_{k+1}G(x_{0:k},\cdot)(x_k,u) \text{ is convex}\ldots\Big)$$

so that, by the revisited Jensen's Lemma, one has

(*i*) $u\mapsto (Q_{k+1}G(x_{0:k},,\cdot)(x_k,u)$ *is* $\downarrow$ *on* $(-\infty,0)$ *and* $\uparrow$ *on* $(0,+\infty)$.

&

(*ii*) *Propagation of the convexity in* $x_{0:k}$.

▷ Assume now that all functions σ_k are ≥ 0 and convex:

Lemma

$$\Big(G:\mathbb{R}^{k+2}\to\mathbb{R}\ \textit{convex}\Big)$$
$$\Downarrow$$
$$\Big((x_{0:k},u)\mapsto \mathbb{E}G(x_{0:k},x_k+uZ_{k+1})=Q_{k+1}G(x_{0:k},\cdot)(x_k,u)\ \textit{is convex}\ldots\Big)$$

so that, by the revisited Jensen's Lemma, one has

$$(i)\quad u\mapsto\big(Q_{k+1}G(x_{0:k},,\cdot)(x_k,u)\ \textit{is}\downarrow\textit{on}\ (-\infty,0)\ \textit{and}\uparrow\textit{on}\ (0,+\infty).$$

&

(ii) Propagation of the convexity in $x_{0:k}$.

- (Second) backward induction $\Longrightarrow$ all functions Φ_k are convex.

- (Third) backward induction $\Longrightarrow \Phi_k \le \Psi_k,\ k = 0 : n-1$.

  First note that $\Phi_n = \Psi_n = F$. If $\Phi_{k+1} \le \Psi_{k+1}$, then

$$\begin{aligned}\Phi_k(x_{0:k}) &= \big(Q_{k+1}\Phi_{k+1}(x_{0,k}, x_k + .)\big)(\sigma_k(x_k))\\ &\le \big(Q_{k+1}\Phi_{k+1}(x_{0:k}, x_k + .)\big)(\theta_k(x_k))\\ &\le \big(Q_{k+1}\Psi_{k+1}(x_{0:k}, x_k + .)\big)(\theta_k(x_k)) = \Psi_k(x_{0:k}).\end{aligned}$$

- When $k = 0$

$$\Phi_0 \ \text{ convex and } \ \Phi_0(x) \le \Psi_0(x) \quad \Longleftrightarrow \quad \mathbb{E}\,F(X_{0:n}) \le \mathbb{E}\,F(Y_{0:n}).$$

  so that

$$\mathbb{E}\,F(X_{0:n}) = \mathbb{E}\,\Phi_0(X_0) \le \mathbb{E}\,\Phi_0(Y_0) \le \mathbb{E}\,\Psi_0(Y_0) = \mathbb{E}\,F(Y_{0:n}).$$

End of discrete time setting

▷ If all $\theta_k \geq 0$ and convex:

This time, one shows that:

- the functions Ψ_k are convex, $k = 0, \ldots, n$
- $\Phi_n \leq \Psi_n \Longrightarrow \Phi_k \leq \Psi_k,\ k = 0, \ldots, n-1$.

Remark. The discrete time setting has its own interest.

Step 2 of the proof: Back to continuous time

▷ Euler scheme(s): Discrete time Euler scheme with step $\frac{T}{n}$, starting at x is an ARCH model. For $X^{(\sigma)}$: for $k = 0, \ldots, n-1$,

$$\bar{X}^{(\sigma),n}_{t^n_{k+1}} = \bar{X}^{(\sigma),n}_{t^n_k} + \sigma(t^n_k, \bar{X}^{(\sigma),n}_{t^n_k})\big(W_{t^n_{k+1}} - W_{t^n_k}\big),\ \bar{X}^{(\sigma),n}_0 = x$$

Set

$$Z_k = W_{t^n_k} - W_{t^n_{k-1}},\ k = 1, \ldots, n,\ i.i.d.$$

$$\Downarrow$$

discrete time setting applies

Remark. Linear growth of σ and θ, implies

$$\forall\, p > 0, \quad \sup_{n\geq 1}\Big\| \sup_{t\in[0,T]} |\bar{X}^{(\sigma),n}_t| \Big\|_p + \sup_{n\geq 1}\Big\| \sup_{t\in[0,T]} |\bar{X}^{(\theta),n}_t| \Big\|_p \leq C(1 + \|X_0\|_p).$$

From discrete to continuous time

▷ Interpolation ($n \geq 1$)

- *Piecewise affine interpolator* defined by

$$\forall\, x_{0:n} \in \mathbb{R}^{n+1},\ \forall\, k = 0, \ldots, n-1,\ \forall\, t \in [t_k^n, t_{k+1}^n], \quad .$$

$$i_n(x_{0:n})(t) = \frac{n}{T}\Big((t_{k+1}^n - t)x_k + (t - t_k^n)x_{k+1}\Big)$$

- $\widetilde{X}^{(\sigma),n} := i_n\Big((\bar{X}^{(\sigma),n}_{t_k^n})_{k=0:n}\Big)$ = piecewise affine Euler scheme.

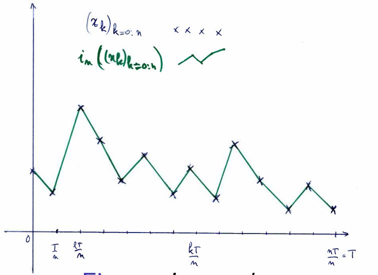

Figure: *Interpolator*

"Strong" solution setting

▷ Let $F:\mathcal{C}([0,T],\mathbb{R})\to\mathbb{R}$ be a Lipschitz convex functional.

$$\forall\, n\geq 1,\qquad F_n:\mathbb{R}^{n+1}\ni x_{0:n}\longmapsto F_n(x_{0:n}):=F\big(i_n(x_{0:n})\big).$$

- Step 1 (Discrete time): $F\big(\widetilde{X}^{(\sigma),n}\big)=F_n\big((\bar{X}^{(\sigma),n}_{t^n_k})_{k=0:n}\big)$ and

$$F\text{ convex}\Longrightarrow F_n\text{ convex},\ \ n\geq 1.$$

Discrete time result implies, since $\sigma(t^n_k,.)\preceq\theta(t^n_k,.)$,

$$\mathbb{E}\,F\big(\widetilde{X}^{(\sigma),n}\big)=\mathbb{E}\,F_n\big((\bar{X}^{(\sigma),n}_{t^n_k})_{k=0:n}\big)\leq\mathbb{E}\,F_n\big((\bar{X}^{(\theta),n}_{t^n_k})_{k=0:n}\big)=\mathbb{E}\,F\big(\widetilde{X}^{(\theta),n}\big).$$

- Step 2 (Transfer in the "strong" Lipschitz setting): We know that

$$\mathcal{W}_1\big(\widetilde{X}^{(\sigma),n},X^{(\sigma)}\big)\leq\Big\|\big\|\widetilde{X}^{(\sigma),n}-X^{(\sigma)}\big\|_{\sup}\Big\|_1\to 0\quad\text{ as } n\to+\infty$$

Hence if $F:\mathcal{C}([0,T],\mathbb{R}^d)\to\mathbb{R}$ is $\|\cdot\|_{\sup}$-Lipschitz

$$\Big|\mathbb{E}\,F\big(\widetilde{X}^{(\sigma),n}\big)-\mathbb{E}\,FX^{(\sigma)}\big)\Big|\leq[F]_{\rm Lip}\mathcal{W}_1\big(\widetilde{X}^{(\sigma),n},X^{(\sigma)}\big)\to 0\quad\text{ as } n\to+\infty$$

Idem for the θ-diffusion, so that

$$\mathbb{E}\,F(X^{(\sigma)})\leq\mathbb{E}\,F(X^{(\theta)}).\qquad\square$$

"Weak" diffusion setting

- Step 2bis (Transfer in the "weak" linear growth continuous setting): See e.g. [Jacod-Shiryaev's book 2^{nd} edition, Theorem 3.39, p.551] ([7]).

$$\widetilde{X}^{(\sigma),n} \stackrel{\mathcal{L}(\|.\|_{\sup})}{\longrightarrow} X^{(\sigma)} \quad \text{and} \quad \widetilde{X}^{(\sigma),n} \stackrel{\mathcal{L}(\|.\|_{\sup})}{\longrightarrow} X^{(\theta)} \quad \text{as } n \to +\infty.$$

- We know that, as $\sigma(t,\cdot)$ and $\theta(t,\cdot)$ have linear growth

$$\Big\| \sup_{t\in[0,T]} |\widetilde{X}^{(\sigma),n}| \Big\|_{1+\eta} + \Big\| \sup_{t\in[0,T]} |\widetilde{X}^{(\theta),n}| \Big\|_{1+\eta} \le C_{\eta,T}(1+\|X_0\|_{1+\eta})$$

Hence, if F is $\|\cdot\|_{\sup}$-Lipschitz, then $F\big(\widetilde{X}^{(\sigma),n}\big)$, $n \ge 1$, is uniformly integrable so that

$$\mathbb{E}\, F(X^{(\sigma)}) = \lim_n \mathbb{E}\, F\big(\widetilde{X}^{(\sigma),n}\big) \qquad (\textit{idem} \text{ for } X^{(\theta)}).$$

- Hence $\quad \mathbb{E}\, F(X^{(\sigma)}) \le \mathbb{E}\, F(X^{(\theta)}).$ □

[7] *Limit theorems for stochastic processes*, Springer, 2010.

Connection between convexity and convex ordering

- Convexity of $x \mapsto \mathbb{E}\, F(X^x)$ can be obtained as a by-product of the proof by "transferring" convexity property from discrete to continuous time...
- but also, a posteriori: in this diffusion framework

$$\text{Convex ordering} \implies \text{Convexity.}$$

- Let $x, y \in \mathbb{R}$, $\lambda \in [0,1]$. One has

$$\delta_{\lambda x+(1-\lambda)y} \preceq_{cvx} \lambda\delta_x + (1-\lambda)\delta_y.$$

Assume $\sigma = \theta$. Let

$X_0^{(\sigma)} = \lambda x + (1-\lambda)y$ and $\widetilde{X}_0^{(\sigma)} = \varepsilon x + (1-\varepsilon)y$, $\varepsilon \sim \mathcal{B}er(\{0,1\}, \lambda) \perp\!\!\!\perp W$.

- Then $\widetilde{X}_0^{(\sigma)} \sim \lambda\delta_x + (1-\lambda)\delta_y$ and $\widetilde{X}^{(\sigma)} = \varepsilon X^x + (1-\varepsilon)X^y$ and $\mathbb{E}\,\varepsilon = \lambda$ so that, for every l.s.c. functional $F : \mathcal{C}([0,T], \mathbb{R}^d) \to \mathbb{R}$,

$$\mathbb{E}\, F(X^{\lambda x+(1-\lambda)y}) = \mathbb{E}\, F(\widetilde{X}^{(\sigma)}) \leq \lambda \mathbb{E}\, F(X^x) + (1-\lambda)\mathbb{E}\, F(X^y).$$

- Same result for monotone convex orders (see later on).

The Euler scheme provides a simulable approximation

which preserves convex order.

Question: Can we get rid of the convexity of σ (at least in one dimension)?

Smooth σ in $1D$ $(d = q = 1)$

- Assume $\sigma : \mathbb{R} \to \mathbb{R}_+$ C^2, Lipschitz ($\|\sigma'\|_\infty < +\infty$).
- True Euler operator, $Z \sim \mathcal{N}(0,1)$:

$$Pf(x) = \mathbb{E}\, f\big(x + \sqrt{h}\sigma(x)Z\big).$$

- Assume w.l.g. (see later on) $f : \mathbb{R}^d \to \mathbb{R}$ C^2 and convex, with bounded derivatives

$$\begin{aligned}(Pf)''(x) &= \mathbb{E}\big[f''\big(x + \sqrt{h}\sigma(x)Z\big)\big(1 + \sqrt{h}\sigma'(x)Z\big)^2\big]\\ &\qquad + \sqrt{h}\sigma''(x)\mathbb{E}\left[f'\big(x + \sqrt{h}\sigma(x)Z\big)Z\right]\\ &= \mathbb{E}\big[f''\big(x + \sqrt{h}\sigma(x)Z\big)\big(1 + \sqrt{h}\sigma'(x)Z\big)^2\big]\\ &\qquad + h\sigma\sigma''(x)\mathbb{E}\big[f''\big(x + \sqrt{h}\sigma(x)Z\big)\big] \quad \text{Stein I.P.}\\ &= \mathbb{E}\Big[f''\big(x + \sqrt{h}\sigma(x)Z\big)\underbrace{\big(\big(1 + \sqrt{h}\sigma'(x)Z\big)^2 + h\sigma\sigma''(x)\big)}_{\text{always } \geq 0\ \forall\, Z(\omega)??}\Big].\end{aligned}$$

- No ! But... If we truncate : $Z \rightsquigarrow Z^h = Z\mathbf{1}_{\{|Z| \leq A_h\}}$, $Pf \rightsquigarrow \tilde{P}^h f$, then...

- Then, the same Stein-I.P. transform yields

$$(\tilde{P}^h f)''(x) = \mathbb{E}\Big[f''(x+\sqrt{h}\sigma(x)Z^h)\underbrace{\Big(\big(1+\sqrt{h}\sigma'(x)Z^h\big)^2 + h\big(1-e^{-(A_h^2-(Z^h)^2)^+}\big)\sigma\sigma''(x)\Big)}_{\text{always } \geq 0 \ \forall\, Z^h(\omega)??}\Big].$$

- YES !! If $A_h = A/\sqrt{h}$ with $A < \frac{1}{\|\sigma'\|_\infty}$ for $h = \frac{T}{n}$ small enough and

$$(\mathcal{S}) \qquad \sup_{x\in\mathbb{R}} \frac{\sigma(\sigma'')^-}{|\sigma'|}(x) < +\infty \qquad (\Longrightarrow \text{Ok if } \sigma \text{ convex!}) \qquad (4)$$

- So we have proved: for every convex C^2-function f with bounded derivatives

$$x \mapsto P^h f(x) \text{ is convex.}$$

- f Lipschitz and convex can be approximated by convolution: let

$$f_\epsilon(x) = \mathbb{E}\, f(x+\epsilon\zeta),\ \zeta \sim \mathcal{N}(0,1).$$

- f_ϵ is convex, $\downarrow f$ as $\epsilon \downarrow 0$ and

$$f_\epsilon'(x) = \frac{1}{\epsilon}\mathbb{E}\big[(f(x+\epsilon\zeta)-f(x))\zeta\big] \quad \text{and} \quad f_\epsilon''(x) = \frac{1}{\epsilon^2}\mathbb{E}\big[(f(x+\epsilon\zeta)-f(x))(\zeta^2-1)\big]$$

  are bounded.

- As $|f_\epsilon(x)| \le |f(x)| + \epsilon\mathbb{E}|\zeta|$,

$$\tilde{P}^h = \lim_{\epsilon\to 0}{}^{\downarrow}\, \tilde{P} f_\epsilon \quad \text{so that} \quad \tilde{P}^h(f) \text{ is convex.}$$

- We still have that $(x,u) \mapsto \tilde{Q}f(x) = \mathbb{E}\, f(x+uZ^h)$ is convex and non-decreasing in u on $\mathbb{R}_+$.

- Let consider the truncated Euler scheme $\tilde{X}^h = \tilde{X}^{(\sigma),h}$ associated with step $h = \frac{T}{n}$ (and $t_k^n = \frac{kT}{n}$), i.e.

$$\widetilde{X}^h_{t^n_{k+1}} = \bar{X}^h_{t^n_k} + \sigma(t^n_k, \widetilde{X}^h_{t^n_k}) Z^h_{k+1}, \quad \widetilde{X}^h_0 = x$$

$$\text{with} \quad Z^h_{k+1} = \sqrt{\frac{n}{T}} \big(W_{t^n_{k+1}} - W_{t^n_k}\big) \mathbf{1}_{\{|W_{t^n_{k+1}} - W_{t^n_k}| \le A\}}.$$

- This scheme satisfies the convex propagation and ordering properties.
- Does it converge strongly in L^p toward to the diffusion $X^{(\sigma)}$? If "yes" then we proved:

  If $\sigma(t,\cdot)$ satisfies $(\mathcal{S})$ uniformly in $t \in [0,T]$ or $\theta(t,\cdot)$ satisfies $(\mathcal{S})$ uniformly in $t \in [0,T]$, if

  $$0 \le \sigma \le \theta \quad \text{and} \quad X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)} \Longrightarrow \forall\, t \in [0,T], \quad X_t^{(\sigma)} \preceq_{cvx} X_t^{(\theta)}$$

  and, when $\sigma(t,\cdot)$ satisfies $(\mathcal{S})$ uniformly in $t \in [0,T]$,

  $$x \mapsto \mathbb{E}\, f(X_T^{(\sigma)}) \quad \text{is convex.}$$

- Functional version in progress (with B. Jourdain).

Proof of convergence of truncated Euler scheme

- Let $(\tilde{X}^h_{t_k})$ be the truncated Euler scheme with step $h = \frac{T}{n}$ i.e. implemented with $Z^h_k := Z_k \mathbf{1}_{\{|Z_k| \le A/\sqrt{h}\}}$, $(Z_k)_{k=1:n}$ i.i.d. $\mathcal{N}(0,1)$. Then, by independence,

$$\begin{aligned}\mathbb{P}(\tilde{X}^h \neq \bar{X}^n) &= \mathbb{P}\big(\exists\, k \in 1:n : |Z_k| \ge A/\sqrt{h}\big)\\ &= 1 - \mathbb{P}\big(|Z| \le A/\sqrt{h}\big)^n \quad \text{since } Z_k \text{ i.i.d.}\\ &= 1 - \big(1 - \mathbb{P}(|Z| \ge A/\sqrt{h})\big)^n.\end{aligned}$$

- Using $\mathbb{P}(|Z| \ge x) \le e^{-\frac{x^2}{2}}$, $x > 0$, (and $h = \frac{T}{n}$)

$$\begin{aligned}\mathbb{P}(\tilde{X}^h \neq \bar{X}^n) &\le 1 - \big(1 - e^{-\frac{An}{2T}}\big)^n\\ &\le 1 - 1 + ne^{-\frac{An}{2T}} = ne^{-\frac{An}{2T}} \to 0 \text{ as } n \to +\infty\end{aligned}$$

  by convexity of $u \mapsto u^n$.

- As a consequence $(\ldots)$, if $X_0 \in L^p(\mathbb{P})$,

$$\Big\| \max_{k=0:n} \big|\tilde{X}^h_{t_k} - \bar{X}^n_{t_k}\big| \Big\|_p \to 0 \quad \text{as} \quad n \to +\infty. \qquad \square$$

Back to non-decreasing convex order ($d = q = 1$)

- Assume $f : \mathbb{R} \to \mathbb{R}$ si smooth convex and non-decreasing.
- If

$$Pf(x) = \mathbb{E}\, f(x + hb(t,x) + \sqrt{h}\sigma(t,x)Z), \quad Z \sim \mathcal{N}(0,1)$$

with $b(t,\cdot)$ and $\sigma(t,\cdot)$ are uniformly Lipschitz then

$$(Pf)'(x) = \mathbb{E}\big[\underbrace{f'(x + hb(t,x) + \sqrt{h}\sigma(t,x)Z)}_{\geq 0}(1 + hb'(t,x) + \sqrt{h}\sigma'_x(t,x)Z)$$

- Note that

$$1 + hb'(t,x) + \sqrt{h}\sigma'_x(t,x)Z \geq 1 - h\|b'_x\|_{\sup} - \sqrt{h}\|\sigma'_x\|_{\sup}|Z|.$$

- Hence, if $0 < h < (2\|b'_x\|_{\sup}\|)^{-1}$ then

$$1 + hb'(t,x) + \sqrt{h}\sigma'_x(t,x)Z \geq 0 \;\text{ on }\; \Big\{|Z| \leq \frac{1}{2\sqrt{h}\|\sigma'_x\|_{\sup}}\Big\}$$

- Etc, like before (the two ideas can be combined...).

A first conclusion and provisional remarks on $1D$ setting

- Relaxing convexity in x of the diffusion coefficient $\sigma(t,x)$ can be seen as a first (partial) extension of Hajek's theorem (for diffusions with no drift).
- This result is deeply one dimensional and cannot be extended to higher dimension at a reasonable level of generality (to our best knowledge).
- The second results for marginal increasing convex ordering for diffusions having convex drifts "$b^\sigma \le b^\theta$" is essentially Hajek's.
- A combination of the two truncations is possible (in progress with B. Jourdain) and would be a first strict improvement of Hajek's theorem. A second improvement is to find a functional version (ongoing work).
- Applications to local volatility models (like CEV) extending results by El Karoui-Jeanblanc-Shreve to continuous time path-dependent options.
- Extension to directionally convex functionals F (see also Rüshendorf & Bergenthum (*AAP, 2006*) though ... "restrictions" are necessary) that is (in discrete time) functionals $f : \mathbb{R}^d \to \mathbb{R}$ such that $\partial^2 x_i x_j f \ge 0$ for every $i \neq j$ ((in progress with B. Jourdain).

Extensions

This provides as systematic approach which successfully works with

- Jump diffusion models,
- Path-dependent American style options,
- BSDE (without "Z" in the driver),
- ...

The case of jump diffusions

$\triangleright$ Lévy process: Let $Z = (Z_t)_{t\in[0,T]}$ be a Lévy process with Lévy measure ν satisfying

- $\displaystyle\int_{0<|z|\leq 1} |z|^2\nu(dz) < +\infty$ of course...
- $\displaystyle\int_{|z|\geq 1} |z|^p\nu(dz) < +\infty,\ p\in[1,+\infty)$ (hence $Z_t \in L^1(\mathbb{P})$, $t\in[0,T]$).
- $\mathbb{E}\,Z_1 = 0$.

Then

$(Z_t)_{t\in[0,T]}$ is an centered $\mathcal{F}^Z$-martingale.

Theorem (P. 2016, $d = q = 1$, "weak version", not yet updated $d, q \geq 1$ but in progress)

Let $\kappa_i \in \mathcal{C}_{lin_x,unif}([0,T]\times\mathbb{R})$, $i = 1, 2$, *be continuous functions Let* $X^{(\kappa_i)} = (X_t^{(\kappa_i)})_{t\in[0,T]}$ *be the diffusion processes, unique weak solutions to*

$$dX_t^{(\kappa_i)} = \kappa_i(t, X_{t_-}^{(\kappa_i)})dZ_t,\ X_0^{(\kappa_i)} \in L^p(\mathbb{P}),\ i = 1, 2.$$

(a) Z_1 *centered: Assume* $\kappa = \kappa_1$ *or* κ_2 *satisfies:* $\forall\, t \in [0,T]$, $\kappa(t, .)$ *convex and that*

$$0 \leq \kappa_1 \leq \kappa_2.$$

(b) Z_1 *radial: If* $Z_1 \stackrel{\mathcal{L}}{=} -Z_1$, $|\kappa|$ *is convex in* x *and* κ_i *satisfy*

$$|\kappa_1| \leq |\kappa_2|.$$

Let $F : \mathbb{D}([0,T],\mathbb{R}) \to \mathbb{R}$ *be a convex Skorokhod-continuous functional with r-polynomial growth,* $r < p$

$$\forall\, \alpha \in \mathbb{D}([0,T],\mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\sup}^r),\ 0 < r < p.$$

Then $$\mathbb{E}\, F(X^{(\kappa_1)}) \leq \mathbb{E}\, F(X^{(\kappa_2)}).$$

Key argument when $d = q = 1$

- Discrete time approach is similar to Brownian diffusions
- Transfer phase is based on the Skorokhod functional weak convergence of the Euler scheme toward the martingale jump diffusion.
- Which in turn relies on functional weak convergence of stochastic integrals (see e.g. [Mémin-Jakubowski-P., *PTRF*, 1989]).
- A "strong" version with Lipschitz coefficients κ_i (uniformly in t) should work, possible without Skorokhod topology.
- Higher dimensions should work too if Z is radial (but not yet proved to our best knowledge).

Discrete time optimal stopping (Bermuda options)...

... of ARCH models in 1-dimension.

▷ Dynamics: Still... $(Z_k)_{1\le k\le n}$ be a sequence of independent, (centered and) symmetric r.v.

$$\begin{aligned} X_{k+1} &= X_k + \sigma_k(X_k)\, Z_{k+1},\ X_0 \in L^1(\mathbb{P}) \\ Y_{k+1} &= Y_k + \theta_k(Y_k)\, Z_{k+1}, \quad 0 \le k \le n-1,\ Y_0 \in L^1(\mathbb{P}) \end{aligned}$$

where σ_k, $\theta_k : \mathbb{R} \to \mathbb{R}$, $k = 0, \ldots, n-1$ with (at most) linear growth.

Snell envelopes

▷ Let $F_k : \mathbb{R}^{k+1} \to \mathbb{R}_+$, $k = 0, \ldots, n$ be a sequence of non-negative *convex* (payoff) functions with r-polynomial growth for the sup norm.

▷ Let $\mathcal{F} = (\mathcal{F}_k)_{0 \le k \le n}$ be a filtration such that Z_k is $\mathcal{F}_k$-adapted and Z_k is independent of $\mathcal{F}_{k-1}$, $k = 1, \ldots n$.

▷ Snell envelopes of the reward processes $\big(F_k(X_{0:k})\big)_{0 \le k \le n}$ and $\big(F_k(Y_{0:k})\big)_{0 \le k \le n}$

$$U_k = \mathbb{P}\text{-esssup}\Big\{\mathbb{E}\big(F_\tau(X_{0:\tau}) \,|\, \mathcal{F}_k\big),\ \tau\ \mathcal{F}\text{-stopping time}, \tau \ge k\Big\}$$

and

$$V_k = \mathbb{P}\text{-esssup}\Big\{\mathbb{E}\big(F_\tau(Y_{0:\tau}) \,|\, \mathcal{F}_k\big),\ \tau\ \mathcal{F}\text{-stopping time}, \tau \ge k\Big\}.$$

▷ These are the lowest super-martingales that dominate the reward processes.

Backward Dynamic programming Principle

Proposition (Backward Dynamic programming Principle (*BDDP*))

(*a*) *The Snell envelope satisfies*

$$U_n = F_n(X_{0:n}), \qquad U_k = \max\big(F_k(X_{0,k}), \mathbb{E}\,(U_{k+1}\,|\,\mathcal{F}_k)\big),\ k = 0 : n-1.$$

(*b*) *One has*

$$U_k = u_k(X_{0:k}) \quad \mathbb{P}\text{-}a.s.,\ k = 0, \ldots, n-1,$$

where the functions $u_k : \mathbb{R}^{k+1} \to \mathbb{R}_+$, $k = 0 : n$, *satisfy the functional* BDDP

$$u_n = F_n, \qquad u_k(x_{0:k}) = \max\Big(F_k\big(x_{0:k}\big), Q_{k+1}u_{k+1}\big(x_{0:k}, x_k + .\big)\big(\sigma_k(x_k)\big)\Big)$$
$$k = 0, \ldots, n-1.$$

- Propagation of the convexity: Note that $(a, b) \mapsto \max(a, b)$ is non-decreasing in a and b and "copy-paste" the proofs for a fixed functional using the "revisited" Jensen's Inequality.

Proposition

(a) Convex ordering. If, either

$$\begin{cases} (*)_\sigma & |\sigma_k| \text{ is convex for every } k = 0 : n-1 \\ \text{or} & \\ (*)_\theta & |\theta_k| \text{ is convex for every } k = 0 : n-1 \end{cases}$$

and

$$|\sigma_k| \leq |\theta_k|,\ k = 0, \ldots, n-1$$

then,

$$u_k(x_{0:k}) \leq v_k(x_{0:k}),\ k = 0, \ldots, n.$$

(b) Convexity. If $(*)_\sigma$ holds then

$$x \longmapsto u_k(x_{0:k}) \text{ is a convex function on } \mathbb{R}^{k+1}.$$

In particular, if $X_0 \preceq_{cvx} Y_0$ then $\mathbb{E}\, U_0 = \mathbb{E}\, u_0(X_0) \leq \mathbb{E}\, u_0(Y_0) \leq \mathbb{E}\, v_0(Y_0) = \mathbb{E}\, V_0$.

...

▷ Idem for $v_k : \mathbb{R}^{k+1} \to \mathbb{R}$ in connection with the $(\mathbb{P}, \mathcal{F})$-Snell envelope V.

▷ Note that u_{k+1} convex still implies

$$\xi \longmapsto \big(Q_{k+1} u_{k+1}(x_{0:k}, \cdot)\big)(x_k, \xi) \quad \text{is non-decreasing on } \mathbb{R}_+.$$

▷ Comparison Principle ($|\sigma_k| \le |\theta_k|$): Backward induction to prove $u_k \le v_k$, $k = 0 : n$ (obvious if $k = n$).

Assume $u_{k+1} \le v_{k+1}$, $k+1 \le n$. For every $x_{0:k} \in \mathbb{R}^{k+1}$

$$\begin{aligned} u_k(x_{0:k}) &\le \max\Big(F_k(x_{0:k}), \big(Q_{k+1} u_{k+1}(x_{0:k}, \cdot)\big)(x_k, \theta_k(x_k))\Big) \\ &\le \max\Big(F_k((x_{0:k}), \big(Q_{k+1} v_{k+1}(x_{0:k}, \cdot)\big)(x_k, \theta_k(x_k))\Big) = v_k(x_{0:k}). \end{aligned}$$

If $k = 0$, we get

$$\mathbb{E}\, U_0 = u_0(x) \le v_0(x) = \mathbb{E}\, V_0. \qquad \square$$

Back to continuous time

▷ Snell envelopes of the Euler schemes of X and Y

$$U^{(n)} = \mathbb{P}\text{-Snell}\big(F_k(\bar{X}^{(\sigma),n}_{0:k})_{k=0:n}\big) \quad V^{(n)} = \mathbb{P}\text{-Snell}\big(F_k(\bar{Y}^{(\theta),n}_{0:k})_{k=0:n}\big).$$

▷ Convergence: In the case of Brownian diffusions, it is a classical result (with convergence rates in fact, see *e.g.* ([8]) that

$$\Big\|\max_{0\le k\le n}\big|U^{(n)}_k - U^X_{t^n_k}\big|\Big\|_p \to 0 \;\text{ and }\; \Big\|\max_{0\le k\le n}\big|V^{(n)}_k - V^Y_{t^n_k}\big|\Big\|_p \to 0 \text{ as } n\to+\infty$$

Etc.

▷ Conclusion: As usual. . .

Theorem (P. 2016)

Under partitioning or dominating assumptions on σ and θ, $F(t,.)$ convex on $\mathcal{C}([0,T],\mathbb{R})$ and F continuous, etc, one has

$$u_0(x) = \mathbb{E}\,U_0^{X^{(\sigma),x}} \le \mathbb{E}\,V_0^{X^{(\theta),x}} = v_0(x).$$

[8] V. Bally-P. ('03), Error analysis of the quantization algorithm for obstacle problems, *Stochastic Processes & Their Applications*, 106(1), 1-40, 2003

Jump martingale diffusions: what makes problem?

▷ Discrete time step: Identical.

▷ From discrete to continuous time: Still the Euler scheme. But we have to make the Snell envelopes converge. . . How to proceed?

Filtration enlargement argument/trick

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration and let Y be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted càdlàg process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ so that

$$\forall\, t \in [0,T], \quad \mathcal{F}_t^Y \subset \mathcal{F}_t$$

We introduce the so-called $\mathcal{H}$-assumption (on the filtration $(\mathcal{F}_t)_{t\in[0,T]}$):

$$(\mathcal{H}) \equiv \forall\, H \in \mathcal{F}_T^Y,\ \text{bounded},\ \mathbb{E}\big(H\,|\,\mathcal{F}_t\big) = \mathbb{E}\big(H\,|\,\mathcal{F}_t^Y\big)\ \mathbb{P}\text{-}a.s.$$

Example: $\mathcal{F}_t = \sigma(\mathcal{F}_t^Y, \Xi)$, $\Xi \perp\!\!\!\perp Y$.

Theorem (Lamberton-P., 1990)

([a]) ▷ *Let $(X^n)_{n\geq 1}$ be a sequence of quasi-left càdlàg processes defined on a probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ of (D)-class and satisfying the Aldous criterion. Let $(\tau_n^*)_{n\geq 1}$ be a sequence of $\big(\mathcal{F}^{X^n}, \mathbb{P}^n\big)$-optimal stopping times. If $(X^n)_{n\geq 1}$ is uniformly integrable and satisfies*

$$X^n \stackrel{\mathcal{L}(Skor)}{\longrightarrow} X,\ \mathbb{P}_X = \mathbb{P} \text{ probability measure on } (\mathbb{D}([0,T],\mathbb{R}), \mathcal{D}_T).$$

▷ *Non-degeneracy of $(\tau_n^*)_{n\geq 1}$: every limiting value $\mathbb{Q}$ of $\mathcal{L}(X^n, \tau_n^*)$ on $\mathbb{D}([0,T],\mathbb{R}) \times [0,T]$ satisfies the $(\mathcal{H})$ property [...], then*

$$\lim_n \mathbb{E}_{\mathbb{P}^n} U_0^{X^n} = \mathbb{E}_{\mathbb{P}} U_0^X.$$

▷ *If the optimal stopping problem related to $(X, \mathbb{Q}, \mathcal{D}^\theta)$ has a unique solution in distribution, say $\mu^*_{\tau^*}$, not depending on $\mathbb{Q}$, then $\tau_n^* \stackrel{[0,T]}{\longrightarrow} \mu^*_{\tau^*}$.*

[a] Sur l'approximation des réduites, *Annales IHP B*, 1990.

Theorem (P. 2012)

Under the usual on κ_i, $i = 1, 2$, $(Z_t)_{t\geq 0}$ (through Z_1 and F (convexity), etc, the "réduites" associated to F and $X^{(\kappa_i),x}$, $i = 1, 2$, satisfy

$$u^{(\kappa_1)}(x) \leq u^{(\kappa_2)}(x)$$

so that the Snell envelopes satisfy $\mathbb{E}\, U_0^{(1)} \leq \mathbb{E}\, V_0^{(1)}$.

All the efforts are focused on showing that the filtration enlargement assumption $(\mathcal{H})$ is satisfied by any limiting distribution $\mathbb{Q}$.

McKean-Vlasov diffusions:

- The *MKV* dynamics. Let $p \geq 1$.

$$(E) \;\equiv\quad dX_t = b(t, X_t, \mu_t)dW_t + \sigma(t, X_t, \mu_t)dW_t, \quad t \in [0, T]$$

with $\mu_t = \mathcal{L}(X_t)$, $W = (W_t)_{t\in[0,T]}$ a standard B.M. and

$$b, \sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \to \mathbb{R} \text{ are continuous satisfying}$$

$$(Lip) \;\equiv\; b(t, \cdot, \cdot),\, \sigma(t, \cdot, \cdot) \textit{ is } \big(|\cdot|, \mathcal{W}_p\big)\textit{-Lipschitz, uniformly in } t \in [0, T].$$

Wasserstein distance: $\mathcal{W}_p^p(\mu, \nu) = \inf\Big\{\int |x-y|^p m(dx, dy),\; m(dx, \mathbb{R}^d) = \mu,\; m(\mathbb{R}^d, dy) = \nu\Big\}.$

$$\Big(= \sup\Big\{\int f d\mu - \int f d\nu, [f]_{\rm Lip} \leq 1\Big\} \text{ when } p = 1\Big).$$

- Under this assumption a strong solution exists for this equation starting from $X_0 \in L^p(\mathbb{P})$, $X_0 \perp\!\!\!\perp W$.
- "Scaled" Martingality "requires" a drift term

$$b(t, X_t, \mu_t) = \alpha(t)(X_t + \beta(t, \mathbb{E}\, X_t))$$

$\alpha(t)$ Hölder-continuous, β Lipschitz in ξ, uniformly in t and $|\beta(t,x) - \beta(s,x)| \leq C(1 + |x|)|t - s|$. (From now on $\alpha = \beta = 0$ for convenience).

Understanding *MKV*

- Vlasov framework ($p = 1$). If σ has the following linear representation in μ

$$\sigma(x,\mu) = \int_{\mathbb{R}} \varpi(x,\xi)\mu(d\xi).$$

- Non linear framework. E.g.

$$\sigma(x,\mu) = \varphi_0\left(\int_{\mathbb{R}} \varpi(x,\xi)\mu(d\xi)\right)$$

  where φ_0 has at most linear growth.

- *MKV* equations were brought back to light through the equilibrium problems arising from the theoretical aspects of mean field game theory (see [Lasry-Lions, 2006], book by [Carmona-Delarue, 2018] ([9]).).

[9] R. Carmona, F. Delarue *Probabilistic Theory of Mean Field Games with Applications I & II*, Springer, 2018

Convex order for MKV: the approach

- Again Discrete time with ARCH models + Backward Dynamic Programming.
- Limit theorem for the (non-simulable) Euler scheme.
- *MKV ARCH* dynamics: Let $(Z_k)_{1\le k\le n}$ be a sequence of independent, radial r.v. in $L^p(\Omega,\mathcal{A},\mathbb{P})$. The two *ARCH* models: $X_0, Y_0 \in L^p(\mathbb{P})$,

$$X_{k+1} = X_k + \sigma_k(X_k,\mu_k)\, Z_{k+1},$$
$$Y_{k+1} = Y_k + \vartheta_k(Y_k,\nu_k)\, Z_{k+1}, \quad k = 0:n-1,$$

with $\mu_k = \mathcal{L}(X_k)$ and $\nu_k = \mathcal{L}(Y_k)$, $k = 0:n$

$$(\mathcal{LG}) \equiv \quad |\sigma_k(x,\mu)| + |\vartheta_k(x,\mu)| \le C\big(1+|x|+\mathcal{W}_p(\mu,\delta_0)\big).$$

- The model is well-defined by induction.

Theorem (Discrete time comparison result)

Let $(X_k)_{k=0:n}$ and $(Y_k)_{k=0:n}$ the two above MKV ARCH models.

(a) *If, either*

$$\begin{cases} (*)_\sigma \;\equiv\; \sigma_k(x,\mu) \preceq\text{-convex in } x,\ \uparrow_{cvx} \text{ in } \mu\in\mathcal{P}_p(\mathbb{R}^d),\ k=0:n-1 \\ or \\ (*)_\vartheta \;\equiv\; \vartheta_k(x,\mu) \preceq\text{-convex in } x,\ \uparrow_{cvx} \text{ in } \mu\in\mathcal{P}_p(\mathbb{R}^d),\ k=0:n-1, \end{cases}$$

$$\sigma_k(x,\mu)\preceq\vartheta_k(x,\mu),\ x\in\mathbb{R}^d,\ \mu\in\mathcal{P}_p(\mathbb{R}^d) \ \text{ and }\ X_0\preceq_{cvx} Y_0$$

then, for every convex function $F:(\mathbb{R}^d)^{n+1}\to\mathbb{R}$, with r-polynomial growth, $r<p$,

$$\mathbb{E}\,F(X_{0:n})\le\mathbb{E}\,F(Y_{0:n}).$$

(b) *If $(*)_\sigma$ holds true then, for every convex function*

$$x\longmapsto\mathbb{E}\,F(X^x_{0:n}) \ \text{ is convex.}$$

Understanding $\uparrow_{cvx}$

- Vlasov framework. If σ has the following linear representation in μ

$$\sigma(x,\mu) = \int_{\mathbb{R}} \varpi(x,\xi)\mu(d\xi)$$

then, ϖ is both convex in x and ξ implies that σ satisfies $(*)_\sigma$.

- Non linear framework. Let $\varphi_0 : \mathbb{R} \to \mathbb{R}$ convex non-decreasing

$$\sigma(x,\mu) = \varphi_0\left(\int_{\mathbb{R}} \varpi(x,\xi)\mu(d\xi)\right).$$

MKV specificity

- If proceeding backward $\mu_k \preceq_{cvx} \nu_k$ not yet proved at time k !
- A first forward preliminary step to prove the marginal convex order

$$\mu_k \preceq_{cvx} \nu_k, \quad k = 0 : n\ ?$$

- Assume $(*)_\sigma$. Define the *MKV ARCH* operators

$$\mathcal{E}_k(x, \mu, z) : x \longmapsto x + \sigma_k(x, \mu)z$$

- Induction: Assume $\mu_k \preceq_{cvx} \nu_k$. Let $f : \mathbb{R} \to \mathbb{R}$ be convex

$$\int f d\mu_{k+1} = \mathbb{E}\, f(X_{k+1}) = \mathbb{E}\, f\big(X_k + \sigma_k(X_k, \mu_k)Z_{k+1})\big)$$

$$= \int_{\mathbb{R}} \mathbb{E}\, f\big(\mathcal{E}_k(x + \sigma_k(x, \mu_k)Z_{k+1})\big)\mu_k(dx) \quad \text{since } X_k \perp\!\!\!\perp Z_{k+1}.$$

MKV specificity

- We know that

$$(x,u) \longmapsto \mathbb{E}\, f(x+uZ_{k+1}) \text{ is convex in } (x,u) \text{ and } \uparrow \text{ in } u.$$

so that $\mu_k \preceq_{cvx} \nu_k$ implies

$$\mathbb{E}\, f\big(x+\sigma_k(x,\mu_k)Z_{k+1}\big) \le \mathbb{E}\, f\big(x+\sigma_k(x,\nu_k)Z_{k+1}\big)$$

and the convexity of $\sigma_k(\cdot,\nu_k)$ implies

$$x \longmapsto \mathbb{E}\, f\big(x+\sigma_k(x,\nu_k)Z_{k+1}\big) \text{ is convex.}$$

- Hence

$$\begin{aligned}\int f d\mu_{k+1} &= \int_{\mathbb{R}} \mathbb{E}\, f\big(x+\sigma_k(x,\mu_k)Z_{k+1}\big)\mu_k(dx)\\ &\le \int_{\mathbb{R}} \mathbb{E}\, f\big(x+\sigma_k(x,\mu_k)Z_{k+1}\big)\nu_k(dx)\\ &\le \int_{\mathbb{R}} \mathbb{E}\, f\big(x+\sigma_k(x,\nu_k)Z_{k+1}\big)\nu_k(dx) = \int f d\nu_{k+1}.\end{aligned}$$

- Same kind of reasoning with ϑ_k satisfying $(*)_\vartheta$.

MKV standardness

- In fact if $F : (\mathbb{R}^d)^{n+1} \times \mathcal{P}_p(\mathbb{R}^d)^{n+1} \to \mathbb{R}$ is space convex and componentwise $\uparrow_{cvx}$ in the distribution variables, then

$$\mathbb{E}\, F\big(X_{0;n}, \mu_{0:n}\big) \le \mathbb{E}\, F\big(Y_{0:n}, \nu_{0,n}\big).$$

- The switch to global convex order by a backward induction is "standard" from the standard ARCH case.

The Euler scheme strikes back

- Under the above assumptions (E) has a unique strong solution.
- The Euler scheme with step $\frac{T}{n}$ is an *MKV* ARCH model. It reads

$$\bar{X}_{k+1} = \bar{X}_k + \underbrace{\sqrt{\frac{T}{n}}\sigma(t_k,}_{\sigma_k(\dots)}\bar{X}_k, \bar{\mu}_k)Z_{k+1}, \quad \bar{X}_0 = X_0,$$

  where $\bar{\mu}_k = \mathcal{L}(\bar{X}_k)$, $k = 0 : n$.
- Its specificity is to be non-simulable, hence supposedly ... useless;
- However, under $(\mathcal{CM})$, it propagates convex order as an *MKV ARCH*.
- ... and its linearly interpolated version strongly converges toward X (with rates) for the sup-norm in L^p:

$$\mathbb{E} \sup_{t\in[0,T]} \left|X_t - \bar{X}^n_t\right|^p \to 0 \quad \text{as } n \to +\infty$$

- Idem for the *MKV SDE*: $dY_t = \theta(t, Y_t, \nu_t)dW_t$.

MKV propagates convex ordering

Theorem (Liu-P., 2019 on ArXiv, to appear *AAP*)

Let $\sigma, \theta \in Lip_{x,\mu,unif}\big([0,T]\times\mathbb{R}\times\mathcal{P}_p(\mathbb{R}), \mathbb{M}_{d,q}(\mathbb{R})\big)$, $p \geq 2$. *Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the* unique *solutions to*

$$dX_t = \sigma(t, X_t, \mu_t)dW_t,\ X_0 \in L^p$$

$$dY_t = \theta(t, Y_t, \nu_t)dW_t,\ Y_0 \in L^p \quad \text{with } (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.}$$

$$\text{If}\quad \begin{cases} (i)_\sigma & \sigma(t,x,\mu) \text{ is } x\text{-}\preceq\text{-convex and } \mu\text{-}\uparrow_{cvx} \text{ for every } t\in[0,T], \\ or & \\ (i)_\theta & \theta(t,x,\mu) \text{ is } x\text{-}\preceq\text{-convex and } \mu\text{-}\uparrow_{cvx} \text{ for every } t\in[0,T], \\ and & \\ (ii) & \sigma(t,x,\mu) \preceq \theta(t,x,\mu) \quad [|\sigma(t,x,\mu)| \leq |\theta(t,x,\mu)| \text{ if } d=1] \end{cases}$$

and $X_0 \leq_{cvx} Y_0$, *then, for every convex functional* $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$,

$$\mathbb{E}\, F(X) \leq \mathbb{E}\, F(Y).$$

Moreover if $(X_0 = x)$ *and* $(i)_\sigma$ *holds, one has* $x \mapsto \mathbb{E}\, F(X^x)$ *is convex.*

Specificity of the proof

- The "regular" Euler scheme is again the main tool . . . although not simulable.
- Specificity for convexity propagation: two steps
  - Forward "marginal " approach necessary
    prior to
  - a backward "functional" approach.
- Convexity cannot be derived from convex ordering comparison but holds true however as a by product of the proof.
- We assume $p \geq 2$ rather than $p = 1$ due to technical limitations in the L^p-convergence of the Euler scheme. To be fixed.

Non-Markovian dynamics: Volterra equations (Jourdain-P. '22))

- Let $(X_t)_{t\in[0,T]}$ be a [strong/weak?] solution to the scaled stochastic Volterra equation

$$X_t = X_0 + \int_0^t K(t,s)\alpha(s)\big(X_s+\beta(s)\big)ds + \int_0^t K(t,s)\sigma(s,X_s)dW_s,\ \ t\in[0,T] \tag{5}$$

where the non-negative kernel $\big(K(t,s)\big)_{0\le s\le t\le T}$ is measurable and integrable, $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{M}_{d,q}$ and $(W_t)_{t\in[0,T]}$ is a standard q-dimensional Brownian motion, $X_0\in L^{??????????}(\mathbb{P}) \perp\!\!\!\perp W$.

- Such a process is centered, $(\mathcal{F}_t^W)$-adapted but is not a martingale (not even a semi-martingale, in general), especially when K is singular like

$$K(s,t) = (t-s)^{H-\frac{1}{2}}, \quad H\in(0,\tfrac{1}{2})$$

(not so) recently brought back to light by the rough vol community.

Back to general Volterra equation. . .

- We consider the equation

$$\forall\, t \in [0, T], \quad X_t = X_0 + \int_0^t K(t,s)b(s,X_s)ds + \int_0^t K(t,s)\sigma(s,X_s)dW_s \quad (6)$$

- where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{M}_{d,q}$ satisfy

$$\exists\, C_{_T} = C_{b,\sigma,T} \ \text{ such that } \ \forall\, t \in [0, T],$$
$$\forall\, x,\, y \in \mathbb{R}^d, \quad |b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \le C_{_T}|x - y|$$

and $\sup_{t\in[0,T]} \big(|b(t,0)| + \|\sigma(t,0)\|\big) < +\infty$. Also assume $X_0 \in L^p(\mathbb{P})$, $p \ge 1$ and $X0 \perp perp$.

- These are standard assumptions in a regular diffusion framework.

Theorem (Existence of a strong solution (see e.g. Zhang, 2005))

Assume that the kernel K satisfies the integrability assumption

$$(\mathcal{K}_\beta^{int}) \qquad \sup_{t\in[0,T]} \int_0^t K(t,s)^{2\beta} ds < +\infty \tag{7}$$

for some $\beta > 1$ and the continuity assumption

$$(\mathcal{K}_\theta^{cont}) \ \exists\, \kappa < +\infty, \forall\, \delta \in (0,T),$$

$$\eta(\delta) := \sup_{t\in[0,T]} \left[\int_0^t \left|K\big((t+\delta)\wedge T, s\big) - K(t,s)\right|^2 ds\right]^{\frac{1}{2}} \leq \kappa\, \delta^\theta \tag{8}$$

for some $\theta \in (0,1]$.

Finally assume that $X_0 \in \bigcap_{p>0} L^p(\mathbb{P})$.

Then the above Volterra equation (5) admits, up to a $\mathbb{P}$-indistinguishability, a unique $(\mathcal{F}_t)$-adapted solution $X = (X_t)_{t\in[0,T]}$, pathwise continuous, in the sense that,

$$\mathbb{P}\text{-}a.s. \left(\forall\, t \in [0,T], \quad X_t = X_0 + \int_0^t K(t,s) b(s, X_s) ds + \int_0^t K(t,s)\sigma(s,X_s) dW_s\right).$$

Theorem (Properties, Jourdain-P. '22)

- *This solution satisfies*

$$\forall\, s,\ t \in [0,T], \quad \|X_t - X_s\|_p \le C_{p,T}(1+\|X_0\|_p)|t-s|^{\theta\wedge\frac{\beta-1}{2\beta}}. \tag{9}$$

- *Moreover,*

$$\forall\, a \in \Big(0, \theta\wedge\frac{\beta-1}{2\beta}\Big), \qquad \Big\|\sup_{s\neq t\in[0,T]}\frac{|X_t-X_s|}{|t-s|^a}\Big\|_p < C_{a,p,T}(1+\|X_0\|_p) \tag{10}$$

for some positive real constant $C_{a,p,T} = C_{a,b,\sigma,K,\theta,p,T}$.

- *In particular*

$$\Big\|\sup_{t\in[0,T]}|X_t|\Big\|_p \le C'_{a,p,T}(1+\|X_0\|_p). \tag{11}$$

- *Finally, if the condition*

$$(\widehat{\mathcal{K}}^{cont}_{\widehat{\theta}})\ \ \exists\,\widehat{\kappa} < +\infty,\ \forall \delta \in (0,T],\ \widehat{\eta}(\delta) := \sup_{t\in[0,T]}\Big[\int_{(t-\delta)^+}^t K_i(t,u)^2 du\Big]^{\frac12} \le \widehat{\kappa}\,\delta^{\widehat{\theta}} \tag{12}$$

is satisfied for some $\widehat{\theta} \in (0,1]$*, then one can replace* $\frac{\beta-1}{2\beta}$ *by* $\widehat{\theta}$ *in* (9) *and* (10).

Main tool: Garsia-Rodemich-Rumsey's lemma (extension of Kolmogorov pathwise continuity criterion).

Extended version

Theorem (Existence of a strong solution (see [ArXiv, Jourdain-P.'22)

for this version)] Assume that the kernel K satisfies the integrability assumption

$$(\mathcal{K}_\beta^{int}) \qquad \sup_{t\in[0,T]} \int_0^t K(t,s)^{2\beta} ds < +\infty \tag{13}$$

for some $\beta > 1$ and the continuity assumption

$$(\mathcal{K}_\theta^{cont}) \ \exists\, \kappa < +\infty, \forall\, \delta \in (0,T),$$

$$\eta(\delta) := \sup_{t\in[0,T]} \left[\int_0^t \left|K\big((t+\delta)\wedge T, s\big) - K(t,s)\right|^2 ds\right]^{\frac{1}{2}} \le \kappa\, \delta^\theta \tag{14}$$

for some $\theta \in (0,1]$.

Finally assume that $X_0 \in L^p(\mathbb{P})$ for some $p \in (0,+\infty)$.

Then the above Volterra equation (100) admits, up to a $\mathbb{P}$-indistinguishability, a unique $(\mathcal{F}_t)$-adapted solution $X = (X_t)_{t\in[0,T]}$, pathwise continuous, in the sense that, $\mathbb{P}$-a.s.,

$$\forall\, t \in [0,T], \quad X_t = X_0 + \int_0^t K(t,s) b(s,X_s) ds + \int_0^t K(t,s)\sigma(s,X_s) dW_s.$$

Representation of the Volterra flow as a Brownian functional

Theorem (Blagoveščenkii-Freidlin like theorem: representation of Volterra's flow)

(*a*) Flow regularity. *Let X^x denotes the solution to the Volterra equation* (100) *starting from $x \in \mathbb{R}^d$ and let $\lambda \in (\frac{1}{2}, 1)$. There exists $p^* = p^*_{\beta,\theta,\lambda,d}$ such that for every $p > p^*$,*

$$\forall\, x, y \in \mathbb{R}^d,\ \Big\| \sup_{t\in[0,T]} |X^x_t - X^y_t| \Big\|_p \le C|x-y|^\lambda$$

for some positive real constant $C = C_{p,b,\sigma,K_1,K_2,\beta,\theta}$.

(*b*) *Representation. There exists a bi-measurable Borel functional $F : \mathbb{R}^d \times \mathcal{C}_0([0,T], \mathbb{R}^q) \ni (x, w) \mapsto F(x, w) \in \mathcal{C}([0,T], \mathbb{R}^d)$, and continuous in x such that,*

$$\forall\, (\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}),\ \forall q\text{-dimensional } (\mathcal{F}_t)_t\text{-B.M. } W,\ \forall\, X_0 \in L^0_{\mathbb{R}^d}(\mathbb{P}, \mathcal{F}_0)$$

the solution to equation (100) *is $X = F(X_0, W)$.*

Euler schemes

- *K-discrete Euler scheme (discrete time):*

$$\bar{X}_{t_k^n} = X_0 + \sum_{\ell=1}^{k} \Big(K(t_k^n, t_{\ell-1}^n) b(t_{\ell-1}^n, \bar{X}_{t_{\ell-1}^n}) \tfrac{T}{n} + K(t_k^n, t_{\ell-1}^n) \sigma(t_{\ell-1}^n, \bar{X}_{t_{\ell-1}^n})(W_{t_\ell^n} - W_{t_{\ell-1}^n}) \Big),\ k = 0 : n. \tag{15}$$

- *K-integrated Euler scheme (discrete time):*

$$\bar{X}_{t_k^n} = X_0 + \sum_{\ell=1}^{k} \Big(\int_{t_{\ell-1}^n}^{t_\ell^n} K(t_k^n, s) ds\, b(t_{\ell-1}^n, \bar{X}_{t_{\ell-1}^n}) + \sigma(t_{\ell-1}^n, \bar{X}_{t_{\ell-1}^n}) \int_{t_{\ell-1}^n}^{t_\ell^n} K(t_k^n, s) dW_s \Big),\ k = 0 : n. \tag{16}$$

- *K-discrete Euler scheme (genuine):* Set $\underline{t} = t_\ell^n$ is $t \in [t_\ell^n, t_{\ell+1}^n)$.

$$\bar{X}_t = X_0 + \int_0^t K_1(t, \underline{s}) b(\underline{s}, \bar{X}_{\underline{s}}) ds + \int_0^t K_2(t, \underline{s}) \sigma(\underline{s}, \bar{X}_{\underline{s}}) dW_s,\ t \in [0, T], \tag{17}$$

- *K-integrated Euler scheme (genuine):*

$$\bar{X}_t = X_0 + \int_0^t K(t, \underline{s}) b(s, \bar{X}_{\underline{s}}) ds + \int_0^t K(t, s) \sigma(\underline{s}, \bar{X}_{\underline{s}}) dW_s,\ t \in [0, T]. \tag{18}$$

Euler schemes (convergence), extension

- See [Zhang], [Richard et al. *SPA 22*] for p "large enough" and [Jourdain-P.'22] for $p \in (0, +\infty)$.

Theorem (K-integrated Euler scheme)

Let $T > 0$ and let $p \in (0, +\infty)$.

(a) Assume the time-space Hölder-Lipschitz continuity assumption for some $\gamma \in (0, 1]$

$$(\mathcal{LH}_\gamma) \quad \exists\, C_{b,\sigma} < +\infty,\ \forall\, s, t \in [0, T],\ \forall\, x, y \in \mathbb{R}^d,$$
$$|b(t, y) - b(s, x)| + \|\sigma(t, y) - \sigma(s, x)\| \le C_{b,\sigma}\big((1 + |x| + |y|)|t - s|^\gamma + |x - y|\big). \quad (19)$$

Assume K satisfies $(\mathcal{K}^{int}_\beta)$ and $(\mathcal{K}^{cont}_\theta)$ for some $\beta > 1$, $\theta \in (0, 1]$. Then the K-integrated Euler scheme $\bar{X}^n$ with time step $\frac{T}{n}$, has a pathwise continuous modification.

(b) Assume furthermore $(\widehat{\mathcal{K}}^{cont}_{\widehat{\theta}})$ holds for some $\widehat{\theta} \in (0, 1]$.

$$\max_{k=0,\ldots,n} \big\|X_{t_k} - \bar{X}^n_{t_k}\big\|_p \le \sup_{t\in[0,T]} \big\|X_t - \bar{X}^n_t\big\|_p \le C(1 + \|X_0\|_p)\big(\tfrac{T}{n}\big)^{\gamma\wedge\theta\wedge\widehat{\theta}}. \quad (20)$$

and, moreover, for every $\varepsilon \in (0, 1)$

$$\Big\|\max_{k=0,\ldots,n} |X_{t_k} - \bar{X}^n_{t_k}|\Big\|_p \le \Big\|\sup_{t\in[0,T]} |X_t - \bar{X}^n_t|\Big\|_p \le C_\varepsilon(1 + \|X_0\|_p)\big(\tfrac{T}{n}\big)^{(\gamma\wedge\theta\wedge\widehat{\theta})(1-\varepsilon)}. \quad (21)$$

- If $K(t,s) = (t-s)^{H-\frac{1}{2}}$, $H > 0$, $\theta \wedge \hat{\theta} = H \wedge 1$ (see [Richard et al.])
- One also has an $L^p(\mathbb{P})$-pathwise regularity

$$\forall\, s,\ t \in [0,T], \qquad \big\|\bar{X}_t - \bar{X}_s\big\|_p \le C(1 + \|X_0\|_p)|t-s|^{\theta\wedge\widehat{\theta}} \qquad (22)$$

  and even a pathwise Hölder regularity.
- For genuine K-discrete Euler scheme the same result holds under slightly more stringent assumptions.

Splitting lemma

Proposition (Splitting lemma)

Assume the assumptions of the $(E\text{-}U)$ *theorem are in force. Let* $\Phi : C([0,T],\mathbb{R}^d)^2 \to \mathbb{R}$ *be a Borel functional and let* $n \in \mathbb{N}$ *such that, for every* $x_0 \in \mathbb{R}^d$,

$$\|\Phi(X^{x_0}, \bar{X}^{n,x_0})\|_{\bar{p}} \leq C_n(1+|x_0|) \quad \text{for some } \bar{p} > 0$$

where X^{x_0} *and* $\bar{X}^{n,x_0}$ *denote the solution of the Volterra equation and any of its (genuine) Euler schemes starting from* x_0*.*
Then, for every $p \in (0,\bar{p}]$ *and every* $X_0 \perp\!\!\!\perp W$, $X = (X_t)_{t\in[0,T]}$ *and the Euler scheme under consideration starting from* X_0 *satisfy*

$$\|\Phi(X, \bar{X}^n)\|_p \leq 2^{(1/p-1)^+} C_n(1+\|X_0\|_p).$$

Proof (sketch of)

- According to our avatar(s) of Blagoveščenkii-Freidlin's theorem

$$X^{X_0} = F\big(X_0, (W_t)_{t\in[0,T]}\big) \quad \text{and} \quad \bar{X}^n = \bar{F}_n\big(X_0, (W_t)_{t\in[0,T]}\big).$$

- This entails that the distribution $\mathbb{P}_{(X,\bar{X}^n)}$ on $\mathcal{C}([0,T],\mathbb{R}^d)^2$ of $(X,\bar{X}^n) = \big(F(X_0,W), \bar{F}_n(X_0,W)\big)$ satisfies

$$\mathbb{P}_{(X,\bar{X}^n)}(dx,d\bar{x}) = \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0)\mathbb{P}_{(X^{x_0},\bar{X}^{n,x_0})}(dx,d\bar{x}).$$

- Using r-monotonicity of $L^r(\mathbb{P})$-norms and pseudo-norms and the elementary inequality $(a+b)^\rho \le a^\rho + b^\rho$, for $a,\ b \ge 0$ with $\rho = \frac{p}{\bar{p}} \in [0,1]$ yields

$$\begin{aligned}
\|\Phi(X,\bar{X}^n)\|_p^p = \mathbb{E}\,|\Phi(X,\bar{X}^n)|^p &= \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0)\mathbb{E}\,|\Phi(X^{x_0},\bar{X}^{n,x_0})|^p \\
&\le \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0)\Big(\mathbb{E}\,|\Phi(X^{x_0},\bar{X}^{n,x_0})|^{\bar{p}}\Big)^{\frac{p}{\bar{p}}} \\
&\le \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0)\Big(C_n^{\bar{p}}(1+|x_0|)^{\bar{p}}\Big)^{\frac{p}{\bar{p}}} \\
&\le C_n^p \int_{\mathbb{R}^d} \mathbb{P}_{X_0}(dx_0)(1+|x_0|^p) = C_n^p(1+\|X_0\|_p^p) \\
&\le 2^{(1-p)^+} C_n^p (1+\|X_0\|_p)^p.
\end{aligned}$$

so that, finally,

$$\|\Phi(X, \bar{X}^n)\|_p \leq 2^{(1/p-1)^+} C(1 + \|X_0\|_p). \qquad \square$$

- In fact, as proved, Zhang's theorem holds true for $p > p_{\beta,\theta} = \frac{1}{\theta} \vee \frac{2\beta}{\beta-1}$.
- Then, the extensions follow from the splitting lemma, once proved that all constants in bounds and estimates are of the form "$C_{X_0} = C(1 + \|X_0\|_p)$" for X and its Euler schemes.
- Proving convex ordering for $X_0 \in L^1(\mathbb{P})$ becomes a realistic project...

Convexity w.r.t. x

- Back to (5) i.e. the scaled Volterra equation

$$X_t = X_0 + \int_0^t K(t,s)\alpha(s)\big(X_s + \beta(s)\big)\,ds + \int_0^t K(t,s)\sigma(s,X_s)dW_s,\ \ t \in [0,T].$$

Theorem (Convexity w.r.t. the starting value)

- *Let* (b,σ) *satisfying* $(\mathcal{LH}_\gamma)$ *for some* $\gamma \in (0,1]$ *and* K *satisfying* $(\mathcal{K}^{int}_\beta)$, $(\mathcal{K}^{cont}_\theta)$ *and* $(\widehat{\mathcal{K}}^{cont}_{\widehat{\theta}})$. *Let* $X^x = (X^x_t)_{t\in[0,T]}$ *denote the solution starting from* $X_0 = x \in \mathbb{R}^d$ *to the above Volterra SDE.*
- *Assume*

$$\forall\, t \in [0,T], \quad x \mapsto \sigma(t,x) \text{ is } \preceq\text{-convex.}$$

- *Then, for every l.s.c. convex functional* $F : \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbb{R}$

$$x \longmapsto \mathbb{E}\, F(X^x) \in (-\infty,+\infty] \quad \text{is convex.}$$

- *If* F *has* $\|\,.\,\|_{\sup}$*-polynomial growth, then it is convex and* $\mathbb{R}$*-valued.*

Functional convex ordering

- Let us consider a siamese equation

$$Y_t = Y_0 + \int_0^t K(t,s)\alpha(s)\big(Y_s + \beta(s)\big)ds + \int_0^t K(t,s)\theta(s, Y_s)dW_s,\ t \in [0, T]$$

Theorem (convex ordering)

If

$$\begin{cases} (i)_\sigma & \sigma(t,x) \text{ is } x\text{-}\preceq\text{-convex for every } t \in [0,T], \\ or & \\ (i)_\theta & \theta(t,x) \text{ is } x\text{-}\preceq\text{-convex for every } t \in [0,T], \\ and & \\ (ii) & \sigma(t,x) \preceq \theta(t,x) \quad [|\sigma(t,x)| \le |\theta(t,x)| \ \text{ if } \ d=1] \end{cases}$$

and $X_0 \preceq_{cvx} Y_0$, *then, for every l.s.c. convex* $F : \mathcal{C}([0,T], \mathbb{R}) \to \mathbb{R}$

$$\mathbb{E}\, F(X) \le \mathbb{E}\, F(Y).$$

- Assumptions cannot be relaxed in dimension $d = q = 1$ (to be compared with regular diffusions).
- Convexity may appears as a consequence $\delta_{\lambda x+(1-\lambda)y} \preceq_{cvx} \lambda\delta_x + (1-\lambda)\delta_y$.

Methods of proof

- ($\alpha = \beta = 0$ for simplicity).
- We consider its Euler scheme with time step $\frac{T}{n}$ ($t_k = \frac{kT}{n}$):

$$\bar{X}_{t_k} = X_0 + \sum_{\ell=0}^{k-1} \sigma(t_\ell, \bar{X}_{t_\ell}) \int_{t_\ell}^{t_{\ell+1}} K(t_k, s)dW_s, \quad \bar{X}_0 = X_0.$$

- Not enough due to lack of Markovianity since $\bar{X}_{t_k}$ is not (in general) a function of $(\bar{X}_{t_{k-1}}, (W_s - W_{t_{k-1}})_{s\in[t_{k-1},t_k]})$.
- Markovianization: introduce for $k \in \{1, \cdots, n\}$, $(X^k_{t_\ell})_{0\le\ell\le k}$ starting from $X^k_0 = X_0$ and evolving inductively according to

$$X^k_{t_{\ell+1}} = X^k_{t_\ell} + \sigma(t_\ell, \bar{X}_{t_\ell}) \int_{t_\ell}^{t_{\ell+1}} K(t_k, s)dW_s, \quad 0 \le \ell \le k-1,$$

  so that $\bar{X}_{t_k} = X^k_{t_k}$ for $k \in \{1, \cdots, n\}$ and $X^n = \bar{X}^n$.
- "Extend" the discrete time backward propagation proof to extended functions

$$F\big((X^n_{t_\ell})_{\ell=0:n}, \ldots, (X^k_{t_\ell})_{\ell=0:k}, \ldots, (X^1_{t_\ell})_{\ell=0:1}, X_0\big).$$

- ...with respect to the discrete time filtration of the Brownian motion $(\mathcal{F}^W_{t_k})_{k=0:n}$ augmented by $\sigma(X_0)$ so that at time $t_0 = 0$ it is $\sigma(X_0)$. Idem for Y.
- Transfer to continuous time by letting $n \to \infty$ (using $L^p(\mathbb{P})$ convergence of K-integrated Euler scheme).
- Then one derives, under the assumptions of the theorem that for Lipschite convex functionals $F : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}$, $x \mapsto \mathbb{E}\, F(X^x)$ is convex and $X \preceq_{cvx} Y$, etc. □
- Extension to (one-dimensional) non-decreasing convex ordering when $d = q = 1$.
  - If the drift $b(t, \cdot)$ is $\preceq$-convex and non-decreasing.
  - the coefficient $|\sigma(t, \cdot)|$ is $\preceq$-convex and non-decreasing.

  then the conclusion of the theorem holds for $\preceq_{icv}$-ordering.
- Still true with two different drifts $b_1(t, x)$ and $b_2(t, x)$ with additional condition $b_1 \leq b_2$.

Applications to Vix options in rough Heston model

- Let us consider the auxiliary variance process in the quadratic rough Heston model (see Gatheral-Jusselin-Rosenbaum'20):

$$V_t = a(Z_t - b)^2 + c \quad \text{with} \quad a,\ b,\ c \geq 0$$

and, for $H \in (0, 1/2)$,

$$Z_t = Z_0 + \int_0^t (t-s)^{H-\frac{1}{2}} \lambda(f(s) - Z_s)ds + \sigma \int_0^t (t-s)^{H-\frac{1}{2}} \sqrt{a(Z_s - b)^2 + c}\, dW_s.$$

- $z \mapsto \sqrt{a(z-b)^2 + c}$ is convex and Lipschitz.
- Let $(Z_t^\sigma)_{t\geq 0}$ be its unique strong solution and V^σ the resulting squared volatility.
- For $\sigma \in (0, \tilde{\sigma}]$, one has $(Z_t^\sigma)_{t\in[0,T]} \preceq_{cvx} (Z_t^{\tilde{\sigma}})_{t\in[0,T]}$.
- Convexity of $L^2(dt)$ norm and (again) of $z \mapsto \sqrt{a(z-b)^2 + c}$ imply that

$$\mathbb{E}\left(\sqrt{\frac{1}{T}\int_0^T V_t^\sigma dt}\right) \leq \mathbb{E}\left(\sqrt{\frac{1}{T}\int_0^T V_t^{\tilde{\sigma}} dt}\right).$$

This is in fact a paradigm:

Propagate convex order in discrete time then transfer to continuous time is easier

(if you know functional limit theorems for the dynamics under consideration)

Bedankt voor je aandacht en bedankt voor de uitnodiging