Robust Risk Management

Carole Bernard

Grenoble EM and VU Brussel

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Part II-a

"Instead of considering marginals and correlations separately ...might also be sensible to consider whether the question of interest permits the estimation problem to be reduced to a one-dimensional one. **For example, if we are really interested in the behaviour of the sum we might consider directly estimating its univariate distribution.**"

Embrechts, McNeil and Straumann (1998), *Correlation and Dependency in Risk Management.*

First Part: Model Risk on the Dependence: Theory and Computational Approach via The rearrangement algorithm

Second Part: Model Risk on the Aggregate Variable

Papers I will focus on in this second part

- Rodrigue Kazzi (2023), "Advancing Model Uncertainty Assessment to Address Actuarial Modelling Challenges." PhD thesis at Vrije Universiteit Brussel.
- Bernard, C., Kazzi R., Vanduffel, S. (2020). *Range Value-at-Risk bounds for unimodal distributions under partial information*. **Insurance: Mathematics and Economics**.
- Bernard, C., Kazzi R., Vanduffel, S. (2022). *Model uncertainty assessment for symmetric and right-skewed distributions.* Working paper.
- Bernard, C., Kazzi R., Vanduffel, S.(2023). *Impact of model misspecification on the Value-at-Risk of unimodal T-symmetric distributions.* Working paper.
- Bernard, C., Kazzi R., Vanduffel, S. (2023). *Incorporating robust information into model risk assessment.* Working paper.
- Bernard, C., Pesenti, S., Vanduffel, S. (2024) *Robust Distortions Measures*, **Mathematical Finance**, forthcoming.

Model risk assessment is indispensable

Actuarial Association of Europe (2017): " ... model risk cannot be disregarded. There will be many models that are consistent with the used data. So, in the end, the specific choice of model will be subjective."

Basel Committee on Banking Supervision (2019): "Banks are encouraged to review and provide evidence on the uncertainty around the outcomes of the capital requirement model ... by identifying the most significant assumptions and estimating uncertainty bounds ..."

A look into model risk

Fully trusted information + Additional assumptions

↓

Adopted model

Fully trusted information + **No** additional assumptions

↙ ↘

Best-case model   Worst-case model

Model risk can be assessed by comparing the value of a risk measure under adopted model to its value under the best-case model and the worst-case model.

A look into model uncertainty

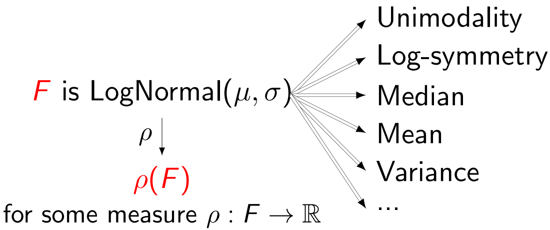
F is LogNormal(μ, σ)

$$\rho \downarrow$$

$\rho(F)$

for some measure $\rho : F \to \mathbb{R}$

A look into model uncertainty



A look into model uncertainty

$\mathcal{V} = \{F : F \text{ is consistent with some assumptions}\}$

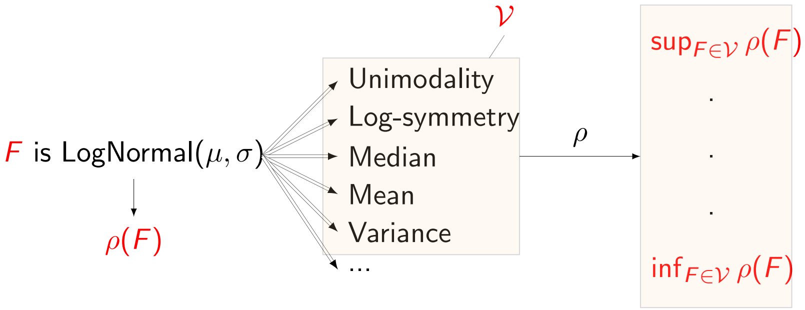
- Unimodality
- Log-symmetry
- Median
- Mean
- Variance
- ...

F is LogNormal(μ, σ)

↓

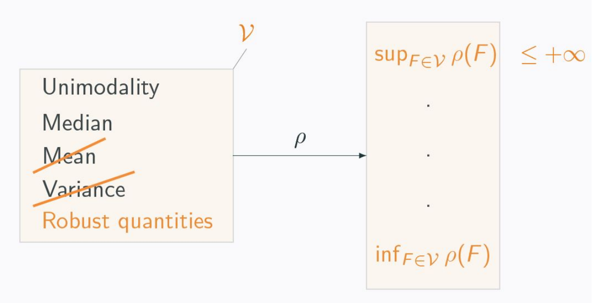
$\rho(F)$

A look into model uncertainty



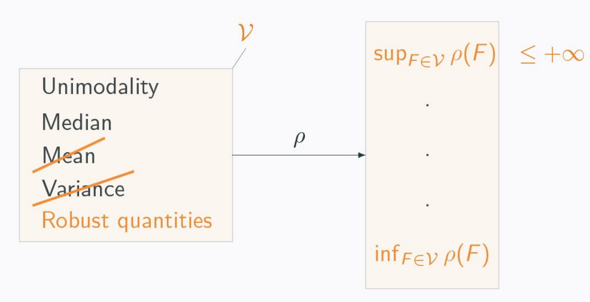
Such problems are dealt with in the first main part of this talk.

A look into model uncertainty for heavy-tailed risks



$\Longrightarrow$ We need to incorporate information on some robust quantities to assess model uncertainty in heavy-tailed distributions

A look into model uncertainty for heavy-tailed risks



$\Longrightarrow$ We need to incorporate information on some robust quantities.

This is the main objective of the last part of this talk.

What was missing?

Omission of assumptions that could be fully trusted

↓

Wide bounds

↓

Not very meaningful

Fully trusting assumptions that are hard to collect

↓

Not very practical

Outline

1. Problem Formulation
2. Developed Methodology
3. Two examples with Risk Bounds with Moments Information
4. Risk Bounds for Heavy-Tailed Risks

VaR, RVAR, ES

For a random variable $X \sim F_X$ and $0 < \alpha < \beta < 1$ we have

- Value-at-Risk:

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

VaR, RVAR, ES

For a random variable $X \sim F_X$ and $0 < \alpha < \beta < 1$ we have

- Value-at-Risk:

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

- Range-Value-at-Risk:

$$\mathrm{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \int_\alpha^\beta \mathrm{VaR}_u(X) \mathrm{d}u.$$

VaR, RVAR, ES

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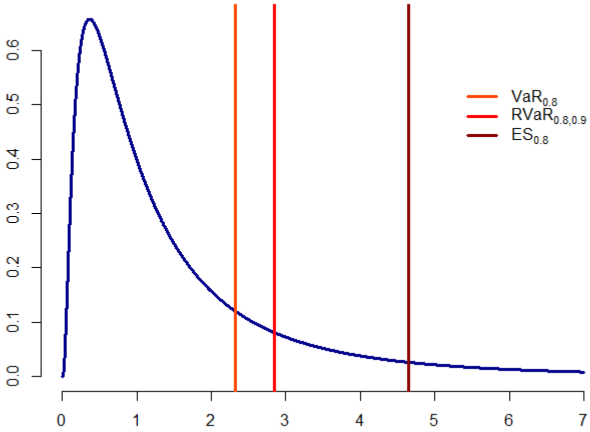
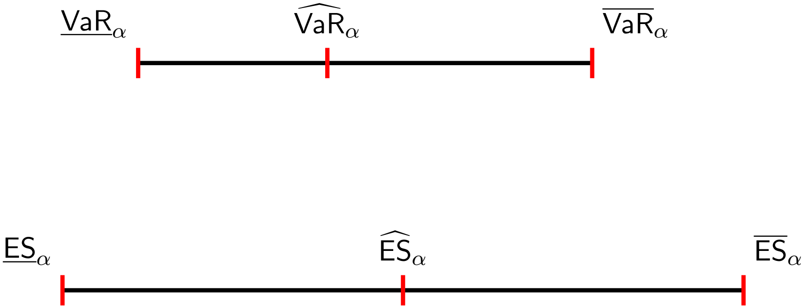
- Expected Shortfall:

$$\mathrm{ES}_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 \mathrm{VaR}_u(X) \mathrm{d}u.$$

VaR, RVAR, ES



VaR and ES


$\underline{\text{VaR}}_\alpha$
$\widehat{\text{VaR}}_\alpha$
$\overline{\text{VaR}}_\alpha$
$\underline{\text{ES}}_\alpha$
$\widehat{\text{ES}}_\alpha$
$\overline{\text{ES}}_\alpha$

Problem Formulation

Problem formulation

Basic Problem

$$\sup_{F \in \mathcal{V}} \rho(F) \qquad \textit{and} \qquad \inf_{F \in \mathcal{V}} \rho(F)$$

for some measure $\rho : F \to \mathbb{R}$ *and set* $\mathcal{V}$ *where*

$$\mathcal{V} = \{F : \ F \textit{ is consistent with some assumptions}\}$$

Problem formulation

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Measures of interest

For some $(\alpha; \beta) \in (0,1) \times (\alpha, 1)$, $(x_1, x_2) \in \mathbb{R} \times (x_1, +\infty)$,

- $VaR_\alpha(F) = F^{-1}(\alpha)$
- $TVaR_\alpha(F)$
- $VaR_\beta(F) - VaR_\alpha(F)$
- $RVaR_{\alpha,\beta}(F) = \frac{1}{\beta-\alpha} \int_\alpha^\beta F^{-1}(p)dp$
- $F(x_2) - F(x_1)$
- $E\left[g(F)\right]$ *for some* $g(.)$

Problem formulation

Basic Problem

$$\sup_{F\in\mathcal{V}} \rho(F) \qquad \text{and} \qquad \inf_{F\in\mathcal{V}} \rho(F)$$

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Measures of interest

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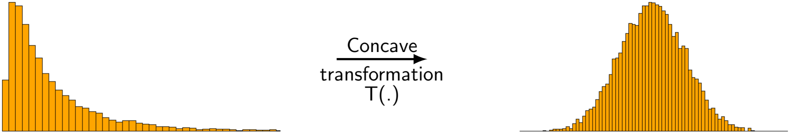
- $F(x_2) - F(x_1)$
- $E[g(F)]$ *for some* $g(.)$

Assumptions of interest



Assumptions

- *Unimodality*
- *Symmetry*
- *T-unimodality*
- *T-symmetry*
- *Non-negativity / Support*
- *Moments on the original distribution*
- *Moments on the transformed distribution*
- *Robust and quantile-based measures*

Examples of robust and quantile-based measures

For $0 < \alpha_1 < \alpha_2 < 1$,

- A specific quantile, e.g., $F^{-1}(0.75)$
- Interpercentile range: $F^{-1}(\alpha_2) - F^{-1}(\alpha_1)$
- Truncated/trimmed moments: $\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} h\left(F^{-1}(p)\right)dp$ for some function h

  E.g., $\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} F^{-1}(p)dp$ and $\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\left(F^{-1}(p)\right)^2 dp$
- Moor's kurtosis: $\frac{F^{-1}(7/8)-F^{-1}(5/8)+F^{-1}(3/8)-F^{-1}(1/8)}{F^{-1}(6/8)-F^{-1}(2/8)}$
- ...

Developed Methodology

Tool: Convex ordering

Convex ordering is a type of **stochastic ordering** that compares the **variability** of risks.

Definition

X is said to be smaller than Y in the convex order, denoted as $X \leqslant_{cx} Y$, if and only if

$$E[v(X)] \leqslant E[v(Y)] \text{ for all convex functions } v : \mathbb{R} \to \mathbb{R},$$

provided the expectations exist.

Tool: Convex ordering

- First result:

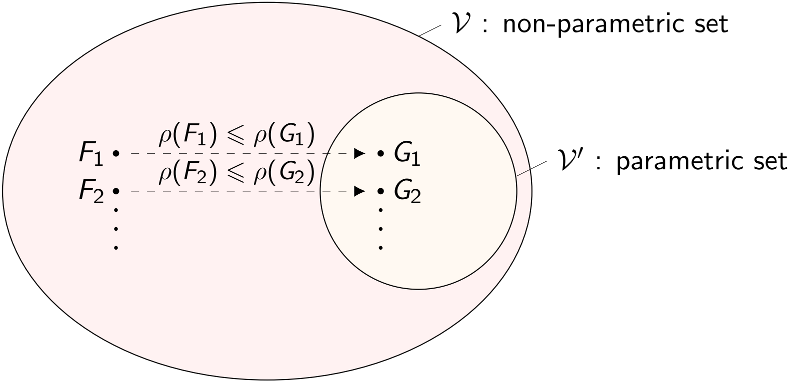
$$X \leqslant_{cx} Y \Rightarrow var[X] \leqslant var[Y]$$

- Second result:

$$\left\{ \begin{array}{c} E[X] = E[Y], \\ \text{and } F_Y^{-1} \text{ up-crosses } F_X^{-1} \text{ exactly once.} \end{array} \right. \Rightarrow X \leqslant_{cx} Y$$

General approach

Mathematical challenge: The optimization is non-parametric
Solution: Reduce it to a parametric optimization via stochastic ordering



Two examples of application of this methodology:

1. VaR bounds with unimodality and moment constraints
2. RVaR bounds

Reduction to a parametric optimization

$V_U(\mu, s)$: set of r.v. with unimodal dist. with first moment μ and maximum variance s^2.
U_R: set of random variables whose quantile is flat-linear.
We show that

$$\forall S^* \in V_U(\mu, s), \text{ there exists } Y_R \in U_R \cap V_U(\mu, s)$$

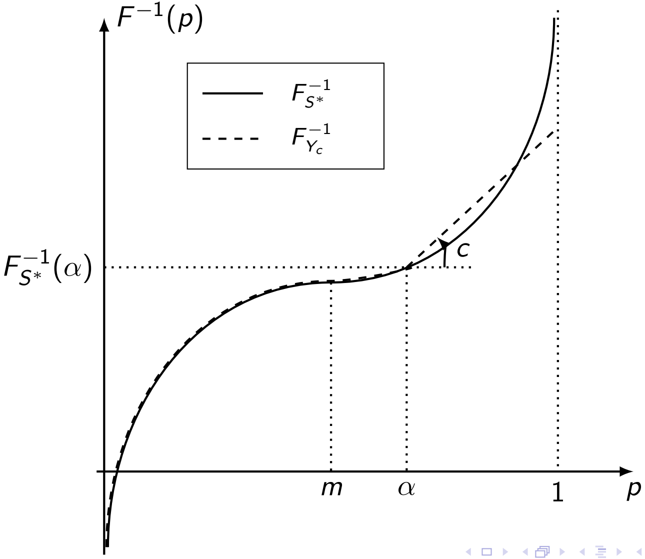
such that

$$VaR_\alpha(Y_R) = VaR_\alpha(S^*).$$

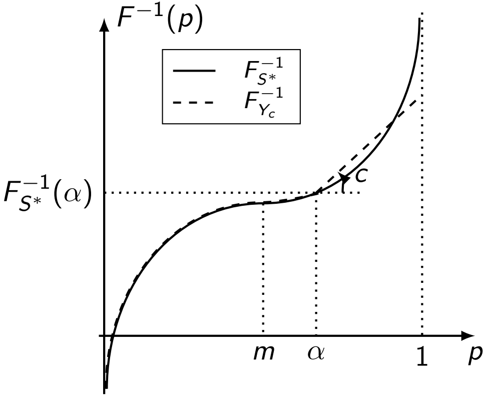
Hence,

$$\sup_{S \in V_U(\mu,s)} VaR_\alpha(S) = \sup_{S \in U_R \cap V_U(\mu,s)} VaR_\alpha(S).$$

Definition of Y_c (case $\alpha > m$)



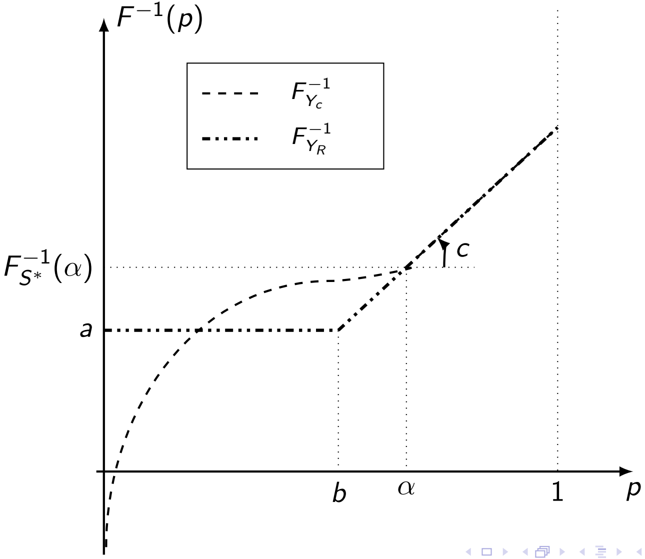
Y_c is smaller than S^* in convex order



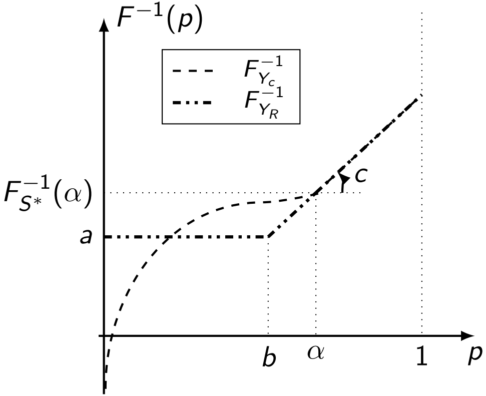
$$\left.\begin{array}{c} E[Y_c] = E[S^*], \\ \text{and } F_{S^*}^{-1} \text{ up-crosses } F_{Y_c}^{-1} \text{ exactly once.} \end{array}\right\} \Rightarrow Y_c \leqslant_{cx} S^*$$

$$\Rightarrow var[Y_c] \leqslant var[S^*] \leqslant s^2$$

Definition of Y_R



Y_R is an element of $V_U(\mu, s)$



$$Y_R \leqslant_{cx} Y_c \Rightarrow var[Y_R] \leqslant var[Y_c] \Rightarrow Y_R \in V_U(\mu, s)$$

and

$$VaR_\alpha(Y_R) = VaR_\alpha(S^*)$$

Further developments

- A similar approach is done for $\alpha < m$. A comparison leads to the VaR upper bounds.
- This method can lead to risk bounds in case we assume
  - Having a non-negative unimodal portfolio loss with known mean and maximum variance
  - Having a non-negative unimodal distribution with known mean and infinite variance
- This method can be amended to derive bounds of other risk measures, like the Range Value-at-Risk (and Tail Value-at-Risk).

Value-at-Risk bounds

$$\sup_{S\in V_U(\mu,s)} \mathrm{VaR}_\alpha(S) = \begin{cases} \mu + s\sqrt{\frac{4}{9(1-\alpha)} - 1} & \text{for } \alpha \in [5/6; 1[, \\ \mu + s\sqrt{\frac{3\alpha}{4-3\alpha}} & \text{for } \alpha \in]0; 5/6[. \end{cases}$$

$$\Downarrow$$

$$\inf_{S\in V_U(\mu,s)} \mathrm{VaR}_\alpha(S) = \begin{cases} \mu - s\sqrt{\frac{1-\alpha}{1/3+\alpha}} & \text{for } \alpha \in]1/6; 1[, \\ \mu - s\sqrt{\frac{4}{9\alpha} - 1} & \text{for } \alpha \in]0; 1/6]. \end{cases}$$

(Li, Shao, Wang and Yang (2018) derived this upper bound for $\alpha \in [5/6; 1[$.)

Maximum distance between the mean and its robust estimator

Denote by Q_α the quantile function, we have that

$$\frac{\left|\frac{Q_\alpha+Q_{1-\alpha}}{2}-\mu\right|}{s} \leqslant \begin{cases} \frac{\sqrt{\frac{4}{9(1-\alpha)}-1}+\sqrt{\frac{1-\alpha}{1/3+\alpha}}}{2} & \text{for } \alpha \in [5/6;1[, \\ \frac{\sqrt{\frac{3\alpha}{4-3\alpha}}+\sqrt{\frac{1-\alpha}{1/3+\alpha}}}{2} & \text{for } \alpha \in]1/6;5/6[, \\ \frac{\sqrt{\frac{3\alpha}{4-3\alpha}}+\sqrt{\frac{4}{9\alpha}-1}}{2} & \text{for } \alpha \in]0;1/6]. \end{cases}$$

For $\alpha = 0.5$, the maximum distance between the median and the mean of a unimodal distribution derived by Basu and Dasgupta (1997) is recovered.

VaR upper bounds for non-negative portfolio losses

Let

$V_U^+(\mu, s) = \{X :\ X \text{ is unimodal, } E[X] = \mu,\ \mathrm{var}[X] \leqslant s^2,\ X \text{ is non-negative}\}$,

and

$W_U^+(\mu) = \{X :\ X \text{ is unimodal, } E[X] = \mu,\ \mathrm{var}[X] \text{ is infinite},\ X \text{ is non-negative}\}$

We have that, for $\alpha \geqslant m$,

$$\sup_{S \in V_U^+(\mu,s)} \mathrm{VaR}_\alpha(S) = \begin{cases} \frac{\mu}{2(1-\alpha)} & \text{for } (\alpha, s) \in \left]\frac{1}{2}; 1\right[\times \left[\mu\sqrt{\frac{\alpha - 1/3}{1-\alpha}}; +\infty\right[, \\ \ldots & \text{for } (\alpha, s) \in \ldots, \\ \mu & \text{for } (\alpha, s) \in \left]0; \frac{1}{2}\right] \times [0; +\infty[\,, \end{cases}$$

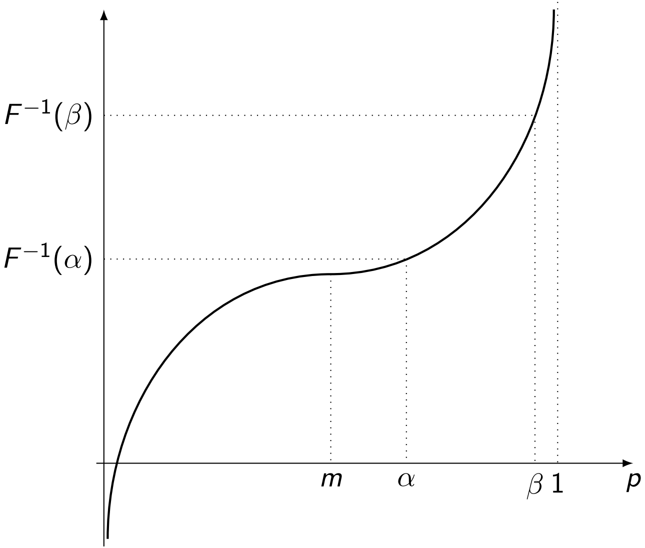
and

$$\sup_{S \in W_U^+(\mu)} \mathrm{VaR}_\alpha(S) \leqslant \begin{cases} \frac{\mu}{2(1-\alpha)} & \text{for } \alpha \in \left]\frac{1}{2}; 1\right[\,, \\ \mu & \text{for } \alpha \in \left]0; \frac{1}{2}\right]. \end{cases}$$

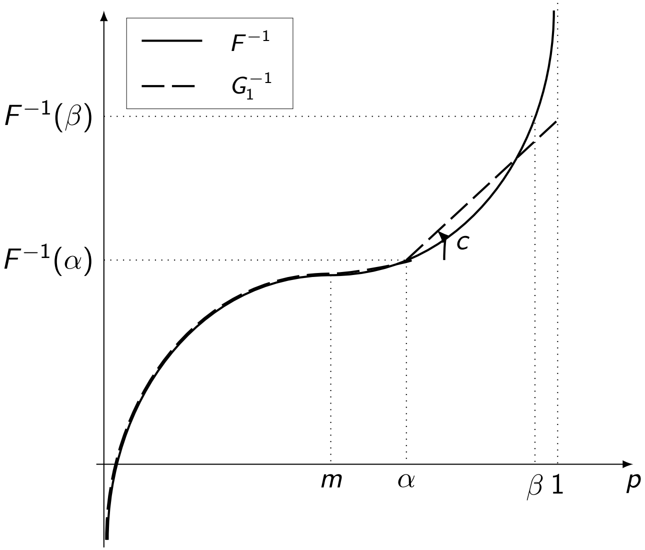
Another example of the reduction technique

- Fully trusted assumptions: unimodality, mean, maximum standard deviation, and non-negativity.
- Risk measure: Range Value-at-Risk.
- The best- and worst-case models correspond to a mixture of a point mass and a uniform.

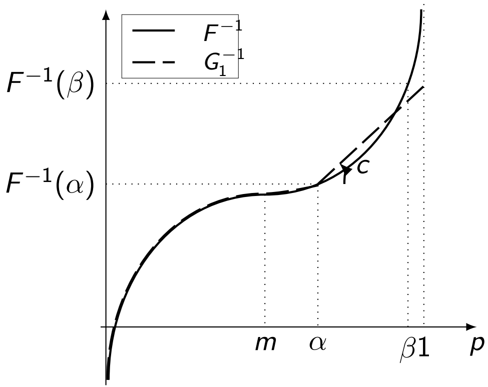
Arbitrary element F of $\mathcal{V}$



Construction of G_1

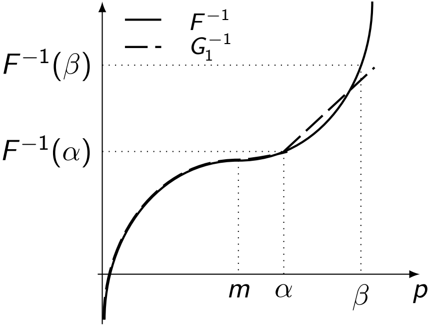


G_1 is smaller than F in convex order



$$\left.\begin{array}{c} E[G_1] = E[F], \\ \text{and } F^{-1} \text{ up-crosses } G_1^{-1} \text{ exactly once.} \end{array}\right\} \Rightarrow G_1 \leqslant_{cx} F$$

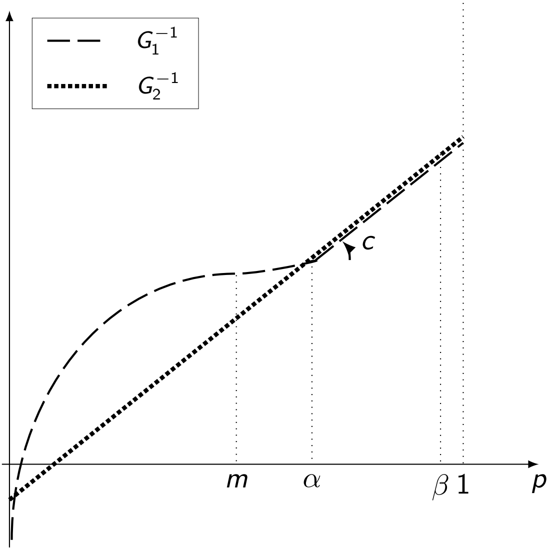
$\mathrm{RVaR}_{\alpha,\beta}(G_1) \geqslant \mathrm{RVaR}_{\alpha,\beta}(F)$



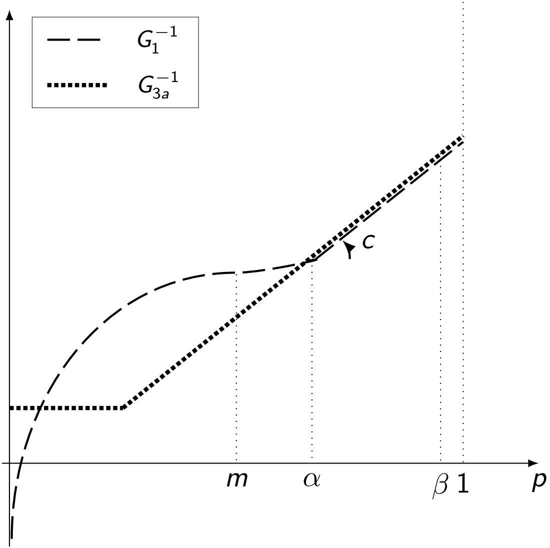
$$(\beta-\alpha)\mathrm{RVaR}_{\alpha,\beta}(G_1) = E[G_1] - \int_0^{\alpha} \mathrm{VaR}_p(G_1)dp - \int_{\beta}^{1} \mathrm{VaR}_p(G_1)dp$$

$$\geqslant E[F] - \int_0^{\alpha} \mathrm{VaR}_p(F)dp - \int_{\beta}^{1} \mathrm{VaR}_p(F)dp$$

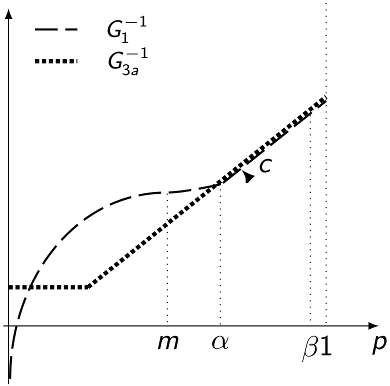
Construction of G_2



If $E[G_2] < E[G_1]$, we construct G_{3a}

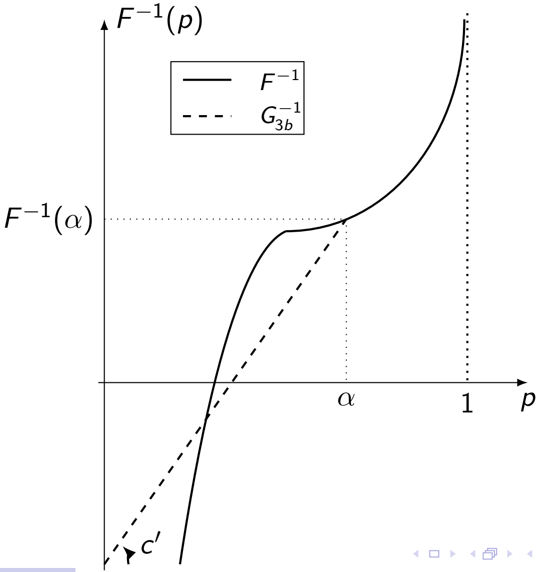


If $E[G_2] < E[G_1]$, we define G_{3a}



- $G_{3a} \leqslant_{cx} G_1 \leqslant_{cx} F$
- $\mathrm{RVaR}_{\alpha,\beta}(G_{3a}) = \mathrm{RVaR}_{\alpha,\beta}(G_1) \geqslant \mathrm{RVaR}_{\alpha,\beta}(F)$
- G_{3a} is non-negative
- G_{3a} is a mixture of a point mass and uniform and hence is unimodal
- G_{3a} is parametric

If $E[G_2] \geqslant E[G_1]$, we construct G_{3b}



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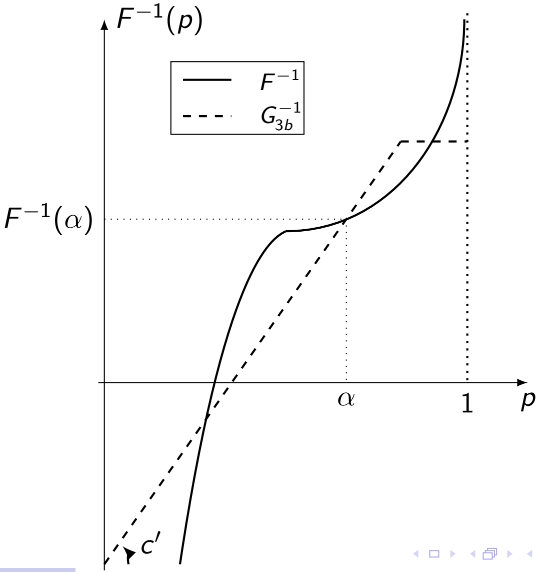

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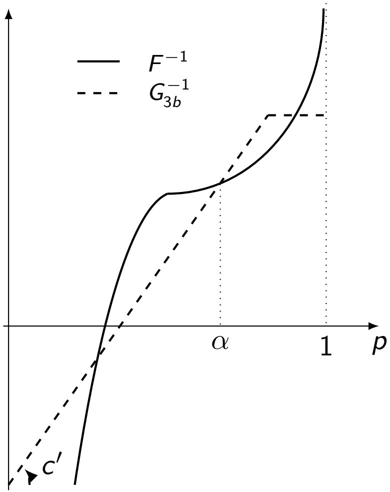
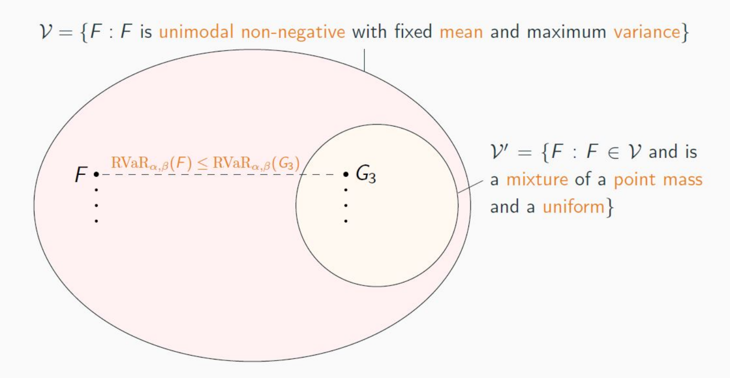


Again, we have

- $G_{3b} \leqslant_{cx} F$
- $\mathrm{RVaR}_{\alpha,\beta}(G_{3b}) \geqslant \mathrm{RVaR}_{\alpha,\beta}(F)$
- G_{3b} is non-negative
- G_{3b} is a mixture of a point mass and uniform and hence is unimodal
- G_{3b} is parametric

End of the proof

The optimization problem can be reduced to a parametric one.



Numerical Example

Numerical example

Characteristics of a credit portfolio:

- $\frac{S}{\text{Total exposure}} \sim$ Beta distribution
- Size $=$ 10000 loans of amount 1 millions Euros each.
- Probability of default on the loan $= 0.1\%$
- $\frac{\sqrt{var[S]}}{E[S]} = 1.3$

Numerical example

	Com. Inf.	μ & s	μ & s & U	+non-neg.	+[0; 0.75]
α	$VaR_\alpha(S)$	$\overline{VaR_\alpha^c}$	$\overline{VaR_\alpha}$	$\overline{VaR_\alpha^+}$	$\overline{VaR_\alpha^p}$
75%	13.546	32.517	24.741	21.465	13.546
90%	26.106	49	34.127	34.127	30.85
95%	36.182	66.666	46.513	46.513	42.94
99.5%	71.290	193.388	131.874	131.874	89.232

Table: **Upper bounds** of the **Value-at-Risk** under different scenarios regarding the distributional information that is available. The first column depicts the "true" risk measure assuming complete information. All figures are in **million Euros**.

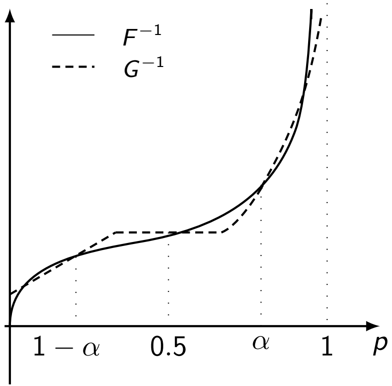
T-unimodal T-symmetric distributions

T-unimodal T-symmetric distributions

- We know that the distribution becomes unimodal symmetric after a concave strictly increasing transformation $T(.)$.
- We can incorporate information on the moments of the original and transformed distribution, as well as information on the median, interquartile range, and the support.
- The best- and worst-case models correspond to a mixture of a point mass and a convex transformation of a uniform.

Unimodal T-symmetric distributions

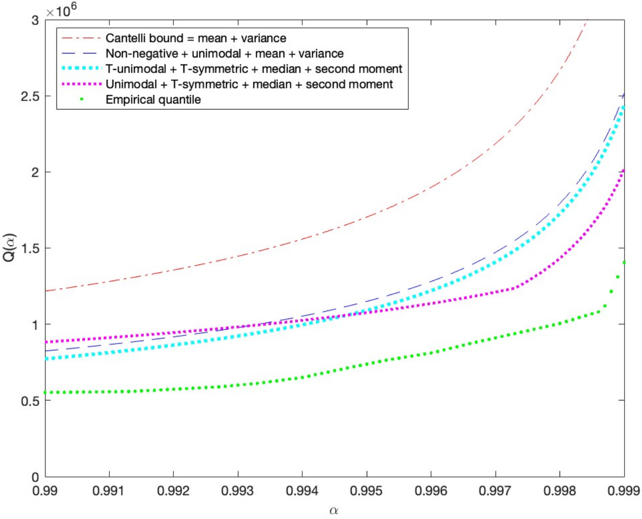
- The distribution is unimodal, T-symmetric.
- Same set of information can be incorporated as in the previous slide.
- Assuming unimodality instead of T-unimodalidy can significantly improve the bounds.
- The optimization of VaR_α for high α's can be reduced to a parametric optimization over distribution functions whose quantiles are of this shape:



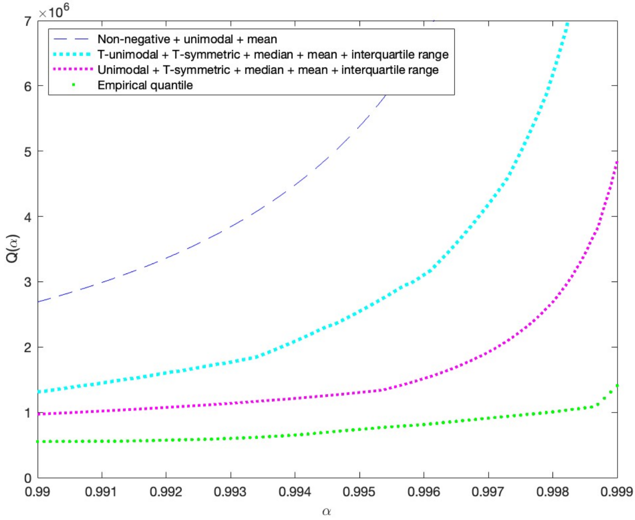
General liability claims dataset (Frees and Valdez (1998))

- The dataset comprises 1,500 general liability claims.
- The loss distribution is unimodal, log-unimodal, and log-symmetric.
- Median $=$ 20,113, Mean $=$ 53,797, Std.dev. $=$ 116,942, and Interquartile Range $=$ 41,720.

Comparison of VaR bounds



Comparison of VaR bounds



Risk bounds for heavy tailed risks

Information that can be incorporated

- Unimodality
- Symmetry
- T-unimodality
- T-symmetry

- Non-negativity / Support
- Moments on the original distribution
- Moments on the transformed distribution
- Robust and quantile-based measures

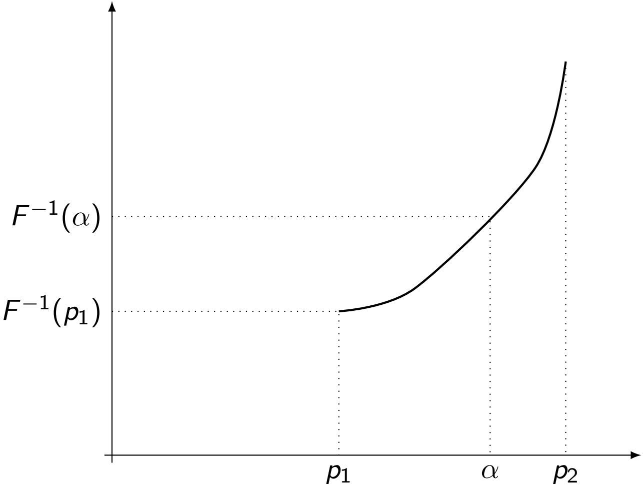
Examples of robust and quantile-based measures

For $0 < \alpha_1 < \alpha_2 < 1$,

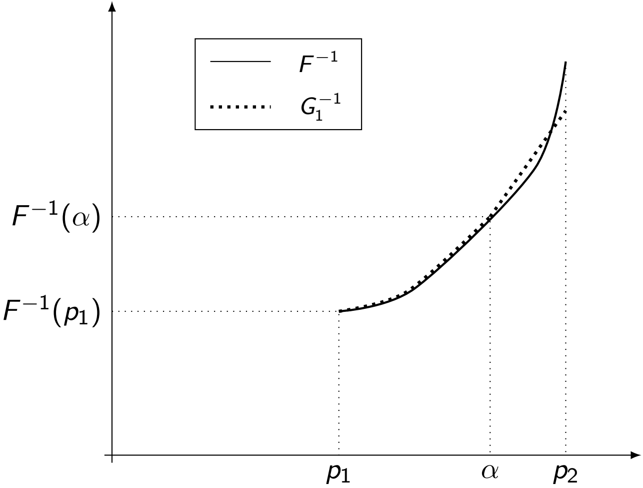
- A specific quantile, e.g., $F^{-1}(0.75)$
- Interpercentile range: $F^{-1}(\alpha_2) - F^{-1}(\alpha_1)$
- Truncated/trimmed moments: $\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} h\left(F^{-1}(p)\right) dp$ for some function h

  E.g., $\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} F^{-1}(p)dp$ and $\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \left(F^{-1}(p)\right)^2 dp$
- Moor's kurtosis: $\frac{F^{-1}(7/8)-F^{-1}(5/8)+F^{-1}(3/8)-F^{-1}(1/8)}{F^{-1}(6/8)-F^{-1}(2/8)}$
- ...

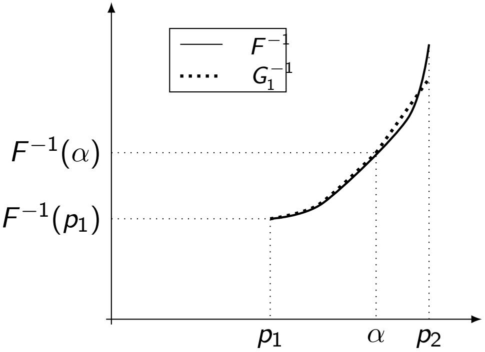
Arbitrary element F of $\mathcal{V}$



Construction of G_1^{-1}



G_1 vs F

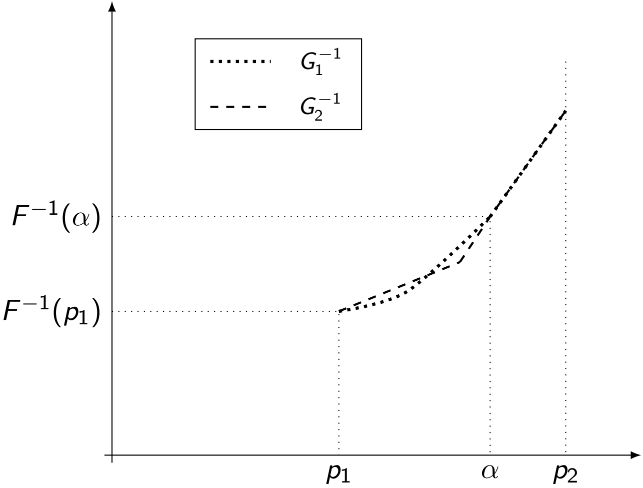


$$\int_{p_1}^{p_2} G_1^{-1}(p)dp = \int_{p_1}^{p_2} F^{-1}(p)dp$$

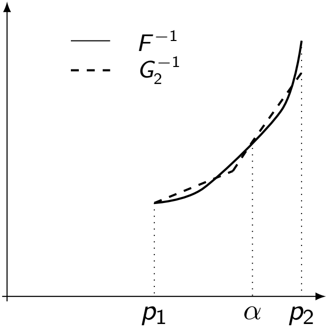
and F^{-1} up-crosses G_1^{-1} exactly once on $[p_1, p_2]$

$$\Longrightarrow \int_{p_1}^{p_2} h\left(G_1^{-1}(p)\right) dp \leqslant \int_{p_1}^{p_2} h\left(F^{-1}(p)\right) dp \text{ for any convex } h$$

Similarly for G_2 vs G_1



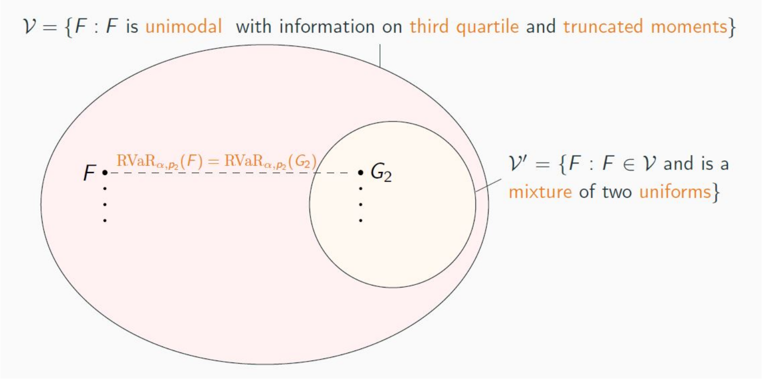
For every $F \in \mathcal{V}$, there exists G_2 such that



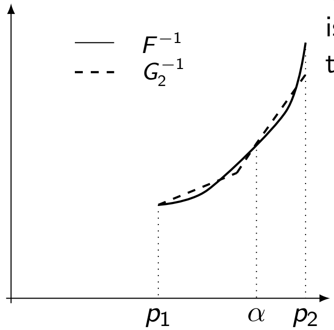
- G_2 is parametric on $[p_1, p_2]$
- $G_2^{-1}(p_1) = F^{-1}(p_1)$
- $\mathrm{VaR}_\alpha(G_2) = \mathrm{VaR}_\alpha(F)$
- $\mathrm{RVaR}_{\alpha,p_2}(G_2) = \mathrm{RVaR}_{\alpha,p_2}(F)$
- $$\int_{p_1}^{p_2} G_2^{-1}(p)dp = \int_{p_1}^{p_2} F^{-1}(p)dp$$
- $$\int_{p_1}^{p_2} h\left(G_2^{-1}(p)\right) dp \leqslant \int_{p_1}^{p_2} h\left(F^{-1}(p)\right) dp$$
  for any convex h

End of the proof

The optimization problem can be reduced to a parametric one.



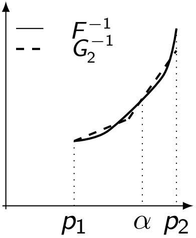
Generalization



The construction of G_2 on $[p_1, p_2]$ is the unique construction such that:

- G_2^{-1} is composed of two consecutive linears on $[p_1, p_2]$ with α belonging to the second linear
- $G_2^{-1}(p_1) = F^{-1}(p_1)$
- $\mathrm{VaR}_\alpha(G_2) = \mathrm{VaR}_\alpha(F)$
- $\mathrm{RVaR}_{\alpha,p_2}(G_2) = \mathrm{RVaR}_{\alpha,p_2}(F)$
- $\int_{p_1}^{p_2} G_2^{-1}(p)dp = \int_{p_1}^{p_2} F^{-1}(p)dp$

Generalization



Let us define two sets of functionals:

- $\mathcal{F}^{\leqslant}(F, \alpha, p_1, p_2) = \{\rho : \rho(G_2) \leqslant \rho(F)\}$
- $\mathcal{F}^{\geqslant}(F, \alpha, p_1, p_2) = \{\rho : \rho(G_2) \geqslant \rho(F)\}$

For example:

- Trimmed moments belong to $\mathcal{F}^{\leqslant}(F, \alpha, p_1, p_2)$
- For $\beta \in (\alpha, p_2)$, $\mathrm{RVaR}_{\alpha,\beta}(.) \in \mathcal{F}^{\geqslant}(F, \alpha, p_1, p_2)$
- $\mathrm{RVaR}_{\alpha,p_2}(.) \in \mathcal{F}^{\leqslant}(F, \alpha, p_1, p_2) \cap \mathcal{F}^{\geqslant}(F, \alpha, p_1, p_2)$

Generalization

Assume information is available on F of the type

- $f(F) \leqslant k \in \mathbb{R}$ for $f \in \mathcal{F}^{\leqslant}(F, \alpha, p_1, p_2)$
- $g(F) \geqslant l \in \mathbb{R}$ for $g \in \mathcal{F}^{\geqslant}(F, \alpha, p_1, p_2)$

And denote by

- $\mathcal{C}_\alpha$ the set of distributions that respect this available information
- $\mathcal{V}_U$ the set of unimodal distributions
- $\mathcal{V}_I$ the set of distribution whose quantiles are as G_2^{-1}

Generalization

Assume information is available on X of the type

- $f(F) \leqslant k \in \mathbb{R}$ for $f \in \mathcal{F}^{\leqslant}(F, \alpha, p_1, p_2)$
- $g(F) \geqslant l \in \mathbb{R}$ for $g \in \mathcal{F}^{\geqslant}(F, \alpha, p_1, p_2)$

And denote by

- $\mathcal{C}_\alpha$ the set of distributions that respect this available information
- $\mathcal{V}_U$ the set of unimodal distributions
- $\mathcal{V}_l(\alpha)$ the set of distribution whose quantiles are as G_2^{-1}

Then

$$\sup_{F \in \mathcal{V}_U \cap \mathcal{C}_\alpha} \rho_1(F) = \sup_{F \in \mathcal{V}_l(\alpha) \cap \mathcal{C}_\alpha} \rho_1(F)$$

and

$$\inf_{F \in \mathcal{V}_U \cap \mathcal{C}_\alpha} \rho_2(F) = \inf_{F \in \mathcal{V}_l(\alpha) \cap \mathcal{C}_\alpha} \rho_2(F),$$

where $\rho_1 \in \mathcal{F}^{\geqslant}(F, \alpha, p_1, p_2)$ and $\rho_2 \in \mathcal{F}^{\leqslant}(F, \alpha, p_1, p_2)$

SAS OpRisk Global dataset

- The dataset contains 39,359 operational losses exceeding \$0.1 million, recorded from March 1971 to April 2023 worldwide.
- The losses are adjusted for inflation and expressed in millions of USD.
- Mean $\approx$ 107, std.dev. $\approx$ 1,022, and 75th percentile $\approx$ 30.
- Truncated moments between 75th and 99.9th percentiles: mean $\approx$ 313 and std.dev. $\approx$ 798.

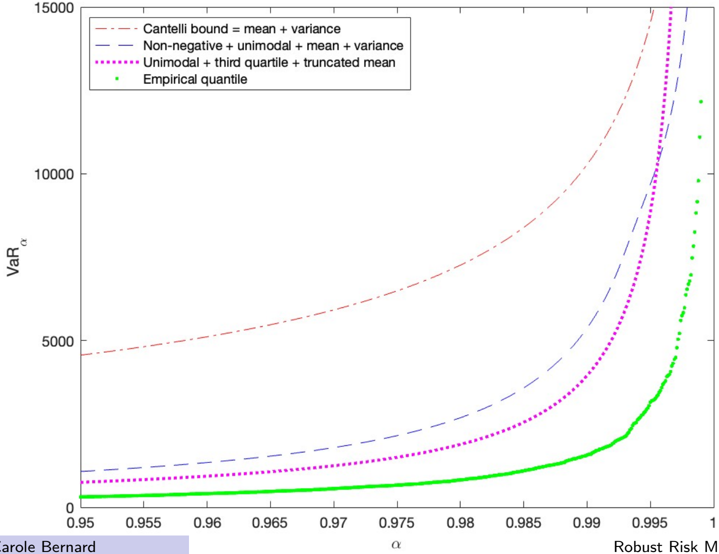
Upper bounds: unimodal + lower quantile + truncated mean

For $0.5 \leqslant \frac{p_1+p_2}{2} < \alpha < \beta \leqslant p_2 < 1$ and $q_1, \mu_{1,t} \in \mathbb{R}^+$, we have that

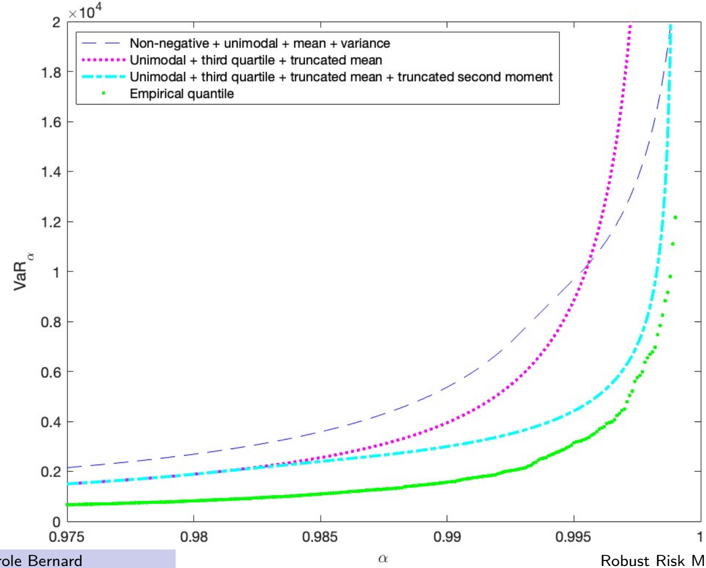
$$\sup_{\substack{F \text{ unimodal} \\ F^{-1}(p_1)=q_1 \\ \int_{p_1}^{p_2} F^{-1}(p)dp \leqslant \mu_{1,t}}} \mathrm{VaR}_\alpha(F) = q_1 \frac{p_2 + p_1 - 2\alpha}{2(p_2-\alpha)} + \mu_{1,t}\frac{p_2-p_1}{2(p_2-\alpha)},$$

$$\sup_{\substack{F \text{ unimodal} \\ F^{-1}(p_1)=q_1 \\ \int_{p_1}^{p_2} F^{-1}(p)dp \leqslant \mu_{1,t}}} \mathrm{RVaR}_{\alpha,\beta}(F) = q_1 \frac{p_2 + p_1 - \alpha - \beta}{2p_2-\alpha-\beta} + \mu_{1,t}\frac{p_2-p_1}{2p_2-\alpha-\beta}.$$

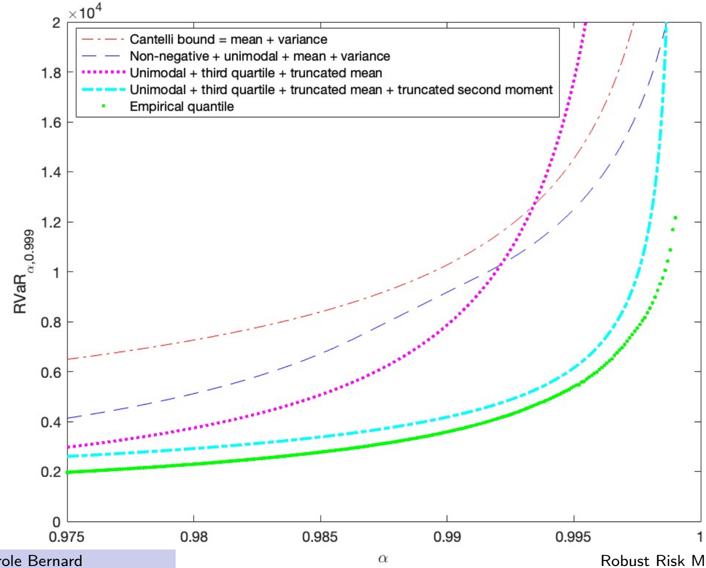
Comparison of VaR upper bounds - 1



Comparison of VaR upper bounds - 2



Comparison of RVaR upper bounds



Main takeaways

- Model uncertainty assessment can accommodate various actuarial modelling contexts and be practical.

Main takeaways

- Model uncertainty assessment can accommodate various actuarial modelling contexts and be practical.
- Using risk bounds, we get to fix the source of model uncertainty.


THANK YOU