Robust Risk Management

Carole Bernard

Grenoble EM and VU Brussel

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Part II-b

Paper I will focus on in this last part

- Bernard, C., Pesenti, S., Vanduffel, S. (2024) *Robust Distortions Measures*, **Mathematical Finance**, forthcoming.

Quantifying Distributional Uncertainty

The set of alternative distribution functions $\mathcal{U}$ or alternative probability measures $\mathcal{P}$ is called the uncertainty / ambiguity set.

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Desirable properties:

- include the "true" distribution
- large enough to account for uncertainty;
- small enough to be useful;
- include additional information / expert opinion.

Optimization Problem for Risk Measures

We consider law-invariant risk measures $\rho\colon \mathcal{X} \to \mathbb{R}$ and aim to solve

$$\sup_{F_X \in \mathcal{U}} \rho(X)\,, \tag{1}$$

Examples of risk measures, VaR, RVaR, ES, coherent risk measures, convex risk measures, distortion risk measures, RDEU ...

Sources of Uncertainty

Uncertainty can stem from

- working with finite samples / empirical distribution
- error in data collection;
- modelling assumptions, e.g., parametric models, stochastic models,
- lack of knowledge, no access to information
- uncertainty on parts of the distribution, e.g, tails
- differing believes, e.g., experts
- $\cdots$

Example of Uncertainty Sets

Moments:

$$\mathcal{P} = \left\{ \nu \in \mathfrak{P} \;\middle|\; \int x\,\nu(dx) = m_0,\; \int x^2\,\nu(dx) = m_1 \right\} \qquad (2)$$

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Distributional information:
symmetry of distribution function, modality, parametric family, tail behaviour, support, ...

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Divergences / Distances:

$$\mathcal{P} = \{\nu \in \mathfrak{P} \mid d(\nu, \mu) \le \delta\}\ , \qquad (3)$$

where $d(\cdot, \cdot)$ is a suitable divergence on probability measures.

Uncertainty sets

Heuristically:

Moment uncertainty sets are

- typically mathematically easier.
- typically results in degenerate (unrealistic) solutions.

Divergence uncertainty sets are

- additional parameter of the radius δ
- $\delta \in [0, +\infty)$ quantifies degree of "uncertainty"
- typically result in more realistic solutions
- inherits information about a "baseline/reference"

2-Wasserstein Distance

The Wasserstein distance of order 2 is defined as

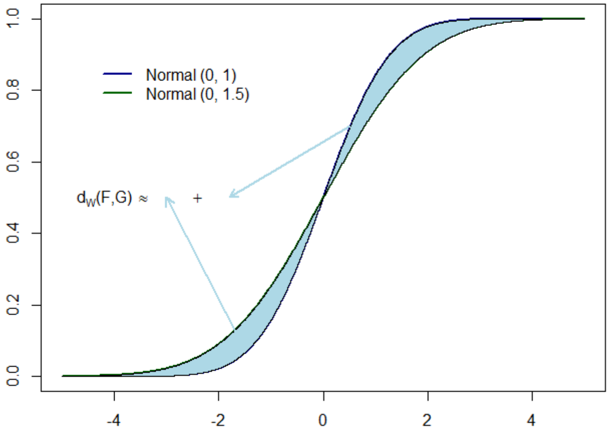
$$d_{W^2}(X,Y) = \left(\inf\left\{\mathbb{E}\left[\,||X-Y\,||^2\right]\ \mid X\sim F_X,\, Y\sim F_Y\}\right)^{\frac{1}{2}} \tag{4}$$

$$= \left(\int_0^1 \left(F_X^{-1}(u) - F_Y^{-1}(u)\right)^2\, du\right)^{\frac{1}{2}} \tag{5}$$

where the last equality holds for distribution on the reals.

Visual interpretation, distance, "tractable"

Wasserstein distance



Worst-case risk measures

Moment constraints

Uncertainty set with fixed first two moments:

$$\mathcal{U}(\mu, \sigma) = \left\{ G \mid \int x \, \mathrm{d}G(x) = \mu, \int x^2 \mathrm{d}G(x) = \mu^2 + \sigma^2, \ \sigma > 0 \right\}$$

Bounds with moment constraints

[Ghaoui et al., 2003, Natarajan et al., 2010, Li et al., 2018]

VaR$_\alpha(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\alpha}{\alpha}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

RVaR$_{\alpha,\beta}(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\beta}{\beta}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

ES$_\alpha(X)$ bounds

$$\left[\mu, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

Bounds with moment constraints

[Ghaoui et al., 2003, Natarajan et al., 2010, Li et al., 2018]

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$\text{RVaR}_{\alpha,\beta}(X)$ bounds

$$\left[\mu - \sigma\sqrt{\frac{1-\beta}{\beta}}, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$\text{ES}_\alpha(X)$ bounds

$$\left[\mu, \quad \mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}}\right]$$

$$\sqrt{\frac{\alpha}{1-\alpha}} = \begin{cases} 3 & \alpha = 0.9 \\ 4.4 & \alpha = 0.95 \\ 9.9 & \alpha = 0.99 \end{cases}$$

! extremely large ! "independent" of X

! worst-case is a two point distribution.

Bounds with moment constraints

	$\underline{\rho(X)}$	$\rho(X)$ Normal	$\rho(X)$ Log-Normal	$\overline{\rho(X)}$
$\text{VaR}_{0.975}$	9.68	13.92	14.46	22.49
$\text{RVaR}_{0.95,0.99}$	9.80	13.82	14.33	18.72
$\text{ES}_{0.95}$	10.00	14.13	14.79	18.72

X has mean 10 and standard deviation 2.

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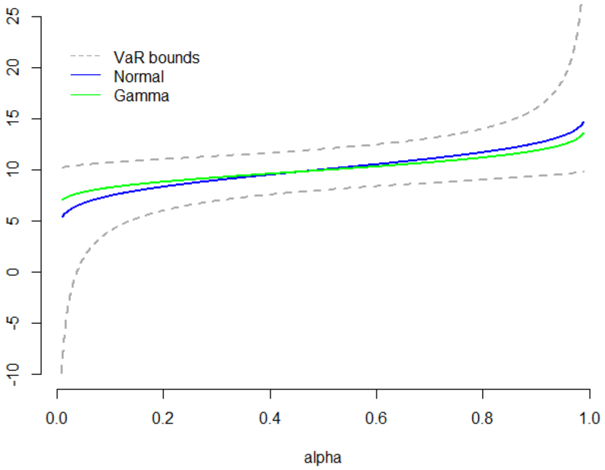
$\Rightarrow$ For any random variable, with mean = 10 and sd = 2, its VaR at level 0.975 belongs to (9.68, 22.49).

VaR bounds; with mean 10 and stdev 2



Deriving worst-case risk measures under moment information

Distortion risk measures

Distortion risk measures:

$$\rho_\gamma(X) = \int_0^1 F_X^{-1}(u)\gamma(u)\,du = \mathbb{E}\left[F_X^{-1}(U)\,\gamma(U)\right]$$

where $U \sim U(0,1)$ and $\gamma\colon [0,1] \to [0,\infty)$ satisfies $\int_0^1 \gamma(u)du = 1$.

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- If γ increasing, then ρ is coherent.

Moment constraints

Let ρ be a coherent distortion risk measure

$$\sup_{F_X \in \mathcal{U}(\mu,\sigma)} \rho_\gamma(X)$$

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$$\sup_{F_X \in \mathcal{U}(\mu,\sigma)} \rho_\gamma(X) = \sup_{F \in \mathcal{U}(\mu,\sigma)} \mathbb{E}[F^{-1}(U)\gamma(U)] \qquad (6)$$

where $U \sim U(0,1)$.

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where $U \sim U(0,1)$.

$$\sup_{F \in \mathcal{U}(\mu,\sigma)} \frac{\mathbb{E}\left[F^{-1}(U)\gamma(U)\right] - \mu}{\sigma\, std(\gamma(U))} =$$

Moment constraints

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By Cauchy-Schwarz the solution is linearly dependent on $\gamma(U)$, and

$$F^{*,-1}(U) := \mu + \sigma\gamma(U)\,, \tag{8}$$

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ES: $\gamma(u) = \frac{1}{1-\alpha}\mathbb{1}_{\{u \geq \alpha\}}$.

Towards better quantification of uncertainty:

$\Rightarrow$ Include further knowledge to the uncertainty set $\mathcal{U}$.

- higher moments [Cornilly et al., 2018]
- symmetric distributions [Zhu & Shao, 2018, Li et al., 2018]
- unimodal distributions [Li et al., 2018],[Bernard et al., 2020], [Kazzi, 2023].

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- unimodal distributions [Li et al., 2018],[Bernard et al., 2020], [Kazzi, 2023].

$\Rightarrow$ only marginal improvements

$\Rightarrow$ worst-case distribution is discrete

Wasserstein Uncertainty

Let F be a reference distribution

$$\mathcal{U}_\varepsilon(\mu,\sigma) := \Big\{ G \mid d_W(F,G)^2 \le \varepsilon \Big\},$$

where $d_W(G,F)$ denotes the Wasserstein distance of order 2, defined for F, G, with finite second moment, by

$$d_W(F,G)^2 = \int_0^1 |F^{-1}(u) - G^{-1}(u)|^2 \mathrm{d}u.$$

Let ρ_γ be a concave distortion (coherent) risk measure

$$\int_0^1 \gamma(u)G^{-1}(u\,)\,du$$

Let ρ_γ be a concave distortion (coherent) risk measure , then

$$\sup_{G\in\mathcal{U}_{\mathcal{W}}} \int_0^1 \gamma(u)G^{-1}(u)\,du = \rho(F) + \sqrt{\varepsilon}\sqrt{\int_0^1 \gamma(u)^2\,du}$$

and the worst-case quantile function is

$$F^{-1,*}(u) := F^{-1}(u) + \frac{\sqrt{\varepsilon}}{\sqrt{\int_0^1 \gamma(u)^2\,du}}\,\gamma(u)\,.$$

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! robust risk measure: constant shift

! constant is independent of F

Moment & Wasserstein uncertainty

Let F be a reference distribution with mean μ and standard deviation $\sigma > 0$.

$$\mathcal{U}_\varepsilon(\mu,\sigma) = \Big\{G \mid \int x \, \mathrm{d}G(x) = \mu, \int x^2 \mathrm{d}G(x) = \mu^2 + \sigma^2,\ \sigma > 0$$

$$\text{and } d_W(F,G)^2 \le \varepsilon\Big\}.$$

Moment & Wasserstein uncertainty

Let ρ_γ be a concave distortion (coherent) risk measure, then

$$\sup_{G \in \mathcal{U}_\varepsilon(\mu,\sigma)} \int_0^1 \gamma(u) G^{-1}(u)\, du$$

is attained by

$i)$ if $0 < \varepsilon < 2\sigma^2(1-c_0)$,

$$F_\lambda^{-1,*}(u) := \mu + \sigma \left(\frac{\gamma(u) + \lambda F^{-1}(u) - a_\lambda}{b_\lambda} \right),$$

such that $F_\lambda^* \in \mathcal{U}(\mu,\sigma)$ and $\lambda > 0$ is the unique solution to $d_W(F, F_\lambda^*) = \sqrt{\varepsilon}$, .

$ii)$ if $2\sigma^2(1-c_0) < \varepsilon$, case $i)$ with $\lambda = 0$ applies.

(Theorem from [Bernard et al., 2024])

Expected Shortfall

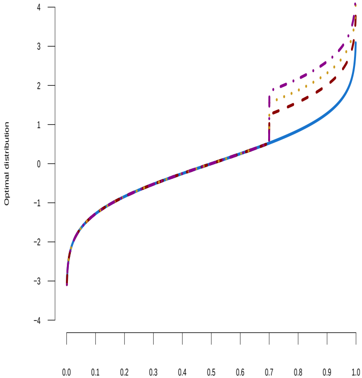
Wasserstein worst-case quantile for ES

The distribution which attains the upper bound, has quantile function

$$\begin{aligned} F^{-1,*}(u) &= F^{-1}(u) + \frac{\sqrt{\varepsilon}}{\sqrt{\int_0^1 \gamma(u)^2\,du}}\,\gamma(u) \\ &= F^{-1}(u) + \frac{\sqrt{\varepsilon}}{\sqrt{1-\alpha}}\,\mathbb{1}_{(\alpha,1]}\,. \end{aligned}$$

Recall for ES, we have $\gamma(u) = \frac{1}{1-\alpha}\mathbb{1}_{(\alpha,1]}$.

Wasserstein worst-case quantile for ES



for $\varepsilon = \{0, 0.16, 0.32, 0.53\}$.

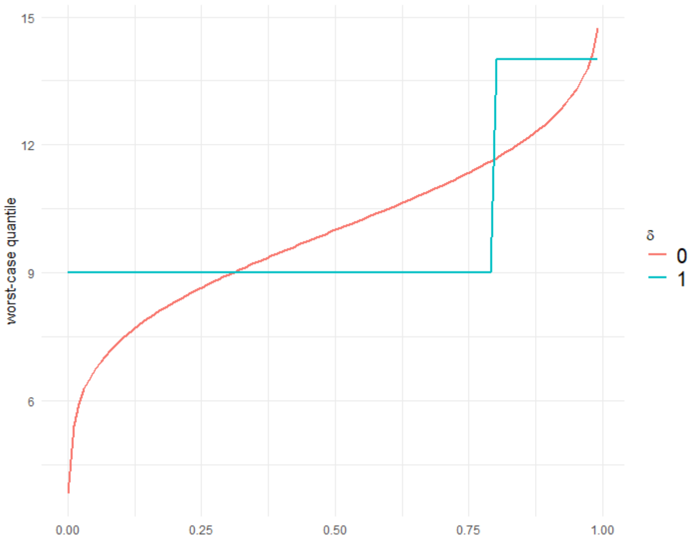
Wasserstein & Moment worst-case quantile for ES

The distribution which attains the upper bound, has quantile function

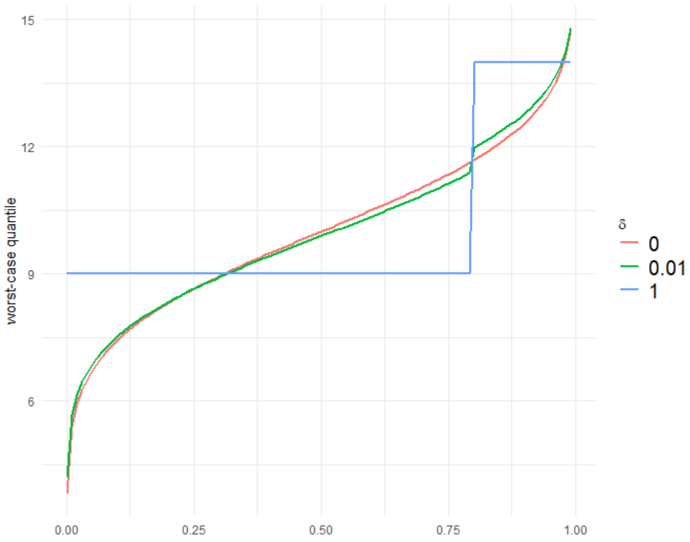
$$F^{*,-1}(u) = a + b\left(\frac{1}{1-\alpha}\mathbb{1}_{(\alpha,1]} + \lambda F^{-1}(u)\right),$$

where a, b are such that the mean and standard deviation constraint is fulfilled.

Wasserstein & Moment worst-case quantile for ES



Wasserstein & Moment worst-case quantile for ES



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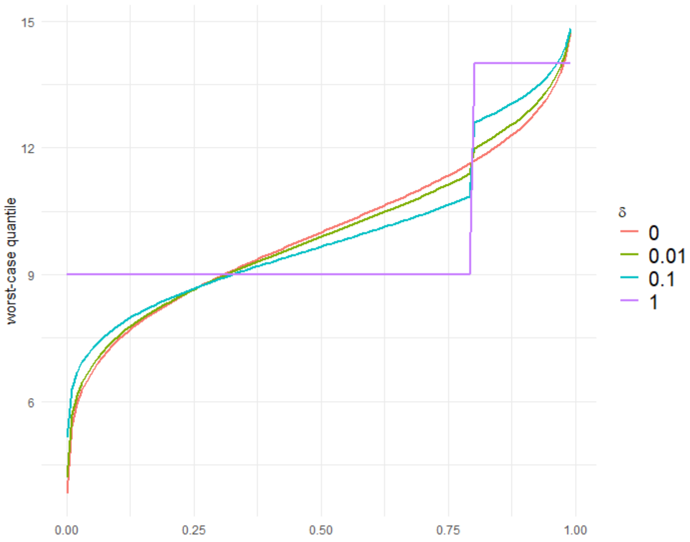

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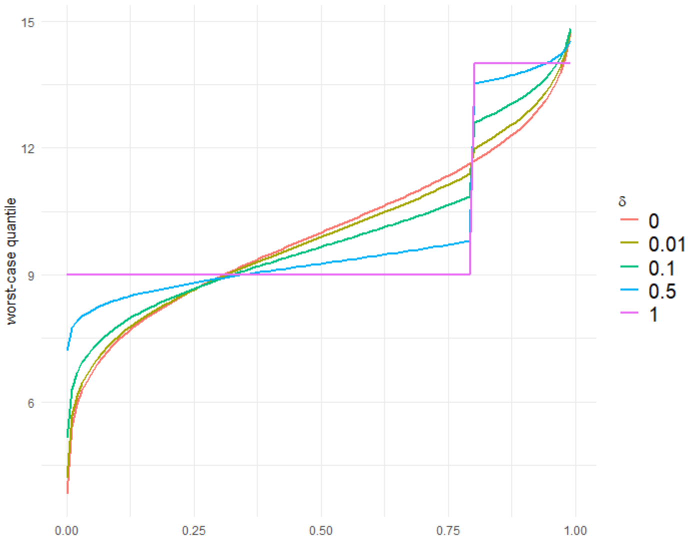
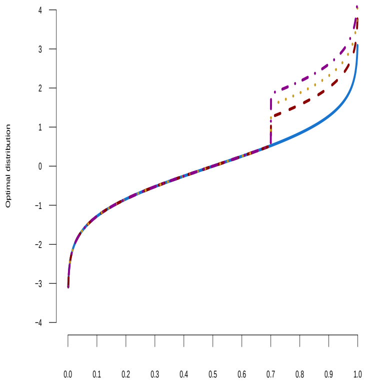
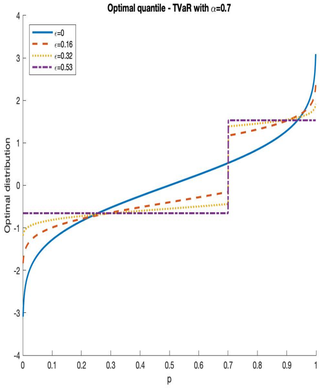


Worst-case Quantiles

Wasserstein

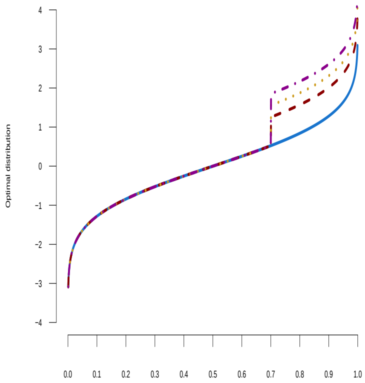


Wasserstein & Moments

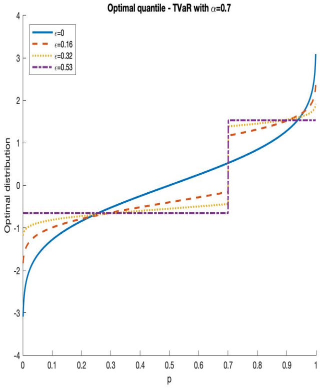


Worst-case Quantiles

Wasserstein



Wasserstein & Moments



Can be generalised to distortion risk measures

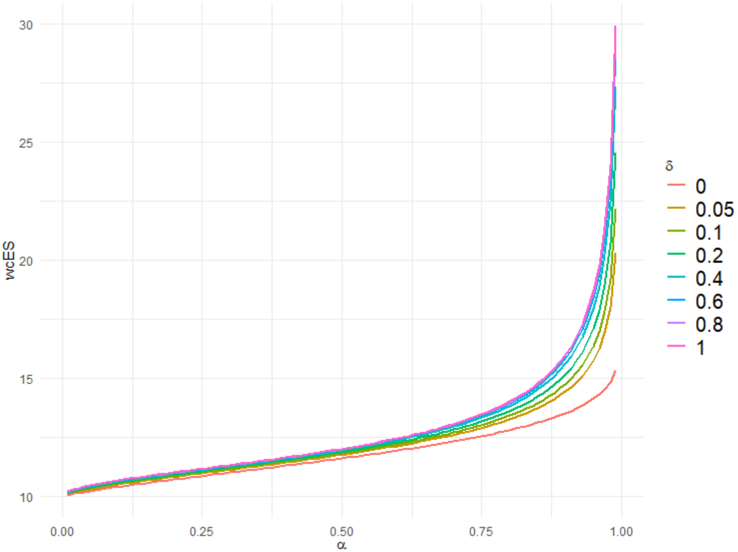
Worst-case ES: $\alpha = 0.99$

$$\sup_{G\in\mathcal{U}_\varepsilon} \mathsf{ES}_\alpha(G) = \mathsf{ES}_\alpha(F) + \sqrt{\varepsilon}\, \underbrace{\sqrt{\frac{1}{1-\alpha}}}_{\approx 10},$$

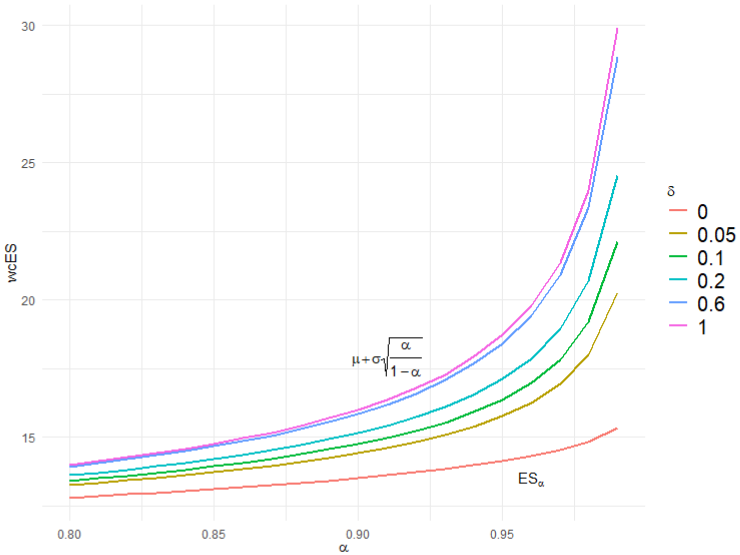
$$\sup_{G\in\mathcal{U}_\varepsilon(\mu,\sigma)} \mathsf{ES}_\alpha(G) = \mu + \sigma \frac{\frac{\alpha}{1-\alpha} + \lambda\left(\mathsf{ES}_\alpha(F) - \mu\right)}{\sqrt{\frac{\alpha}{1-\alpha} + 2\lambda\left(\mathsf{ES}_\alpha(F) - \mu\right) + \lambda^2\sigma^2}},$$

$$\sup_{G\in\mathcal{U}_\infty(\mu,\sigma)} \mathsf{ES}_\alpha(G) = \mu + \sigma\, \underbrace{\sqrt{\frac{\alpha}{1-\alpha}}}_{\approx 9.9},$$

Wasserstein & Moment upper bound for ES



Wasserstein & Moment upper bound for ES



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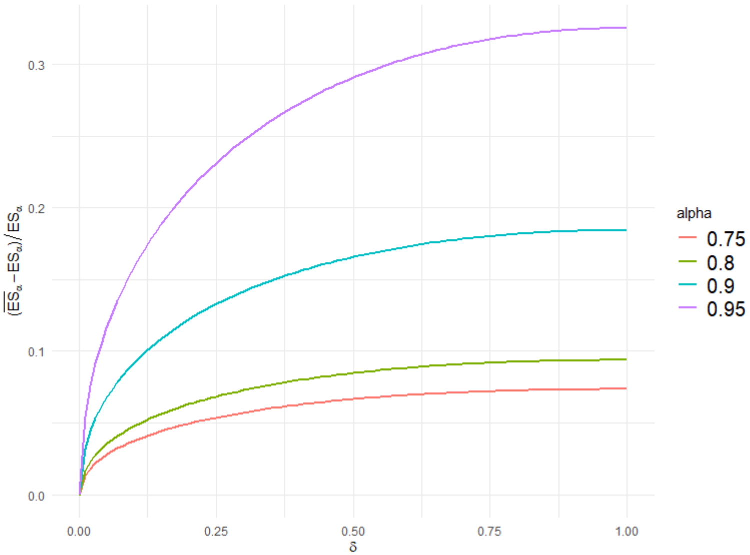

Robust portfolio optimisation

Robust portfolio optimisation

Let ρ be a concave distortion risk measure.

$$\min_{\boldsymbol{x}} \max_{\substack{G_{\boldsymbol{R}} \in \mathcal{U}(\boldsymbol{\mu}, \Sigma) \\ d_W(F_{\boldsymbol{x}}, G_{\boldsymbol{x}})^2 \leq \varepsilon_{\boldsymbol{x}}}} \rho\big(G_{\boldsymbol{x}}\big)$$

- $G_{\boldsymbol{R}}$ mult. distribution of returns $\boldsymbol{R}$
- $G_{\boldsymbol{x}}$ aggregate return of portfolio $\boldsymbol{x}$, i.e. of $-\boldsymbol{x}^\intercal \boldsymbol{R}$, with mean $-\mu_{\boldsymbol{x}} = -\boldsymbol{x}^\intercal \boldsymbol{\mu}$, and variance $\sigma_{\boldsymbol{x}}^2 = \boldsymbol{x}^\intercal \Sigma \boldsymbol{x}$.
- Benchmark model: $F_{\boldsymbol{x}}^{-1}(u) = -\mu_{\boldsymbol{x}} + \sigma_{\boldsymbol{x}} F^{-1}(u)$, location-scale family

Robust portfolio optimisation

Then, the portfolio optimisation is

$$\varepsilon_{\boldsymbol{x}} = 0 \qquad \min_{\boldsymbol{x}} \; -\mu_{\boldsymbol{x}} + \sigma_{\boldsymbol{x}} \, \rho(F)$$

$$0 < \varepsilon_{\boldsymbol{x}} < \frac{c}{\sigma_{\boldsymbol{x}}^2} \qquad \min_{\boldsymbol{x}} \; -\mu_{\boldsymbol{x}} + \sigma_{\boldsymbol{x}} \, h(\varepsilon_{\boldsymbol{x}})$$

$$\frac{c}{\sigma_{\boldsymbol{x}}^2} \leq \varepsilon_{\boldsymbol{x}} \qquad \min_{\boldsymbol{x}} \; -\mu_{\boldsymbol{x}} + \sigma_{\boldsymbol{x}} \sqrt{\int_0^1 (\gamma(u) - 1)^2 du}$$

$h(\cdot)$ increasing satisfying $h(0) = \rho(F)$, $h(\frac{c}{\sigma_{\boldsymbol{x}}^2}) = \sqrt{\int_0^1 (\gamma(u) - 1)^2 du}$.

Then, the portfolio optimisation is

$$\varepsilon_{\boldsymbol{x}} = 0 \qquad \min_{\boldsymbol{x}} \; -\mu_{\boldsymbol{x}} + \sigma_{\boldsymbol{x}} \, \rho(F)$$

$$0 < \varepsilon_{\boldsymbol{x}} < \frac{c}{\sigma_{\boldsymbol{x}}^2} \qquad \min_{\boldsymbol{x}} \; -\mu_{\boldsymbol{x}} + \sigma_{\boldsymbol{x}} \, h(\varepsilon_{\boldsymbol{x}})$$

$$\frac{c}{\sigma_{\boldsymbol{x}}^2} \leq \varepsilon_{\boldsymbol{x}} \qquad \min_{\boldsymbol{x}} \; -\mu_{\boldsymbol{x}} + \sigma_{\boldsymbol{x}} \sqrt{\int_0^1 (\gamma(u) - 1)^2 du}$$

$h(\cdot)$ increasing satisfying $h(0) = \rho(F)$, $h(\frac{c}{\sigma_{\boldsymbol{x}}^2}) = \sqrt{\int_0^1 (\gamma(u) - 1)^2 du}$.

- ! larger $\varepsilon_{\boldsymbol{x}}$, we recover the result of [Li, 2018] (Wasserstein distance constraint become irrelevant).
- ! choose $\varepsilon_{\boldsymbol{x}}$ proportional to $\sigma_{\boldsymbol{x}}^2$
- ! second order cone program

Another Example

In the paper, we have a second application on how to choose ε.

- Pareto-Clayton model fitted with given mean and variance
- 7 experts provides 7 alternative models with same mean and variance.
- ε is chosen as the max distance between the model and alternative propositions.

Aggregation of experts' opinion : ε is chosen big enough such that all the models suggested by experts are all in the admissible set of distributions.

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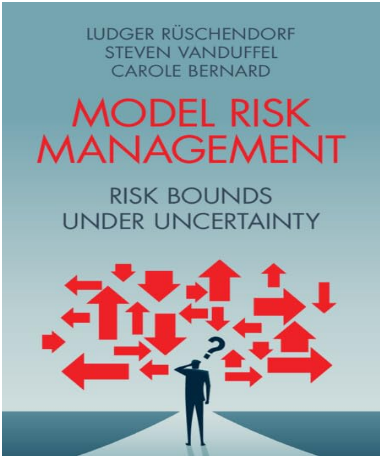
Conclusions

- Robust Risk Management - Approach 1: uncertainty on the dependence / copula
- Robust Risk Management - Approach 2: uncertainty on the aggregate claim / portfolio

$\Rightarrow$ Material presented here can be found in the book... and much more.




THANK YOU