Signatures methods in finance

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partly based on a course given jointly with Sara Svaluto-Ferro

University of Vienna

Mini course

Soesterberg, January 2024

Part I

Introduction to the theory of signature

- partly based on Chapter 7 of “Multidimensional stochastic processes as rough paths - Theory and Applications” by Friz & Victoir (2010)
- We refer to the slides from January 22, 2024.

Part II

Signature methods in Stochastic Portfolio Theory

based on joint work with Janka Möller.

Overview on Stochastic Portfolio Theory (SPT)

Major goals of Stochastic Portfolio Theory (SPT) are

- ... to specify only a few normative assumptions on the market (not necessarily absence of arbitrage);
- ... to analyze the relative performance of a portfolio with respect to the market portfolio, corresponding to major indices like S&P500;
- ... to develop and analyze models which allow for relative arbitrage with respect to the market portfolio;
- ... to understand various aspects of relative arbitrages, in particular properties of portfolios generating them, e.g., so-called functionally generated portfolios.

A (very incomplete) literature overview of SPT

- The first instance of the ideas of SPT is the article "Stochastic Portfolio Theory and Stock Market Equilibrium" by Robert Fernholz and Brian Shay.
- Robert Fernholz further developed it in several papers and the monograph "Stochastic Portfolio Theory" (2002).

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- Robert Fernholz further developed it in several papers and the monograph “Stochastic Portfolio Theory” (2002).
- Since then a lot of research has been conducted in this area, in particular by Adrian Banner, Daniel Fernholz, Robert Fernholz, Ioannis Karatzas, Constantinos Kardaras, Martin Larsson, Soumik Pal, Johannes Ruf, etc., which is partly summarized in the...
- ... overview articles and recent book
  - Stochastic Portfolio Theory: an Overview (2009) by Robert Fernholz Ioannis Karatzas;
  - Topics in Stochastic Portfolio Theory (2015) by Alexander Vervuurt;
  - Portfolio Theory and Arbitrage: A Course in Mathematical Finance (2021) by Ioannis Karatzas and Constantinos Kardaras.

Basic definitions of Stochastic Portfolio Theory (SPT)

- Consider a finite time-horizon $T > 0$ and some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$.
- Market capitalizations of d companies given by a vector $S = (S^1, ..., S^d)$ of d positive continuous semimartingales.
- Portfolio: a vector $\pi = (\pi^1, ..., \pi^d)$ of predictable processes such that $\sum_{i=1}^d \pi_t^i \equiv 1$ for all $t \in [0, T]$. Each π_t^i represents the proportion of current wealth invested at time t in the i^{th} asset for $i \in \{1, \dots, d\}$
- Market Portfolio: $\mu = (\mu^1, ..., \mu^d)$ with

$$\mu_t^i = \frac{S_t^i}{S_t^1 + ... + S_t^d}, \quad t \in [0, T].$$

- Denote the simplex of dimension d by

$$\Delta^d := \{(x^1, ..., x^d) \in \mathbb{R}^d | x^1 \geq 0, ..., x^n \geq 0 \text{ and } \sum_{i=1}^d x^i = 1\}.$$

Relative wealth process

- For a portfolio π the relative wealth process with respect to the market portfolio is given by

$$Y^\pi := \frac{V^\pi}{V^\mu}, \quad Y_0^\pi = 1,$$

where V^π (V^μ resp.) denotes the wealth process generated by the portfolio π (μ resp.).

- In this multiplicative setting, the dynamics of this relative wealth process are given by

$$\frac{dY_t^\pi}{Y_t^\pi} = \sum_{i=1}^{d} \pi_t^i \frac{d\mu_t^i}{\mu_t^i}, \quad Y_0^\pi = 1,$$

in perfect analogy with the usual wealth process dynamics where we have μ^i instead of S^i.

Relative arbitrage and functionally generated portfolios

Definition (Relative arbitrage opportunity)

A portfolio π is said to generate a relative arbitrage opportunity with respect to the market μ over the time horizon $[0, T]$ if

$$\mathbb{P}[Y_T^\pi \geq 1] = 1 \quad \text{and} \quad \mathbb{P}[Y_T^\pi > 1] > 0.$$

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Under certain conditions on the market, e.g. diversity and ellipticity or sufficient volatility, so-called functionally generated portfolios have been shown to generate such relative arbitrage opportunities.

Definition (Functionally Generated Portfolios (Fernholz '02))

Consider a C^2-function $G : U \supset \Delta^d \to \mathbb{R}_+$ such that $x_i D_i \log G(x)$ is bounded on Δ^d. Then G defines the functionally generated portfolio via

$$\pi_t^i = \mu_t^i (D_i \log G(\mu_t) + 1 - \sum_{j=1}^d \mu_t^j D_j \log G(\mu_t)).$$

If G is concave, it holds that $\pi_t^i \geq 0$ for all $i \in \{1, ..., d\}$ and $t \in [0, T]$.

Fernholz's master equation

Proposition (Pathwise version of Fernholz's master equation)

Let π be a functionally generated by G and $(\mu_t)_{t\in[0,T]}$ a continuous path admitting a continuous S_+^d-valued quadratic variation $[\mu]$ along a refining sequence of partitions (in the sense of Föllmer).

Then the relative wealth process $(Y_t^\pi)_{t\geq 0}$ satisfies

$$\log(Y_t^\pi) = \log(G(\mu(t))) - \log(G(\mu(0))) + \mathfrak{g}_t, \quad t \in [0, T],$$

where $\mathfrak{g}_t = \int_0^t -\frac{1}{2G(\mu(t))} \sum_{i,j} D^{ij} G(\mu(t)) d[\mu^i, \mu^j]_t$.

Remark: Under certain market conditions it can be shown that after a sufficiently long time horizon t^*, the term $\mathfrak{g}_{t^*}$ dominates $\log(G(\mu(t))) - \log(G(\mu(0)))$ and thus creates relative arbitrage.

Signature portfolios

- Inspired by functionally generated portfolios and control problems in finance solved via signature methods (e.g. Kalsi et al. ('19) or Bayer et al. ('21)), we introduce path functional portfolios and signature portfolios.
- We denote here and throughout the signature of X by $\mathbb{X}_t := \mathbb{X}_{0,t}$.

Definition (Path-functional portfolios)

Consider a continuous semimartingale $(X_t)_{t\in[0,T]}$ and let $\hat{X}_t = (t, X_t)$. We define two types of path-functional portfolios, denoted by η and θ,

$$\eta_t^i = \mu_t^i(F^i(\hat{X}_{[0,t]}) + 1 - \sum_{j=1}^{d} \mu_t^j F^j(\hat{X}_{[0,t]})), \qquad (\eta\text{-portfolio})$$

$$\theta_t^i = F^i(\hat{X}_{[0,t]}) + \mu_t^i(1 - \sum_{j=1}^{d} F^j(\hat{X}_{[0,t]})). \qquad (\theta\text{-portfolio})$$

If $F^i(\hat{X}_{[0,t]}) = \sum_{0\leq|I|\leq n} \alpha_I^{(i)} \langle \epsilon_I, \hat{\mathbb{X}}_t \rangle$, then the path functional portfolio is called signature portfolio.

Optimizing performance functionals - logarithmic utility

- The goal is now to optimize certain performance functionals within the class of signature portfolios.

- We start with logarithmic utility for the relative wealth process, i.e. the goal is to optimize $\mathbb{E}[\log Y_t^\eta]$, by finding optimal parameters $\{\alpha_I^i\}_{0 \le I \le n, i \in \{1,\dots,d\}}$. A similar method also works for the θ-portfolio.

- Note that it is the same to optimize the (absolute) log portfolio wealth or the relative log portfolio wealth (w.r.t the market) as

$$\left(\max_{\{\alpha_I^i\}_{0 \le I \le n, i \in \{1,\dots,d\}}} \mathbb{E}[\log V_t^\eta] \right) \Leftrightarrow \left(\max_{\{\alpha_I^i\}_{0 \le I \le n, i \in \{1,\dots,d\}}} \mathbb{E}[\log V_t^\eta] - \mathbb{E}[\log V_t^\mu] \right)$$

$$\Leftrightarrow \left(\max_{\{\alpha_I^i\}_{0 \le I \le n, i \in \{1,\dots,d\}}} \mathbb{E}\left[\log Y_t^\eta\right] \right).$$

Optimizing logarithmic utility within signature portfolios

Theorem (C.C., Janka Möller ('23))

Consider a market of d stocks, let X and μ be a $\mathbb{R}^n$-valued and Δ^d-valued continuous semimartingales. Let $t_0 \geq 0$ be the time at which we start to invest. Consider an arbitrary but fixed labelling function $\mathcal{L}$. Then

$$\max_{\{\alpha_I^{(i)}\}_{i\in\{1,\dots,d\},0\leq|I|\leq n}} \mathbb{E}\left[\log\left(Y_t^{\eta}\right)\right] \Leftrightarrow \min_{\mathrm{x}} \frac{1}{2}\mathrm{x}^{\mathsf{T}}\mathbb{E}[Q(t)]\mathrm{x} - \mathbb{E}[\mathrm{c}(t)]^{\mathsf{T}}\mathrm{x}$$

where x, $\mathrm{c}(t)$ *are vectors and* $Q(t)$ *is a matrix with coefficients*

$$\mathrm{x}_{\mathcal{L}(I,i)} = \alpha_I^{(i)}$$

$$(\mathrm{c}(t))_{\mathcal{L}(I,i)} = \int_{t_0}^{t} \langle \epsilon_I, \hat{\mathbb{X}}_s \rangle d\mu_s^i, \; (Q(t))_{\mathcal{L}(I,i),\mathcal{L}(J,j)} = \int_{t_0}^{t} \langle \epsilon_I \sqcup\!\sqcup \epsilon_J, \hat{\mathbb{X}}_s \rangle d[\mu^i, \mu^j]_s.$$

The optimization task is a convex quadratic optimization problem.

Sketch of the proof and remarks

- By the form of the η-portfolio the log relative wealth process is given by

$$\log\left(Y_t^{\eta}\right) = \sum_{i=1}^{d} \int_{t_0}^{t} \frac{\eta_s^i}{\mu_s^i} d\mu_s^i - \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_{t_0}^{t} \frac{\eta_s^i}{\mu_s^i} \frac{\eta_s^j}{\mu_s^j} d[\mu^i, \mu^j]_s$$

$$= \sum_{i=1}^{d} \int_{t_0}^{t} F^i(\hat{X}_{[0,s]}) d\mu_s^i - \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_{t_0}^{t} F^i(\hat{X}_{[0,s]}) F^j(\hat{X}_{[0,s]}) d[\mu^i, \mu^j]_s.$$

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$$= \sum_{i=1}^d \int_{t_0}^t F^i(\hat{X}_{[0,s]}) d\mu_s^i - \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \int_{t_0}^t F^i(\hat{X}_{[0,s]}) F^j(\hat{X}_{[0,s]}) d[\mu^i, \mu^j]_s.$$

  The linearity of F and the shuffle property of the signature yields the above convex quadratic optimization problem.

- If $X = \mu$, then the components of $c(t)$ and $Q(t)$ are linear functions of the signature of $t \mapsto \widehat{\mu}_t = (t, \mu_t)$, whose expected value can then often easily be computed.

- Note that in practice the optimization is performed along the observed trajectory, i.e. without expected values. This allows to detect (path-)functionally generated relative arbitrages if they exist.

Remarks

- Suppose that μ has dt characteristics with drift b_t and diffusion matrix C_t. The general log-optimal portfolio is found by solving the quadratic optimization task

$$\inf_{\pi} \mathbb{E}[\int_{t_0}^{t} \frac{1}{2}(\frac{\pi_t}{\mu_t})^\top C_t(\frac{\pi_t}{\mu_t}) - b_t^\top \frac{\pi_t}{\mu_t})dt],$$

where the inf is taken over predictable processes with $\sum \pi_t^i = 1$.

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  where the inf is taken over predictable processes with $\sum \pi_t^i = 1$.
- This optimization problem on the level of π is translated to a quadratic optimization problem over signature coefficients without constraints.
- A similar convex quadratic optimization problem (with $Q(t)$ of slightly different form) is obtained by replacing F^i by any linear function of some features, corresponding e.g. to
  - randomized signature (C.C., Gonon, Grigoryeva, Ortega, Teichmann);
  - random neural networks (Herrera, Krach, Teichmann).

General structure

Corollary (Quadratic Optimization Tasks)

Consider an optimization problem of the form

$$\inf_{\beta} \mathbb{E}[\int_{t_0}^{t} \beta_s^\top C_s \beta_s \nu_1(ds) - \int_{t_0}^{t} b_s^\top \beta_s \nu_2(ds)] \qquad (*)$$

over predictable processes β with values in $\mathbb{R}^d$, where b and C are stochastic processes with values in $\mathbb{R}^d$ and $\mathbb{S}^d$ resp., ν_i denotes signed measures on $[t_0, t]$.

If the controls β are parametrized via $\beta_t^i = \sum_{p \in \mathcal{P}} \alpha_p^i \varphi^p(t, X_{[0,t]})$, where $\{\varphi^p\}_{p \in \mathcal{P}}$ is a collection of feature maps and $\alpha_p^i \in \mathbb{R}$ are constant optimization parameters, then (*) is a quadratic optimization problem in $\{\alpha_p^i\}_{1 \leq i \leq d, p \in \mathcal{P}}$.

- A choice for φ^p is a version of randomized signature, $\varphi^p = \langle A^p, \widehat{\mathbb{X}}_t^N \rangle$, where A^p denotes the p-th row of a Johnson-Lindenstrauss projection matrix.
- Beside the log-optimal portfolio, a mean-variance type portfolio optimization can be cast into this framework.

Approximation by signature portfolios

Define the space of lifted stopped paths $\Lambda_T^2 = \bigcup_{t\in[0,T]}\{(\hat{\mathbb{X}}^2_{[0,t]})(\omega) \mid X \text{ cont. semi-mart.}, \hat{X}_s = (s, X_s),\ s \in [0,t]\}$ and equip it with an appropriate α-Hölder norm for $\alpha \in (1/3, 1/2)$.

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Proposition (C.C., Janka Möller ('23))

Consider for $t \in [0,T]$ path-functional portfolios of η- and θ-type of the form

$$\pi_t^i = \mu_t^i(f^i(\hat{\mathbb{X}}^2_{[0,t]}) + 1 - \sum_j \mu_t^j f^j(\hat{\mathbb{X}}^2_{[0,t]})) \quad \text{and} \quad \pi_t^i = f^i(\hat{\mathbb{X}}^2_{[0,t]}) + \mu_t^i(1 - \sum_j f^j(\hat{\mathbb{X}}^2_{[0,t]})),$$

where f^i are continuous non-anticipating path functionals on Λ_T^2 for every i.

- *Then portfolios of η- and θ-type can be approximated arbitrarily well by signature portfolios η^{Sig} (θ^{Sig} resp) uniformly in time and on compacts of Λ_T^2.*
- *Moreover, if $\mathbb{E}[\exp(\beta\|\hat{\mathbb{X}}_{[0,T]}\|^{\gamma}_{CC,\alpha})] < \infty$ for $\beta > 0$ and $\gamma > 1$, then for any $\varepsilon, \delta > 0$, there exists a signature portfolio η^{Sig} (θ^{Sig} resp) such that*

$$\mathbb{P}[\sup_{t\in[0,T]} \|\pi_t - \eta_t^{Sig}\| > \varepsilon] < \delta.$$

Approximation of the log-optimal portfolio

Proposition (C. C., Janka Möller ('23))

Consider a market model, where for all $i \in \{1, ..., d\}$

$$dS_t^i = S_t^i \left(a^i \left(\hat{\mathbb{X}}^2_{[0,t]} \right) dt + \sum_{j=1}^m B^{ij} \left(\hat{\mathbb{X}}^2_{[0,t]} \right) dW_t^j \right),$$

with $m \geq d$ such that $(BB^T)^{-1}$ exists (and some integrability cond. are satisfied). Assume that for all $i \in \{1, ..., d\}$, $j \in \{1, ..., m\}$ a^i, B^{ij} are continuous non-anticipating path-functionals on Λ_T^2.

- *Then the log-optimal portfolio can be approximated arbitrarily well by signature portfolios θ^{Sig} uniformly in time and on compact sets of Λ_T^2.*
- *Moreover, if $\mathbb{E}[\exp(\beta \|\hat{\mathbb{X}}_{[0,T]}\|^{\gamma}_{CC,\alpha})] < \infty$ for $\beta > 0$ and $\gamma > 1$, then for any $\varepsilon, \delta > 0$, there exists a signature portfolio θ^{Sig} such that*

$$\mathbb{P}[\sup_{t \in [0,T]} \|\pi_t - \theta_t^{Sig}\| > \varepsilon] < \delta.$$

Learning the log-optimal portfolio

1. Correlated Black-Scholes Market:

$$dS_t^i = S_t^i(a^i dt + \sum_{j=1}^{m} B^{ij} dW_t^j), \quad 1 \le i \le d.$$

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2. Volatility Stabilized Market:

$$\frac{dS_t^i}{S_t^i} = \frac{1+\gamma}{2}\frac{1}{\mu_t^i}dt + \sqrt{\frac{1}{\mu_t^i}}dW_t^i \quad 1 \leq i \leq d.$$

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3. Signature Market:

$$dS_t^i = S_t^i(a_t^i dt + \sum_{j=1}^{m} B^{ij}\, dW_t^j) \quad 1 \leq i \leq d$$

where $(a_t^i) = \sum_{0 \leq |I| \leq N} \lambda_I^{(i)} \langle \epsilon_I, \hat{\mu} \rangle_t$ and $B \in \mathbb{R}^{d \times m}$.

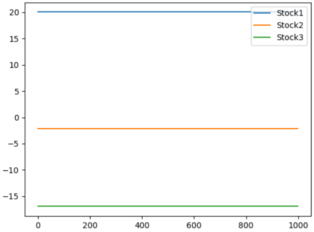
Optimization procedure

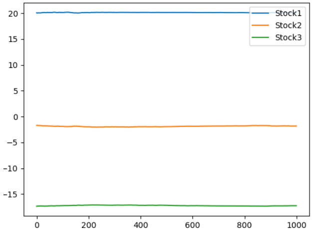
For each market:

- We use a Monte-Carlo type optimization. Note
$$\left(\max_{\eta} \; \frac{1}{M}\sum_{m=1}^{M} \log Y_T^{\eta}(\omega_m)\right) \Leftrightarrow \left(\min_{x} \; -x^T\tilde{c}(T) + \frac{1}{2}x^T\tilde{Q}(T)x\right),$$
for $\omega_1, \ldots, \omega_M \in \Omega$ and where $\tilde{Q}(T) = \frac{1}{M}\sum_{m=1}^{M} Q(T, \omega_m)$ and $\tilde{c}(T) = \frac{1}{M}\sum_{m=1}^{M} c(T, \omega_m)$.
- We take here $d = 3$.
- Simulate $M \approx 100000$ in-sample trajectories to create $\tilde{Q}(T)$, $\tilde{c}(T)$.
- Evaluate performance on 100000 test samples and compare it to the respective theoretical log-optimal portfolio.
- Log-optimal weights are never shown to signature portfolios during training!

Results: Black-Scholes Market

- We learned a signature portfolio of type η of degree three.
- Mean log-relative wealth equals 9.0115 in the theoretical log-optimal portfolio (left), while in the learned signature portfolio (right) it is 9.0122.





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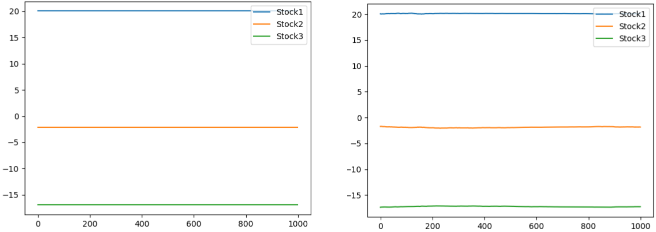

- The log-optimal portfolio in the B&S model, is a signature portfolio of type θ, but as we approximate it with an η-portfolio, the approximation task is actually

$$F^{(BS),i}(\mu_{[0,t]}) \approx \frac{c_i}{\mu_t^i}.$$

Results: Volatility Stabilized Market

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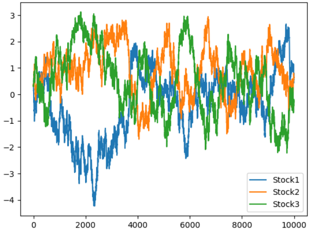
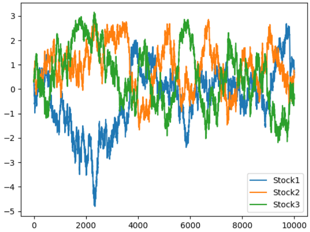
- Mean log-relative wealth equals 8.7619 in the theoretical log-optimal portfolio (left), while in the learned signature portfolio (right) it is 8.7417.





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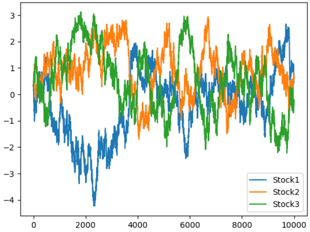
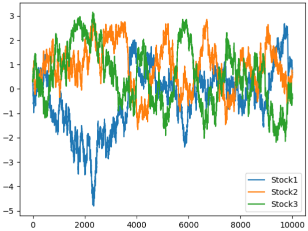


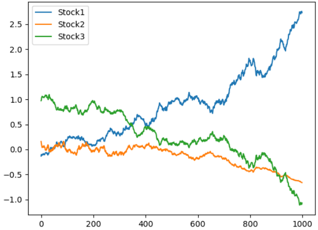


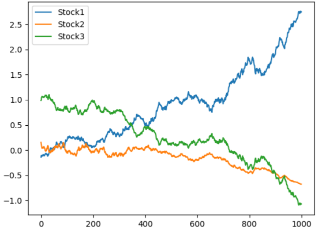
- The approximation task is here

$$F^{(Vol),i}(\mu_{[0,t]}) \approx \frac{\alpha+1}{2\mu_t^i} + \frac{d}{2}(\alpha-1).$$

Results: Signature Market

- We learned a signature portfolio of type θ of degree two.
- Mean log-relative wealth equals 0.2357 in the theoretical log-optimal portfolio (left), while in the learned signature portfolio (right) it is 0.2355.





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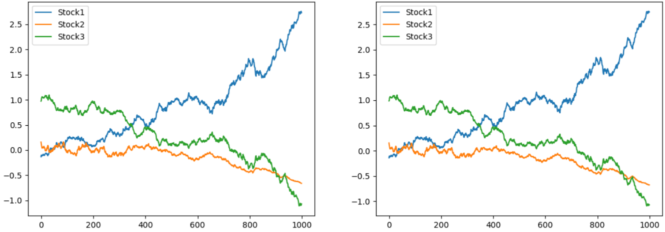

- Here, the log-optimal portfolio is a signature portfolio of type θ.

NASDAQ market

- We here consider the 100 dimensional NASDAQ market.
- Note that when working with real market data, we only have one realization available. Hence, we optimize just along the past observed trajectory (in other words we replace expectations by time averages).
- We choose X to be the ranked market weights.
- We apply a Johnson-Lindenstrauss projection of dimension 50 to the signature computed up to order 3 and then replace F^i in the η-portfolio by a linear map of this randomized signature.
- We perform both the log-utility and the mean-variance optimization with different risk aversion parameters.
- We take as an in-sample period 2000 trading days and as an out-of-sample period the following 750 trading days. The training is performed on historical data without estimating any drift or volatiltiy.

Results NASDAQ Market

We present out-of-sample results here without transaction costs.

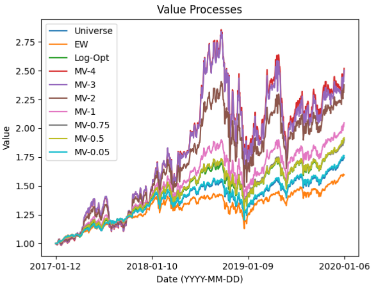

Figure: Left: Out-of-sample wealth processes entire NASDAQ, equally weighted portfolio, randomized signature portfolios optimizing log-utility and mean-variance.
Right: Average weights

Results S&P500 market

- We apply a similar procedure to the S&P 500, this time by choosing X to be the name-based market weights and by adding transaction costs.
- To keep the convex quadratic optimization structure we add the penalization term $\frac{\beta}{T}\sum_{t=0}^{T-1}\sum_i \left(\frac{\pi^i_{t+1}}{\mu^i_{t+1}} - \frac{\pi^i_t}{\mu^i_t}\right)^2$ accounting for transaction costs.

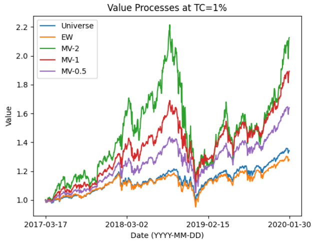

Figure: Out of sample wealth process with 1% prop. trans. costs, S&P500, equally weighted and randomized signature portfolio optimizing mean-variance.

- This picture suggests that a (strong) relative arbitrage opportunity even under transaction costs has been detected at least in this testing period.

Conclusion

- Related literature: by Owen Futter, Blanka Horvath, and Magnus Wiese: mean-variance optimization with an additive approach where the trading strategies correspond to numbers of shares (inclusion of bank account is necessary to guarantee selfinancing)

Conclusion

- Related literature: by Owen Futter, Blanka Horvath, and Magnus Wiese: mean-variance optimization with an additive approach where the trading strategies correspond to numbers of shares (inclusion of bank account is necessary to guarantee selfinancing)
- Signature portfolios can approximate a large class of path-functional portfolios including
  - classical functionally generated portfolios
  - log-optimal portfolios in a large class of non-Markovian markets.

  In some markets the log-optimal portfolios are exactly signature portfolios.
- Despite their versatility, optimizing the log-utiltiy or mean variance within the class of (randomized) signature portfolio leads to a convex quadratic optimization problem.
- Inclusion of transaction costs is possible, while preserving tractability of the optimization problem.
- The application to real data points towards out-performance during the out-of-sample testing period we considered, also under transaction costs.