Some topics related to stochastic mortality and/or interest rates in the valuation of life insurance products

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joint work with

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Outline

Part 1:
Randomization in Finance and Insurance in regime-switching models: e.g. Guaranteed Minimum Death Benefits (GMDB)

Part 1: Randomization and GMDBs

In this talk, we will mainly focus upon the Randomization and guaranteed minimum death benefits in a general regime switching model:

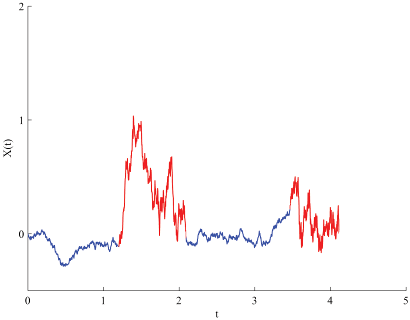
- We focus upon an underlying financial return process X which follows a regime switching Brownian motion with two-sided phase-type jumps (see also Asmussen [2003], Jiang and Pistorius [2008]) while the density functions of the random payments times τ can be approximated by a Laguerre series expansion or a combination of Erlang distributions

- This talk is based upon:
  Deelstra, G., Hieber, P., (2023), Randomization and the valuation of guaranteed minimum death benefits, European journal of operational research, Vol. 309, Issue 3, 1218-1236.

1) Introduction: Regime-Switching Models

Regime-switching models

- Hamilton (1989): financial models should account for the cyclical pattern of boom and recession.
- allow the model (or the model parameters) to switch at certain times by means of a Markov process whose states represent the different regimes or "phases".



Introduction: Regime-Switching Models

Regime-switching models

- turn out to be convenient in a lot of fields: optimal control in Finance (see, e.g., Korn et al. (2017), Jin et al. (2020), cyclical patterns in temperature and/or electricity modeling (see, e.g. Elias et al. (2014), Benth, Deelstra, Kozpınar S. (2023, 202x)), or GMDB or pension fund modeling in Insurance (see, e.g., Hainaut (2014), Deelstra and Hieber (2023)).
- have been extensively used for option pricing:
  - Elliott et al. (2005), Elliott and Siu (2009), Konikov and Madan (2002), Elliott and Osakwe (2006), Ramponi (2012), Elliott and Lian (2013), Shen and Siu (2013a,b,c), Chen et al. (2014), Deelstra and Simon (2017), Fan et al. (2017), Cao et al. (2018), Deelstra, Simon, Kozpinar (2018), Tour et al. (2018), Deelstra, Latouche and Simon (2020), Bao and Zhao (2019), Xie and Deng (2022),...

Randomization, Erlangization

- Randomization: In Finance, Carr [1998] approximates a fixed maturity T by an Erlang random time $\tau_{N,\mu}$ for high N.

- The method has also been used for both American-type and barrier option pricing in a no regime-switching framework, see e.g. Avram et al. (2002) and Boyarchenko and Levendorski (2012).
- In finance, it is also referred to as the "Canadization" method, see Mijatović et al. (2015).
- The technique is also known in risk theory as Erlangization (see e.g. Asmussen and Albrecher (2010), Ch. IX.8).

  - Deelstra, Latouche and Simon [2020] apply randomizaton to study the pricing of path-dependent options like digital options and down-and-out call options in a Markov modulated Brownian motion framework in the presence of two-sided phase-type jumps.
  - We replace the maturity T by a random variable $q \sim$ Erlang$(N, \frac{N}{T})$ where $N \in \mathbb{N}_0$. The expectation of q equals T and its variance T^2/N goes to zero as N goes to infinity.
  - Using fluid embedding (see e.g. Jiang and Pistorius [2008]) and Erlangization to obtain explicit expressions for different quantities related to the path properties of the MMBM up to time q, the approximating option prices followed. Compared to other existing methods, this approach does not require the inversion of Laplace (or Fourier) transforms.
  - By choosing a large enough number of Erlangization intervals, the obtained precision turns out to be very high.

Discounted density and GMDB

- Recall: In Finance, Carr [1998] approximates a fixed maturity T by an Erlang random time $\tau_{N,\mu}$ for high N.
- In Insurance, the contract payoffs often depend on a financial risk process while claim dates are random events like death or the occurence of a claim or natural catastrophe.
- Gerber et al. [2012], [2013] introduced the discounted density approach for GMDB valuation.
- Several frameworks and generalizations have been studied for GMDBs: regime-switching jumps and volatility (e.g. Siu et al. [2015], Ciu et al. [2017]), different types of payoffs (e.g. Kirkby [2021]) and different types of random time approximations (Zhang and Yong [2019]).

In this talk

- In this talk, we focus upon an underlying financial return process X which follows a regime switching Brownian motion with two-sided phase-type jumps while the density functions of the random payments times τ can be approximated by a Laguerre series expansion or a combination of Erlang distributions, see e.g. Zhang and Yong [2019].
- We obtain the (discounted) density of X_τ in closed-form by a Laurent series. (European-type guarantees, also risk measures, e.g. VaR)
- We deal with path-dependent GMDBs semi closed-form via Sylvester equations (quadratic, easy-to-solve).
- We avoid any Fourier/Laplace inversion, and obtain very fast calculations.

Motivation: Basic calculus and Black-Scholes model

- This is simple calculus following Gerber et al. [2012], Asmussen [2003].
- Consider a guaranteed amount G and an underlying stock $\{S_t\}_{t\geq 0}$ with Black-Scholes dynamics, namely

$$S_t = S_0 e^{X_t} \text{ with } X_t \sim \mathcal{N}\big((r - \frac{\sigma^2}{2})t, \sigma^2 t\big).$$

- The guaranteed amount is due at the time of death τ which follows an exponential distribution $\tau \sim \text{Exp}(\mu)$ and is independent of $\{S_t\}_{t\geq 0}$.
- The valuation is done the standard way, assuming independence between τ and X:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau}\max(G - S_\tau,\, 0)\right] = \underbrace{\int_0^\infty \underbrace{\mathbb{E}_{\mathbb{Q}}[e^{-rt}\max(G - S_t,\, 0)]}_{\text{financial risk integration}} \cdot f_\tau(t)\,\mathrm{d}t}_{\text{insurance risk integration}}\ .$$

Motivation: Basic calculus and Black-Scholes model

- **Can we do better than that?**
- (Discounted) Laplace transform obtained as (β in suitable range):

$$
\begin{aligned}
\varphi(\beta) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau} e^{\beta X_\tau}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau} e^{\beta X_\tau} \,\middle|\, \tau = t\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{\left(\left(r-\frac{\sigma^2}{2}\right)\beta+\frac{1}{2}\sigma^2\beta^2-r\right)\tau}\right] \\
&= \frac{\mu}{(\mu+r) - \left(\left(r-\frac{\sigma^2}{2}\right)\beta + \frac{1}{2}\sigma^2\beta^2\right)} \\
&= \frac{\mu}{-\frac{1}{2}\sigma^2(\alpha_1-\beta)(\beta_1-\beta)} \\
&= \frac{-\mu}{(\mu+r)(\beta_1-\alpha_1)}\left(\alpha_1 \cdot \frac{\beta_1}{\beta_1-\beta} - \beta_1 \cdot \frac{\alpha_1}{\alpha_1-\beta}\right).
\end{aligned}
$$

where $\alpha_1 < 0$ and $\beta_1 > 0$ are roots of $-\frac{\sigma^2\beta^2}{2} - \left(r-\frac{\sigma^2}{2}\right)\beta + (\mu+r) = 0$.

Motivation: Basic calculus and Black-Scholes model

- Using $-\frac{1}{2}\sigma^2\alpha_1\beta_1 = \mu + r$ (constant in quadratic equ.), we arrived to:

$$\varphi(\beta) = \frac{-\mu}{(\mu + r)(\beta_1 - \alpha_1)}\left(\alpha_1 \cdot \frac{\beta_1}{\beta_1 - \beta} - \beta_1 \cdot \frac{\alpha_1}{\alpha_1 - \beta}\right).$$

- This shows that the corresponding density is composed of two exponential densities – for the negative, respectively positive, part.

- With a little bit of algebra, this leads to the (discounted) density $f^{(r)}_{X_\tau}(x) = \int_0^\infty e^{-rt} f_{X_t}(x) f_\tau(t) dt$ (see Gerber et al. [2012]):

$$f^{(r)}_{X_\tau}(x) = \begin{cases} \underbrace{\frac{\mu}{\mu + r}\frac{-\alpha_1\beta_1}{\beta_1 - \alpha_1}}_{\text{constant } C} \cdot e^{-\alpha_1 x}, & \text{if } x \leq 0, \\ \underbrace{\frac{\mu}{\mu + r}\frac{-\alpha_1\beta_1}{\beta_1 - \alpha_1}}_{\text{constant } C} \cdot e^{-\beta_1 x}, & \text{if } x > 0. \end{cases}$$

Motivation: Basic calculus and Black-Scholes model

Under the hypothesis of independence of X and τ:

$$\begin{aligned}
\varphi(\beta) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau} e^{\beta X_\tau}\right] \\
&= \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-rt} e^{\beta x} f_{X_t}(x) f_\tau(t) dt dx \\
&= \int_{-\infty}^{\infty} e^{\beta x} f_{X_\tau}^{(r)}(x) dx
\end{aligned}$$

Motivation: Basic calculus and Black-Scholes model

- Let's look at the OTM put option with $S_0 > G$:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau}\max(G-S_\tau,\,0)\right] &= \int_{-\infty}^{\infty}\max(G-S_0\cdot e^x,\,0)\,f^{(r)}_{X_\tau}(x)\,\mathrm{d}x\\ &= S_0\cdot C\int_{-\infty}^{\ln(G/S_0)}\left(\frac{G}{S_0}-e^x\right)e^{-\alpha_1 x}\,\mathrm{d}x\\ &= \frac{C\cdot G}{\alpha_1(\alpha_1-1)}\left(\frac{G}{S_0}\right)^{-\alpha_1}.\end{aligned}$$

- Compare this to:

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau}\max(G-S_\tau,\,0)\right] = \underbrace{\int_0^{\infty}\underbrace{\mathbb{E}_{\mathbb{Q}}[e^{-rt}\max(G-S_t,\,0)]}_{\text{financial risk integration}}\cdot f_\tau(t)\,\mathrm{d}t}_{\text{insurance risk integration}}.$$

- Computation times in Matlab (using `tictoc` function):

expression	**computation time**
$Ge^{-rt}\Phi\left(\frac{\ln(\frac{G}{S_0})-(r-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right) - S_0\Phi\left(\frac{\ln(\frac{G}{S_0})-(r+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right)$	6.4ms
$\int_0^\infty \mathbb{E}_{\mathbb{Q}}[e^{-rt}\max(G-S_t,\,0)]\cdot f_\tau(t)\,\mathrm{d}t$	33.4ms
$\frac{C\cdot G}{\alpha_1(\alpha_1-1)}\left(\frac{G}{S_0}\right)^{-\alpha_1}$	3.2ms

$\Longrightarrow$ This is about **10-times faster** using the discounted density approach.

- The ITM put option: via the call-put parity

- We can generalize the exponential distributions to Erlang distributions (= sum of N independent exponentials):

$$f_{\tau_{N,\mu}}(t) = \frac{\mu(\mu t)^{N-1}}{(N-1)!} e^{-\mu t}\,, \quad t > 0\,. \tag{1}$$

- If τ is an Erlang(n, μ) r.v. independent of X, then the (discounted) density is still available analytically (see Gerber et al. [2012]):

$$f^{(r)}_{X_\tau}(x) = \begin{cases} C^n e^{-\alpha_1 x} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}(-x)^{j-1}}{(j-1)!(\beta_1-\alpha_1)^{n-j}}\,, & \text{if } x \leq 0\,, \\ C^n e^{-\beta_1 x} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}(x)^{j-1}}{(j-1)!(\beta_1-\alpha_1)^{n-j}}\,, & \text{if } x > 0\,. \end{cases}$$

Phase-type distributions

The distribution function of a Phase-type distributed random variable $Y \sim PH(\boldsymbol{\alpha}, A)$ with A a square matrix and $\boldsymbol{\alpha}$ and $\boldsymbol{a} = -A\mathbf{1}$ vectors (with the same number of components as rows in A) is

$$\mathbb{P}(Y \leq t) = 1 - \boldsymbol{\alpha} e^{At}\mathbf{1},$$

and its density function is

$$f_Y(t) = \boldsymbol{\alpha} e^{At}\boldsymbol{a} \qquad \text{for } t \geq 0.$$

Here and in the following we will use the notation $\mathbf{1}$ and $\mathbf{0}$ for vectors with each component equal to 1 and 0, respectively.

$\boldsymbol{e}_j$ a vector where the i-th component is the Kronecker delta δ_{ji}.

The matrix exponential of a matrix $\boldsymbol{B} \in \mathbb{C}^{k \times k}$ is defined via the power series $\exp(\boldsymbol{B}) := \sum_{n=0}^{\infty} \boldsymbol{B}^n/n!$

Phase-type distributions

- Let $\tilde{\varphi} = \{\tilde{\varphi}(t) | t \in \mathbb{R}^+\}$ be a Markov process defined on a state space $\mathcal{S} \cup \{\star\}$, where $\mathcal{S}$ contains a finite number states, all transient, and $\star$ is an absorbing state. The generator of $\tilde{\varphi}$ is of the form

$$G = \left[\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \boldsymbol{a} & A \end{array}\right] \tag{2}$$

where A is a square $|\mathcal{S}| \times |\mathcal{S}|$ matrix containing the transition rates between the transient states and $\boldsymbol{a}$ is the vector containing the transition rates from the transient states to the absorbing state.

- Denote by $\tau_\star$ the absorption time in this process:

$$\tau_\star = \inf\{t \geq 0 \mid \tilde{\varphi}(t) = \star\}$$

Phase-type distributions

- Let $\boldsymbol{\alpha}$ be the initial probability vector of $|\mathcal{S}|$ components with

$$\alpha_i = \mathbb{P}(\tilde{\varphi}(0) = i) \quad \forall i \in \mathcal{S}.$$

- We say that a random variable Y has a phase-type distribution with parameters $\boldsymbol{\alpha}$ and A if Y is distributed as $\tau_\star$:

$$Y \sim PH(\boldsymbol{\alpha}, A)$$

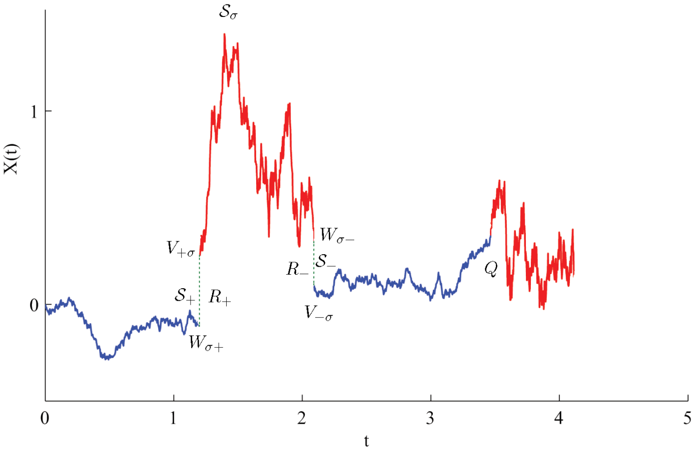
Phase-type distributions

- Phase-type distributions have been introduced by Neuts (1975, 1981).
- The most basic example is the exponential distribution $\mathrm{Exp}(\mu)$, for which $A = -\mu$ and $\boldsymbol{\alpha} = 1$.
- Another classical example is the Erlang distribution $\mathrm{Erlang}(N, \mu)$ with parameters $N \in \mathbb{N}$ and μ, which can be interpreted as the time needed by a Markov process $\tilde{\varphi}$ to go through N states, the sojourn time in each of them being distributed as an $\mathrm{Exp}(\mu)$.

2) Regime-switching model: two-sided phase-type jumps

- $S_t = S_0 e^{X_t}$.
- A process $\varphi = \{\varphi_t\}_{t \geq 0}$ governs the diffusion states of the process X. It is defined on a finite state space with $M \in \mathbb{N}$ phases, that is at any time $t > 0$, $\varphi_t = j$, where $j \in \mathcal{S}_\sigma := \{1, 2, \ldots, M\}$.
- When $\varphi_t = j$, the level X evolves like a Brownian motion with drift $d_j \in \mathbb{R}$ and variance $\sigma_j^2 > 0$.
- We assume that the process X_t starts in a diffusion state and that φ_0 has initial distribution $\boldsymbol{\pi} \in \mathbb{R}^{M \times 1}$.
- When $\varphi_t = j \in \mathcal{S}_\sigma$, two kinds of transitions are possible: or jumps or instantaneous transitions from j to a different diffusion state $v \in \mathcal{S}_\sigma$ at a rate $\{\boldsymbol{Q}\}_{jv}$, which are collected in the subgenerator matrix $\boldsymbol{Q}$.
- Jumps can be positive or negative; we group the different jumps in two state spaces $\mathcal{S}_+ = \{s_1^+, s_2^+, \ldots, s_n^+\}$ and $\mathcal{S}_- = \{s_1^-, s_2^-, \ldots, s_m^-\}$ for $n, m \in \mathbb{N}$.

A MMBM with two-sided Phase-type jumps



A MMBM X with two-sided phase-type jumps

- Regime-switching model with two-sided phase-type jumps.

$$X_t = X_0 + \int_0^t d_{\varphi_s} \mathrm{d}s + \int_0^t \sigma_{\varphi_s} \mathrm{d}B_s + \int_0^t J^+_{\varphi_s} \mathrm{d}N_s^{\varphi_s,+} - \int_0^t J^-_{\varphi_s} \mathrm{d}N_s^{\varphi_s,-}$$

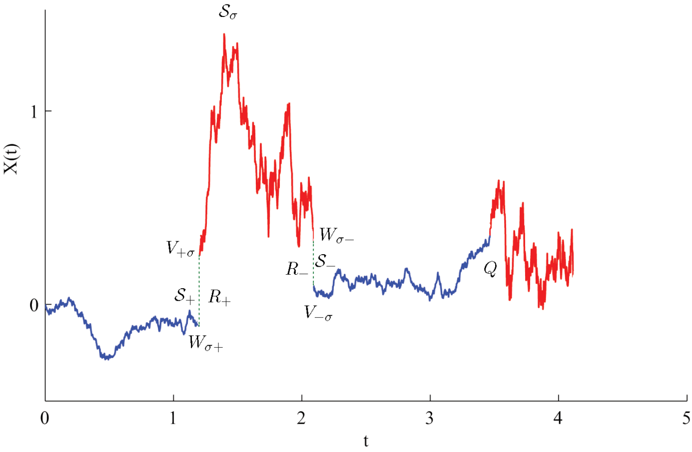
If J^+_j and J^-_j represent the absolute size of an upward and downward jump that occurred in phase j, then for all $x \geq 0$,

$$\mathbb{P}(J^+_j \in \mathrm{d}x, \varphi = i \text{ after the jump}) = \frac{1}{(\boldsymbol{W}_{\sigma+}\mathbf{1})_j} \left(\boldsymbol{W}_{\sigma+} e^{\boldsymbol{R}_+ x} \boldsymbol{V}_{+\sigma}\right)_{ji} \mathrm{d}x \,,$$

$$\mathbb{P}(J^-_j \in \mathrm{d}x, \varphi = i \text{ after the jump}) = \frac{1}{(\boldsymbol{W}_{\sigma-}\mathbf{1})_j} \left(\boldsymbol{W}_{\sigma-} e^{\boldsymbol{R}_- x} \boldsymbol{V}_{-\sigma}\right)_{ji} \mathrm{d}x \,.$$

- In the diffusion state $j \in \mathcal{S}_\sigma$, the processes $\{N_t^{j,+}\}_{t\geq 0}$ and $\{N_t^{j,-}\}_{t\geq 0}$ define the arrival of jumps. More specifically, the arrival rate of an upward jump $k \in \mathcal{S}_+$ (respectively $k \in \mathcal{S}_-$ for a downward jump) is the constant $\{\boldsymbol{W}_{\sigma+}\}_{jk}$ (respectively $\{\boldsymbol{W}_{\sigma-}\}_{jk}$).
- The jumps may be accompanied by a change in diffusion state.
- If a jump $k \in \mathcal{S}_+$ appears, $\{\boldsymbol{V}_{+\sigma}\}_{ki}$ is the rate at which the jump terminates and the process returns to the diffusion state $i \in \mathcal{S}_\sigma$ (analogous the rate is $\{\boldsymbol{V}_{-\sigma}\}_{ki}$ after a downward jump $k \in \mathcal{S}_-$).
- The upward jumps have phase-type distribution represented by a subgenerator matrix $\boldsymbol{R}_+ \in \mathbb{R}^{n\times n}$ on the state space $\mathcal{S}_+$, and the downward jumps have phase-type distribution represented by a subgenerator matrix $\boldsymbol{R}_- \in \mathbb{R}^{m\times m}$ on the state space $\mathcal{S}_-$.

A MMBM with two-sided Phase-type jumps



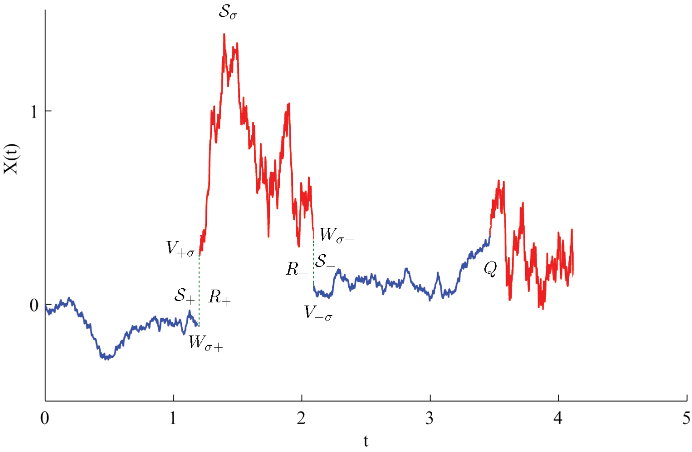
- For later use, we also define the transition matrices $\boldsymbol{W} \in \mathbb{R}^{M\times(n+m)}$ and $\boldsymbol{V} \in \mathbb{R}^{(n+m)\times M}$:

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_{\sigma+} & \boldsymbol{W}_{\sigma-} \end{bmatrix}, \quad \boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_{+\sigma} \\ \boldsymbol{V}_{-\sigma} \end{bmatrix} \quad .$$

  The process does not contain an absorbing state, that is the diagonal entries of $\boldsymbol{Q}$ are determined such that $[\boldsymbol{Q}\ \boldsymbol{W}]\mathbf{1} = \mathbf{0}$.

- Finally, we define the drift $\boldsymbol{D} = \text{diag}(d_j)_{j\in\mathcal{S}_\sigma}$ and volatility matrix $\boldsymbol{\Sigma} = \text{diag}(\sigma_j)_{j\in\mathcal{S}_\sigma}$.

A MMBM with two-sided Phase-type jumps



Example 1 (Regime switching Kou model)

In Kou's model, in state $j \in \mathcal{S}_\sigma$, the MMBM process X has dynamics

$$\mathrm{d}X_t = d_j \mathrm{d}t + \sigma_j \mathrm{d}W_t + \mathrm{d}J_t^{(j)}\,, \tag{3}$$

where $\{W_t\}_{t\geq 0}$ denotes a standard Brownian motion and $\{J_t^{(j)}\}_{t\geq 0}$ is an independent compound Poisson process with a constant arrival rate $\lambda_j \geq 0$ and random double-exponential jump sizes

$$\nu_j(\mathrm{d}y) = \left(p_j \alpha_{-,j} e^{\alpha_{-,j} y} \mathbb{1}_{y<0} + (1-p_j)\alpha_{+,j} e^{-\alpha_{+,j} y} \mathbb{1}_{y\geq 0}\right) \mathrm{d}y\,,$$

Positive and negative jump sizes are exponentially distributed with intensity $\alpha_{+,j} > 0$ and $\alpha_{-,j} > 0$, and with probability $p_j \in [0,1]$ jumps are negative.

In our notation, the regime switching Kou model is obtained as $\boldsymbol{V}_{+\sigma} = -\boldsymbol{R}_+ = \mathrm{diag}(\alpha_{+,j})_{j\in\mathcal{S}_\sigma}$, $\boldsymbol{V}_{-\sigma} = -\boldsymbol{R}_- = \mathrm{diag}(\alpha_{-,j})_{j\in\mathcal{S}_\sigma}$, $\boldsymbol{W}_{\sigma-} = \mathrm{diag}(p_j\lambda_j)_{j\in\mathcal{S}_\sigma}$, $\boldsymbol{W}_{\sigma+} = \mathrm{diag}((1-p_j)\lambda_j)_{j\in\mathcal{S}_\sigma}$ and $\boldsymbol{Q} = \boldsymbol{Q}_0 - \mathrm{diag}(\boldsymbol{W}\mathbf{1}) = \boldsymbol{Q}_0 - \mathrm{diag}(\lambda_j)_{j\in\mathcal{S}_\sigma}$. Given the matrix $\boldsymbol{Q}$ introduced earlier, the matrix $\boldsymbol{Q}_0 := \boldsymbol{Q} + \mathrm{diag}(\boldsymbol{W}\mathbf{1})$ is a generator matrix.

Example 2 (A MMBM with Phase type downward jumps)

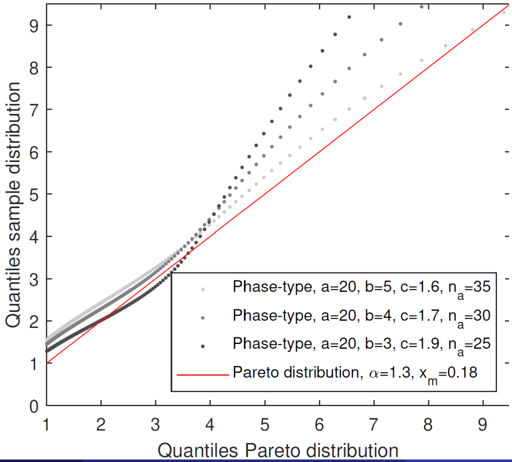
See also Robert and Boudec [1997] and Deelstra et al. [2020] for a more detailed analysis and motivation. We consider two phases ($M = 2$). The jump with transition from phase 1 to phase 2 is defined by a more general phase-type distribution with subgenerator matrix with size n_a:

$$\boldsymbol{R}_{-} = \begin{bmatrix} -(c + s_a) & 1/a & (1/a)^2 & \cdots & (1/a)^{n_a-1} \\ b/a & -b/a & 0 & \cdots & 0 \\ (b/a)^2 & 0 & -(b/a)^2 & \cdots & 0 \\ \vdots & & & & 0 \\ (b/a)^{n_a-1} & 0 & 0 & \cdots & -(b/a)^{n_a-1} \end{bmatrix}$$

with $n_a \in \mathbb{N}$, $a > \max(1, b)$, $b, c > 0$ and $s_a = 1/a + 1/a^2 + \ldots 1/a^{n_a-1}$. The other matrices are chosen as follows for parameters $\lambda > 0$, $q_1 > 0$, $q_2 > 0$, $\boldsymbol{R}_{+} = -\lambda$, $\varphi_0 = 1$, $\boldsymbol{V}_{+\sigma} = \begin{bmatrix} \lambda & 0 \end{bmatrix}$, $\boldsymbol{V}_{-\sigma} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{R}_{-}\mathbf{1} \end{bmatrix}$,

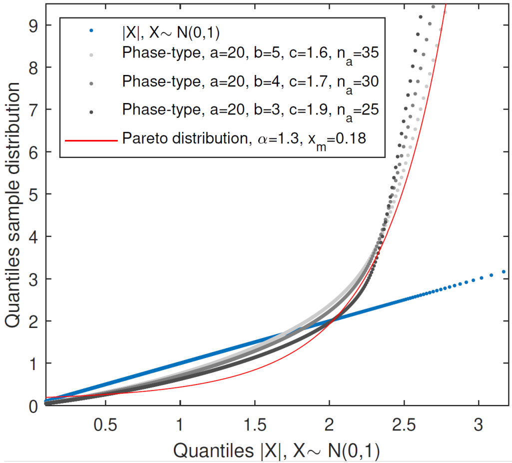
$$\boldsymbol{Q} = \begin{bmatrix} -q_1 & 0 \\ 0 & -q_2 \end{bmatrix}, \quad \boldsymbol{W}_{\sigma-} = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \boldsymbol{W}_{\sigma+} = \begin{bmatrix} 0 \\ q_2 \end{bmatrix}.$$

In the figure, the quantiles of a Pareto distribution with density $f(x) = \alpha x_m^\alpha / x^{\alpha+1} \mathbb{1}_{x \geq x_m}$, $\alpha > 1$, are compared to three phase-type approximations $PH(\boldsymbol{e}_1, R_-)$ with parameters (a, b, c, n_a). The means of the distribution are chosen to be equal: $\mathbb{E}[|X|] = \sqrt{2/\pi} = \alpha x_m / (\alpha - 1) = \frac{1}{c} \sum_{l=0}^{n_a - 1} \left(\frac{1}{b}\right)^l$, see also Deelstra et al. [2020].



Comparison with absolute value of a stand. Normal distrib.

In the figure, the quantiles of the absolute value of a standard Normal distribution are compared to a Pareto distribution with density $f(x) = \alpha x_m^\alpha / x^{\alpha+1} \mathbb{1}_{x \geq x_m}$, $\alpha > 1$, and three phase-type approximations with parameters (a, b, c, n_a). The means of the distribution are chosen to be equal to $\mathbb{E}[|X|] = \sqrt{2/\pi} = \alpha x_m/(\alpha - 1) = \frac{1}{c} \sum_{l=0}^{n_a - 1} \left(\frac{1}{b}\right)^l$.



3) Derivatives: GMDB payoff

- Remaining lifetime for a person (currently) aged x, denoted by T_x, independent from financial risk and also the underlying Markov process.
- We are interested in evaluating quantities of the form

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^{T_x}\theta^{(\varphi_s)}\mathrm{d}s}b(S_{T_x}, T_x, M_{T_x}, m_{T_x})\right], \tag{4}$$

where b is a payoff function and the running minimum and maximum of the process X_t is defined as

$$M_t := \sup_{s\in[0,t]} X_s\,, \quad m_t := \inf_{s\in[0,t]} X_s\,. \tag{5}$$

- Here, one uses given a vector of constants $\boldsymbol{\theta} := (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(M)}) \in \mathbb{R}^M$, the process

$$\theta_t = \sum_{j\in\mathcal{S}_\sigma} \theta^{(j)} \cdot \mathbb{1}_{\varphi_t=j} = \theta^{(\varphi_t)}\,, \qquad \text{where } \varphi_t \in \mathcal{S}_\sigma\,.$$

that is constant in each phase φ_t. We define $\boldsymbol{\Theta} = \mathrm{diag}(\theta^{(j)})_{j\in\mathcal{S}_\sigma}$.
- If $\theta^{(\varphi_s)}$, for $s \geq 0$, is the (regime-dependent) risk-free rate, this corresponds to the valuation of, for example, European, digital and lookback options under a given risk-neutral measure under a martingale condition.

Discounted Laplace transform

For $\beta \in \mathbb{R}$, we denote the discounted Laplace transform of the process X as:

$$\phi_t^{(j)}(\beta) := \mathbb{E}\left[e^{-\int_0^t \theta^{(\varphi_s)}\,\mathrm{d}s} e^{\beta X_t} \,\middle|\, \varphi_0 = j\right]. \tag{6}$$

Lemma 3 (Discounted Laplace transform)

Set $\varphi_0 = j \in \mathcal{S}_\sigma$. Let λ_0^+ be the largest eigenvalue of the subgenerator matrix $\boldsymbol{R}_+$, that is $\lambda_0^+ := \max\{\lambda : \lambda$ eigenvalue of $\boldsymbol{R}_+\}$. For $\beta < -\lambda_0^+$, the matrix discounted Laplace transform (6) *is given by*

$$\phi_t^{(j)}(\beta) = \boldsymbol{e}_j' \exp\left(\boldsymbol{\Psi}(\beta, \boldsymbol{\Theta})t\right) \boldsymbol{1}\,, \tag{7}$$

with Laplace exponent matrix:

$$\boldsymbol{\Psi}(\beta,\boldsymbol{\Theta}) = \boldsymbol{Q} + \boldsymbol{D}\beta - \boldsymbol{\Theta} + \frac{1}{2}\boldsymbol{\Sigma}^2\beta^2 + \boldsymbol{W}_{\sigma-}(\beta\boldsymbol{I}_m - \boldsymbol{R}_-)^{-1}\boldsymbol{V}_{-\sigma} - \boldsymbol{W}_{\sigma+}(\beta\boldsymbol{I}_n + \boldsymbol{R}_+)^{-1}\boldsymbol{V}_{+\sigma} \tag{8}$$

Martingale condition

Lemma 4 (Martingale condition)

If the model parameters satisfy the relation

$$\mathbf{\Psi}(1, \mathbf{\Theta})\, \mathbf{1} = \mathbf{0}\,, \tag{9}$$

where $\mathbf{\Psi}(\beta, \mathbf{\Theta})$ *is as in* (8) *with* $\mathbf{\Theta} = \mathrm{diag}(\theta^{(j)})_{j \in \mathcal{S}_\sigma}$, *then the process* $\{e^{-\int_0^t \theta^{(\varphi_s)} \mathrm{d}s} S_t\}_{t \geq 0}$ *is a martingale, that is* $\mathbb{E}\big[e^{-\int_0^t \theta^{(\varphi_s)}\, \mathrm{d}s} S_t \,\big|\, \varphi_0 = j\,\big] = S_0$.

Example 5 (Regime switching Kou model (continued))

Given the Laplace exponent matrix

$$
\begin{aligned}
&\boldsymbol{\Psi}(\beta, \boldsymbol{\Theta}) \\
=&\boldsymbol{Q} + \boldsymbol{D}\beta - \boldsymbol{\Theta} + \frac{1}{2}\boldsymbol{\Sigma}^2\beta^2 + \boldsymbol{W}_{\sigma-}(\beta\boldsymbol{I}_M - \boldsymbol{R}_-)^{-1}\boldsymbol{V}_{-\sigma} - \boldsymbol{W}_{\sigma+}(\beta\boldsymbol{I}_M + \boldsymbol{R}_+)^{-1}\boldsymbol{V}_{+\sigma} \\
=&\boldsymbol{Q}_0 + \boldsymbol{D}\beta - \boldsymbol{\Theta} + \frac{1}{2}\boldsymbol{\Sigma}^2\beta^2 + \mathrm{diag}\left(\lambda_j p_j \frac{\alpha_{-,j}}{\alpha_{-,j} + \beta} + \lambda_j(1 - p_j)\frac{\alpha_{+,j}}{\alpha_{+,j} - \beta} - \lambda_j\right),
\end{aligned}
$$

the martingale condition $\boldsymbol{\Psi}(1, \boldsymbol{\Theta})\,\mathbf{1} = \mathbf{0}$ is simplified to:

$$
d_j = \theta^{(j)} - \frac{1}{2}\sigma_j^2 - \left(\lambda_j p_j \frac{\alpha_{-,j}}{\alpha_{-,j} + 1} + \lambda_j(1 - p_j)\frac{\alpha_{+,j}}{\alpha_{+,j} - 1} - \lambda_j\right) \qquad (10)
$$

for $j \in \mathcal{S}_\sigma$.

4) Distribution of remaining lifetime

In the following, we denote by f_{T_x} the density of the remaining lifetime T_x.

1) Approximation by a combination of Erlang densities

$$f_{T_x}(t) \approx \sum_{k=0}^{K_B} B_k \cdot f_{\tau_{n_k,\mu_k}}(t) =: \hat{f}_{T_x}(t)\,, \tag{11}$$

for constants $K_B \in \mathbb{N}$, $B_k \in \mathbb{R}$ with $\sum_{k=0}^{K_B} B_k = 1$.

Using the independence of T_x and X, one find for a European-type payoff $b(S_{T_x})$:

$$\begin{aligned}
\mathbb{E}\Big[e^{-\int_0^{T_x} \theta^{(\varphi_s)}\mathrm{d}s} b\big(S_{T_x}\big)\Big] &\approx \sum_{k=0}^{K_B} B_k \int_0^\infty \mathbb{E}\Big[e^{-\int_0^t \theta^{(\varphi_s)}\mathrm{d}s} b\big(S_t\big)\Big] f_{\tau_{n_k,\mu_k}}(t)\,\mathrm{d}t \\
&= \sum_{k=0}^{K_B} B_k \mathbb{E}\Big[e^{-\int_0^{\tau_{n_k,\mu_k}} \theta^{(\varphi_s)}\mathrm{d}s} b\big(S_{\tau_{n_k,\mu_k}}\big)\Big]\,,
\end{aligned}$$

for $n_k \in \mathbb{N}$ and $\mu_k > 0$.

As in for example Zhang and Yong [2019], we calibrate these approximations to a life table, minimizing the root mean squared error between the true data and the approximations, that is we solve

$$\operatorname{argmin}_{(B_k,n_k,\mu_k)\in\mathbb{R}^3,k=1,2,\dots,K_B} \sum_{t=1}^{L} \left| F_{T_x}(t) - \sum_{k=1}^{K_B} B_k \cdot F_{\tau_{n_k,\mu_k}}(t) \right|^2, \tag{12}$$

subject to $\sum_{k=1}^{K_B} B_k = 1$, where $F_{T_x}(t)$ is the distribution function corresponding to $f_{T_x}(t)$ and the distribution function of an Erlang random variable is $F_{\tau_{N,\mu}}(t) = 1 - \sum_{k=1}^{N-1} \frac{1}{k!} e^{-\mu t} (\mu t)^k$.

2) Approximation by Laguerre series expansion

The advantage of the Laguerre series expansion is that it allows for an error analysis of the truncation error, see also Zhang and Yong [2019].

- Form a complete orthonormal basis of L^2.
- Error control possible.
- Coefficients can be computed (rather) easily (no optimization).

Laguerre functions are defined as

$$\Psi_k(t) := \sqrt{2\mu}\, e^{-\mu t} \sum_{N=0}^{k} (-1)^N \binom{k}{N} \frac{(2\mu t)^N}{N!} = \sum_{N=0}^{k} \underbrace{(-2)^N \sqrt{\frac{2}{\mu}} \binom{k}{N}}_{\text{“weights“}} \underbrace{f_{\tau_{N+1,\mu}}(t)}_{\text{Erlang densities}},$$

for $k = 1, 2, \ldots$ and $t > 0$.

We can expand the density $f_{T_x} \in L^2(\mathbb{R}_+)$ as:

$$f_{T_x}(t) = \sum_{k=0}^{\infty} A_k \cdot \Psi_k(t) \approx \sum_{k=0}^{K_A} A_k \cdot \Psi_k(t) =: \tilde{f}_{T_x}(t)\,. \tag{13}$$

This approximation by Laguerre series is a also a combination of Erlang distributions.

Indeed, we exploit that the optimal coefficients $A_k = \langle \Psi_k(t), f_{T_x}(t) \rangle$ in (13) can be computed explicitly. For a discrete life table, we obtain:

$$A_k = \left\langle \Psi_k(t), f_{T_x}(t) \right\rangle = \sqrt{2\mu} \sum_{N=0}^{k} \binom{k}{N} \frac{(-2\mu)^N}{N!} \int_0^{\omega - x} t^N e^{-\mu t} f_{T_x}(t) \mathrm{d}t$$

$$\approx \sqrt{2\mu} \sum_{N=0}^{k} \binom{k}{N} \frac{(-2\mu)^N}{N!} \sum_{t=1}^{\omega - x} t^N e^{-\mu t} \mathbb{P}\big(T_x \in (t-1, t]\big)$$

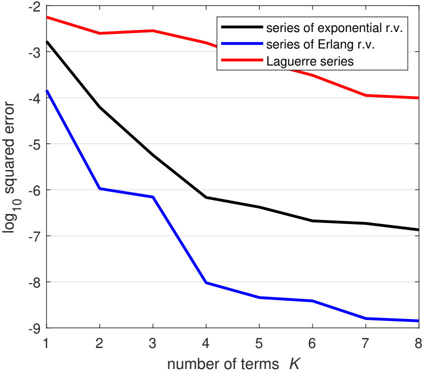
see also Zhang and Yong [2019]. Here, ω denotes the maximum possible age in the life table and x the (current) age of the person.

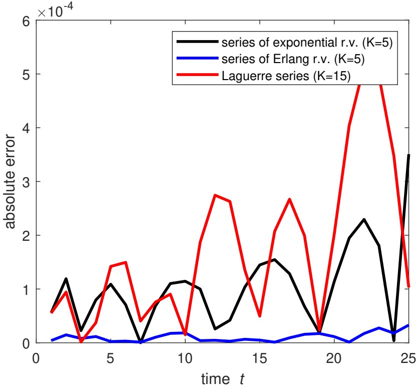
The fact that Laguerre polynomials are uniformly bounded and form an orthonormal basis allows to get theoretical bounds for the approximation error. It holds that:

$$\left| f_{T_x}(t) - \tilde{f}_{T_x}(t) \right|^2 \leq \sum_{k=K_A+1}^{\infty} A_k^2 \, ,$$

see also Zhang and Su [2018] and Zhang and Yong [2019]. We can use this result to provide an upper bound for the total calibration error $\sum_{t=1}^{L} |f_{T_x}(t) - \tilde{f}_{T_x}(t)|^2 \leq L \cdot \sum_{k=K_A+1}^{\infty} A_k^2$.

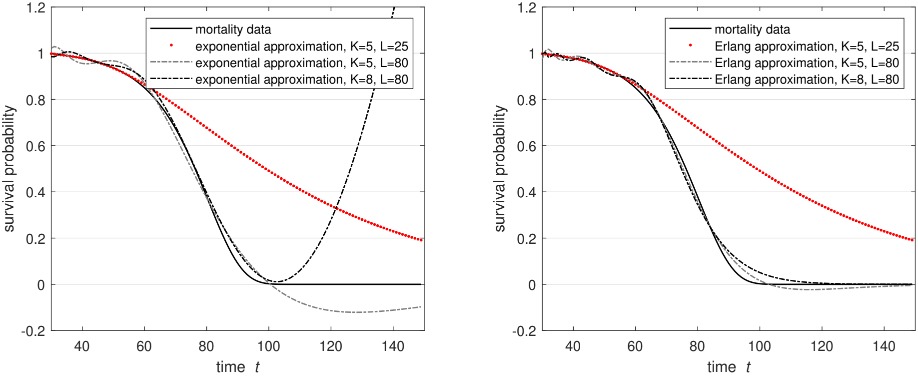
Calibration to a mortality table





We calibrate sums of exponential / Erlang density via $\sum_{k=1}^{K_B} B_k = 1$.

Calibration to a mortality table



5) European-type GMDBs and Laurent series expansion

In the following, we focus on the case where the remaining lifetime T_x is an Erlang distributed random variable $\tau_{N,\mu}$.

At time $T_x = \tau_{N,\mu}$, the payoff is a function b of the risky asset price $S_{T_x} = S_{\tau_{N,\mu}}$. The time-0 value of this product is

$$P_{\mathsf{V}}(S_0) = \mathbb{E}\left[e^{-\int_0^{\tau_{N,\mu}} \theta^{(\varphi_s)} \mathrm{d}s} b(S_{\tau_{N,\mu}})\right] . \qquad (14)$$

An example is a simple guarantee product with guarantee level $K \geq 0$, that is

$$b(S_{\tau_{N,\mu}}) = \max(S_{\tau_{N,\mu}} - K, 0).$$

We want to first obtain the density of $X_{\tau_{N,\mu}} = \ln(S_{\tau_{N,\mu}}/S_0)$.

Lemma 6 (Laplace transform of Erlang-subordinated process)

Consider an Erlang random variable (r.v.) $\tau_{N,\mu}$. Assume that $\beta \leq -\lambda_0^+$ where $\lambda_0^+ := \max\{\lambda : \lambda$ eigenvalue of $\boldsymbol{R}_+\}$. Further assume that the eigenvalues of $\boldsymbol{\Psi}(\beta, \boldsymbol{\Theta})$ have nonpositive real part only. For $j \in \mathcal{S}_\sigma$:

$$\phi^{(j)}_{\tau_{N,\mu}}(\beta) := \mathbb{E}\left[e^{-\int_0^{\tau_{N,\mu}} \theta^{(\varphi_s)}\, \mathrm{d}s} e^{\beta X_{\tau_{N,\mu}}} \,\middle|\, \varphi_0 = j\right] = \boldsymbol{e}_j' \left(\mu^N \left(\mu \boldsymbol{I}_M - \boldsymbol{\Psi}(\beta, \boldsymbol{\Theta})\right)^{-N}\right) \mathbf{1} \, . \qquad (15)$$

Lemma 7 (Laurent series expansion of $\phi^{(j)}_{\tau_{N,\mu}}(\beta)$)

(a) *The Laplace transform*

$$\phi^{(j)}_{\tau_{N,\mu}}(\beta) = \boldsymbol{e}'_j \left(\mu^N \left(\mu \boldsymbol{I}_M - \boldsymbol{\Psi}(\beta, \boldsymbol{\Theta}) \right)^{-N} \right) \mathbf{1}$$

can be written as the quotient $p(\beta)/q(\beta)$ of two polynomials $p(\beta)$ and $q(\beta)$.

(b) *Denote by $\{\alpha_i\}$ and $\{\beta_k\}$ the (by assumption simple) roots of the polynomial $q(\beta)$ in (a) with negative and positive real part, respectively. We can expand $\phi^{(j)}_{\tau_{N,\mu}}(\beta)$ in terms of its Laurent series:*

$$\phi^{(j)}_{\tau_{N,\mu}}(\beta) = \sum_i \sum_{z=1}^{N} \frac{a_{iz}}{(\alpha_i - \beta)^z} + \sum_k \sum_{z=1}^{N} \frac{b_{kz}}{(\beta_k - \beta)^z} . \tag{16}$$

The coefficients a_{iz}, b_{kz} are uniquely determined solving:

$$a_{iz} = \frac{(-1)^z}{(N-z)!} \lim_{\beta \to \alpha_i} \frac{\mathrm{d}^{N-z}}{\mathrm{d}\beta^{N-z}} \left((\beta - \alpha_i)^N \phi^{(j)}_{\tau_{N,\mu}}(\beta) \right), \quad \textit{and} \tag{17}$$

$$b_{kz} = \frac{(-1)^z}{(N-z)!} \lim_{\beta \to \beta_k} \frac{\mathrm{d}^{N-z}}{\mathrm{d}\beta^{N-z}} \left((\beta - \beta_k)^N \phi^{(j)}_{\tau_{N,\mu}}(\beta) \right). \tag{18}$$

Theorem 8 (Laurent series expansion)

(a) *If $\mathbf{\Theta} = \mathbf{0}$, the density of $X_{\tau_{N,\mu}}$ is a series of Erlang densities:*

$$f_{X_{\tau_{N,\mu}}}(x) = \begin{cases} \sum_i \sum_{z=1}^{N} (-a_{iz}) \frac{x^{z-1}}{(z-1)!} e^{-\alpha_i x}, & x < 0 \\ \sum_k \sum_{z=1}^{N} b_{kz} \frac{x^{z-1}}{(z-1)!} e^{-\beta_k x}, & x \geq 0 \end{cases} \tag{19}$$

The coefficients a_{iz}, b_{kz} are uniquely determined by (17)-(18)*.*

(b) *If $\mathbf{\Theta} \neq \mathbf{0}$, i.e. if at least one of the $\theta^{(j)}$s is positive, the Markov chain is absorbing and the absorption probability is given by $\mathbb{P}(\varphi_{\tau_{N,\mu}} = \star) = 1 - \int_{\mathbb{R}} f_{X_{\tau_{N,\mu}}}(x)\,\mathrm{d}x$ with $f_{X_{\tau_{N,\mu}}}(x)$ as in* (19)*. In this case, the density of $X_{\tau_{N,\mu}}$ is composed of a point mass at the absorbing state $\varphi_{\tau_{N,\mu}} = \star$ and the (defective) density* (19) *of the "survived" paths.*

- The proof follows arguments as in Asmussen (2003), Gerber et al. (2013), Eustice and Klamkin (1979).
- With this, we compute risk measures (as e.g. VaR or CTE), and values for any maturity guarantee product.
- This includes results of e.g. Gerber et al. (2012) and of Siu et al. (2015).

5) European-type GMDBs

European-type GMDBs with payoff $b(S_t)$, paid at an Erlang random time $t = \tau_{N,\mu}$, can be written as:

$$
\begin{aligned}
V(\mu, r) &:= \mathbb{E}\Big[e^{-\int_0^{\tau_{N,\mu}} \theta^{(\varphi_s)} \mathrm{d}s} b\big(S_{\tau_{N,\mu}}\big)\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[e^{-\int_0^{\tau_{N,\mu}} \theta^{(\varphi_s)} \mathrm{d}s} b\big(S_0 e^{X_{\tau_{N,\mu}}}\big) \,\Big|\, \varphi,\, \tau_{N,\mu}\Big]\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[0 \cdot \big(1 - e^{-\int_0^{\tau_{N,\mu}} \theta^{(\varphi_s)} \mathrm{d}s}\big) + \int_{\mathbb{R}} b\big(S_0 e^{x}\big) f_{X_{\tau_{N,\mu}}}(x) \,\mathrm{d}x \,\Big|\, \varphi,\, \tau_{N,\mu}\Big]\Big] \\
&= \int_{\mathbb{R}} b\big(S_0 e^{x}\big) f_{X_{\tau_{N,\mu}}}(x) \,\mathrm{d}x \,,
\end{aligned}
\tag{20}
$$

with the (defective) density $f_{X_{\tau_{N,\mu}}}(x)$ from (19).

Theorem 9 (European-type GMDBs)

Consider European-type GMDBs with payoff $b(S_t)$, paid at an Erlang random time $t = \tau_{N,\mu}$. Their fair value is given by

$$V(\mu, r) := \mathbb{E}\Big[e^{-\int_0^{\tau_{N,\mu}} \theta^{(\varphi_s)} \mathrm{d}s} b\big(S_{\tau_{N,\mu}}\big)\Big] = \sum_i \sum_{z=1}^{N} \int_{-\infty}^{0} b(S_0 e^x) \cdot (-a_{iz}) \frac{x^{z-1}}{(z-1)!} e^{-\alpha_i x} \, \mathrm{d}x$$
$$+ \sum_k \sum_{z=1}^{N} \int_0^{\infty} b(S_0 e^x) \cdot b_{kz} \frac{x^{z-1}}{(z-1)!} e^{-\beta_k x} \, \mathrm{d}x \,. \tag{21}$$

The case of one state $M = 1$ relates to the discounted density approach by Gerber et al. (2012, 2013).

Note that our case of a regime-dependent discount factor leads to a dependence between discount factor and asset value $S_{\tau_{N,\mu}}$.

Out-of-the-money call option valuation

Example 10

For $b(S_t) = \max(S_t - K, 0)$, $Re(h) > 1$ and $S_0 \leq K$, we obtain:

$$\begin{aligned} C(h,z) &:= \int_0^\infty e^{-hx} \frac{x^{z-1}}{(z-1)!} \max(S_0 e^x - K, 0) \mathrm{d}x \\ &= \int_{\ln(K/S_0)}^\infty e^{-hx} \frac{x^{z-1}}{(z-1)!} (S_0 e^x - K) \mathrm{d}x \\ &= S_0 \cdot \eta\big(\ln(K/S_0), h-1, z\big) - K \cdot \eta\big(\ln(K/S_0), h, z\big)\,, \end{aligned}$$

where we use that for $y \geq 0$, we can apply partial integration to obtain

$$\eta(y,h,z) = \int_y^\infty e^{-hx} \frac{x^{z-1}}{(z-1)!} \mathrm{d}x = \sum_{i=1}^{z} e^{-hy} \frac{1}{h^{z+1-i}} \frac{y^{i-1}}{(i-1)!}\,.$$

From (21), we finally obtain:

$$C(S_0) := \mathbb{E}\Big[e^{-\int_0^{\tau_{N,\mu}} \theta_s \mathrm{d}s} b\big(S_{\tau_{N,\mu}}\big)\Big] = \sum_k \sum_{z=1}^{N} b_{kz}\, C(\beta_k, z)\,, \tag{22}$$

where the coefficients b_{kz} are given by (18) and β_k are the (non-removable) singularities with positive real part of $\phi_{\tau_{N,\mu}}(\beta)$. Note that the singularities β_k and the coefficients b_{kz} depend on the discount factor via the matrix $\boldsymbol{\Theta}$.

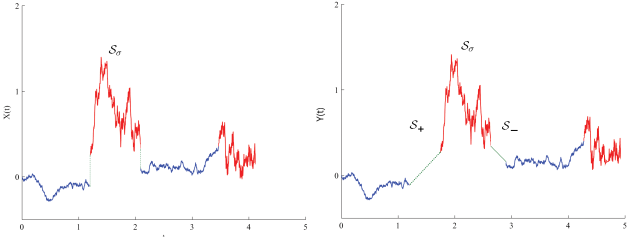
	Fourier price $P_V(S_0)$		time	Our price $P_V(S_0)$		time	True price $P_V(S_0)$	
	$\varphi_0 = 1$	$\varphi_0 = 2$		$\varphi_0 = 1$	$\varphi_0 = 2$		$\varphi_0 = 1$	$\varphi_0 = 2$
$K = 100$	1.8476	2.7552	0.4583s	1.8476	2.7552	0.0310s	1.8848	2.7624
$K = 105$	2.0492	3.0207	0.4198s	2.0492	3.0207	0.0295s	2.0823	3.0231
$K = 110$	2.2667	3.2964	0.4346s	2.2667	3.2964	0.0295s	2.2912	3.2928
$K = 115$	2.4998	3.5819	0.4346s	2.4998	3.5819	0.0303s	2.5113	3.5712
$K = 120$	2.7474	3.8767	0.4240s	2.7474	3.8767	0.0297s	2.7421	3.8580
$K = 125$	3.0081	4.1805	0.5004s	3.0082	4.1805	0.0298s	2.9831	4.1529
$K = 130$	3.2808	4.4929	0.4463s	3.2808	4.4929	0.0291s	3.2338	4.4555

Table: Values of Eur. GMDBs, OTM call options following a 2-dim regime switching Kou model and a sum-of-Erlang remaining lifetime distribution ($K_B = 5, L = 80$).

Parameter set from Siu et al. (2005):
$S_0 = 100$, $\theta_1 = \theta_2 = r = 0.05$, $\sigma_1 = 0.1$, $\alpha_{+,1} = 40$, $\alpha_{-,1} = 60$, $p_1 = 0.25$, $\lambda_1 = 2$, $\sigma_2 = 0.4$, $\alpha_{+,2} = 60$, $\alpha_{-,2} = 70$, $p_2 = 0.75$, $\lambda_2 = 0.5$, and $\boldsymbol{Q}_0 = [-0.1\ 0.1; 0.2\ -0.2]$.

Fluid embedding technique

The process X_t is converted into a continuous process Y_t for which jumps are replaced by straight lines . We extend the underlying Markov process $\varphi(t)$ regulating the diffusion part to a process ζ_t with phases in $\mathcal{S}_\sigma \cup \mathcal{S}_+ \cup \mathcal{S}_-$.



See e.g. Rogers (1994), Jiang and Pistorius (2008), Asmussen and Albrecher (2010), Deelstra et al. (2020)

Exponential time $\tau_{1,\mu}$

- Define the upper $(+)$ and lower $(-)$ first-passage time of our (fluidized) asset process (Y_t, ζ_t), for $x \geq 0$:

$$\tau_x^{\pm}(i) := \inf\left\{t > 0 \,|\, Y_t = 0,\, Y_0 = \mp x,\, \zeta_0 = i \in \mathcal{S}\right\},$$

- For $i \in \mathcal{S}_\sigma \cup \mathcal{S}_+$ and $j \in \mathcal{S}_\sigma \cup \mathcal{S}_-$, we further introduce the limiting cases $\lim_{x \to 0} \tau_x^+(i) = \tau_0^+(i) = 0$ and $\lim_{x \to 0} \tau_x^-(j) = \tau_0^-(j) = 0$, respectively.
- For $j \in \mathcal{S}$, define

$$\{\boldsymbol{E}_{\pm}^{(1)}(x)\}_{ij} := \mathbb{E}\big[e^{-\int_0^{\tau_x^{\pm}(i)} \theta(\varphi_s)\mathrm{d}s} \mathbb{1}_{\zeta_{\tau_x^{\pm}(i)}=j}\big]$$

  for $x \geq 0$, $i, j \in \mathcal{S}_\sigma \cup \mathcal{S}_+$ and $i, j \in \mathcal{S}_\sigma \cup \mathcal{S}_-$, respectively.
- Let us define the matrices

$$\left\{\boldsymbol{\Psi}_{\pm}^{(1)}\right\}_{ij} = \left\{\boldsymbol{E}_{\pm}^{(1)}(0)\right\}_{ij} = \mathbb{E}\left[e^{-\int_0^{\tau_0^{\pm}(i)} \theta(\varphi_s)\mathrm{d}s} \mathbb{1}_{\zeta_{\tau_0^{\pm}(i)}=j}\right],$$

  for $i \in \mathcal{S}$, $j \in \mathcal{S}_\sigma \cup \mathcal{S}_+$ and $j \in \mathcal{S}_\sigma \cup \mathcal{S}_-$, respectively.

- We also introduce the parameterization:

$$\boldsymbol{E}_{\pm}^{(1)}(x) := \exp\left(\boldsymbol{U}_{\pm}^{(1)} x\right) := \exp\left(\begin{bmatrix} \boldsymbol{U}_{\sigma\sigma}^{(1)} & \boldsymbol{U}_{\sigma\pm}^{(1)} \\ \boldsymbol{U}_{\pm\sigma}^{(1)} & \boldsymbol{U}_{\pm\pm}^{(1)} \end{bmatrix} x\right), \tag{23}$$

  where $\exp(\,\cdot\,)$ denotes the matrix exponential.
- The matrices $\boldsymbol{U}_{\pm}^{(1)}$ and $\boldsymbol{\Psi}_{\pm}^{(1)}$ can be derived in terms of Sylvester equations.
- Again, we have the (discounted) Laplace transform: convenient for exponential / Erlang time change.

Theorem 11 (Sylvester equations: Exponential time $\tau_{1,\mu}$)

The matrices $(\boldsymbol{\Psi}_-^{(1)}, \boldsymbol{\Psi}_+^{(1)}, \boldsymbol{U}_-^{(1)}, \boldsymbol{U}_+^{(1)})$ *are uniquely defined by a system of Sylvester equations*

$$\Upsilon\big(\boldsymbol{U}_-^{(1)}, \boldsymbol{\Psi}_-^{(1)}, \mathcal{P}(\boldsymbol{\Theta}), \hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{D}}\big) = \mathbf{0}, \quad \Upsilon\big(-\boldsymbol{U}_+^{(1)}, \boldsymbol{\Psi}_+^{(1)}, \mathcal{P}(\boldsymbol{\Theta}), \hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{D}}\big) = \mathbf{0}\,, \tag{24}$$

where

$$\Upsilon\big(\boldsymbol{U}, \boldsymbol{\Psi}, \boldsymbol{\mathcal{P}}, \hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{D}}\big) = \frac{1}{2}\hat{\boldsymbol{\Sigma}} \cdot \boldsymbol{\Psi} \cdot \boldsymbol{U}^2 + \hat{\boldsymbol{D}} \cdot \boldsymbol{\Psi} \cdot \boldsymbol{U} + \boldsymbol{\mathcal{P}} \cdot \boldsymbol{\Psi}\,,$$

$$\boldsymbol{\Psi}_-^{(1)} = \begin{bmatrix} \boldsymbol{I}_M & \mathbf{0} \\ \boldsymbol{\Psi}_{+\sigma}^{(1)} & \boldsymbol{\Psi}_{+-}^{(1)} \\ \mathbf{0} & \boldsymbol{I}_m \end{bmatrix}, \boldsymbol{\Psi}_+^{(1)} = \begin{bmatrix} \boldsymbol{I}_M & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_n \\ \boldsymbol{\Psi}_{-\sigma}^{(1)} & \boldsymbol{\Psi}_{-+}^{(1)} \end{bmatrix}, \hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \boldsymbol{\Sigma}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \hat{\boldsymbol{D}} = \begin{bmatrix} \boldsymbol{D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{I}_m \end{bmatrix}.$$

Proof: See the work by Rogers, Asmussen, Jiang, Pistorius, Ivanovs and others.

Generalization to our purpose

- Sylvester equations can be solved by fixed point iterations or eigenvalue algorithms (pre-implemented in many software like Matlab).
- Discounting is used as an absorbing state (very convenient).
- This can be generalized to Erlang random variables.
- Deelstra, Latouche, Simon [2020]: Barrier-type financial derivatives with high values of N (much larger matrix dimension for Sylvester equations; necessary for approximating fixed maturity dates).
- We apply this approach and extend results to lookback GMDBs, dynamic fund protection, dynamic withdrawal benefits.

7) Discussion and conclusion

- We discuss the valuation and risk management of death-linked contingent claims.
- Based on Laurent series, respectively Sylvester equations, we **avoid numerical Fourier inversion** (see the motivating example).
- Mortality table is approximated by Erlang distributions and by Laguerre series (convenient error control, parameters known).
- "Merging" mortality and financial risk makes computations **(semi) closed-form** and **much faster**.

More details in:

Deelstra, G., Hieber, P., (2023), Randomization and the valuation of guaranteed minimum death benefits, European journal of operational research, Vol. 309, Issue 3, 1218-1236.

Thanks

Thank you for your attention!

Back

References I

S. Asmussen. *Applied Probability and Queues*. Springer, New York, 2003.

S. Asmussen and H. Albrecher. *Ruin Probabilities*. World Scientific, 2nd edition, 2010.

S. Asmussen, P. J. Laub, and H. Yang. Phase-type models in life insurance: Fitting and valuation of equity-linked benefits. *Risks*, Vol. 7, No. 1:pp. 1–17, 2019.

F. Avram, M. R. Pistorius, and M. Usabel. The two barriers ruin problem vie a Wiener Hopf decomposition approach. *Annals of University of Craiova*, Vol. 30:pp. 38–44, 2003.

J. Bao and Y. Zhao. Option pricing in markov-modulated exponential Lévy models with stochastic interest rates. *Journal of Computational and Applied Mathematics*, 357:146–160, 2019.

F. E. Benth, G. Deelstra, and S. Kozpınar. Pricing energy quanto options in the framework of markov-modulated additive processes. *IMA Journal of Management Mathematics*, 34(1): 187–220, 2023.

N. L. Bowers, H. U. Gerber, J. C. Hickman, D. A. Jones, and C. J. Nesbitt. *Actuarial Mathematics*. Society of Actuaries, Schaumburg, Illinois, 1997.

S. Boyarchenko and S. Levendorskii. Valuation of continuously monitored double barrier options and related securities. *Mathematical Finance*, Vol. 22, No. 3:pp. 419–444, 2012.

J. Cao, T. R. N. Roslan, and W. Zhang. Pricing variance swaps in a hybrid model of stochastic volatility and interest rate with regime-switching. *Methodology and computing in applied probability*, 20(4):1359–1379, 2018.

References II

P. Carr. Randomization and the American put. *Review of Financial Studies*, Vol. 11, No. 3:pp. 597–626, 1998.

S.-N. Chen, M.-H. Chiang, P.-P. Hsu, and C.-Y. Li. Valuation of quanto options in a Markovian regime-switching market: A Markov-modulated gaussian HJM model. *Finance Research Letters*, 11(2):161–172, 2014.

Z. Cui, J. L. Kirkby, and D. Nguyen. Equity-linked annuity pricing with cliquet-style guarantees in regime-switching and stochastic volatility models with jumps. *Insurance: Mathematics & Economics*, Vol. 74:pp. 46–62, 2017.

Z. Cui, J. L. Kirkby, and D. Nguyen. A general framework for time-changed Markov processes and applications. *European Journal of Operational Research*, Vol. 273, No. 1:pp. 785–800, 2019.

G. Deelstra and P. Hieber. Randomization and the valuation of guaranteed minimum death benefits. *European Journal of Operational Research*, 309(3):1218–1236, 2023.

G. Deelstra and M. Simon. Multivariate European option pricing in a Markov-modulated lévy framework. *Journal of Computational and Applied Mathematics*, 317:171–187, 2017.

G. Deelstra, S. Kozpınar, and M. Simon. Spread and basket option pricing in a markov-modulated Lévy framework with synchronous jumps. *Applied Stochastic Models in Business and Industry*, 34(6):782–802, 2018.

G. Deelstra, G. Latouche, and M. Simon. On barrier option pricing by Erlangization in a regime-switching model with jumps. *Journal of Computational and Applied Mathematics*, 371, 2020.

References III

R. S. Elias, M. I. M. Wahab, and L. Fang. A comparison of regime-switching temperature modeling approaches for applications in weather derivatives. *European Journal of Operational Research*, Vol. 232:pp. 549–560, 2014.

R. Elliott, L. Aggoun, and J. Moore. *Hidden Markov Models: Estimation and Control*. Springer: New York, 1994.

R. Elliott, L. Chan, and T. Siu. Option pricing and Esscher transform under regime switching. *Annals of Finance*, 1(4):423–432, 2005.

R. J. Elliott and G.-H. Lian. Pricing variance and volatility swaps in a stochastic volatility model with regime switching: discrete observations case. *Quantitative Finance*, 13(5):687–698, 2013.

R. J. Elliott and C.-J. U. Osakwe. Option pricing for pure jump processes with Markov switching compensators. *Finance and Stochastics*, 10:250–275, 2006.

R. J. Elliott and T. K. Siu. On Markov-modulated exponential-affine bond price formulae. *Applied Mathematical Finance*, 16(1):1–15, 2009.

D. Eustice and M. S. Klamkin. On the coefficients of a partial fraction decomposition. *American Mathematical Monthly*, 86(6):478–480, 1979.

K. Fan, Y. Shen, T. K. Siu, and R. Wang. An FFT approach for option pricing under a regime-switching stochastic interest rate model. *Communications in Statistics - Theory and Methods*, 46(11):5292–5310, 2017.

References IV

H. U. Gerber, E. S. Shiu, and H. Yang. Valuing equity-linked death benefits and other contingent options: a discounted density approach. *Insurance: Mathematics and Economics*, 51(1):73–92, 2012.

H. U. Gerber, E. S. W. Shiu, and H. Yang. Valuing equity-linked death benefits in jump diffusion models. *Insurance: Mathematics & Economics*, Vol. 53:pp. 615–623, 2013.

D. Hainaut. Impulse control of pension fund contributions, in a regime switching economy. *European Journal of Operational Research*, Vol. 239:pp. 1024–1042, 2014.

J. D. Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, Vol. 57, No. 2:pp. 357–384, 1989.

P. Hieber and M. Scherer. A note on first-passage times of continuously time-changed Brownian motion. *Statistics & Probability Letters*, Vol. 82, No. 1:pp. 165–172, 2012.

J. Ivanovs. Markov-modulated Brownian motion with two reflecting barriers. *Journal of Applied Probability*, 47(4):1034–1047, 2010.

Z. Jiang and M. R. Pistorius. On perpetual American put valuation and first-passage in a regime-switching model with jumps. *Finance and Stochastics*, Vol. 12, No. 3:pp. 331–355, 2008.

Z. Jin, G. Liu, and H. Yang. Optimal consumption and investment strategies with liquidity risk and lifetime uncertainty for Markov regime-switching jump diffusion models. *European Journal of Operational Research*, Vol. 280:pp. 1130–1143, 2020.

References V

J. L. Kirkby and D. Nguyen. Equity-linked guaranteed minimum death benefits with dollar cost averaging. *Insurance: Mathematics & Economics*, Vol. 100:pp. 408–428, 2021.

M. Konikov and D. B. Madan. Option pricing using variance gamma markov chains. *Review of Derivatives Research*, 5:81–115, 2002.

R. Korn, Y. Melnyk, and F. T. Seifried. Stochastic impulse control with regime-switching dynamics. *European Journal of Operational Research*, Vol. 260, No. 3:pp. 1024–1042, 2017.

S. G. Kou and H. Wang. First passage times of a jump diffusion process. *Advances in Applied Probability*, Vol. 35, No. 2:pp. 504–531, 2003.

A. Mijatović, M. Pistorius, and J. Stolte. Randomisation and recursion methods for mixed-exponential Lévy models, with financial applications. *Journal of Applied Probability*, 52 (4):1076–1096, 2015.

A. Ramponi. Fourier transform methods for regime-switching jump-diffusions and the pricing of forward starting options. *International Journal of Theoretical and Applied Finance*, 15(05): 1250037, 2012.

S. Robert and J.-Y. L. Boudec. New models for pseudo self-similar traffic. *Performance Evaluation*, 30:57–68, 1997.

L. Rogers. Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Annals of Applied Probability*, 4(2):390–413, 1994.

Y. Shen and T. K. Siu. Pricing bond options under a markovian regime-switching Hull–White model. *Economic Modelling*, 30:933–940, 2013a.

References VI

Y. Shen and T. K. Siu. Longevity bond pricing under stochastic interest rate and mortality with regime-switching. *Insurance: Mathematics and Economics*, 52(1):114–123, 2013b.

Y. Shen and T. K. Siu. Pricing variance swaps under a stochastic interest rate and volatility model with regime-switching. *Operations Research Letters*, 41(2):180–187, 2013c.

C. C. Siu, S. C. P. Yam, and H. Yang. Valuing equity-linked death benefits in a regime-switching framework. *ASTIN Bulletin*, 45(2):355–395, 2015.

J. J. Sylvester. On the equation to the secular inequalities in the planetary theory. *Philosophical Magazine Series*, 5:267–269, 1883.

G. Tour, N. Thakoor, A. Khaliq, and D. Tangman. COS method for option pricing under a regime-switching model with time-changed Lévy processes. *Quantitative Finance*, 18(4): 673–692, 2018.

Y. Xie and G. Deng. Vulnerable european option pricing in a Markov regime-switching Heston model with stochastic interest rate. *Chaos, Solitons & Fractals*, 156:111896, 2022.

Z. Zhang and W. Su. A new efficient method for estimating the Gerber-Shiu function in the classical risk model. *Scandinavian Actuarial Journal*, 2018(5):426–449, 2018.

Z. Zhang and Y. Yong. Valuing guaranteed equity-linked contracts by Laguerre series expansion. *Journal of Computational and Applied Mathematics*, Vol. 357:pp. 329–348, 2019.

Z. Zhang, Y. Yong, and W. Yu. Valuing equity-linked death benefits in general exponential Lévy models. *Insurance: Mathematics and Economics*, Vol. 365, 2021.