Term structure modeling beyond stochastic continuity

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The LIBOR reform

- London Interbank Offered Rate (LIBOR), computed as the trimmed average of rates reported by a panel of banks, for five currencies (CHF, EUR, GBP, JPY, USD) and seven tenors (1D, 1W, 1M, 2M, 3M, 6M, 1Y).
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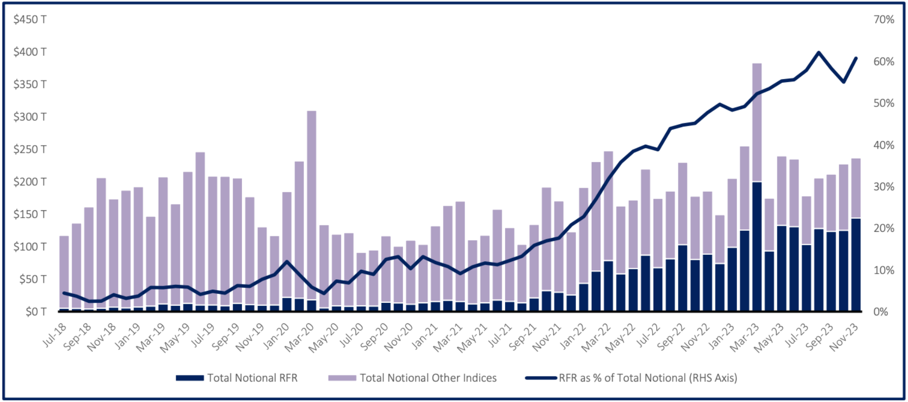
- July 2017: *The future of LIBOR* speech by Andrew Bailey (FCA): LIBOR discontinuation after 2021.
- Transition towards transaction-based overnight rates as benchmark rates. ARRC, June 2017: Secured Overnight Funding Rate (SOFR) in the US.
- FCA, March 2021: complete LIBOR cessation after June 2023.

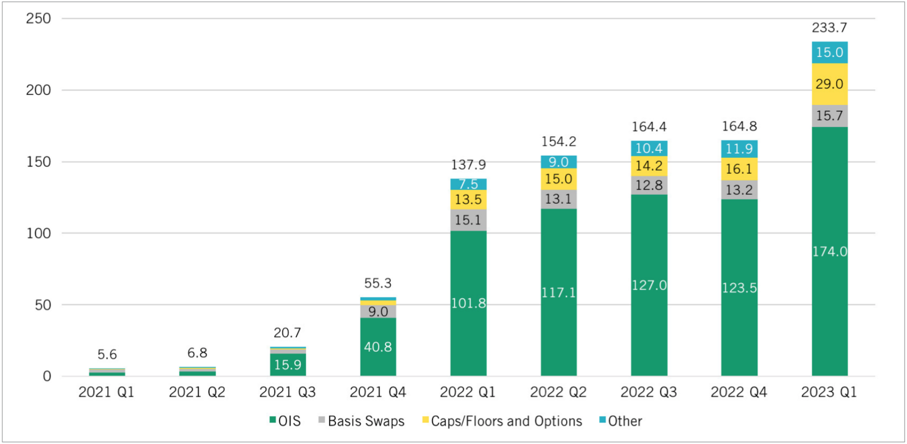
Adoption of overnight rates



Source: ISDA-Clarus RFR adoption indicator, November 2023.

SOFR

Chart 5: SOFR Trade Count by Product (thousands)



Source: DTCC SDR

Alternative risk-free rates and stochastic discontinuities

- Alternative risk-free rates (RFRs) are (nearly) risk-free overnight rates;
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- Upward/downward spikes at regulatory reporting dates:
  SOFR was on average 20.25 bps higher at quarter-ends compared to other dates (source: Klingler and Syrstad (2021), period: 08/2014 - 12/2019).

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These facts provide evidence of **stochastic discontinuities**:
new information arriving at pre-determined dates that affects overnight rates.

SOFR: spikes and hikes

SOFR time series from 01/01/2018 until 12/12/2022 (source: Refinitiv).

SOFR: spikes and hikes

- Let us consider the spike observed on 17/09/2019.
  According to Anbil et al. (2020):

  > *Strains in money markets in September seem to have originated from routine market events, including a corporate tax payment date and Treasury coupon settlement. The outsized and unexpected moves in money market rates were amplified by a number of factors.*

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- Presence of stochastic discontinuities in the RFR dynamics.
  This phenomenon is playing an important role in recent works:
  - Andersen and Bang (2020): spikes in the SOFR dynamics, both at expected and unexpected times.
  - Gellert and Schlögl (2021): diffusive HJM model for instantaneous forward rates, with jumps/spikes at fixed times in the short rate.
  - Brace et al. (2022): diffusive HJM model with stochastic volatility.
  - Backwell and Hayes (2022): short-rate model for the SONIA rate, based on a pure jump process with expected and unexpected jump times.
  - Schlögl et al. (2023): joint model for policy and overnight benchmark rates.
  - Kim and Wright (2014): short rate model with jumps at fixed times.

Outline of the talk

1. Numéraire, backward-looking and forward-looking rates;
2. an extended HJM framework;
3. the affine semimartingale setup;
4. hedging via local risk-minimization.

The RFR numéraire

- We consider a continuous-time RFR process $\rho = (\rho_t)_{t \geq 0}$. In line with empirical evidence, ρ is allowed to have expected and unexpected jumps.
- Numéraire:
$$S_t^0 = \exp\left(\int_{(0,t]} \rho_u \, \eta(du) \right),$$
where $\eta(du) = du + \sum_{n \in \mathbb{N}} \delta_{\{t_n\}}(du)$.
- The set $\mathcal{T} := \{t_n; n \in \mathbb{N}\}$ of roll-over dates, at which S^0 is expected to jump.

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- Depending on the specification of ρ and η, this setup includes:
  - classical short-rate approach (corresponding to $\mathcal{T} = \emptyset$);
  - discretely updated bank account at overnight frequency:
$$S_t^0 = \prod_{t_{n+1} \leq t} \big(1 + r_{t_n}(t_{n+1} - t_n)\big),$$
where r_{t_n} is the overnight rate for the time interval $[t_n, t_{n+1}]$.
- $P(t, T)$: zero-coupon bond (ZCB) price at time t for maturity T.

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$$R(S,T) := \frac{1}{T-S}\left(\prod_{n\in N(S,T)} \frac{1}{P(t_n,t_{n+1})} - 1\right),$$

  where $N(S,T) := \{n \in \mathbb{N} : S \leq t_n < t_{n+1} \leq T\}$.
- According to the ISDA protocol, $R(S,T)$ is adopted as LIBOR fallback, up to an additive spread determined from historical data.
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- Forward-looking rate $F(S,T)$: rate K such that the single-period swap (SPS) delivering $R(S,T) - K$ at maturity T has zero value at time S.
- CME Term SOFR and Refinitiv Term SONIA are forward-looking rates. 12/29/2021: ARRC endorsed CME term SOFR as forward-looking rate.
- The use of term SOFR for derivatives is currently restricted by ARRC, but there is increasing market demand for derivatives referencing term SOFR.

Forward term rates

As in Lyashenko and Mercurio (2019), we can consider two types of forward rates:

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Consequence of the above definitions:

$$F(t,S,T) = R(t,S,T), \qquad \text{for all } t \in [0,S].$$

The forward-looking forward rate $F(t,S,T)$ stops evolving at time S, while the backward-looking forward rate $R(t,S,T)$ continues to evolve until time T, with

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Forward-looking and backward-looking forward rates can be consolidated into a single process $R(\cdot, S, T)$. We call this process the forward term rate.

Forward term rates

Payoff $1 + (T - S)R(S, T)$ at maturity T can be statically replicated as follows:

- buy-and-hold strategy in one ZCB with maturity S;
- at time S, invest 1 in a roll-over strategy remunerated at the overnight rate.

This implies the following (classical) representation of forward term rates:

$$R(t, S, T) = \frac{1}{T - S}\left(\frac{P(t, S)}{P(t, T)} - 1\right),$$

for all $t \in [0, T]$, extending ZCB bond prices beyond maturity by setting

$$P(t, S) = \frac{P(t, t_{n(t)})}{P(t_{n(t)-1}, t_{n(t)})} \prod_{n \in N(S,t)} \frac{1}{P(t_n, t_{n+1})}, \qquad \text{for } t > S,$$

with $n(t) := \inf\{n \in \mathbb{N} : t_n > t\}$.

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Similarly to classical (single-curve) interest rate models, the family of ZCB prices $\{P(\cdot, T); T > 0\}$ constitutes the fundamental basis of the term structure model.

An extended HJM framework

We start by specifying ZCB prices as follows:

$$P(t,T) = \exp\Big(-\int_{(t,T]} f(t,u)\eta(du)\Big),$$

where we recall that $\eta(dt) = dt + \sum_{n\in\mathbb{N}} \delta_{\{t_n\}}(dt)$ and assume that

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \varphi(s,T)dW_s + V(t,T),$$

with W a d-dim. Brownian motion and $V(\cdot,T)$ a pure jump process such that

$$\{\Delta V(\cdot,T) \neq 0\} \subseteq \Omega \times \mathcal{S}, \qquad \text{where } \mathcal{S} = \{s_1,\dots,s_M\}.$$

The set $\mathcal{S}$ contains expected jump dates, i.e., dates at which the RFR ρ and forward term rates are expected to jump.

Remarks:

- Lévy-type jumps can be included;
- we do not exclude the case $\mathcal{S} \cap \mathcal{T} \neq \emptyset$;
- $\mathcal{S}$ can be generalized to a countable family of predictable times.

Martingale representation

In the representation of forward rates, we are implicitly using the following.

Assumption

There exists a family $(\xi_1, \ldots, \xi_M)$ of random variables taking values in $(X, \mathcal{B}_X)$ such that ξ_i is $\mathcal{F}_{s_i}$-measurable, for each $i = 1, \ldots, M$, and every local martingale $N = (N_t)_{t \geq 0}$ can be represented as

$$N = N_0 + \int_0^{\cdot} \theta_t dW_t + \sum_{i=1}^{M} f_i(\xi_i) \mathbf{1}_{[\![s_i, +\infty [\![},$$

where $f_i(\cdot) : \Omega \times X \to \mathbb{R}$ is a $(\mathcal{F}_{s_i-} \otimes \mathcal{B}_X)$-measurable function such that

$$E[f_i(\xi_i) | \mathcal{F}_{s_i-}] = 0 \quad \text{a.s.}$$

Technical assumptions

The following conditions hold a.s.:

(i) the *initial forward curve* $T \to f(0,T)$ is $(\mathcal{F}_0 \otimes \mathcal{B}_{\mathbb{R}_+})$-measurable, real-valued and satisfies $\int_0^T |f(0,u)|du < +\infty$, for all $T > 0$;

(ii) the *drift process* $\alpha : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is progressively measurable, satisfies $\alpha(t,T) = 0$ for $T < t$, and

$$\int_0^T \int_0^u |\alpha(s,u)|ds\,\eta(du) < +\infty, \qquad \text{for all } T > 0;$$

(iii) the *volatility process* $\varphi : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}^d$ is progressively measurable and satisfies $\varphi(t,T) = 0$ for $T < t$, and

$$\sum_{i=1}^d \int_0^T \left(\int_0^u |\varphi^i(s,u)|^2 ds \right)^{1/2} \eta(du) < +\infty, \qquad \text{for all } T > 0;$$

(iv) the *stochastic discontinuity process* $V(\cdot,T)$ satisfies $\int_0^T |\Delta V(s,u)|du < +\infty$ for all $s \in \mathcal{S}$ and $\Delta V(t,T) = 0$ for $T < t$.

An extended HJM framework

Goal: characterize when Q is a risk-neutral measure, i.e., S^0-denominated ZCB prices are local martingales under Q. This ensures absence of arbitrage in the sense of *no asymptotic free lunch with vanishing risk* (NAFLVR, see Cuchiero et al. (2016)), with respect to the numéraire S^0.

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As a preliminary to the next result, let us define

$$\bar{\alpha}(t,T) := \int_{[t,T]} \alpha(t,u)\eta(du),$$

$$\bar{\varphi}(t,T) := \int_{[t,T]} \varphi(t,u)\eta(du),$$

$$\bar{V}(t,T) := \int_{[t,T]} \Delta V(t,u)\eta(du).$$

HJM-type conditions

Theorem

Q is a risk-neutral measure if and only if (suitable integrability properties hold) and the following four conditions are satisfied:

(i)
$$f(t,t) = \rho_t,$$

(ii)
$$\bar{\alpha}(t,T) = \frac{1}{2}\|\bar{\varphi}(t,T)\|^2$$

(iii) for every $j = 1, \ldots, N$ it holds that
$$f(t_j-, t_j) = \rho_{t_j-} - \log\bigl(E[e^{-\Delta\rho_{t_j}}|\mathcal{F}_{t_j-}]\bigr),$$

(iv) for every $i = 1, \ldots, M$ it holds that
$$E\left[e^{-\Delta\rho_{s_i}\delta_{\mathcal{T}}(s_i)}\bigl(e^{-\int_{(s_i,T]}\Delta V(s_i,u)\eta(du)} - 1\bigr)\Big|\mathcal{F}_{s_i-}\right] = 0.$$

Remark: if $\mathcal{S} \cap \mathcal{T} = \emptyset$, then conditions (i) and (iii) can be jointly written as
$$f(t,t) = \rho_t, \qquad \eta(dt) \otimes dQ\text{-a.e.}$$

Example: a Cheyette-type model

An extension of the Cheyette model with stochastic discontinuities:

- for simplicity, no roll-over dates ($\mathcal{T} = \emptyset$), so that $S^0 = \exp(\int_0^\cdot r_u du)$;
- instantaneous forward rates:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \varphi(s,T)dW_s + \sum_{s_i \leq t} \big(\alpha_i(T) + \xi_i g_i(T)\big),$$

  with independent $\xi_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, for $i = 1, \dots, M$;
- separable volatility structure (1-factor, for illustration):

$$\varphi(t,T) = \frac{a(T)}{a(t)} b(t) \qquad \text{and} \qquad g_i(T) = a(T)B_i.$$

- Under this volatility structure, it holds that

$$f(t,T) = f(0,T) + \frac{a(T)}{a(t)} X_t + U(t,T),$$

  where X is a mean-reverting Gaussian Markov process with mean-reversion speed $\partial_t \log(a(t))$, diffusion coefficient b and jumps at dates $\{s_1, \dots, s_M\}$, and $U(t,T)$ is a deterministic function.

The affine framework

The presence of expected jump times requires an extension of affine processes: affine semimartingales generalize affine processes by allowing for jumps at fixed times with possibly state-dependent jump sizes (see Keller-Ressel et al. (2019)).

An affine semimartingale $X = (X_t)_{t \geq 0}$ taking values in $\mathbb{R}_+^m \times \mathbb{R}^n$ satisfies

$$E\left[e^{\langle u, X_T \rangle} | \mathcal{F}_t\right] = \exp\left(\phi_t(T, u) + \langle \psi_t(T, u), X_t \rangle\right),$$

for all $u \in \mathcal{U} = \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n$, where the functions $\phi_t(T, u)$ and $\psi_t(T, u)$ satisfy generalized Riccati equations.

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Short-rate approach: let the RFR be given by

$$\rho_t = \ell(t) + \langle \Lambda, X_t \rangle, \qquad \text{for all } t \geq 0,$$

where the function ℓ fits the initially observed term structure.

Proposition

The joint process $(X, \int_0^{\cdot} \rho_u \, \eta(du))$ is an affine semimartingale.

- Similar to the enlargement of the state-space approach of Duffie et al. (2003).
- Fourier-based methods for pricing a variety of interest rate derivatives.

An example: an extended Hull-White model

Assume that $\rho = (\rho_t)_{t \geq 0}$ satisfies

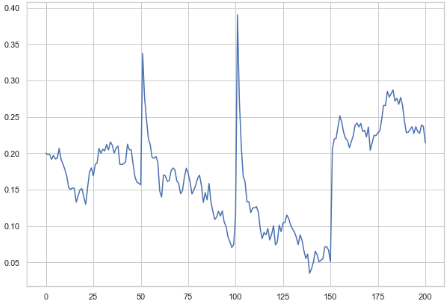
$$d\rho_t = \big(\alpha(t) + \beta\rho_t\big)dt + \sigma dW_t + dJ_t,$$

where J is a pure jump process independent of W:

$$J = \sum_{i=1}^{M} \xi^i \mathbf{1}_{[\![s_i,+\infty[\![},$$

In the Gaussian case (i.e., $(\xi_i)_{i=1,\dots,M}$ independent and Gaussian):

- explicit formula for ZCB prices;
- Black-type formula for post-Libor caplets/floorlets.



Hedging with stochastic discontinuities

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Recall that, in our setup, every local martingale N can be represented as

$$N = N_0 + \int_0^{\cdot} \theta_t dW_t + \sum_{i=1}^{M} f_i(\xi_i) \mathbf{1}_{[\![s_i,+\infty[\![}.$$

- Suppose that the market contains a single risky asset with price process

$$X = X_0 + A + M,$$

  where
  - A is process of finite variation;
  - $M = \int_0^{\cdot} \eta_t dW_t + \sum_{s_i \leq \cdot} w_i(\xi_i)$ is a square-integrable martingale, with $\eta > 0$.
- For instance, X can represent the price process of a SOFR future contract (currently the most liquid SOFR contract).

Hedging with stochastic discontinuities

Let $H \in L^2$ be an $\mathcal{F}_T$-measurable payoff. We denote by Θ the set of predictable processes ζ such that $E[\int_0^T \zeta_u^2 d\langle M\rangle_u + (\int_0^T |\zeta_u dA_u|)^2] < +\infty$.

Definition

- An *H-admissible strategy* is a pair $\varphi = (\zeta, V)$, where $\zeta = (\zeta_t)_{t\in[0,T]} \in \Theta$ and $V = (V_t)_{t\in[0,T]}$ is an adapted process such that $V_T = H$ a.s.
- An H-admissible strategy $\varphi = (\zeta, V)$ is *locally risk-minimizing* if the associated cost process

$$C_t(\varphi) := V_t - \int_0^t \zeta_u dX_u, \qquad t \in [0,T],$$

is a square-integrable martingale orthogonal to M.

Remarks:

- ζ_t and V_t represent respectively the position in the traded asset and the portfolio value at time t, for all $t \in [0,T]$;
- if X satisfies the structure conditions and A is continuous, this definition can be shown to be equivalent to the original definition of Schweizer (1991).

Hedging with stochastic discontinuities

- By absence of arbitrage, there exists a predictable process $\boldsymbol{\lambda}$ such that $A = \int_0^\cdot \lambda_u \, d\langle M \rangle_u$. This implies that

$$\Delta A_{s_i} = \lambda_{s_i} E[(\Delta M_{s_i})^2 | \mathcal{F}_{s_i-}] \text{ a.s., for all } i = 1, \ldots, M.$$

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- Assume that $\widehat{\boldsymbol{Z}} := \mathcal{E}(-\int_0^\cdot \lambda_u dM_u)$ is a strictly positive square-integrable martingale and define the minimal martingale measure by $d\widehat{Q} = \widehat{Z}_T dP$.
- We can then define the $\widehat{Q}$-martingale $\widehat{\boldsymbol{H}} = (\widehat{H}_t)_{t \in [0,T]}$ by

$$\widehat{H}_t := \widehat{E}[H | \mathcal{F}_t], \qquad \text{for all } t \in [0, T],$$

where we denote by $\widehat{E}$ the expectation with respect to $\widehat{Q}$.

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- By absence of arbitrage, there exists a predictable process $\boldsymbol{\lambda}$ such that $A = \int_0^{\cdot} \lambda_u \, d\langle M \rangle_u$. This implies that
$$\Delta A_{s_i} = \lambda_{s_i} E[(\Delta M_{s_i})^2 | \mathcal{F}_{s_i-}] \text{ a.s., for all } i = 1, \dots, M.$$
- Assume that $\widehat{\boldsymbol{Z}} := \mathcal{E}(-\int_0^{\cdot} \lambda_u dM_u)$ is a strictly positive square-integrable martingale and define the minimal martingale measure by $d\widehat{Q} = \widehat{Z}_T dP$.
- We can then define the $\widehat{Q}$-martingale $\widehat{\boldsymbol{H}} = (\widehat{H}_t)_{t \in [0,T]}$ by
$$\widehat{H}_t := \widehat{E}[H | \mathcal{F}_t], \qquad \text{for all } t \in [0, T],$$
where we denote by $\widehat{E}$ the expectation with respect to $\widehat{Q}$.
- By Bayes' formula, $\widehat{H} = N / \widehat{Z}$, with $N_t := E[\widehat{Z}_T H | \mathcal{F}_t]$, for all $t \in [0, T]$.
- As a consequence of the martingale representation assumption, we have that
$$N = N_0 + \int_0^{\cdot} \theta_u dW_u + \sum_{s_i \le \cdot} \Delta N_{s_i}.$$
- We can then write
$$H = \widehat{H}_T = \widehat{H}_0 + \int_0^T \zeta_u^H dX_u + \sum_{s_i \le T} (\Delta \widehat{H}_{s_i} - \zeta_{s_i}^H \Delta X_{s_i}) = \widehat{H}_0 + \int_0^T \boldsymbol{\zeta_u^H} dX_u + \boldsymbol{L_T^H}.$$

Hedging with stochastic discontinuities

Theorem

Let H be an $\mathcal{F}_T$-measurable random variable such that $\sup_{t\in[0,T]} \widehat{H}_t \in L^2$. Define the predictable process

$$\zeta_t^H := \big(\widehat{Z}_{t-}^{-1}\eta_t^{-1}\theta_t + \widehat{H}_{t-}\lambda_t\big)\delta_{\mathcal{S}^c}(t) + \frac{E[\Delta\widehat{H}_t\Delta M_t|\mathcal{F}_{t-}]}{E[(\Delta M_t)^2|\mathcal{F}_{t-}]}\delta_{\mathcal{S}}(t).$$

If $\zeta^H \in \Theta$, then the strategy $\varphi^H = (\zeta^H, V^H)$ is locally risk-minimizing, where $V_t^H = \widehat{H}_t$, for all $t \in [0,T]$.

Remarks:

- perfect replication at all times $t \in [0,T] \setminus \mathcal{S}$, when the only active source of randomness is the Brownian motion W;
- at the expected jump dates $\mathcal{S} = \{s_1, \dots, s_M\}$, the strategy $\zeta_{s_i}^H$ is determined by a linear regression of $\Delta\widehat{H}_{s_i}$ onto ΔX_{s_i}, conditionally on $\mathcal{F}_{s_i-}$:

$$\zeta_{s_i}^H = \frac{\mathrm{Cov}(\Delta\widehat{H}_{s_i}, \Delta X_{s_i}|\mathcal{F}_{s_i-})}{\mathrm{Var}(\Delta X_{s_i}|\mathcal{F}_{s_i-})},$$

- One can then obtain, e.g., explicit formula for the locally risk-minimizing strategy of a term SOFR caplet with respect to a SOFR future.

Thank you for your attention

For more information:

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