Knightian Uncertainty in Economics and Finance

22nd Winter School on Mathematical Finance
January 20-22, 2025
Soesterberg

Frank Riedel

Bielefeld University

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Outline

Overview

- Frank Knight and the Discovery of Uncertainty as a Relevant Economic Factor

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- Re-Thinking Economics under the new Paradigm

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  - Interactions: Equilibrium in Markets

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  - Robust Finance

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- Knight identifies "proper uncertainty" as a source of profit

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IMW

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- "The mathematical type of probability is practically never met with in business." (p.215)
- In business, no law of large numbers that allows to estimate the probability of success with accuracy.

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- Knight distinguishes risk (measurable uncertainty) from uncertainty (unmeasurable uncertainty)
- "the income of an entrepreneur is larger ... as there is a scarcity of self-confidence in society combined with the power to make effective guarantees to employees." (p.283)
- excess profit is the result of confronting uninsurable uncertainty

The Bayesian Paradigm

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- but was this way the right way to go, and if not, what other options do we have?

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4. Imprecise Probabilistic Information: irreducible uncertainty

   Research of the last 20 years allows to deal with that -- Our lecture series

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5. see *Lo, Mueller*, Journal of Investment Management, 2010

Knightian Uncertainty as a new paradigm

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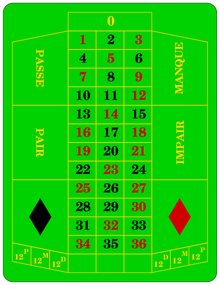
- The last 20 years have seen a huge development in economics, finance, and mathematics
- Knightian uncertainty (microeconomics), model uncertainty (finance), robustness (macroeconomics)
- we next consider the basic paradigmatic decision situations under Knightian uncertainty

Roulette Bets: You win 1 Euro if

- ‘Rouge’
- ‘Manque’ (1-18)
- ‘Colonne 34’ (1, 4, 7, ... , 34)
- ’Plein’ (one particular number)



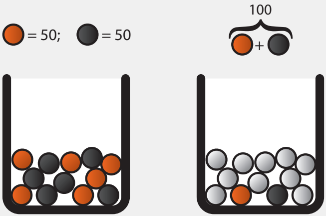
’Almost’ everyone agrees that

$$\text{Rouge} \sim \text{Manque} \succ \text{Colonne 34} \succ \text{Plein}$$

with $\sim$ meaning “I am indifferent”, $\succ$ meaning “I prefer”

Decision Situations under Uncertainty I: Ellsberg's Experiments

Ellsberg's Thought Experiment 1



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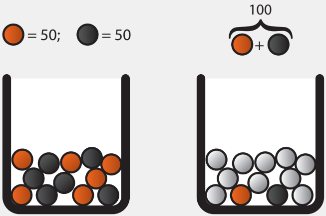

Literature

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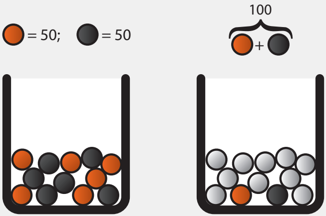

Literature

- *Daniel Ellsberg*, Risk, Ambiguity, and the Savage Axioms, Quarterly Journal of Economics, 1961

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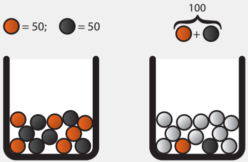

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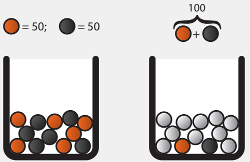

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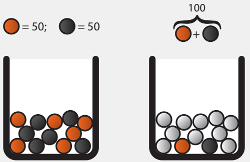

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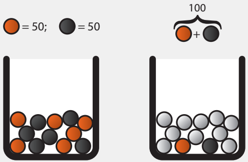

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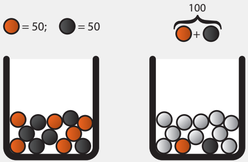

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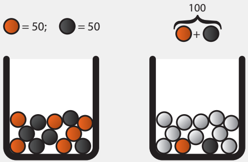

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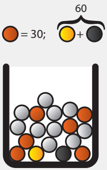

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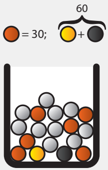

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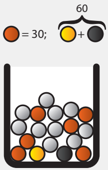

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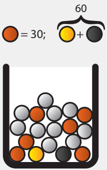

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- 'Red or Yellow' or 'Black or Yellow' ?

Complex Decision Situations under Uncertainty: The Real World

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Order the bets according to your beliefs! Do you know the probabilities? Do you think it is possible to know the probabilities or to obtain estimates?

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- there is a reason why the odds are $1:1$ for ‘Rouge’ and $1:2$ for ‘Colonne 34’
- if someone does not agree with the ordering

$$\text{Rouge} \succ \text{Colonne 34}$$

we can offer him bets and make money in the long run (whiteboard)

Consequences for Rational Choice

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- it is sufficient to know how to order probability distributions on $\mathcal{X}$; we write $\Delta = \Delta\mathcal{X}$ for the set of all probability distributions on $\mathcal{X}$

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- it is sufficient to know how to order probability distributions on $\mathcal{X}$; we write $\Delta = \Delta\mathcal{X}$ for the set of all probability distributions on $\mathcal{X}$
- we call such acts with known probabilities lotteries

Literature

von Neumann--Morgenstern Theory

Literature

- *John von Neumann, Oscar Morgenstern*, Theory of Games and Economic Behavior, 1944

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Basic Assumption

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Basic Assumption

- In the probabilistic world, a 'rational' man orders lotteries $P, Q \in \Delta$ over some set of outcomes $\mathcal{X}$ with the help of a complete and transitive ordering $\succeq$, a preference relation

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- In the probabilistic world, a 'rational' man orders lotteries $P, Q \in \Delta$ over some set of outcomes $\mathcal{X}$ with the help of a complete and transitive ordering $\succeq$, a preference relation
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Basic Assumption

- In the probabilistic world, a 'rational' man orders lotteries $P, Q \in \Delta$ over some set of outcomes $\mathcal{X}$ with the help of a complete and transitive ordering $\succeq$, a preference relation
- that satisfies the linear rules of mixing lotteries (independence)
- and that is continuous

Zur Theorie der Gesellschaftsspiele[1]).

Von

J. v. Neumann in Berlin.

Einleitung.

1. Die Frage, deren Beantwortung die vorliegende Arbeit anstrebt, ist die folgende:

n Spieler, $S_1, S_2, \ldots, S_n$, spielen ein gegebenes Gesellschaftsspiel $\mathfrak{G}$. Wie muß einer dieser Spieler, S_m, spielen, um dabei ein möglichst günstiges Resultat zu erzielen?

Die Fragestellung ist allgemein bekannt, und es gibt wohl kaum eine Frage des täglichen Lebens, in die dieses Problem nicht hineinspielte; trotzdem ist der Sinn dieser Frage kein eindeutig klarer. Denn sobald $n > 1$ ist (d. h. ein eigentliches Spiel vorliegt), hängt das Schicksal eines jeden Spielers außer von seinen eigenen Handlungen auch noch von denen seiner Mitspieler ab; und deren Benehmen ist von genau denselben egoistischen Motiven beherrscht, die wir beim ersten Spieler bestimmen möchten. Man fühlt, daß ein gewisser Zirkel im Wesen der Sache liegt.

Wir müssen also versuchen, zu einer klaren Fragestellung zu kommen. Was ist zunächst ein Gesellschaftsspiel? Es fallen unter diesen Begriff sehr viele, recht verschiedenartige Dinge: von der Roulette bis zum Schach, vom Bakkarat bis zum Bridge liegen ganz verschiedene Varianten des Sammelbegriffes „Gesellschaftsspiel" vor. Und letzten Endes kann auch irgendein Ereignis, mit gegebenen äußeren Bedingungen und gegebenen Handelnden (den absolut freien Willen der letzteren vorausgesetzt), als Gesellschaftsspiel angesehen werden, wenn man seine Rückwirkungen auf die in ihm handelnden Personen betrachtet[2]). Was ist nun das gemeinsame Merkmal aller dieser Dinge?

---

[1]) Der Inhalt dieser Arbeit ist (mit einigen Kürzungen) am 7. XII. 1926 der Göttinger Math. Ges. vorgetragen worden.

[2]) Es ist das Hauptproblem der klassischen Nationalökonomie: was wird, unter gegebenen äußeren Umständen, der absolut egoistische „homo œconomicus" tun?

Independence

Axiom (Independence Axiom)

The ordering $\succeq$ is linear, i.e. for all probability distributions P, Q, R on $\mathcal{X}$ and for all $\alpha \in (0,1)$ we have

$$P \succeq Q \longleftrightarrow \alpha P + (1-\alpha)R \succeq \alpha Q + (1-\alpha)R$$

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Exercise

Define the strict relations $\succ$ and $\prec$ and the indifference relation $\sim$. Show that $\succ$ and $\sim$ satisfy the independence axiom as well.

Continuity

Axiom (Archimedean Continuity Axiom)

The ordering $\succeq$ is continuous; for all probability distributions P, Q, R on $\mathcal{X}$ we have the following. If $P \succ Q \succ R$, then there exist $\alpha, \beta \in (0, 1)$ with

$$\alpha P + (1 - \alpha)R \succ Q \succ \beta P + (1 - \beta)R$$

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Axiom (Archimedean Continuity Axiom)

The ordering $\succeq$ *is continuous; for all probability distributions* P, Q, R *on* $\mathcal{X}$ *we have the following. If* $P \succ Q \succ R$*, then there exist* $\alpha, \beta \in (0, 1)$ *with*

$$\alpha P + (1 - \alpha)R \succ Q \succ \beta P + (1 - \beta)R$$

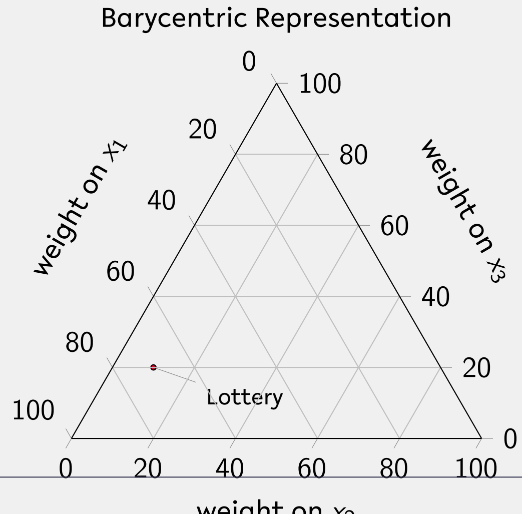
Example

Continuity implies the following: if you prefer 100 Euro over 10 Euro over -10000 Euro, then you also prefer the lottery that yields 100 Euro with, say, 99 %, and -10000 Euro with 1 % to 10 Euro for sure.

One could argue against continuity. “Lexicographic preferences” violate continuity.

Graphical Representations

When there are only three possible outcomes, $\mathcal{X} = \{x_1, x_2, x_3\}$, one can represent lotteries $p = (p_1, p_2, p_3)$ as points in the simplex $\Delta = \{(p_1, p_2, p_3) \in \mathbb{R}^3_+ : p_1 + p_2 + p_3 = 1\}$.



Exercise

Exercise

The independence axiom implies that indifference curves over the simplex are linear.

von Neumann--Morgenstern Theory

Definition

Let Δ be the set of all probability measures over $\mathcal{X}$. We say that a function $U : \Delta \to \mathbb{R}$ is a utility function for $\succeq$ if we have

$$P \succeq Q \longleftrightarrow U(P) \geq U(Q)$$

for all $P, Q \in \mathcal{X}$.

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for all $P, Q \in \mathcal{X}$.

Theorem (Expected Utility Theorem, von Neumann-Morgenstern)

A preference relation over Δ that satisfies the independence and continuity axiom admits a utility function U of the form

$$U(P) = \sum_{x \in \mathcal{X}} u(x)P(x)$$

for some Bernoulli utility function $u : \mathcal{X} \to \mathbb{R}$.

IMW

Proof of the vNM-Theorem by Experiment

Experiment 1

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- We assume that there are three outcomes $x_1 = 100$, $x_2 = 50$, $x_3 = 0$.

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- Clearly, $(1, 0, 0) \succ (0, 1, 0) \succ (0, 0, 1)$

Proof of the vNM-Theorem by Experiment

Experiment 1

- We assume that there are three outcomes $x_1 = 100$, $x_2 = 50$, $x_3 = 0$.
- Clearly, $(1, 0, 0) \succ (0, 1, 0) \succ (0, 0, 1)$
- Make a list of lotteries $(0.9, 0, 0.1), (0.8, 0, 0.2), \ldots, (0.1, 0.0.9)$ and compare them to $(0, 1, 0)$!

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Sketch of Proof for the EU Theorem

The utility assigns values 1 and 0 to the best resp. worst outcome. The utility $u(x)$ is an indifference probability. $u(x)(1, 0, \ldots, 0) + (1 - u(x))(0, 0, \ldots, 1) \sim$ x for sure

Finding your Utility Function by Experiment

Experiment 2

EXCEL-File Erwartungsnutzenbestimmen.xls

Uniqueness of Bernoulli Utility and Cardinality

Theorem

Suppose that the preference relation $\succeq$ admits an expected utility function of the form

$$U(P) = \sum_{x \in \mathcal{X}} u(x)P(x)$$

for some Bernoulli utility function $u : \mathcal{X} \to \mathbb{R}$. Suppose that

$$V(P) = \sum_{x \in \mathcal{X}} v(x)P(x)$$

is another expected utility function for $\succeq$. Then there is a number $\lambda > 0$ and a number $m \in \mathbb{R}$ such that for all $x \in \mathcal{X}$ we have

$$v(x) = \lambda u(x) + m.$$

Bernoulli utility functions are unique up to affine transformations.

Uniqueness of Bernoulli Utility and Cardinality

Remark

Expected utility theory is cardinal - the Bernoulli utilities have a "measurable" meaning.

Generalization: Mixture Space Theorem

A mixture space is a set with an operation that allows you to take convex combinations.

Definition

Let $\mathcal{Z}$ be a nonempty set and let $\oplus$ be an operation that maps $\alpha \in [0, 1]$ and $y, z \in \mathcal{Z}$ to an element in $\mathcal{Z}$ such that for all $\alpha, \beta \in [0, 1]$ and $y, z \in \mathcal{Z}$

$$1 \cdot y \oplus 0 \cdot z = y \qquad \text{(sure mix)}$$

$$\alpha y \oplus (1 - \alpha) z = (1 - \alpha) z \oplus \alpha y \qquad \text{(commutativity)}$$

$$\alpha (\beta y + (1 - \beta) z) \oplus (1 - \alpha) z = \alpha\beta y \oplus (...) z \qquad \text{(distributivity)}$$

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Example

- The set of all lotteries Δ is a mixture space
- Convex subsets of vector spaces are mixture spaces
- non-convex mixture spaces (*Mongin*, Dec. Econ. Fin. 2000)

Mixture Space Axioms

Axiom (Mixture Space Continuity Axiom)

For all $a, b, c \in \mathcal{Z}$*, the sets* $\{\mu \in [0,1] : \mu \cdot a \oplus (1-\mu) \cdot b \succeq c\}$ *and* $\{\mu \in [0,1] : c \preceq \mu \cdot a \oplus (1-\mu) \cdot b\}$ *are closed.*

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Remark

- *Mixture Space Continuity implies Archimedean Continuity.*

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Remark

- *Mixture Space Continuity implies Archimedean Continuity.*
- *The Archimedean continuity axiom and the independence axiom over lotteries of finite prizes imply the mixture space continuity axiom.*

Mixture Space Axioms

Axiom (Mixture Space Independence Axiom)

For all $a, b, c \in \mathcal{Z}$ *and all* $\mu \in (0, 1)$ *we have*

$$a \succeq b \longleftrightarrow \mu \cdot a \oplus (1-\mu) \cdot c \succeq \mu \cdot b \oplus (1-\mu) \cdot c.$$

Mixture Space Theorem

Theorem (Herstein, Milnor 1953)

A preference relation $\succeq$ on a mixture space $\mathcal{Z}$ satisfies the mixture space continuity and the mixture space independence axiom if and only if it admits a linear utility function U.

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Exercise

Derive the von Neumann--Morgenstern theorem from the mixture space theorem.

Mixture Space Theorem: Proof Idea

The proof uses the same idea that we encountered in our small experiment.

- Fix two elements $a, b \in \mathcal{Z}$ with $a \succ b$.

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- Fix two elements $a, b \in \mathcal{Z}$ with $a \succ b$.
- For any c with $a \succeq c \succeq b$, there is a unique (!) number $\lambda \in [0, 1]$ with $\lambda \cdot a \oplus (1 - \lambda) \cdot b \sim c$.

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- For existence, we need the continuity axiom.
- For uniqueness, the independence axiom.

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- For any c with $a \succeq c \succeq b$, there is a unique (!) number $\lambda \in [0, 1]$ with $\lambda \cdot a \oplus (1 - \lambda) \cdot b \sim c$.
- For existence, we need the continuity axiom.
- For uniqueness, the independence axiom.
- We define $U(c) = \lambda$. λ is called indifference probability.

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- For any c with $a \succeq c \succeq b$, there is a unique (!) number $\lambda \in [0, 1]$ with $\lambda \cdot a \oplus (1 - \lambda) \cdot b \sim c$.
- For existence, we need the continuity axiom.
- For uniqueness, the independence axiom.
- We define $U(c) = \lambda$. λ is called indifference probability.
- We then extend linearly to the whole set $\mathcal{Z}$.

Mixture Space Theorem: Proof

In the following, fix $a, b \in \mathcal{Z}$ with $a \succ b$.

Lemma

For each $c \in \mathcal{Z}$ with $a \succeq c \succeq b$, there is a unique number $\lambda \in [0, 1]$ with $\lambda \cdot a \oplus (1 - \lambda) \cdot b \sim c$.

Mixture Space Theorem: Proof

In the following, fix $a, b \in \mathcal{Z}$ with $a \succ b$.

Lemma

For each $c \in \mathcal{Z}$ with $a \succeq c \succeq b$, there is a unique number $\lambda \in [0,1]$ with $\lambda \cdot a \oplus (1-\lambda) \cdot b \sim c$.

Proof.

We first show existence: Let $T = \{\lambda \in [0,1] : \lambda \cdot a \oplus (1-\lambda) \cdot b \succeq c\}$ and $W = \{\lambda \in [0,1] : \lambda \cdot a \oplus (1-\lambda) \cdot b \preceq c\}$.
As $\succeq$ is complete, we have $T \cup W = [0,1]$. By the mixture continuity axiom, T and W are closed. T and W are nonempty because $1 \in T$ and $0 \in W$. As $[0,1]$ is connected, it cannot be decomposed into two nonempty, disjoint closed sets. So $W \cap T \neq \emptyset$. For $\lambda \in T \cap W$, we have $\lambda \cdot a \oplus (1-\lambda) \cdot b \sim c$. □

Mixture Space Theorem: Proof

For uniqueness, we need to work a little bit more. We first show that the independence axiom implies a certain monotonicity of preferences: higher probabilities on good outcomes are preferred.

Lemma

We have $1 \geq \lambda > \mu \geq 0$ *if and only if* $\lambda \cdot a \oplus (1-\lambda) \cdot b \succ \mu \cdot a \oplus (1-\mu) \cdot b$.

This lemma yields uniqueness (why?)

Mixture Space Theorem: Proof of Monotonicity Lemma

We start with $1 \geq \lambda > \mu \geq 0$. In this case, the independence axiom yields

$$c := \lambda \cdot a \oplus (1-\lambda) \cdot b \succ \lambda \cdot b \oplus (1-\lambda) \cdot b = b.$$

Now let $\gamma = \mu/\lambda \in [0, 1)$. We apply the independence axiom again and use the mixture space rules:

$$c \succ \gamma \cdot c \oplus (1-\gamma) \cdot b = \mu \cdot a \oplus (1-\mu) \cdot b.$$

Now we do the reverse direction. Suppose that $\lambda \cdot a \oplus (1-\lambda) \cdot b \succ \mu \cdot a \oplus (1-\mu) \cdot b$. We need to show that $\lambda > \mu$. Suppose that $\lambda < \mu$. Then we could apply the first part of the proof to conclude $\lambda \cdot a \oplus (1-\lambda) \cdot b \prec \mu \cdot a \oplus (1-\mu) \cdot b$, a contradiction. On the other hand, $\lambda = \mu$ is also impossible because then $\lambda \cdot a \oplus (1-\lambda) \cdot b \sim \mu \cdot a \oplus (1-\mu) \cdot b$.

Mixture Space Theorem: Definition of Utility

1. If $a \succeq c \succeq b$, we set $U(c) = \lambda$ with $\lambda \in [0, 1]$ satisfying

$$\lambda \cdot a \oplus (1 - \lambda) \cdot b \sim c.$$

Exercise: check that U is linear!

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   (Exchange the roles of c and a.) We set $U(c) = 1/\gamma$.

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2. If $c \succ a$, there exists $\gamma \in [0, 1]$ with
$$\gamma \cdot c \oplus (1 - \lambda) \cdot b \sim a.$$
(Exchange the roles of c and a.) We set $U(c) = 1/\gamma$.
3. For $b \succ c$, there exists $\gamma \in [0, 1]$ with
$$\gamma \cdot a \oplus (1 - \lambda) \cdot c \sim b.$$
Set $U(c) = -\gamma/(1 + \gamma)$.

Exercise: check that U is linear!

Mixture Space Theorem: End of Proof

It remains to be shown that U is indeed a linear utility function for $\succeq$. We do this for the case in which a is maximal and b is is minimal in $\mathcal{Z}$, i.e.

$$\mathcal{Z} = \{c \in \mathcal{Z} : a \succeq c \succeq b\}.$$

Whiteboard.

Infinite Outcome Sets

In applications the set $\mathcal{X}$ is usually not finite. For example, we could have $\mathcal{X} = \mathbb{R}$. In this case, $\mathcal{Z} = \Delta\mathcal{X}$ consists of all (Borel) probability measures on the real line. In this case, we have to strengthen the topological requirements to obtain a similar theorem.

For details, see *Föllmer, Schied*, Stochastic Finance, Chapter 2.2.

Outline

Motivation

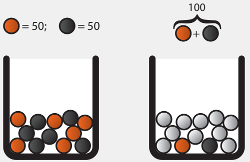
We now understand the structure of linear preferences. Our next aim is to try to understand if we can deal with the Ellsberg experiments.

Ellsberg Experiment 1

Ellsberg Bets: You win 1 Euro if

- red ball is drawn in Urn 1, R1
- black ball is drawn in Urn 1, B1,
- red ball is drawn in Urn 2, R2,
- black ball is drawn in Urn 1, B2

Ellsberg's Thought Experiment 1



Vote

- R1 or B1 ?
- R2 or B2 ?
- R1 or R2 ?
- B1 or B2 ?

Analysis of Ellsberg's Experiments

- It is not irrational to order bets as follows in Experiment 1:

Analysis of Ellsberg's Experiments

- It is not irrational to order bets as follows in Experiment 1:
- $R1 \sim B1$, $R2 \sim B2$, $R1 \succ R2$, $B1 \succ B2$

Analysis of Ellsberg's Experiments

Bayesian Subjective EU Approach

Analysis of Ellsberg’s Experiments

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- A ‘rational’ man’s betting should reflect his beliefs for ‘red’ or ‘black’

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Analysis of Ellsberg's Experiments

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- for example, by Laplace's principle of insufficient reason, $p_R = .5$ in first experiment. In this case, we would have $R1 \sim B1 \sim R2 \sim B2$
- if $p < .5$, then we would have $R1 \succ R2$, $B1 \prec B2$.

Analysis of Ellsberg's Experiments

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- Urn 1: clearly 50 % (objective probability, not a belief)
- Urn 2: beliefs for red, black, yellow
- for example, by Laplace's principle of insufficient reason, $p_R = .5$ in first experiment. In this case, we would have $R1 \sim B1 \sim R2 \sim B2$
- if $p < .5$, then we would have $R1 \succ R2$, $B1 \prec B2$.
- the Bayesian approach would yield opposite orderings as those that we considered to be rational. It does not capture the aversion of not knowing the probabilities (because it simply assumes that you have a subjective probability for the second urn).

Ambiguity--Averse Approach

Sophisticated Maxmin Approach

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IMW

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- *Klibanoff, Marinacci, Mukerji*, A Smooth Model of Decision Making under Ambiguity, Econometrica 2005

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- *Maccheroni, Marinacci, Rustichini*, Ambiguity Aversion, Robustness, and the Robust Representation of Preferences, Econometrica 2006

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Literature

- *Bewley*, Knightian Decision Theory, Decisions in Economics and Finance, 2002

Outline

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- we allow that horse races pay off in terms of lottery tickets

Ellsberg Experiment 1 in the Anscombe-Aumann Framework

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  We know the probabilities for urn 1, we do not know them for urn 2.
- The bets are defined as follows:

$$\begin{aligned} r_1(\omega) &= 1 && \text{if } \omega_1 = red \\ b_1(\omega) &= 1 && \text{if } \omega_1 = black \\ r_2(\omega) &= 1 && \text{if } \omega_2 = red \\ b_2(\omega) &= 1 && \text{if } \omega_2 = black \end{aligned}$$

Acts form a mixture space

We define the operation $\oplus$ pointwise for acts.

$$(\alpha f \oplus (1-\alpha)g)(\omega) = \alpha f(\omega) + (1-\alpha)g(\omega)$$

Lemma

The set of acts $\mathcal{Z}$ with the operation $\oplus$ is a mixture space.

Independence and Continuity for Acts

The preference ordering $\succeq$ is defined over acts in $\mathcal{Z}$

Axiom (Independence Axiom)

The ordering $\succeq$ is linear, i.e. for all acts $f, g, h \in \mathcal{Z}$ we have for all $\alpha \in (0, 1)$

$$f \succeq g \longleftrightarrow \alpha f \oplus (1-\alpha)h \succeq \alpha g \oplus (1-\alpha)h$$

Axiom (Mixture Space Continuity Axiom)

For all $a, b, c \in \mathcal{Z}$, the sets $\{\mu \in [0,1] : \mu \cdot a \oplus (1-\mu) \cdot b \succeq c\}$ and $\{\mu \in [0,1] : c \preceq \mu \cdot a \oplus (1-\mu) \cdot b\}$ are closed.

Theorem

If $\succeq$ satisfies the Independence and (Mixture Space) Continuity Axiom, then for every state $\omega \in \Omega$ there exist a Bernoulli utility function $u_\omega : \mathcal{X} \to \mathbb{R}$ such that

$$U(f) = \sum_{\omega \in \Omega} \sum_{x \in \mathcal{X}} u_\omega(x) f(\omega)(x)$$

is a utility function for $\succeq$.

Proof. Mixture Theorem. Linear functions look like that!

State--Independent Tastes

Definition

Let $P \in \Delta$ be given. For an act $f : \Omega \to \Delta$, and state $\omega_k \in \Omega$, we define $f_k^P(\omega_k) = P$ and $f_k^P(\omega_l) = f(\omega_l)$ for $l \neq k$.

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Axiom

For all $P, Q \in \Delta$, all acts f, and all states ω_k, ω_l we have

$$f_k^P \succeq f_k^Q \textit{ iff } f_l^P \succeq f_l^Q.$$

State--Independent Tastes

If we are willing to assume state--independent tastes, we get more:

Theorem

If $\succeq$ satisfies the Independence and Continuity Axiom, and if $\succeq$ has state--independent tastes, then there exists a probability measure μ over Ω and a Bernoulli utility function $u : \mathcal{X} \to \mathbb{R}$ such that

$$U(f) = \sum_{\omega \in \Omega} \left(\sum_{x \in \mathcal{X}} u(x) h(\omega)(x) \right) \mu(\omega)$$

is a utility function for $\succeq$.

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- the theorem is the master piece of decision theory
- yet, it imposes a lot of assumptions on the “rational man”
- even from a rational point of view, the Ellsberg experiments suggest that rational decisions need not be based on subjective beliefs

What is wrong with the Independence Axiom in Ellsberg's Experiments?

We agreed that it can be rational to have the following (Ellsberg) preferences: $r_1 \succ r_2$ and $b_1 \succ b_2$.

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- for acts, mixing is more like splitting your money on two different tickets
- for example $(1/2 \cdot r_2 \oplus 1/2 \cdot b_2)(\omega) = 1/2 = r_1$; putting half of your money on betting red and putting the other half on betting black hedges the uncertainty of urn 2 --- the Knightian uncertainty is gone

What is wrong with the Independence Axiom in Ellsberg's Experiments?

If you have $r_1 \succ r_2$, then the independence axiom implies

$$(1/2 \cdot r_1 \oplus 1/2 b_2) \succ (1/2 \cdot r_2 \oplus 1/2 b_2) = r_1$$

So

$$(1/2 \cdot r_1 \oplus 1/2 b_2) \succ (1/2 \cdot r_1 \oplus 1/2 r_1)$$

We apply the independence axiom again and get

$$b_2 \succ r_1 \sim b_1$$

in contradiction to the Ellsberg ordering.

How to get the other utility representations? We need to relax one of the crucial axioms in Ancombe--Aumann.

1. Completeness: either $f \succ g$ or $f \prec g$ or $f \sim g$

One of the axioms needs to be relaxed if we want to allow Ellsberg choices

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2. if we give up completeness, we get Bewley's incomplete EU theory
3. Independence Axiom
4. if we weaken the independence axiom, we get the maxmin or smooth or variational model, depending on how we replace it

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Gilboa--Schmeidler Axioms

Replace the independence axiom by the following two axioms.

Axiom (Uncertainty aversion, Preference for Hedging)

$f \sim g$ implies for all $\alpha \in (0,1)$

$$\alpha f \oplus (1-\alpha) g \succeq g\,.$$

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Axiom (Certainty Independence)

Let $P \in \Delta$ be a lottery and f, g be acts. $f \succeq g$ implies for all $\alpha \in (0, 1)$

$$\alpha f \oplus (1 - \alpha) P \succeq \alpha g \oplus (1 - \alpha) P$$

and vice versa.

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Gilboa--Schmeidler Axioms

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- *For $x, y \in \mathcal{X}$ with $x > y$, we have $\delta_x \succ \delta_y$.*

Gilboa--Schmeidler Theorem

Theorem

Let $\succeq$ *be a preference relation over acts that is (mixture)--continuous and satisfies the axioms of monotonicity, uncertainty aversion, and certainty independence. Then there exists a Bernoulli utility function* $u : \mathcal{X} \to \mathbb{R}$ *and a set of probability measures* $\mathcal{M}$ *on* Ω *such that* $\succeq$ *is represented by the utility function*

$$U(f) = \min_{\mu \in \mathcal{M}} E^{\mu} u(f)$$

with

$$u(f(\omega)) = \sum_{k=1}^{m} u(x_k) f_k(\omega).$$

Gilboa--Schmeidler Theorem: Proof

- the preference relation $\succeq$ induces a preference relation $\succeq_0$ over lotteries; for a lottery $P \in \Delta$, let id^P denote the constant act $id^P(\omega) = P$. Set

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- We have $P \sim \delta_{c(P)}$.

Gilboa--Schmeidler Theorem: Proof

- Now let f be an act, i.e. $f : \Omega \to \Delta$. We construct a non-randomized act $g : \Omega \to \mathcal{X}$ with $f \sim g$. Let $g(\omega) = c(f(\omega))$. By the monotonicity axiom, $g \sim f$.

Gilboa--Schmeidler Theorem: Proof

Lemma

There exists a utility function of the form

$$U(f) = J\left(\sum_x u(x) f(\omega)(x)\right)$$

for some function $J : \mathcal{Z}_0 \to \mathbb{R}$.
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Proof of the Lemma

- wlog $0 \in \mathcal{X}$ and $u(0) = 0$. Let $g \in \mathcal{Z}_0$ and $\lambda \in (0, 1)$. We want to show $J(\lambda g) = \lambda J(g)$. Choose $g_0 \in \mathcal{Z}$ with $u(g_0) = g$.

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- Hence, $U(h) = U(f)$, or $J(u(h)) = J(u(f))$, or $J(\lambda g) = J(u(f))$.

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- Hence,

$$J(u(f)) = U(f) = U(\lambda P + (1 - \lambda)\delta_0) = \lambda u(P) = \lambda J(g)$$

Foundations for the Smooth Model

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- The smooth model corresponds to a double Bayesian approach with an uncertainty-averse twist
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- we have subjective expected utility over "second-order acts" (bets on models); this yields the second-order Bernoulli utility function ϕ

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Intertemporal Choice and BSDEs

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- time can be modeled in discrete steps, $t = 0, 1, 2, \ldots$, or as continuous time, $t \in [0, T]$ for a fixed horizon T, or $t \in [0, \infty[$, infinite horizon
- if there is uncertainty given by a measurable space $(\Omega, \mathcal{F})$, the evolution of information about the state of the world is typically modeled by a filtration $(\mathcal{F}_t)$

Consumption Plans

- World 1: a consumption plan is a sequence $c = (c_t)_{t=0,1,2,\ldots}$ of nonnegative numbers, or a mapping $c : [0, T] \to \mathbb{R}_+$

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- world 2-4: an adapted sequence or stochastic process $c = (c_t)$

Additively Separable Expected Utility

Samuelson, 1937, A Note on the Measurement of Utility, Review of Economic Studies

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IMW

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- analogy to expected utility: independence and stationarity (*Koopmans* 1960, Stationary Ordinal Utility and Impatience, Econometrica)

A B(S)DE for Additively Separable Expected Utility

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  - *Kreps, Porteus*, Econometrica 1978, Temporal Resolution of Uncertainty and Dynamic Choice Theory

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- See *El Karoui, Peng, Quenez*, Backward Stochastic Differential Equations in Finance, Mathematical Finance 1997

Intertemporal Utility under Knightian Uncertainty

- In the light of Gilboa-Schmeidler utility and the usual additively separable intertemporal utility, it is natural to write down the following version of utility under Knightian uncertainty

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- *Epstein, Schneider*, Journal of Economic Theory 2003, *Riedel*, Stochastic Processes and Their Applications, 2004

Intertemporal Utility under Drift Uncertainty

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  solves the BSDE

$$-dV_t = \left(u(c_t) - \rho V_t - \max_{\theta \in \mathcal{K}} \theta \cdot Z_t \right) dt - Z_t dW_t$$

  for some (endogenous) volatility process Z

Intertemporal Utility under Volatility Uncertainty

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- $$V_t = \inf_{P \in \mathcal{P}} \mathbb{E}^P \left[\int_t^T \exp(-\rho(s-t)) u(c_s) ds \,\middle|\, \mathcal{F}_t \right]$$
- but now the set if priors contains mutually singular probability measures, so we need quasi--sure analysis, see *Denis, Hu, Peng*, Potential Analysis 2011