UNIVERSITÄT
BIELEFELD

Faculty of Business Administration
and Economics

IMW

# Knightian Uncertainty in Economics and Finance

22nd Winter School on Mathematical Finance
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Soesterberg


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Uncertainty Sharing

# Outline

1. Model Uncertainty:
   Real World, Decision Models, Identifiability

2. Insuring Model Uncertainty - Efficient Uncertainty Sharing

3. Linear Risk Tolerance Economies

4. Asset Pricing Implications: The Pricing Kernel Puzzle

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## Climate change is making it harder to be a young farmer

"We have less options to work with, so we have to get more creative."

SimonSkafar / Getty Images

grist.org: "With climate change, it's hard to put your finger on single events," says Ben Whalen, ... at Bumbleroot Organic Farm near Portland, Maine. "But we're accepting the reality that the weather is just going to get more extreme and unpredictable. That's the mindset that we're adopting as we start planning for the future of the farm."

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- climate change is Knightian uncertainty

## SCIENCE, UNCERTAINTY AND THE COVID-19 RESPONSE


Boris Johnson Coronavirus Press Conference, by Pippa Fowles / Number 10 (CC by-nc-nd 2.0)

March 16th, 2020 Ian Scoones 5 Comments

# A Virus

- epidemiological models give probability forecast contingent on assumptions on rate of reproduction, mode of transmission, infectious period
- etc. initially unknown
- can be identified ex post

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- Beissner, R., Finance Stoch. 2018 show fundamental incompleteness of the market
- the model is identifiable because the quadratic variation of a Brownian motion is observable

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- the parameters can be identified (ex post) by events in $\Omega$.

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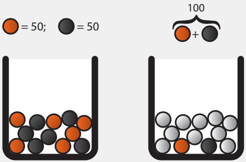
- $(\Omega, \mathcal{F})$ measurable space, states of the world
- $\mathcal{P}$ set of probability measures on $(\Omega, \mathcal{F})$, models
- $\mathcal{P}$ is identifiable, i.e. there exists a measurable mapping $k : \Omega \to \mathcal{P}$ with

$$k = P \qquad P - a.s.$$

for all $P \in \mathcal{P}$

# Examples

## Ellsberg's Thought Experiment 1




## Ellsberg Urn

- An urn contains 100 blue and red balls in unknown proportions, verifiable ex post
- $\omega = (c(olor), n(umberofredballs))$
- $P_n$: the urn contains $n$ red balls
- $k(\omega) = P_n$

## I.I.D. Experiments

- Sequence of independent and identical experiments with outcome $(X_n)$
- $E^{P_m} X_n = m$, mean $m$ unknown
- Let

$$\tilde{m} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Then $k = P_{\tilde{m}}$ identifies the unknown law

# Examples

## Volatility Uncertainty

- $(\Omega, \mathcal{F})$ Wiener space
- Family of probability measures $P^\sigma$ where $\sigma$ is an adapted process taking values in some convex, compact subset of $\mathbb{R}^d$, unknown
- Construction: $P^0$ Wiener measure on the canonical Wiener space with Brownian motion $W$

$$P^\sigma = \text{law}\left(\int_0^\cdot \sigma_u dW_u\right)$$

- the model is identifiable because

$$k(\omega) = (\langle W \rangle_t)_t = \int_0^t \sigma_s^2 ds \qquad P^\sigma - a.s.$$

# Preferences: The Smooth Model

How shall an agent evaluate uncertain consumption plans under uncertainty?

- Subjective Expected Utility: choose a belief $Q \in \mathcal{P}$ and take

$$U(X) = \mathbb{E}^Q \, u(X)$$

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- The smooth (second-order Bayesian approach): take a prior $\mu$ over $\mathcal{P}$, an ambiguity index $\phi$ and set

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- Denti,Pomatto, ECMA 21 show that in identifiable models, the preference parameters can be uniquely identified from observed choices

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- Expected utility over certainty equivalents

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- we analyze how optimal allocations look like
- we do not ask if markets can achieve these allocations

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- Constant absolute risk aversion ...
- the solution is comonotone, i.e. $Y_1$ and $Y_2$ are both monotone functions of $X$

# Linear Risk Sharing

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- utilities are given by:

$$u_i(\xi) = \begin{cases} \frac{(a_i+b\xi)^{1-1/b}}{1/b(1-1/b)} & \text{if} \quad b \neq 0, \quad b \neq 1 \\ -a_i e^{-\xi/a_i} & \text{if} \quad b = 0 \\ \log(a_i + \xi) & \text{if} \quad b = 1 \end{cases} \tag{1}$$

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- in other words: we can make consumption model-contingent

# Efficiency

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- efficient if there is no feasible allocation $(Y_i)_i$ such that $U_i(X_i) \leq U_i(Y_i)$ for every $i$, with at least one strict inequality.
- $P$-conditionally efficient if for $P \in \mathcal{P}$, the allocation $\left(X_i^P\right)_i$ is Pareto efficient under model $P$, that is, there is no feasible allocation $\left(Y_i^P\right)_i$ such that

$$E^P\left(u_i\left(X_i^P\right)\right) \leq E^P\left(u_i\left(Y_i^P\right)\right)$$

for every $i$, with at least one strict inequality.

$\left(X_i^P\right)_{P,i}$ is conditionally efficient if it is $P$-conditionally efficient for all $P \in \mathcal{P}$.

# The Optimization Problem

The following utilitarian welfare maximization problem characterizes efficient allocations for suitable individual weights $\lambda_i \geq 0$.

$$V(\bar{X}) = \max_{(X_i)_i} \sum_i \lambda_i U_i(X_i) \tag{2}$$

$$\text{subject to} \sum_i X_i \leq \bar{X} \tag{3}$$

We call $V$ the utility of the representative agent.

# Conditionally Efficient Allocations

Recall the following results for expected utility economies

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- the allocation is comonotone
- if aggregate endowment is constant, efficient allocations are constant (full insurance)

# Efficient Allocations

- As we allow model-contingent consumption, the problem separates across $P$

$$\max_{(X_i^P)_{P,i}} \sum_i \lambda_i U_i((X_i^P)_P)) = \int_{\mathcal{P}} \max_{(X_i^P)_{P,i}} \sum_i \lambda_i \phi_i \left( E^P u_i \left( X_i^P \right) \right) \mu(dP)$$

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- monotone transformation of welfare functional
- efficient allocations are conditionally efficient allocations!

# Efficient Allocations

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Three immediate consequences are that

- Efficient uncertainty sharing is efficient risk sharing model by model
- if the aggregate endowment $\bar{X}$ is unambiguous, then efficient allocations are also unambiguous.
- with no aggregate uncertainty, efficient allocations are full insurance allocations

# First-Order Conditions

Let $P_0$ be a dominating probability measure for the family $\mathcal{P}$.

- $$\psi(P,\omega) = \lambda_i \phi_i' \left( E^P u_i \left( X_i^P \right) \right) u_i' \left( X_i^P(\omega) \right) \frac{dP}{dP_0}(\omega) \qquad (4)$$

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- The first-order necessary and sufficient condition for a feasible allocation $\left(X_i^P\right)_{P,i}$ to be conditionally efficient

$$\psi^P(\omega) = \eta_i^P u_i'\left(X_i^P(\omega)\right) \tag{5}$$

# Representative Agent

**Theorem**

*Define the utility possibility set*

$$\mathcal{U}(P, \overline{X}) := \{v \in \mathbb{R}^I : \textit{there exists a feasible allocation } (X_i) \textit{ such that } v_i \leq E^P(u_i(X_i))\}.$$

*For weights* $\lambda_i > 0$*, define the function*

$$\Phi(P, \overline{X}) := \max_{(v_i) \in \mathcal{U}(P, \overline{X})} \sum_i \lambda_i \phi_i(v_i).$$

*The representative agent's utility function has the form*

$$V(\overline{X}) = \int_{\mathcal{P}} \Phi(P, \overline{X}) \, \mu(dP) \, .$$

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# Linear Risk Tolerance Economies

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# Linear Risk Tolerance Economies

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- risk tolerance, the inverse of risk aversion, is linear and the parameter $b$ is common,

$$-\frac{u_i'(\xi)}{u_i''(\xi)} = a_i + b\xi, i = 1, \ldots, I$$

$$u_i(\xi) = \begin{cases} \frac{(a_i+b\xi)^{1-1/b}}{1/b(1-1/b)} & \text{if} \quad b \neq 0, \quad b \neq 1 \\ -a_i e^{-\xi/a_i} & \text{if} \quad b = 0 \\ \log(a_i + \xi) & \text{if} \quad b = 1 \end{cases} \tag{6}$$

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- so $\phi_i(t) = -(-t)^{\gamma_i/\alpha_i}$.

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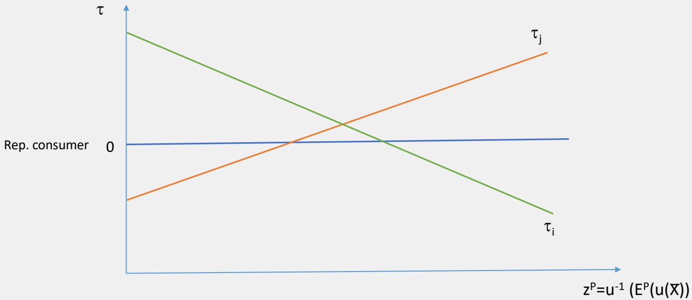
3. *The representative consumer's utility*

$$V(\not{X}) = \int_{\mathcal{P}} \phi(E^P u(\not{X}))\, \mu(dP)$$

*where $\phi$, and $v$ are CARA.*

# Model Insurance Payments in the CARA Case

Less ambiguity-averse consumers should be protected from the model uncertainty (the variability of the certainty equivalents of the aggregate consumption) by making their model-contingent constant term $\tau_i^P$ move in opposite directions to the certainty equivalents




i has a larger coefficient of amb. aversion than the rep. consumer. Receives a higher transfer in less optimistic models
j has a smaller coefficient of amb. aversion than the rep. consumer. Receives a higher transfer in more optimistic models

# General CRRA-like Case

**Theorem**

*Let $((X_i))_i$ be an efficient allocation. Let $\zeta = \sum_i \zeta_i$. Then*

$$X_i^P = \theta_i^P(\not{X} - \zeta) + \zeta_i.$$

# A Nested Negishi--Approach For LRT Economies

- Recall that

$$U_i\left(X_i\right) = \int_{\mathcal{P}} \phi_i\left(E^P u_i(X_i^P)\right)\mu(dP).$$

- Define $v_i = \phi_i \circ u_i$, then

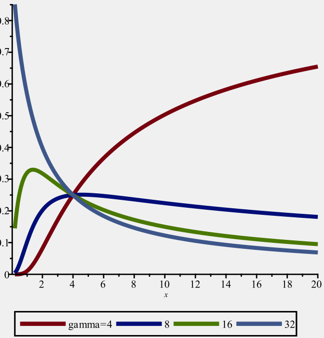
$$U_i(X_i) = \int_{\mathcal{P}} v_i\left(u_i^{-1}(E^P u_i(X_i^P))\right)\mu(dP)$$

- For linear risk tolerance, at the second-order level, one has to solve model by model

$$\Phi(P, \not{X}) := \max_{(v_i):\sum c^i = c} \sum_i \lambda_i v_i(c_i) \tag{8}$$

where $c$ is the certainty equivalent of aggregate endowment under model $P$

# Shares of Aggregate in the Heterogeneous CRRA-Case



# Shares $\theta_i^P$ in the Heterogeneous LRT case

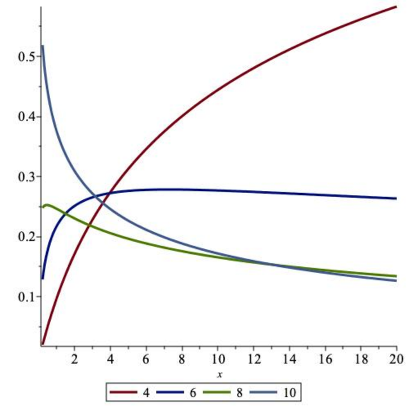



Figure 1: Four consumer economy with heterogeneous ambiguity aversion and common relative risk aversion 2

# Outline

1. Model Uncertainty:
   Real World, Decision Models, Identifiability

2. Insuring Model Uncertainty - Efficient Uncertainty Sharing

3. Linear Risk Tolerance Economies

4. Asset Pricing Implications: The Pricing Kernel Puzzle

# Pricing Kernel Puzzle

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- empirical studies (Jackwerth (2000), Ait-Sahalia and Lo (2000)) suggest that this monotone relation does not hold true

# Pricing Kernel Puzzle

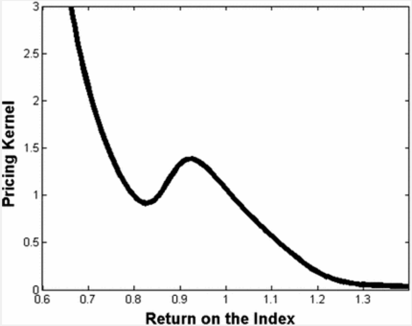


Figure 2: Rosenberg, J. and Engle, R. (2002), Empirical pricing kernels, Journal of Financial Economics

See also Figlewski, Risk-Neutral Densities, Annual Review of Financial Economics, 2018

# Pricing Kernel Puzzle

- representative agent with smooth utility
- class $\mathcal{P}$ dominated by a measure $P_0$
- state price

$$\psi(s) = \int_{\mathcal{P}} \phi' \left( E^P u(\bar{X}) \right) u'(\bar{X}(s)) \frac{dP}{dP_0}(s) \mu(dP)$$

# A Regime-Switching Model

- Let us assume that we have two regimes. A good regime in which the mean is high and the volatility is low, and a bad regime in which the mean is low and the volatility is high.

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- Aggregate endowment is lognormal. We consider a two person economy in which one agent is ambiguity neutral and the other one is very ambiguity averse.

# Graph of the pricing kernel, two regimes

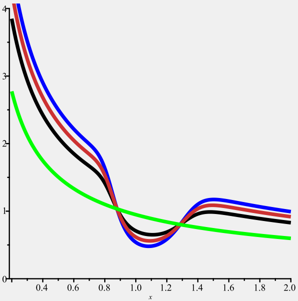


Figure 3: Pricing kernel in three economies: ambiguity-neutral, single agent ambiguity-averse (6 and 12), and mixed. Regime 1: mean 15 %, volatility 1 %, Regime 2: mean -0.15 %, volatility 11 %.

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- In Bayesian Statistics, it is common to work with the precision, the inverse of the variance. For the precision, one commonly assumes a Gamma-distribution because the normal and the Gamma distributions form "conjugate priors"; the posterior of the precision is then also Gamma-distributed.
- in ongoing work with Marco Spengemann, we study mean-variance mixture models (Barndorf--Nielsen) closer calibration to observed kernels, extension to dynamic models

# U--Shaped Pricing Kernels

Sichert, T., The Pricing Kernel is often U--shaped, 2023

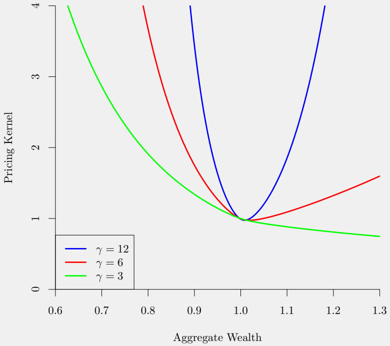


Figure 4: A U--Shaped Pricing kernel in the Mean-Variance-Normal-Mixture Model.

# Conclusion

- We discuss efficient risk and uncertainty sharing under identifiable Knightian Uncertainty
- model-contingent trade is allowed
- efficient allocations are conditionally efficient, thus comonotone
- discussion of sharing rules under linear risk and ambiguity tolerance
- asset pricing implications