UNIVERSITÄT BIELEFELD
Faculty of Business Administration and Economics

IMW

Knightian Uncertainty in Economics and Finance

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Soesterberg

Frank Riedel

Bielefeld University

Lecture 5: Equilibrium

Outline

1. General Equilibrium under Risk

2. Dynamic Equilibrium in Financial Markets

3. Impossibility of Implementation under Knightian Uncertainty

4. Knightian Uncertainty in Prices

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- $\mathbb{E}$ denotes the expectation under $\mathbb{P}$

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- by the Riesz representation theorem, it can be written as

$$\Psi(c) = \mathbb{E}[\psi c]$$

for some state price $\psi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$

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- Properties of u^i:
  - strictly increasing
  - strictly concave (risk aversion)
  - twice continuously differentiable with

$$\lim_{x \downarrow 0} (u^i)'(x) = \infty, \lim_{x \to \infty} (u^i)'(x) = 0$$

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- endowments are sufficiently far away from zero
- can be slightly weakened, yet potential equilibrium prices need to be in the dual space

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1. An allocation $(c^i) \in L^2_+(\Omega, \mathcal{F}, \mathbb{P})^I$ is feasible if $\sum_i (c^i - e^i) \leq 0$

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2. An *Arrow--Debreu equilibrium* for the risk economy consists of a state price $\psi \in L^2_+(\Omega, \mathcal{F}, \mathbb{P})$ and a feasible allocation (c^i) such that c^i maximizes $U^i(c) = \mathbb{E}u^i(c)$ subject to the budget constraint $\mathbb{E}\psi(c - e^i) \leq 0$.

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Remark

It is common to call finite-dimensional equilibria Walras equilibria, and the corresponding equilibria in infinite--dimensional spaces Arrow---Debreu equilibria.

Efficient Allocations

Write $e = \sum_{i=1}^{I} e^i$ for the aggregate endowment.

Definition

A feasible allocation $(c^i) \in L^2_+(\Omega, \mathcal{F}, \mathbb{P})^I$ is called (Pareto) efficient if there is no other feasible allocation (d^i) with $U^i(d^i) > U^i(c^i)$ for all agents i.

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Lemma

An feasible allocation is efficient f and only if it maximizes the weighted sum

$$\sum_{i=1}^{I} \alpha^i U^i(c^i) \tag{1}$$

over feasible allocations for some weights $\alpha^i \geq 0$.
We call the (unique) solution $c_\alpha = (c^i_\alpha)$ of (1) the α-efficient allocation.

Efficient Allocations are Comonotone

Theorem

Let weights $\alpha^i \geq 0$, $i = 1, \ldots, I$ *be given. There exist monotone, continuous functions* $f_\alpha^i : \mathbb{R}_+ \to \mathbb{R}_+$ *with*

$$\sum_{i=1}^{I} f_\alpha^i(x) = x$$

such that

$$c_\alpha^i = f_\alpha^i\cancel{(e)}.$$

Remark

- *efficient allocations are comonotone*
- *and independent of* $\mathbb{P}$

First Welfare Theorem

Theorem

Let $(\psi, (c^i))$ be an Arrow-Debreu equilibrium. Then (c^i) is efficient.

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- *for uniqueness, one uses "gross substitutes property"*

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- idea: individuals face risk, yet risk washes out in the aggregate by the law of large numbers
- the society should be able to remove all individual risk
- can markets achieve this outcome?

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Proof.

First Welfare Theorem. □

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- utility is time--additive expected utility

  $$U^i(c) = \mathbb{E} \int_0^T \exp(-\delta^i t) u^i(c_t) dt$$

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- the previous results on existence, efficiency etc. apply

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  Radner, R. (1972) 'Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets', Econometrica
- main insight: we obtain the same allocation as in an Arrow--Debreu equilibrium if financial markets are dynamically complete

Exogenous versus Endogenous Asset Prices

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- in real asset markets, assets pay off in terms of consumption goods and their prices are endogenous
- consequence: if you want to understand the relation between consumption prices and asset prices, you need to study models with endogenous asset prices (mathematically much more complex)

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$$V_t^{(c,\overline{\theta})} = \overline{\theta}_t \cdot \overline{S}_t$$

- and satisfies the intertemporal budget constraint

$$dV_t^{(c,\overline{\theta})} = \theta_t dS_t + \psi_t(e^i - c_t)dt$$

and $V_0^{(c,\overline{\theta})} = 0$

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- and agents maximize utility subject to their budget constraint: c^i maximizes agent i's utility over all budget--feasible consumption--portfolio strategies.

- Let $((c^i), \psi)$ be an Arrow--Debreu equilibrium.

Equivalence Theorem I: Nominal Assets

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- Can we construct a financial market and a Radner equilibrium with the same (efficient) allocation (c^i)?

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- The basic idea is dynamic completeness and martingale representation.
- If one can find a set of d martingales such that every $(\mathcal{F}_t)$-martingale can be written as a stochastic integral with respect to these martingales, then one can do the construction.

Equivalence Theorem I: Nominal Assets

Theorem (Duffie, Huang 1985)

Suppose that $(\mathcal{F}_t)$ is the completed Brownian filtration of a d--dimensional Brownian motion W.
Suppose that $\psi(e^i - c^i)$ are square--integrable.
Let $S^0_t = 1$ (numéraire) and $S^d = W^d$, $d = 1, \ldots, D$ (Bachelier model of finance)).
Then there exist trading strategies θ^i such that $((c^i, \theta^i), \psi)$ form a Radner equilibrium.

Duffie--Huang Theorem and Martingale Representation

Theorem

Every square integrable random variable $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ can be written as a stochastic integral:

$$X = \mathbb{E}X + \int_0^T \theta_t dW_t$$

for some square--integrable adapted process θ.

Existence of Equilibria in Financial Models: Discrete Time

Endogenous Asset Prices

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- existence of equilibria in continuous time with incomplete real assets open question

Existence of Financial Equilibria with Potentially Complete Markets

based on *Herzberg, Riedel*, J. Math. Econ. 2013 and *Anderson, Raimondo*, Econometrica 2005, see also *Hugonnier et al., Econometrica 2012*, *Kramkov*, Finance and Stochastics 2015

Analytic Markov Economy

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Analytic Markov Economy

- Information generated by a diffusion (X_t)

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- Asset dividends are linearly independent at maturity T
- main point: asset prices are analytic, and hence, the linear independence carries over from terminal payoffs prices $\longrightarrow$ dynamically complete market

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- uses analyticity to show dynamic completeness
- implement Arrow--Debreu as a Radner equilibrium

Time and Information

- the state variable is a diffusion X_t with values in $\mathbb{R}^K$ driven by a K--dimensional Brownian motion W:

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- for Lipschitz continuous functions

$$b : \mathbb{R}^K \to \mathbb{R}^K$$

and

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- The diffusion matrix satisfies the uniform ellipticity condition

$$\|x \cdot a(x)x\| \geq \varepsilon \|x\|^2$$

for some $\varepsilon > 0$. b and σ are analytic functions. b and σ as well as all derivatives up to second order are bounded.

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- agents' endowment $e_t^i = e^i(t, X_t)$ is an analytic function of time and state; Aggregate endowment $e = \sum_i e^i$ is bounded and bounded away from zero.

Nice Bernoulli Utilities

The period utility functions u^i are continuous on $[0, T] \times \mathbb{R}_{++}$ and analytic on $(0, T) \times \mathbb{R}_{++}$. They are differentiably strictly increasing and differentiably strictly concave in consumption on $[0, T] \times \mathbb{R}_{++}$, i.e.

$$\frac{\partial u^i}{\partial c}(t, c) > 0, \frac{\partial^2 u^i}{\partial c^2}(t, c) < 0 .$$

They satisfy the Inada conditions

$$\lim_{c \downarrow 0} \frac{\partial u^i}{\partial c}(t, c) = \infty$$

and

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uniformly in $t \in [0, T]$.

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  - g^k analytic on $(0, T) \times \mathbb{R}^K$.
- Asset 0 is a real zero--coupon bond with maturity T, $A_T = 1$,

Financial Market: Independence Assumption at Maturity

On a nonempty open set $V \subset \mathbb{R}^K$, the dividend of the zero--th asset is strictly positive at maturity,

$$g^0(T, x) > 0, \qquad (x \in V).$$

The functions $h^k : x \mapsto \frac{g^k(T,x)}{g^0(T,x)}$ are continuously differentiable on V for $k = 1, \dots, K$ and the Jacobian matrix

$$Dh(x) = \begin{pmatrix} \frac{\partial h^1(T,x)}{\partial x_1} & \cdots & \frac{\partial h^1(T,x)}{\partial x_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial h^K(T,x)}{\partial x_1} & \cdots & \frac{\partial h^K(T,x)}{\partial x_K} \end{pmatrix}$$

has full rank on V.

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$$G_t^k = S_t^k + \int_{[0,t)} A_s^k \psi_s \nu(ds), \qquad (0 \leq t \leq T)$$

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  the (nominal) gain process for asset k.
- A portfolio process is a predictable process θ with values in $\mathbb{R}^{K+1}$ that is G--integrable
- A portfolio is admissible for agent i if its present value plus the present value of the agent's endowment is nonnegative, or

$$V_t + \mathbb{E}^P \left[\int_{t+}^{T} e^i_s \psi_s \nu(ds) \,\middle|\, \mathcal{F}_t \right] \geq 0\,.$$

Radner Equilibrium

- A portfolio θ finances a consumption plan $c \in \mathcal{X}_+$ for agent i if θ is admissible for agent i and the intertemporal budget constraint is satisfied for the associated value process V:

$$V_t = n^i \cdot S_0 + \int_0^t \theta_u dG_u + \int_0^t \left(e_u^i - c_u\right) \psi_u \nu(du) .$$

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- A Radner equilibrium consists of asset prices S, a consumption price ψ, portfolios θ^i and consumption plans $c^i \in \mathcal{X}_+$ for each agent i such that θ^i is admissible for agent i and finances c^i, c^i maximizes agent i's utility over all such i--feasible portfolio/consumption pairs, and markets clear, i.e. $\sum_{i=1}^I c^i = e$ and $\sum_{i=1}^I \theta^i = N$.

Main Theorem

Theorem

There exists a Radner equilibrium $\left(S, \psi, (\theta^i, c^i)_{i=1,\dots,I}\right)$ *with a dynamically complete market* (S, A, ψ)*; the prices and dividends are linked by the present value relation*

$$S_t^k = \mathbb{E}^P\left[\int_t^T A_s^k \psi_s \,\nu(ds) \,\middle|\, \mathcal{F}_t\right]. \tag{2}$$

Step 1: Arrow--Debreu Equilibrium

Assumption

For each agent, the marginal utility of his endowment belongs to the price space Ψ:

$$\frac{\partial}{\partial c} u^i(t, \varepsilon_t^i) \in \Psi \,.$$

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Theorem

There exists an Arrow--Debreu equilibrium $\left(\psi, (c^i)_{i=1,\dots,I}\right)$ *such that*

$$\psi_t = \psi(t, X_t), c^i_t = c^i(t, X_t)$$

for continuous functions ψ, c^i *that are analytic on* $(0, T) \times \mathbb{R}^K$.

Step 2: Analytic Prices and Completeness

Lemma

The Markov process X has a transition density $P[X_{s+t} \in dy | X_s = x] = p(t, x, y)\, dy$ for a continuous function

$$p : (0, T] \times \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}_+$$

that is analytic on $(0, T) \times \mathbb{R}^K \times \mathbb{R}^K$. Moreover, the transition density p is bounded on $(\eta, T] \times \mathbb{R}^K \times \mathbb{R}^K$ for all $\eta > 0$.

Theorem

Define $S^k_t = \mathbb{E}^P\left[\int_t^T A^k_s \psi_s \, \nu(ds) \,\middle|\, \mathcal{F}_t\right]$. *There exist continuous functions* $s : [0, T] \times \mathbb{R}^K \to \mathbb{R}_+$ *that are analytic on* $(0, T) \times \mathbb{R}^K$ *and*

$$S_t = s(t, X_t) \,.$$

The first derivatives with respect to x, $\frac{\partial s}{\partial x_l}$ *are continuous on* $[0, T] \times \mathbb{R}^K$ *and we have*

$$\lim_{t \uparrow T} \frac{\partial s}{\partial x_l}(t, x) = \frac{\partial s}{\partial x_l}(T, x) = \frac{\partial g}{\partial x_l}(T, x)$$

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- our paper led to a subsequent analysis of this problem in *Kramkov*, Finance and Stochastics 2015

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- by analyticity, the volatility matrix cannot vanish

Outline

1. General Equilibrium under Risk

2. Dynamic Equilibrium in Financial Markets

3. Impossibility of Implementation under Knightian Uncertainty

4. Knightian Uncertainty in Prices

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based on *Beissner, Riedel*, Finance and Stochastics 2018

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- Important: the measures are not dominated by one common measure
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Quasi--sure Analysis necessary: An event is negligible for agents if it is null simultaneously under all P^σ

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- Ambiguity washes out in the aggregate - possibility for insurance

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Dynamic (Radner) Equilibrium

Agents trade dynamically in a financial market with asset prices $S = \left(S_t^d\right)$, , $d = 0, \dots, D$, $t \geq 0$; the spot price of consumption at time T is ψ.

1. agents finance net demand $c^i - e^i$, i.e. there are S-integrable portfolio processes θ^i such that

$$\psi(c^i - e^i) = \int_0^T \theta^i dS^d$$

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Agents trade dynamically in a financial market with asset prices $S = \left(S_t^d\right),\quad, d = 0, \ldots, D, t \geq 0$; the spot price of consumption at time T is ψ.

1. agents finance net demand $c^i - e^i$, i.e. there are S-integrable portfolio processes θ^i such that

$$\psi(c^i - e^i) = \int_0^T \theta^i dS^d$$

2. asset markets clear : $\sum_{i=1}^I \theta^i = 0$

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2. asset markets clear : $\sum_{i=1}^I \theta^i = 0$
3. c^i maximizes utility U^i over all c that can be financed with trading dynamically S

Duffie--Huang Theorem (Repetition)

Let $((c^i), \Psi)$ be an Arrow--Debreu equilibrium.

Ψ can be identified with a positive, suitably bounded random variable ψ

Can we find a Radner equilibrium with the same (efficient) allocation?

Under risk, in diffusion settings, the answer is yes!

- If the filtration has a martingale generator $M^d, d = 1, \dots, D$, then we can set $S_t^0 = 1$ (numéraire) and $S^d = M^d, d = 1, \dots, D$

Our claim: "usually" this result breaks down under Knightian (volatility) uncertainty.

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- In Brownian settings, one can thus take the Brownian motion itself

  Bachelier model of finance

Our claim: "usually" this result breaks down under Knightian (volatility) uncertainty.

Theorem

Every efficient allocation (c^i) is ambiguity--free.

It satisfies the probability--free characterization of identical marginal rates of substitution among agents: for some weights $\alpha^i > 0$ we have

$$\alpha^i u^{i\prime}(c^i) = \alpha^j u^{j\prime}(c^j)$$

As a consequence, $c^i = f^i(e)$ for some monotone, continuous function f^i.

Static Equilibrium

We denote $\mathcal{E}^P$ the expected utility economy with homogenous priors P.

Theorem

Let $((c^i), \psi)$ *be an AD equilibrium in the expected utility economy* $\mathcal{E}^P$*. Then* $((c^i), \Psi)$ *with*

$$\Psi(X) = E^P(X\psi)$$

is an AD equilibrium in the economy $\mathcal{E}$*.*

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Remark

The market chooses P and state-price ψ.
Ψ is not unique in general.
Indeterminacy

Implementation under no Aggregate Uncertainty

$e = 1$, no aggregate uncertainty

We use two financial assets, a riskless one with price 1, and the G--Brownian motion W as the "uncertain" asset

Under risk, these assets suffice to span a complete market

Theorem

Implementation of an Arrow--Debreu equilibrium $((c^i), \Psi)$ is possible if and only if the net trade values $(c^i - e^i)\psi$ are mean--ambiguity--free.

In particular, if some individuals face proper Knightian uncertainty in the mean, implementation will not be possible.

Intuition: It is possible to hedge perfectly under each P^σ, but impossible to do so under all P^σ simultaneously

Proof

If implementation is possible, we can write

$$(c^i - e^i)\psi = \int \theta^i dW$$

Stochastic integrals are symmetric martingales
mean--ambiguity--free

Proof, Relation to Martingale Representation

Martingale Representation Theorem of *Soner, Touzi, Zhang, 2011*, see also *Mu, Ji, Peng, Song 2014*

One can decompose the net consumption vale as follows:

$$(c^i - e^i)\psi = \int \theta^i dW - K^i$$

for some increasing martingale K^i

Consequence: market clearing implies that all $K^i = 0$ quasi--surely

$K^i = 0$ is equivalent to no ambiguity in the mean

“Usually” Impementation Fails

Prevalence (*Hunt, Sauer, Yorke*, *Anderson, Zame*): a measure--theoretic notion of “large sets” for infinite--dimensional spaces

$A \subset X$ is (finitely) prevalent if there is a finite--dimensional subspace V of X such that for all $x \in X$ the complement of A has Lebesgue measure zero in $x + V$.

Theorem

The set of economies for which no Arrow--Debreu equilibrium can be implemented is (finitely) prevalent.

The Message

Financial Markets can efficiently deal with risk, not with uncertainty

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- Asset markets work well when we are faced with risk and diffusions

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- Asset markets work well when we are faced with risk and diffusions
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- asset markets are inefficient when there are jumps (known)
- new: when there is Knightian uncertainty about volatility, even the "nice" asset markets can be inefficient
- open question: how do inefficient market equilibria look like?

Outline

1. General Equilibrium under Risk

2. Dynamic Equilibrium in Financial Markets

3. Impossibility of Implementation under Knightian Uncertainty

4. Knightian Uncertainty in Prices

Overview

Beissner, Riedel, Equilibria under Knightian price uncertainty, Econometrica 2019

- We consider markets

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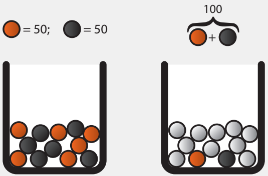
- Discontinuity of Equilibrium Correspondence

Imprecise Probabilistic Information

- *Ellsberg (1961)* experiments: agents choose between a risky urn and an uncertain urn
- risky urn: the composition is exactly known, e.g. 50 red, 50 black balls
- uncertain urn: the composition is known only up to some bounds, e.g. 100 balls, at least 30, at most 80 red, rest black
- probability for drawing a red ball from risky urn is 0.5
- probability for drawing a red ball from uncertain urn is in the intervall $[0.3, 0.8]$



- Knightian uncertainty modeled by a set of probabilities $\mathcal{P}$

Expectations and Forward Prices

Ω is a finite set of states of nature, $\mathbb{X} = \mathbb{R}^\Omega$ commodity space of contingent plans

Definition

We call $\mathbb{E} : \mathbb{X} \to \mathbb{R}$ a *(Knightian) expectation* if it satisfies the following properties:

1. $\mathbb{E}$ preserves constants: $\mathbb{E}c = c$ for all $c \in \mathbb{R}$,
2. $\mathbb{E}$ is monotone: $\mathbb{E}X \leq \mathbb{E}Y$ for all $X, Y \in \mathbb{X}$ with $X \leq Y$,
3. $\mathbb{E}$ is sub-additive: $\mathbb{E}[X + Y] \leq \mathbb{E}X + \mathbb{E}Y$ for all $X, Y \in \mathbb{X}$,
4. $\mathbb{E}$ is homogeneous: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}X$ for $\lambda > 0$ and $X \in \mathbb{X}$,
5. $\mathbb{E}$ is relevant: $\mathbb{E}[-X] < 0$ for all $X \in \mathbb{X}_+ \setminus \{0\}$.

Expectations and Forward Prices

Lemma

We have $\mathbb{E}X = \max_{P \in \mathbb{P}} E^P X$ *for a convex and compact set* $\mathbb{P}$ *of probability measures on* Ω *with* $\mathbb{P} \subset \operatorname{int} \Delta$*.*

The set $\mathbb{P}$ captures the imprecision of the available information about the model.

The Economy with Sublinear Forward Price

Definition

An Knightian economy (on Ω) is a triple $\mathcal{E} = \left(I, \left(e^i, U^i\right)_{i \in \mathbb{I}}, \mathbb{E}\right)$ where

- *$I \geq 1$ denotes the number of agents,*
- *$e^i \in \mathbb{X}_+$ is the endowment of agent i,*
- *$U^i : \mathbb{X}_+ \to \mathbb{R}$ agent i's utility function,*
- *and $\mathbb{E}$ is a Knightian expectation.*

For a state price ψ, we call $\Psi(X) = \mathbb{E}(\psi X)$ the forward price of a plan $X \in \mathbb{X}$

Interpretation of the Model

- agents trade contingent plans on a forward market at time 0 as in Debreu's original model of trade under uncertainty

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    - cautious market maker who has imprecise probabilistic information about the states of the world, described by $\mathbb{P}$. The market maker then computes the maximal expected present value over this set of models to stay on the safe side. ("stress testing")
    - agents in the Knightian economy $\mathcal{E}^{\mathbb{P}}$ consider only *robustly affordable* plans

Sublinear Prices in Related Economies

- incomplete financial markets (“superhedging”) (*Araujo, Châteauneuf, Faro, Econ.Theory, 2012*)
- in insurance markets (“model risk”) (*Castagnoli, Maccheroni, Marinacci, Ins.Math.Econ., 2002*)
- in markets with transaction costs (*Jouini, Kallal, J. Math. Econ., 1995*)

The papers cited above discuss properties related to sublinear functionals, but do not study equilibrium. Our paper completes this gap in the literature.

Assumptions on Preferences I

Assumption

Each agent's endowment e_i *is strictly positive. Each utility function* $U_i : \mathbb{X}_+ \to \mathbb{R}$ *is*

- *continuous,*
- *monotone, i.e. if* $x \geq y$ *then* $U_i(x) \geq U_i(y)$*,*
- *semi--strictly quasi--concave, i.e. for all* $x, y \in \mathbb{X}_+$ *with* $U(x) > U(y)$ *we have for all* $\lambda \in (0,1)$

$$U(\lambda x + (1-\lambda)y) > U(y)\,.$$

- *and non--satiated, i.e. for* $y \in \mathbb{X}_+$ *there exists* $x \in \mathbb{X}_+$ *with* $U_i(x) > U_i(y)$*.*

Multiple--prior expected utilities

- "Rational expectations for pessimistic agents"
- $\mathbb{P}$ common knowledge and (*Gilboa--Schmeidler*)-agents

$$U^i(c) = \min_{P \in \mathbb{P}} E^P u^i(c)$$

  for $u^i : \mathbb{R}_+ \to \mathbb{R}$ continuous, strictly increasing, strictly concave
- subjective reactions to imprecise probabilistic information (*Gajdos, Hayashi, Tallon, Vergnaud*): for $\phi^i(\mathbb{P}) \subset \mathbb{P}$

$$U^i(c) = \min_{P \in \phi_i(\mathbb{P})} E^P u^i(c)$$

Examples

Smooth Ambiguity model

- (*Klibanoff, Marinacci, Mukerji, ECMA 2005*)
- second--order prior μ^i over $\mathbb{P}$
- continuous, monotone, strictly concave ambiguity index $\phi^i : \mathbb{R} \to \mathbb{R}$
- $$U^i(c) = \int_{\mathbb{P}} \phi^i \left(E^P u^i(c) \right) \mu^i(\mathrm{d}P)$$

Examples

Anchored Preferences, Variational Preferences

- *Dana, Riedel, JET 2013* study preferences anchored at endowments:

$$U^i(c) = \min_{P \in \mathbb{P}} E^P[u^i(c) - u(e^i)]$$

- special case of variational preferences (*Maccheroni, Marinacci, Rustichini, ECMA, 2006*) of the form

$$U_i(c) = \inf_{Q \in \mathbb{P}} E^Q u_i(c) + \alpha(Q)$$

for a suitable penalty function $\alpha : \Delta \to \mathbb{R}_+ \cup \{\infty\}$.

Common Features of Examples

Rigotti, Shannon, Strzalecki, ECMA 2008 introduce subjective beliefs

$$\pi_i(c) \;=\; \Big\{ Q \in \Delta : E^Q[y] \geq E^Q[c] \text{ for all } y \text{ with } U_i(y) \geq U_i(c) \Big\}.$$

- Each U_i is translation invariant at certainty: For all $h \in \mathbb{X}$ and all constant bundles $c, c' > 0$, if $U_i(c + \lambda h) \geq U_i(c)$ for some $\lambda > 0$, then there exists $\lambda' > 0$ such that $U_i(c' + \lambda' h) \geq U_i(c')$.
- Consequence: beliefs at certainty independent of level c; we denote the subjective beliefs of agent i at any constant bundle $c > 0$ by π_i.
- Preferences are consistent with the set of priors $\mathbb{P}$

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- *Preferences are consistent with the set of priors $\mathbb{P}$, i.e. we have $\pi_i \subset \mathbb{P}$, and agents share some common subjective belief at certainty: $\bigcap_{i=1}^{I} \pi_i \neq \emptyset$.*

Equilibrium

Definition

We call a pair (ψ, c) *of a state--price* $\psi : \Omega \to \mathbb{R}_+$ *and an allocation* $c = (c^i)_{i=1,\dots,I} \in \mathbb{X}_+^I$ *a Knight--Walras equilibrium if*

1. *the allocation* c *is feasible, i.e.* $\sum_{i=1}^I (c^i - e^i) \leq 0$.
2. *for each agent* i, c^i *is optimal in the* Knight-Walras *budget set*

$$\mathbb{B}(\psi, e^i) = \left\{ c \in \mathbb{X}_+ : \mathbb{E}\psi(c - e^i) \leq 0 \right\} ,$$

i.e. if $U^i(d) > U^i(c^i)$ *then* $d \notin \mathbb{B}(\psi, e^i)$.

Remarks

1. For $\mathbb{P} = \{P_0\}$, back to Arrow--Debreu equilibrium; equilibrium allocations are efficient.
2. For $\mathbb{P} = \Delta$ and ψ strictly positive, the budget sets consist of all plans c with $c \leq e^i$ in all states.
   - There is no trade in equilibrium.
   - Equilibrium allocations are inefficient, in general, and equilibrium prices are indeterminate.

Existence

Theorem

Knight--Walras Equilibria exist under our first Assumption.

Game--theoretic proof with a “Walrasian” and a “Knightian auctioneer”

Proof

The social game

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- Knightian Auctioneer maximizes value of excess demand over priors $P \in \mathbb{P}$: $\arg\max_{P \in \mathbb{P}} E^P \left[\psi \sum_{i \in \mathbb{I}} \left(x_i - e_i \right) \right]$

Arbitrage

Do Sublinear prices induce arbitrage opportunities?

- *Aliprantis, Florenzano, Tourky, 2005*: an arbitrage is a consumption plan $c \in \mathbb{X}_+ \setminus \{0\}$ with $\Psi(c) = 0$.
- splitting a bundle x into two bundles y and z and selling or buying them separately

No Arbitrage in Equilibrium

Theorem

Let $(\psi, (\widehat{c}_i)_{i\in\mathbb{I}})$ be a Knight--Walras equilibrium. The following absence of arbitrage conditions hold true.

1. *We have $\Psi(c) > 0$ for all $c \in \mathbb{X}_+ \setminus \{0\}$.*
2. *Let $x = y + z$ for $x, y, z \in \mathbb{X}$. Buying (selling) x and selling (buying) y and z separately yields no profits. We have*

$$\Psi(x) \geq -\Big(\Psi(-y)+\Psi(-z)\Big) \quad \textit{and} \quad \Psi(y)+\Psi(z) \geq -\Psi(-x).$$

(Non--) Equivalence to Arrow--Debreu Equilibrium

Under what conditions are Knight--Walras equilibria the same as Arrow--Debreu equilibria for some fixed P?

Definition

Fix a convex, compact, nonempty set of priors $\mathbb{P}$. We call a plan $\xi \in \mathbb{X}(\mathbb{P})$--ambiguity free in mean if ξ has the same expectation for all $Q \in \mathbb{P}$, i.e. there is a constant $k \in \mathbb{R}$ with $E^Q\xi = k$ for all $Q \in \mathbb{P}$.

(Non--) Equivalence to Arrow--Debreu Equilibrium

Theorem

Fix a prior $P \in \mathbb{P}$. Let $(\psi, (c^i))$ be an Arrow--Debreu equilibrium for the (linear) economy $\mathcal{E}^{\{P\}}$.
Then $(\psi, (c^i))$ is a Knight--Walras equilibrium for $\mathcal{E}^{\mathbb{P}}$ if and only if the value of net demands $\xi^i = \psi(c^i - e^i)$ are $\mathbb{P}$--ambiguity free in the mean for all agents i.

Generic Non--Equivalence with No Aggregate Uncertainty

- The benchmark case of no aggregate uncertainty is archetypical to discuss economic institutions (*Mirrlees, RES 1971*)

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- *Rigotti, Shannon, Strzalecki, ECMA 2008* generalize to translation-invariance at certainty
- **Efficient allocations are full insurance allocations.**

Generic Non--Equivalence with No Aggregate Uncertainty

Theorem

Assume that $\mathbb{E}$ is not linear. Generically in endowments, Arrow--Debreu equilibria of $\mathcal{E}^{\{P\}}$ for some $P \in \mathbb{P}$ are not Knight--Walras equilibria of $\mathcal{E}^{\mathbb{P}}$.

Generic Non--Equivalence with No Aggregate Uncertainty

More precisely: let $M = \left\{ (e_i)_{i=1,\ldots,I} \in \mathbb{X}_{++}^{I} : \sum e_i = 1 \right\}$ be the set of economies without aggregate uncertainty normalized to 1. Let N be the subset of elements (e_i) of M for which there exists $P \in \mathbb{P}$ and an Arrow--Debreu equilibrium $(\psi, (c_i))$ of the economy $\mathcal{E}^{\{P\}}$ which is also a Knight--Walras equilibrium of the economy $\mathcal{E}^{\mathbb{P}}$. N is a Lebesgue null subset of the $(I-1) \cdot \#\Omega$--dimensional manifold M.

Generic Non--Equivalence with No Aggregate Uncertainty

Crucial step in the proof

Lemma

The set of plans $\xi \in \mathbb{X}$ which are $\mathbb{P}$--ambiguity--free in mean forms a subspace of $\mathbb{X}$. We denote this subspace by $\mathbb{L}$ or $\mathbb{L}^{\mathbb{P}}$. If $\#\mathbb{P} > 1$, $\mathbb{L}$ has a strictly smaller dimension than $\mathbb{X}$ and satisfies $1_\Omega \in \mathbb{L}$.

Generic Inefficiency

Theorem

Next to our Assumptions, assume that the utility functions U_i are differentiable at certainty. Under no aggregate uncertainty, generically in endowments, Knight--Walras equilibrium allocations of $\mathcal{E}^{\mathbb{P}}$ are inefficient.

Uncertainty--Neutral Efficiency

Definition

Let $\mathcal{E} = \left(I, (e^i, U^i)_{i\in\mathbb{I}}, \mathbb{E}\right)$ *be a Knightian economy. Let* $c = (c^i)_{i\in\mathbb{I}}$ *be a feasible allocation. Let* ψ *be a state--price density. We call the allocation* c uncertainty neutral efficient *(given* ψ *and* $\mathbb{E}$*) if there is no other feasible allocation* $d = (d^i)_{i=1,\dots,I}$ *with*

$$\eta^i = \psi\left(d^i - e^i\right) \in \mathbb{L}$$

and $U^i(d^i) > U^i(c^i)$ *for all* $i \in \mathbb{I}$*.*

Uncertainty--Neutral Efficiency

Theorem

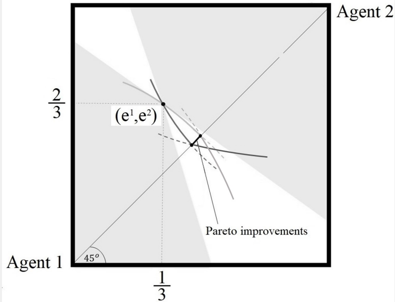
Let (ψ, c) be a Knight--Walras equilibrium of the Knightian economy $\mathcal{E} = \left(I, \left(e^i, U^i\right)_{i \in \mathbb{I}}, \mathbb{E}\right)$. Then c is uncertainty neutral efficient (given ψ and $\mathbb{E}$).

Sensitivity of Arrow--Debreu Equilibria with respect to Knightian Price Uncertainty

Example

$\Omega = \{1, 2\}$. $\mathbb{P}_\varepsilon = \{p \in \Delta : p_1 \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]\}$ *no aggregate ambiguity*

two agents $I = 2$ *with multiple--prior utilities and uncertain endowments* $e^1 = (1/3, 2/3)$ *and* $e^2 = (2/3, 1/3)$. *There is no trade in Knight--Walras equilibrium for every* $\varepsilon > 0$.



Sensitivity of Arrow--Debreu Equilibria with respect to Knightian Price Uncertainty

Equilibrium correspondence

$$\mathcal{KW}(\mathbb{P}) = \left\{ (\psi, c) \in \mathbb{X}_+^{I+1} : (\psi, c) \text{ is a KW--equilibrium in } \mathcal{E}^{\mathbb{P}} \right\}.$$

Theorem

Let $\mathbb{P} : [0,1) \to\to \Delta$ be a continuous correspondence with $\mathbb{P}(0) = \{P_0\}$ for some $P_0 \in \text{int}(\Delta)$. For $0 < \varepsilon < 1$, assume $P_0 \in \text{int}\,\mathbb{P}(\varepsilon)$ and (e^i) not $\mathbb{P}(\varepsilon)$--ambiguity--free.
The Knight--Walras equilibrium correspondence

$$\varepsilon \mapsto \mathcal{KW}(\mathbb{P}(\varepsilon), e)$$

is discontinuous in zero.

No Trade with Sufficiently Large Ambiguity

Theorem

If ambiguity is sufficiently large, every Knight--Walras--equilibrium is a no--trade equilibrium: There is a $\mathbb{P}' \in \mathbb{K}(\Delta)$ such that for every $\mathbb{P}'' \in \mathbb{K}(\Delta)$ with $\mathbb{P}'' \supset \mathbb{P}'$, initial endowment is the unique equilibrium allocation,

Conclusion

- Equilibrium model for Knightian uncertainty about state prices
- Under no aggregate uncertainty, generic inefficiencies
- a small amount of Knightian uncertainty can lead to no trade
- no trade also under "large" uncertainty