Convergence of the Markovian iteration for coupled FBSDEs via a differentiation approach

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Outline

1. Background
2. The Markovian iteration for coupled FBSDEs
3. Generalize the Markovian iteration to FBSDEs with Y and Z coupling
4. Numerical Examples

Background

Motivations

- FBSDEs have various applications in finance
    - coupled FBSDE: stochastic control problem depends on the ways of coupling
    - decoupled FBSDE: option pricing, hedging, etc.
    - non-standard FBSDE: mean-field game/control, optimal stopping, etc.
- Difficult to find an analytical solution in general
    - numerical method is needed

Forward-backward stochastic differential equation

An **"almost" fully coupled FBSDE** is given by

$$\begin{cases} X_t = x_0 + \int_0^t b(s, X_s, Y_s, Z_s)\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s)\mathrm{d}W_s, \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \end{cases} \tag{1}$$

where the solution is a triple of $(\mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q\times m})$ valued, $\mathcal{F}_t$ adapted stochastic processes $\{(X_t, Y_t, Z_t)\}_{0\leq t\leq T}$ solve above almost surely and satisfy natural integrability conditions.

Other classes of FBSDEs:

- Decoupled FBSDE: $b \equiv b(t, X_t)$ and $\sigma \equiv \sigma(t, X_t)$
- Coupled FBSDE through Y: $b \equiv b(t, X_t, Y_t)$ and $\sigma \equiv \sigma(t, X_t, Y_t)$
- Fully coupled FBSDE: $b \equiv b(t, X_t, Y_t, Z_t)$ and $\sigma \equiv \sigma(t, X_t, Y_t, Z_t)$

Connection with PDEs: Feynman-Kac formula

The FBSDE (1) is associated with the following system of quasi-linear parabolic PDEs,

$$\begin{cases} \partial_t u^i + \frac{1}{2}\partial_{xx} u^i : \sigma\sigma^\top(t,x,u) + \partial_x u^i b(t,x,u,\partial_x u\sigma(t,x,u)) \\ \qquad + f^i(t,x,u,\partial_x u\sigma(t,x,u)) = 0, \quad \forall i = 1,\cdots,q, \\ u(T,x) = g(x). \end{cases}$$

Under suitable regularity assumptions on b, σ, f and g, one can show that the solutions are connected by non-linear Feynman-Kac formula

$$Y_t = u(t,X_t), \quad Z_t = \partial_x u(t,X_t)\sigma(t,X_t,u(t,X_t)) \coloneqq v(t,X_t)$$

$\Rightarrow$ The functions u and v are called decoupling fields in the context of FBSDE.

The Backward Euler Scheme for BSDE

Consider a decoupled FBSDE case, i.e. $b(t, X_t)$ and $\sigma(t, X_t)$.

The Backward Euler scheme is given by

- Step 1: Sampling X_i^π for all i in a forward fashion;
- Step 2: For $i = N$, set $(Y_N^\pi, Z_N^\pi) = (g(X_N^\pi), 0)$
- Step 3: For $i = N-1, N-2, \ldots, 0$, compute

$$\begin{cases} Z_i^\pi = E_{t_i}\left(h^{-1} Y_{i+1}^\pi \Delta W_i\right) \\ Y_i^\pi = E_{t_i}\left(Y_{i+1}^\pi + hf\left(X_i^\pi, Y_{i+1}^\pi, Z_i^\pi\right)\right) \end{cases}$$

- To compute the conditional expectations, use L^2 projections properties and set up optimization problems,

$$Z_i^\pi = \arg\min_z E[(h^{-1} Y_{i+1}^\pi \Delta W_i - z)^2]$$

$$Y_i^\pi = \arg\min_y E[(Y_{i+1}^\pi + hf(X_i^\pi, Y_{i+1}^\pi, Z_i^\pi) - y)^2]$$

The Backward Euler Scheme

What if the forward SDE of X couples with Y and Z?

The Markovian iteration for coupled FBSDEs

Markovian iteration for FBSDEs with Y coupling

Let m be the number of iterations. Set $u_i^{\pi,0}(\cdot) = 0$, $v_i^{\pi,0}(\cdot) = 0$, and $X_0^{\pi,m} \triangleq x_0$ for all m. Bender et al. 2008 Introduced the following scheme,

$$\begin{cases} X_{i+1}^{\pi,m} = X_i^{\pi,m} + b(t_i, X_i^{\pi,m}, u_i^{\pi,m-1}(X_i^{\pi,m}))h \\ \qquad\qquad + \sigma(t_i, X_i^{\pi,m}, u_i^{\pi,m-1}(X_i^{\pi,m}))\Delta W_i, \\ Y_N^{\pi,m} = g(X_N^{\pi,m}), \\ Z_i^{\pi,m} = h^{-1} E_{t_i}(Y_{i+1}^{\pi,m} \Delta W_i), \quad v_i^{\pi,m}(X_i^{\pi,m}) = Z_i^{\pi,m} \\ Y_i^{\pi,m} = E_{t_i}(Y_{i+1}^{\pi,m} + f(t_i, X_i^{\pi,m}, Y_{i+1}^{\pi,m}, Z_i^{\pi,m})h), \quad u_i^{\pi,m}(X_i^{\pi,m}) = Y_i^{\pi,m} \end{cases} \tag{2}$$

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- We have the dependency:

$$Z_i^{\pi,m} \Rightarrow (X_i^{\pi,m}, Y_{i+1}^{\pi,m}), \quad Y_i^{\pi,m} \Rightarrow (X_i^{\pi,m}, Y_{i+1}^{\pi,m}, Z_i^{\pi,m}) \Rightarrow (X_i^{\pi,m}, Y_{i+1}^{\pi,m})$$

and due to the coupling in the SDE of X,

$$Y_i^{\pi,m} \Rightarrow (X_i^{\pi,m}, Y_{i+1}^{\pi,m}) \Rightarrow (X_{i-1}^{\pi,m}, Y_{i-1}^{\pi,m-1}, Y_{i+1}^{\pi,m})$$

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- **Good:** $Y^{\pi,m} \Rightarrow Y^{\pi,m-1} \Rightarrow \ldots \Rightarrow Y^{\pi,0}$. Need to control the Lipschitz constant of $u_i^{\pi,m}$ over time steps i and iteration steps m

Existing convergence results

With the Lipschitz and Holder continuity assumptions, weak and monotonicity conditions, Bender et al. 2008 show that, for sufficiently small h

Theorem (Bender et al. 2008, Thm 5.1 and Thm 6.3)

There is an unique solution u^{π} to the discretized scheme (2) and there are constants $C > 0$ and $0 < c < 1$ such that

$$\max_{0 \le i \le N} \left| u_i^{\pi,m}(x) - u_i^{\pi}(x) \right|^2 \le C \left(|x|^2 + m \right) c^m$$

and, the associated PDE admits a viscosity solution $u(t, x)$ and we have the estimate

$$\max_{0 \le i \le N} \left| u_i^{\pi}(x) - u(t_i, x) \right|^2 \le C \left(1 + |x|^2 \right) h$$

Existing convergence results

Consequently, an error estimate can be derived,

Theorem (Bender et al. 2008, Thm 6.5)

The continuous FBSDE with Y coupling has a unique solution (X, Y, Z) and there are constants $C > 0$ and $0 < c < 1$ such that, for sufficiently small h,

$$\sup_{1\leq i\leq N} E\left\{\sup_{t\in[t_{i-1},t_i]}(|X_t - X_{i-1}^{\pi,m}|^2 + |Y_t - Y_{i-1}^{\pi,m}|^2)\right\} + \sum_{i=1}^{N} E\left\{\int_{t_{i-1}}^{t_i} |Z_t - Z_{i-1}^{\pi,m}|^2 dt\right\}$$

$$\leq C(1 + |x_0|^2)(mc^m + h)$$

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$$\le C(1+|x_0|^2)(mc^m + h)$$

What if the forward SDE also couples with Z?

A similar analysis for Z which is parallel with the Y coupled case?

Challenge: Intertwining of Y and Z

If in addition X also couples with Z,

$$\begin{cases} X_{i+1}^{\pi,m} = X_i^{\pi,m} + b(t_i, X_i^{\pi,m}, u_i^{\pi,m-1}(X_i^{\pi,m}), v_i^{\pi,m-1}(X_i^{\pi,m}))h \\ \qquad\qquad + \sigma(t_i, X_i^{\pi,m}, u_i^{\pi,m-1}(X_i^{\pi,m}))\Delta W_i, \\ Y_N^{\pi,m} = g(X_N^{\pi,m}), \\ Z_i^{\pi,m} = h^{-1}E_{t_i}(Y_{i+1}^{\pi,m}\Delta W_i), \quad v_i^{\pi,m}(X_i^{\pi,m}) = Z_i^{\pi,m} \\ Y_i^{\pi,m} = E_{t_i}(Y_{i+1}^{\pi,m} + f(t_i, X_i^{\pi,m}, Y_{i+1}^{\pi,m}, Z_i^{\pi,m})h), \quad u_i^{\pi,m}(X_i^{\pi,m}) = Y_i^{\pi,m} \end{cases}$$

then through $X_i^{\pi,m}$ we have

$$Y_i^{\pi,m} \Rightarrow (X_i^{\pi,m}, Y_{i+1}^{\pi,m}) \Rightarrow (X_{i-1}^{\pi,m}, Y_{i-1}^{\pi,m-1}, Z_{i-1}^{\pi,m-1}, Y_{i+1}^{\pi,m},)$$
$$Z_i^{\pi,m} \Rightarrow (X_i^{\pi,m}, Y_{i+1}^{\pi,m}) \Rightarrow (X_{i-1}^{\pi,m}, Y_{i-1}^{\pi,m-1}, Z_{i-1}^{\pi,m-1}, Y_{i+1}^{\pi,m})$$

- $Y_i^{\pi,m}$ and $Z_i^{\pi,m}$ are intertwined over m and i;
- Not easy to separate them such that $Y_i^{\pi,m}$ only depends on previous Y and $Z_i^{\pi,m}$ only depends on previous Z.

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- $Y_i^{\pi,m}$ and $Z_i^{\pi,m}$ are intertwined over m and i;
- Not easy to separate them such that $Y_i^{\pi,m}$ only depends on previous Y and $Z_i^{\pi,m}$ only depends on previous Z.

Why not combine them?

Generalize the Markovian iteration to FBSDEs with Y and Z coupling

Combining Y and Z: a differentiation approach

Construct $u_i^{\pi,m}$ as in Bender et al. 2008 and denote $L(u_i^{\pi,m})$ the Lipchitz constant.
Inspired by the Feynman-Kac formulas, we simply construct $v_i^{\pi,m}$ by

$$v_i^{\pi,m}(x) = \partial_x u_i^{\pi,m}(x)\sigma(t, x, u_i^{\pi,m}(x))$$

With this setting, we can show that, for some $C > 0$

$$|v_i^{\pi,m}(x_1) - v_i^{\pi,m}(x_2)|^2 \leq CL(u_i^{\pi,m})|x_1 - x_2|^2$$

- Obtain $v_i^{\pi,m}$ by direct computation;
- Controlling two "independent" and intertwined Lipschitz constants reduces to controlling one object.
- Convergence of $u_i^{\pi,m}$ leads to convergence of $v_i^{\pi,m}$

Convergence result: FBSDEs with Y and Z coupling

Let $b := b(t, X_t, Y_t, Z_t)$ and $\sigma := \sigma(t, X_t, Y_t)$, we can show that

Theorem 1 (Convergence for FBSDEs coupled with Y and Z)

In addition to the assumptions in Bender et al. 2008, if $u \in C_b^2$ and σ is bounded, with our differentiation setting there exist constants $C > 0$ and $0 < c < 1$ such that for sufficiently small h

$$\sup_{1\leq i\leq N} E\left\{\sup_{t\in[t_{i-1},t_i]}(|X_t - X_{i-1}^{\pi,m}|^2 + |Y_t - Y_{i-1}^{\pi,m}|^2)\right\} + \sum_{i=1}^{N} E\left\{\int_{t_{i-1}}^{t_i} |Z_t - Z_{i-1}^{\pi,m}|^2 dt\right\}$$

$$\leq C(1 + |x_0|^2)(mc^m + h)$$

- Proofs are based on a fixed-point argument, which naturally gives algorithm and the well-posedness of the discretized FBSDE
- Z-coupling $\Rightarrow$ can treat FBSDEs formulated via dynamic programming

Combining Y and Z: a differentiation approach

Finally, recall that we still want to have the relations

$$\begin{cases} Z_i^\pi = E_{t_i}\left(h^{-1} Y_{i+1}^\pi \Delta W_i\right) \\ Y_i^\pi = E_{t_i}\left(Y_{i+1}^\pi + hf\left(X_i^\pi, Y_{i+1}^\pi, Z_i^\pi\right)\right) \end{cases}$$

$\Rightarrow$ Reformulating the two regression problems into one, i.e. we solve

$$(Y_i^{\pi,m}, Z_i^{\pi,m}) = \arg\min_{y,z} E\left(\left|Y_{i+1}^{\pi,m} - (y - hf(X_i^{\pi,m}, Y_{i+1}^{\pi,m}, z) + z\Delta W_i)\right|^2\right)$$

- Price to pay: a more complicated optimization problem
- Similar to the one used in Huré et al. 2020 for decoupled case, but we consider coupled FBSDEs and set f explicit in Y.

Numerical Examples

Implementation setting

- We approximate $u_i^{\pi,m}$ by basis functions and obtain $v_i^{\pi,m}$ by direct computation, i.e.

$$u_i^{\pi,m}(x) = \sum_k w_k \beta_k(x), \quad v_i^{\pi,m}(x) = \sum_k w_k \partial_x \beta_k(x) \sigma(t_i, x, u_i^{\pi,m}(x))$$

$\Rightarrow$ Could also use other function approximators, e.g. neural networks.

- Hyper-parameters: Basis functions, batch size, optimizer.
- The errors corresponding to the discretized equation are defined as:

$$\text{error}(X) := \max_{0 \le n \le N} E\left[\left\| X_{t_n}^{\pi} - X_{t_n} \right\|^2\right], \quad \text{error}(Y) := \max_{0 \le n \le N} E\left[\left\| Y_{t_n}^{\pi} - Y_{t_n} \right\|^2\right],$$

$$\text{error}(Z) := T/N \sum_{n=0}^{N-1} E\left[\left\| Z_{t_n}^{\pi} - Z_{t_n} \right\|^2\right], \quad \text{total} = \text{error}(X) + \text{error}(Y) + \text{error}(Z).$$

Example 1: FBSDEs with Z coupling only

We adopt and modify a one-dimensional FBSDE example with a time-dependent solution from Ruijter et al. 2015,

$$\begin{aligned} b(t,x,y,z) &= \kappa_z z + \kappa_b, \\ \sigma(t,x,y) &= 1, \\ f(t,x,y,z) &= yz - z + 2.5y - \sin(x+t)\cos(x+t) - 2\sin(x+t) \\ &\quad - \cos(x+t)\kappa_z z - \cos(x+t)\kappa_b, \\ g(x) &= \sin(x+T), \end{aligned}$$

and the solution is given by

$$Y_t = \sin(X_t + t), \quad Z_t = \cos(X_t + t)$$

- FBSDE with Z coupling but no Y coupling in the forward SDE, which doesn't fall in the framework of Bender et al. 2008;
- But it falls in our analysis framework;

Example 1: FBSDEs with Z coupling only

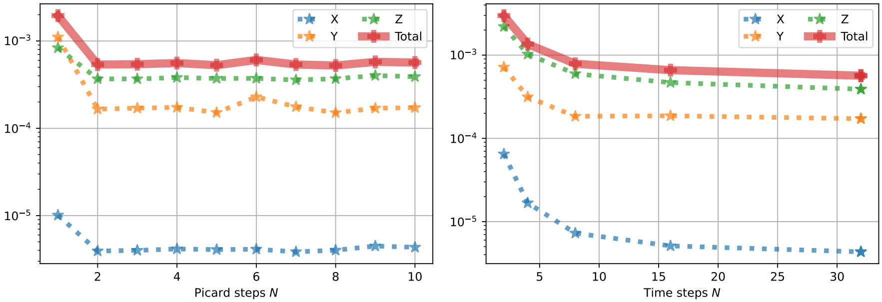

Figure: Example 1. **Left**: Errors versus Iteration steps. **Right**: Errors versus time steps. $T = 0.25, X_0 = 1$.

Example 2: An "almost" fully coupled FBSDE

We modify a coupled FBSDE originating from Bender et al. 2008, and obtain an "almost" fully coupled FBSDE:

$$b(t,x,y,z) = \kappa_y \bar{\sigma} y \mathbf{1}_d + \kappa_z z^\top, \quad \sigma(t,x,y) = \bar{\sigma} y I_d, \quad g(x) = \sum_{i=1}^{d} \sin(x_i),$$

$$\begin{aligned} f(t,x,y,z) = & -ry + 1/2 e^{-3r(T-t)} \bar{\sigma}^2 (\sum_{i=1}^{d} \sin(x_i))^3 \\ & - \kappa_y \sum_{i=1}^{d} z_i - \kappa_z \bar{\sigma} e^{-3r(T-t)} \sum_{i=1}^{d} \sin(x_i) \sum_{i=1}^{d} \cos^2(x_i), \end{aligned}$$

with dimensions $q = 1, d = m = 4$. The solution to the BSDE is given by

$$y(t,x) = e^{-r(T-t)} \sum_{i=1}^{d} \sin(x_i), \quad z_i(t,x) = e^{-2r(T-t)} \bar{\sigma} (\sum_{j=1}^{d} \sin(x_j)) \cos(x_i).$$

- FBSDE with fully coupled drift and partial coupled diffusion;
- it satisfies the assumptions we need;

Example 2: An "almost" fully coupled FBSDE

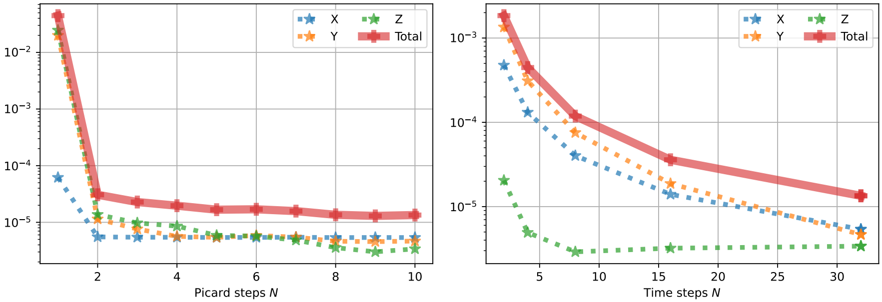

Figure: Example 2. **Left**: Errors versus Iteration steps. **Right**: Errors versus time steps. $T = 0.25, X_0 = (\pi/4, \ldots, \pi/4)$.

Summary and conclusion

- We generalize the Markovian iteration by Bender et al. 2008 to FBSDEs with both Y and Z coupling via a differentiation setting.
- Convergence results are based on a fixed-point argument for the discretized scheme, which naturally gives the well-posedness and an algorithm.
- Enable the treatment of FBSDEs formulated via the dynamic programming.
- The numerical results have verified our theoretical findings.

End

Thanks for your attention!

This talk is based on a working paper (coming soon!):

Zhipeng Huang, and Cornelis W. Oosterlee (2024). Convergence of the Markovian iteration for coupled FBSDEs via a differentiation approach.

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