Variational Quantum Eigensolver
Quantum Approximate Optimisation Algorithm

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Structure

An (NP-hard) optimisation problem

Problem: Given $f : \{0,1\}^n \to \mathbb{R}$, $\min_{z \in \{0,1\}^n} f(z)$. (1)

- Hamiltonian formulation: $\mathcal{H}_F := \sum_{z \in \{0,1\}^n} f(z) |z\rangle \langle z|$.
- If $(|z_i\rangle)$ are eigenvectors of $\mathcal{H}_F$, then

$$\begin{aligned}
\mathcal{H} |z_i\rangle &= \left(\sum_{z \in \{0,1\}^n} f(z) |z\rangle \langle z| \right) |z_i\rangle \\
&= \left(\sum_{z \in \{0,1\}^n \setminus \{z_i\}} f(z) |z\rangle \langle z| \right) |z_i\rangle \quad + \Big(f(z_i) |z_i\rangle \langle z_i| \Big) |z_i\rangle \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad + f(z_i) |z_i\rangle \langle z_i|z_i\rangle \\
&= f(z_i) |z_i\rangle ,
\end{aligned}$$

  so that $(f(z_i))$ are eigenvalues of $\mathcal{H}_F$.
- Solving (1) amounts to finding the smallest eigenvalues (minimum energy) of $\mathcal{H}_F$.
- Problem: it is often difficult to find them.

Variational Quantum Eigensolver

Variational Quantum Eigensolver

The Variational Quantum Eigensolver (VQE) is a PQC-based algorithm that aims to find the smallest eigenvalue (the lowest energy) of a problem Hamiltonian.

Objective functions of many NP-hard combinatorial optimisation problems can be encoded in the Hamiltonians of the quantum systems – finding the ground state of the Hamiltonian gives us the minimum of the objective function.

The variational part of the algorithm refers to the systematic search for the best possible approximation of the ground state by trying various PQC ansatzes and configurations of adjustable PQC parameters – the variational approach.

How does the algorithm work?

The *characteristic equation* for the Hamiltonian $\mathcal{H}$ is

$$\mathcal{H}|\psi_i\rangle = \lambda_i|\psi_i\rangle,$$

where $|\psi_i\rangle$ is an eigenstate associated with the eigenvalue λ_i.

The objective is to find the smallest eigenvalue λ_0 (the lowest energy) of $\mathcal{H}$ corresponding to the ground state (the lowest energy state) $|\psi_0\rangle$.

The eigenvalue (energy) of $\mathcal{H}$ is the expectation of $\mathcal{H}$:

$$\langle\psi_i|\mathcal{H}|\psi_i\rangle = \langle\psi_i| \cdot \lambda_i \cdot |\psi\rangle_i = \lambda_i \cdot \langle\psi_i|\psi_i\rangle = \lambda_i.$$

This expectation can be calculated on a quantum computer.

Therefore, the task is to use the PQC to construct a quantum state that is as close as possible to the ground state of the problem Hamiltonian.

We do it by trying many candidate states and choosing the one that minimises the expectation value.

Motivation

The variational approach is motivated by the Spectral theorem. Since a Hamiltonian operator $\mathcal{H}$ is Hermitian, it can then be expressed as

$$\mathcal{H} = \sum_i \lambda_i |\psi_i\rangle \langle\psi_i| = \sum_i \lambda_i \mathcal{H}_i,$$

with $\mathcal{H}_i = |\psi_i\rangle \langle\psi_i|$ the projection onto the eigenspace of $\mathcal{H}$ corresponding to λ_i (We assume the eigenvalues to be ordered: $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots$).

Born's rule

When measuring an observable $\mathcal{H}$ in the system ψ, the probability of obtaining a given eigenvalue λ_i in is equal to $\langle\psi|\mathcal{H}_i|\psi\rangle$.

We can compute

$$\langle\psi|\mathcal{H}|\psi\rangle = \langle\psi| \left(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \right) |\psi\rangle = \sum_i \lambda_i \langle\psi|\,\psi_i\rangle\langle\psi_i\,|\psi\rangle = \sum_i \lambda_i |\langle\psi|\,\psi_i\rangle|^2.$$

But, with $|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle$, then $\langle\psi|\,\psi_i\rangle = \alpha_i^*$ for all i, and hence

$$\langle\psi|\mathcal{H}|\psi\rangle = \sum_i \lambda_i |\langle\psi|\,\psi_i\rangle|^2 = \sum_i \lambda_i |\alpha_i|^2 \geq \lambda_0 \sum_i |\alpha_i|^2 = \lambda_0.$$

The hybrid quantum-classical protocol

The role of PQC is to produce the candidate states $|\psi\rangle$.

The variational part of the algorithm consists of iterative improvements of the candidate state (iterative updates of the adjustable parameters).

The quantum part of the algorithm consists of running the PQC and then measuring $\mathcal{H}$ on the constructed quantum state to obtain the expectation of $\mathcal{H}$.

The variational approach allows us to solve hard optimisation problems encoded in the Hamiltonian on the digital gate model quantum computer – an alternative to the adiabatic quantum computing since not all optimisation problems can be efficiently formulated in a QUBO format.

Computing expectations on a quantum computer

Start with the simplest case of a one-qubit system. Since any 2×2 unitary and Hermitian matrix can always be decomposed into a sum of the Pauli matrices $\mathtt{X}$, $\mathtt{Y}$, $\mathtt{Z}$ and an identity matrix $\mathtt{I}$, we can represent any 1-qubit Hamiltonian as

$$\mathcal{H} = a\mathtt{X} + b\mathtt{Y} + c\mathtt{Z} + d\mathtt{I},$$

where a, b, c and d are some real coefficients. For a given $|\psi\rangle$, the expectation value of a Hamiltonian is given by the expression:

$$\langle \mathcal{H} \rangle := \langle \psi | \mathcal{H} | \psi \rangle = a \langle \psi | \mathtt{X} | \psi \rangle + b \langle \psi | \mathtt{Y} | \psi \rangle + c \langle \psi | \mathtt{Z} | \psi \rangle + d \langle \psi | \mathtt{I} | \psi \rangle .$$

The expectation is computed by adding the expectation values of each term.

Thus we can compute the expectation values of the Pauli terms independently and then sum them up to obtain $\langle \mathcal{H} \rangle$. We can do it by first constructing state $|\psi\rangle$ with the help of a PQC and then performing measurement in the computational basis. The process of constructing the state and performing measurement should be repeated sufficiently many times to obtain accurate statistics.

Expectation values of Pauli operators

Expectation value of I is 1:

$$\langle\psi|\,\mathtt{I}\,|\psi\rangle = \langle\psi\,|\psi\rangle = 1.$$

This term will contribute d to $\langle\mathcal{H}\rangle$.

The next term is cZ. Recall that Z-gate is the PHASE gate ($\mathtt{Z}\,|1\rangle = -\,|1\rangle$, $\mathtt{Z}\,|0\rangle = |0\rangle$). The measurement is performed in the z-basis, where $|\psi\rangle$ can be represented as a superposition of the basis states $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha_z\,|0\rangle + \beta_z\,|1\rangle\,,$$

for some coefficients α_z and β_z. The expectation $\langle\psi|\,\mathtt{Z}\,|\psi\rangle$ is then calculated as

$$\begin{aligned}
\langle\psi|\,\mathtt{Z}\,|\psi\rangle &= \Big(\alpha_z^*\,\langle 0| + \beta_z^*\,\langle 1|\,\Big)\mathtt{Z}\Big(\alpha_z\,|0\rangle + \beta_z\,|1\rangle\,\Big) \\
&= |\alpha_z|^2\,\langle 0|\,\mathtt{Z}\,|0\rangle + \alpha_z^*\beta_z\,\langle 0|\,\mathtt{Z}\,|1\rangle + \alpha_z\beta_z^*\,\langle 1|\,\mathtt{Z}\,|0\rangle + |\beta_z|^2\,\langle 1|\,\mathtt{Z}\,|1\rangle \\
&= |\alpha_z|^2\,\langle 0|0\rangle - \alpha_z^*\beta_z\,\langle 01\rangle + \alpha_z\beta_z^*\,\langle 10\rangle - |\beta_z|^2\,\langle 11\rangle \\
&= |\alpha_z|^2 - |\beta_z|^2,
\end{aligned}$$

where $|\alpha_z|^2$ and $|\beta_z|^2$ are the probabilities that after z-basis measurement the quantum state $|\psi\rangle$ will become $|0\rangle$ or $|1\rangle$ respectively.

Expectation value of Z

If we run the quantum circuit (to construct state $|\psi\rangle$) and perform measurement N times, the probability of finding qubit in state $|0\rangle$ can be estimated as $\frac{n_0}{N}$, where n_0 is the number of state $|0\rangle$ measurements.

Similarly, the probability of finding qubit in state $|1\rangle$ can be estimated as $\frac{n_1}{N}$, where n_1 is the number of state $|1\rangle$ measurements ($n_0 + n_1 = N$).

Therefore, the contribution of the Z-term to $\langle \mathcal{H} \rangle$ is given by

$$c \langle \psi | \, \mathrm{Z} \, | \psi \rangle = c \frac{n_0 - n_1}{N}.$$

Expectation values of X and Y

First, note that states $|0\rangle$ and $|1\rangle$ are the eigenstates of operator Z with the corresponding eigenvalues being $+1$ and -1:

$$\mathtt{Z}\,|0\rangle = |0\rangle\,, \quad \mathtt{Z}\,|1\rangle = -\,|1\rangle\,.$$

The eigenstates of operator X are

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

and the eigenstates of operator Y are

$$|\mathrm{R}\rangle = \frac{|0\rangle + \mathrm{i}\,|1\rangle}{\sqrt{2}} \quad \text{and} \quad |\mathrm{L}\rangle = \frac{|0\rangle - \mathrm{i}\,|1\rangle}{\sqrt{2}}.$$

Their corresponding eigenvalues also are $+1$ and -1:

$$\mathtt{X}|+\rangle = |+\rangle, \quad \mathtt{X}|-\rangle = -|-\rangle, \quad \mathtt{Y}|\mathrm{R}\rangle = |\mathrm{R}\rangle, \quad \mathtt{Y}|\mathrm{L}\rangle = -|\mathrm{L}\rangle.$$

Expectation values of X and Y (continued)

Therefore, the quantum state $|\psi\rangle$ can also be decomposed into the superposition of the basis states $\{|\mathrm{R}\rangle, |\mathrm{L}\rangle\}$ (y-basis) and $\{|+\rangle, |-\rangle\}$ (x-basis):

$$|\psi\rangle = \alpha_x|+\rangle + \beta_x|-\rangle = \alpha_y|\mathrm{R}\rangle + \beta_y|\mathrm{L}\rangle.$$

If we can perform measurement in x-basis and y-basis, the expectations $\langle\psi|\,\mathtt{X}\,|\psi\rangle$ and $\langle\psi|\,\mathtt{Y}\,|\psi\rangle$ can be calculated in exactly the same way as expectation $\langle\psi|\,\mathtt{Z}\,|\psi\rangle$:

$$a\,\langle\psi|\,\mathtt{X}\,|\psi\rangle = a\frac{n_+ - n_-}{N}, \quad b\,\langle\psi|\,\mathtt{Y}\,|\psi\rangle = b\frac{n_\mathrm{R} - n_\mathrm{L}}{N}.$$

Here, n_+ and n_- are numbers of measurements in the x-basis that correspond, respectively, to the $|+\rangle$ and $|-\rangle$ outcomes. And n_R and n_L are numbers of measurements in the y-basis that correspond, respectively, to $|\mathrm{R}\rangle$ and $|\mathrm{L}\rangle$ outcomes.

Expectation values of X and Y (continued)

In the case when we can only perform measurement in the *z*-basis, we need to apply some additional operators (gates) to state $|\psi\rangle$ before the measurement, such that the probability of measuring $|0\rangle$ in *z*-basis is the same as the probability of measuring $|+\rangle$ in *x*-basis if we are calculating $\langle\psi|\,\mathtt{X}\,|\psi\rangle$, or the probability of measuring $|0\rangle$ in *z*-basis is the same as the probability of measuring $|\mathrm{R}\rangle$ in *y*-basis if we are calculating $\langle\psi|\,\mathtt{Y}\,|\psi\rangle$:

$$\mathtt{H}\,|\psi\rangle = \mathtt{H}\,(\alpha_x|+\rangle + \beta_x|-\rangle) = \alpha_x\,|0\rangle + \beta_x\,|1\rangle\,,$$

with $\mathtt{H}|+\rangle = |0\rangle$ and $\mathtt{H}|-\rangle = |1\rangle$.

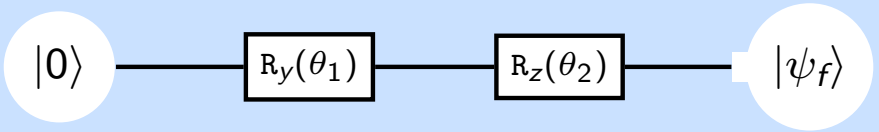
$$\mathtt{G}\,|\psi\rangle = \mathtt{G}\,(\alpha_y|\mathrm{R}\rangle + \beta_y|\mathrm{L}\rangle) = \alpha_y\,|0\rangle + \beta_y\,|1\rangle\,,$$

with $\mathtt{G}|\mathrm{R}\rangle = |0\rangle$ and $\mathtt{G}|\mathrm{L}\rangle = |1\rangle$.

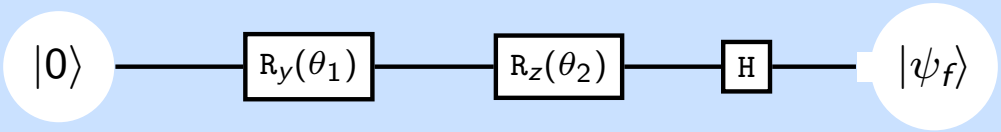
The operators H and G are the following one-qubit rotations:

$$\mathtt{H} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \quad \mathtt{G} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -\mathrm{i}\\ 1 & \mathrm{i}\end{bmatrix}.$$

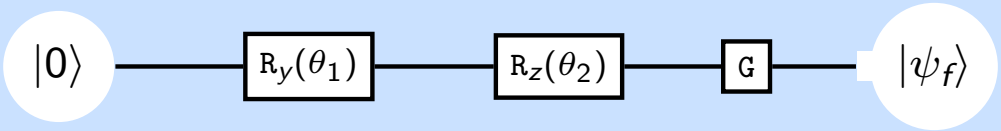
Constructing the PQC (continued)



PQC for a one-qubit system to calculate $\langle Z \rangle$.



PQC with H gate to calculate $\langle X \rangle$.



PQC with G gate to calculate $\langle Y \rangle$.

Quantum Approximate Optimisation Algorithm

The time evolution

The dynamics of the quantum mechanical system is governed by the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \mathcal{H}|\psi(t)\rangle,$$

where $|\psi(t)\rangle$ is the quantum state at time t and $\mathcal{H}$ is the *time-independent* Hamiltonian. The solution of Schrödinger equation is given by the following expression:

$$|\psi(t)\rangle = \mathcal{U}(0,t)|\psi(0)\rangle,$$

where the operator $\mathcal{U}(t)$ is obtained from the Hamiltonian $\mathcal{H}$ via

$$\mathcal{U}(0,t) = \exp\left(\frac{-\mathrm{i}\mathcal{H}t}{\hbar}\right).$$

In the following we will be working with the units where Planck's constant $\hbar$ is set equal to 1 and the system dynamics is given by the expression

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\mathcal{H}t}|\psi(0)\rangle.$$

Time-dependent Hamiltonian

If the initial state of the system, $|\psi(0)\rangle$, is known then the state of the system at time t is also known and is determined by the action of Hamiltonian $\mathcal{H}$ over period of time t.

We would like to work with the time-dependent Hamiltonians of the form

$$\mathcal{H}(t) = \left(1 - \frac{t}{T}\right)\mathcal{H}_0 + \frac{t}{T}\mathcal{H}_F,$$

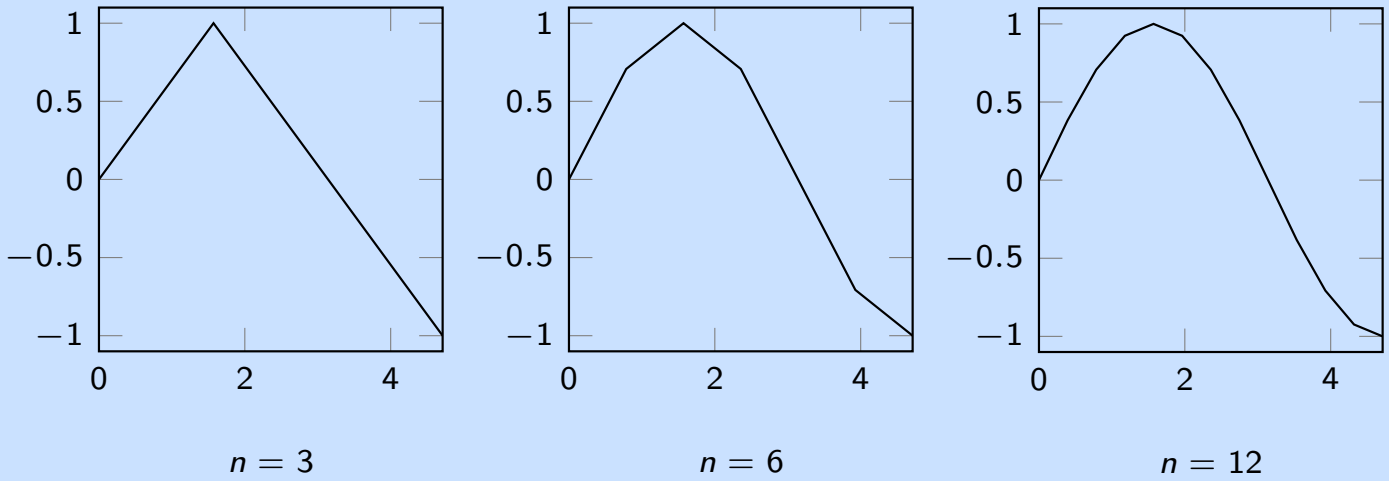
where $\mathcal{H}_0$ is the *initial* Hamiltonian and $\mathcal{H}_F$ is the *final* or *problem* (i.e., encoding the optimisation problem) Hamiltonian.

How do we reconcile the time-independent Hamiltonian of the Schrödinger equation and the time-dependent Hamiltonian describing the evolution of our system?

We can *approximate* the time-dependent Hamiltonian $\mathcal{H}(t)$ that transforms the state on interval $[0, T]$ by a sequence of time-independent Hamiltonians, $\{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n\}$, transforming the state in corresponding shorter time intervals $\{[t_0 = 0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n = T]\}$.

Approximation

A good analogy can be an approximation of continuous function (e.g., $\sin(t)$) by a piecewise linear function. The more granular the time intervals $[t_{i-1}, t_i]$, the better the approximation.



Piecewise linear approximation of $\sin(t)$.

Similarly, we can approximate operator $\mathcal{U}(0, T)$ as

$$\mathcal{U}(0, T) = \mathcal{U}(t_{n-1}, t_n = T)\mathcal{U}(t_{n-2}, t_{n-1}) \cdots \mathcal{U}(t_2, t_1)\mathcal{U}(t_0 = 0, t_1).$$

Larger n gives us better approximation.

The Suzuki-Trotter expansion

A particularly useful approximation of $\mathcal{U}(0, T)$ can be obtained using the *Suzuki-Trotter expansion*. If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ are operators that do not necessarily *commute* with each other, then

$$\mathrm{e}^{\mathcal{A}_1+\mathcal{A}_2+\ldots+\mathcal{A}_p} = \lim_{m\uparrow\infty} \left(\mathrm{e}^{\frac{\mathcal{A}_1}{m}} \mathrm{e}^{\frac{\mathcal{A}_2}{m}} \cdots \mathrm{e}^{\frac{\mathcal{A}_p}{m}} \right)^m .$$

Two operators $\mathcal{A}$ and $\mathcal{B}$ are said to commute with each other if

$$\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = 0,$$

or equivalently $\mathcal{A}\mathcal{B}\,|\psi\rangle = \mathcal{B}\mathcal{A}\,|\psi\rangle$ for any quantum state $|\psi\rangle$.

In general, operators do not commute; for example, rotations around different axes do not.

If they do, we can *measure* them in an arbitrary order and get the same answer. The Suzuki-Trotter expansion, however, does not require operators to commute. This has important implications for QAOA.

QAOA Hamiltonian

If $\mathcal{U}(0, T)$ has the form $\mathrm{e}^{\mathcal{A}+\mathcal{B}}$, then the Suzuki-Trotter expansion yields

$$\mathrm{e}^{\mathcal{A}+\mathcal{B}} = \lim_{m\uparrow\infty} \left(\mathrm{e}^{\frac{\mathcal{A}}{m}} \mathrm{e}^{\frac{\mathcal{B}}{m}}\right)^m.$$

The time evolution of $\mathcal{A}+\mathcal{B}$ can be approximated by applying alternatively $\mathcal{A}$ and $\mathcal{B}$ for time intervals T/m.

The Hamiltonian terms $\mathcal{H}_0$ and $\mathcal{H}_F$ have the following general form:

$$\mathcal{H}_0 = \sum_i \sigma_x^i \qquad \text{and} \qquad \mathcal{H}_F = \sum_i a_i \sigma_z^i + \sum_{ij} b_{ij} \sigma_z^i \sigma_z^j,$$

where a_i and b_{ij} are some coefficients.

The initial Hamiltonian $\mathcal{H}_0$ is the operator $\mathcal{A}$ (the *mixing* Hamiltonian) and the final Hamiltonian $\mathcal{H}_F$ is the operator $\mathcal{B}$ (the *phase* Hamiltonian).

QAOA Hamiltonian (continued)

The initial state is selected to be the equal superposition state of all possible solutions, and can easily be constructed with the help of Hadamard gates:

$$\begin{aligned}|\psi(0)\rangle &= \frac{1}{\sqrt{2^n}}\Big(|0\ldots00\rangle + |0\ldots01\rangle + \ldots + |1\ldots11\rangle\Big) \\ &= \mathrm{H}^{\otimes n}\,|0\rangle^{\otimes n} \\ &= (\mathrm{H}\,|0\rangle) \otimes \cdots (\mathrm{H}\,|0\rangle).\end{aligned}$$

Quantum Approximate Optimisation Algorithm

Algorithm 1: Quantum Approximate Optimisation Algorithm

Result: Optimal solution

- A parameterised quantum state, $|\psi(\beta,\gamma)\rangle$, is created by alternately applying operators $\mathcal{A}$ and $\mathcal{B}$ (mixing and phase Hamiltonians) for m rounds, where the duration in round i, $i = 1,\ldots,m$, is specified by parameters β_i and γ_i respectively:

$$|\psi(\beta,\gamma)\rangle = \mathrm{e}^{-\mathrm{i}\beta_m\mathcal{A}}\mathrm{e}^{-\mathrm{i}\gamma_m\mathcal{B}}\ldots\mathrm{e}^{-\mathrm{i}\beta_2\mathcal{A}}\mathrm{e}^{-\mathrm{i}\gamma_2\mathcal{B}}\mathrm{e}^{-\mathrm{i}\beta_1\mathcal{A}}\mathrm{e}^{-\mathrm{i}\gamma_1\mathcal{B}}.$$

- A computational basis (z-basis) measurement is performed on the obtained state, which returns a candidate solution. Repeating the above state preparation and measurement, the expected value of the cost function f over the returned solution samples is given by

$$\langle f\rangle = \langle\psi(\beta,\gamma)|\mathcal{B}|\psi(\beta,\gamma)\rangle,$$

  which can be statistically estimated from the samples produced.
- The above steps may then be repeated with the updated sets of time parameters β and γ – the variational part of the algorithm – within the classical optimisation loop that aims to minimise the expectation of the cost function $\langle f\rangle$.
- The algorithm returns the best found solution.

Operators $\mathcal{A}$ and $\mathcal{B}$ do not commute

It is important to apply operators $\exp(-i\beta\mathcal{A})$ and $\exp(-i\gamma\mathcal{B})$ alternately to ensure that we are not trappped in a local minimum.

It is also important that operators $\mathcal{A}$ and $\mathcal{B}$ do not commute. This is because by applying only operator $\exp(-i\gamma\mathcal{B})$ (the phase Hamiltonian) we are running into a danger of getting into a state which is the eigenstate of the phase Hamiltonian.

If this happens we will be trapped in this state: any further application of a linear operator to its eigenvector may change its length but not its direction.

The same consideration applies to alternating between two commuting operators: if $\mathcal{A}$ and $\mathcal{B}$ commute, then we can come up with a set of basis states that are eigenstates of both $\mathcal{A}$ and $\mathcal{B}$, and once we get into one of these eigenstates we will be trapped in it. However, since σ_x and σ_z do not commute:

$$\sigma_x\sigma_z - \sigma_z\sigma_x = -2i\sigma_y \neq 0,$$

there is always a chance to escape from the local minimum.

The Max-Cut problem

GOAL: divide the vertices of a graph in two groups such that either the maximum possible number of edges going between the two groups are "cut" (if all edges have the same weight) or the total weight of these edges is maximised (if they have different weights).

The cost function for the edge connecting vertices i and j:

$$c_{ij} = \frac{1}{2} w_{ij}(1 - s_i s_j),$$

where s_i and s_j are classical spin variables taking values $\{+1, -1\}$ and w_{ij} is the weight associated with the edge connecting vertices i and j. The two groups of vertices are those where the spin variables take the same values (either $+1$ or -1).

The cost function for the whole graph:

$$L(\mathbf{s}) = \sum_{\{ij\} \in G} \frac{1}{2} w_{ij}(1 - s_i s_j),$$

where $\mathbf{s} = (s_1, \ldots, s_N)$ is the set of decision variables associated with the N-node graph G and the sum goes over all pairs of nodes connected by the graph edges.

QAOA gates

The mixing Hamiltonian $\mathcal{A}$ and the phase Hamiltonian $\mathcal{B}$ that correspond to the Max-Cut cost function are

$$\mathcal{A} = \sum_{i=1}^{N} \sigma_x^i$$

and

$$\mathcal{B} = \sum_{\{ij\} \in G} \frac{1}{2} w_{ij} \left(1 - \sigma_z^i \sigma_z^j\right),$$

where the spin variables s are replaced by the corresponding Pauli operators σ. Therefore, we need to find the quantum gate representation of the following operators:

$$\exp\left(-\mathrm{i}\beta\sigma_x^i\right) \quad \text{and} \quad \exp\left(-\frac{1}{2}\mathrm{i}\gamma\sigma_z^i\sigma_z^j\right).$$

QAOA gates (continued)

In order to find the quantum gate representation of the operators

$$\exp\left(-\mathrm{i}\beta\sigma_x^j\right) \quad \text{and} \quad \exp\left(-\frac{1}{2}\mathrm{i}\gamma\sigma_z^i\sigma_z^j\right),$$

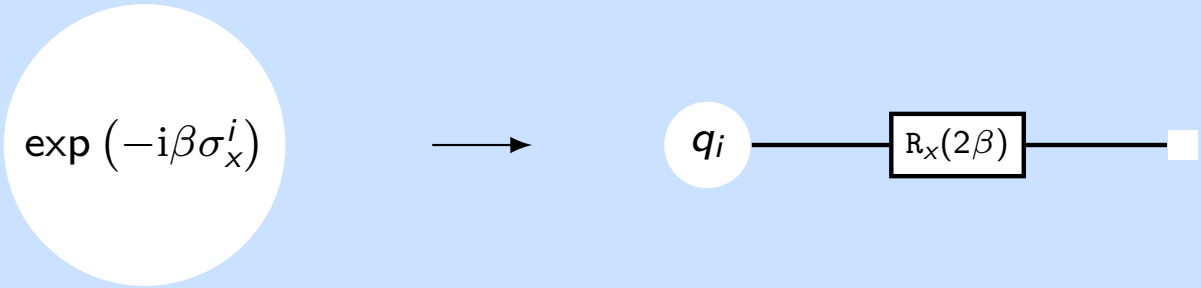
we should use the following theorem:

The following equation holds for any unitary Hermitian operator $\mathcal{H}$:

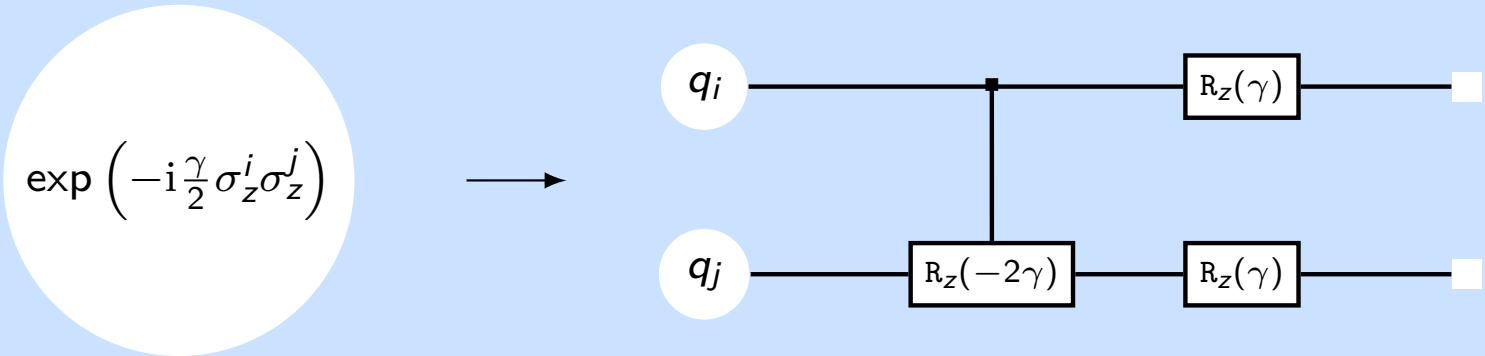
$$R_\theta(\mathcal{H}) := \exp\left(-\mathrm{i}\frac{\theta}{2}\mathcal{H}\right) = \cos\left(\frac{\theta}{2}\right)\mathcal{I} - \mathrm{i}\sin\left(\frac{\theta}{2}\right)\mathcal{H},$$

where $\mathcal{I}$ is the identity operator.

QAOA gates (continued)



Gate representation of operator $\exp\left(-i\beta\sigma_x^i\right)$.

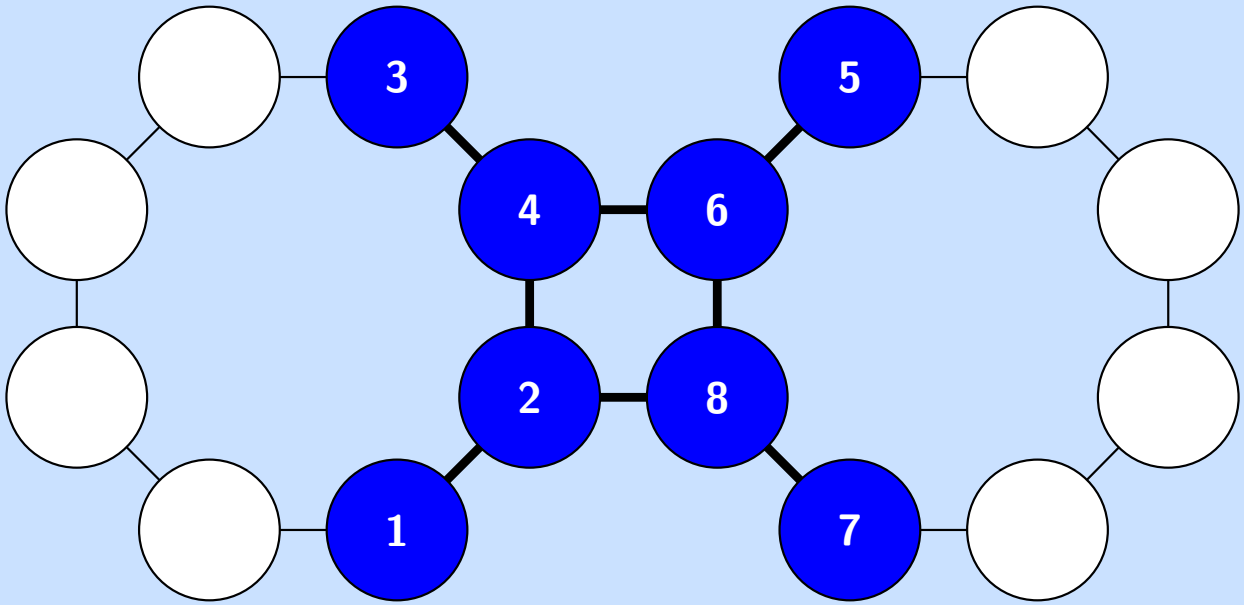


Gate representation of operator $\exp\left(-i\frac{\gamma}{2}\sigma_x^i\right)$.

Sample Max-Cut problem

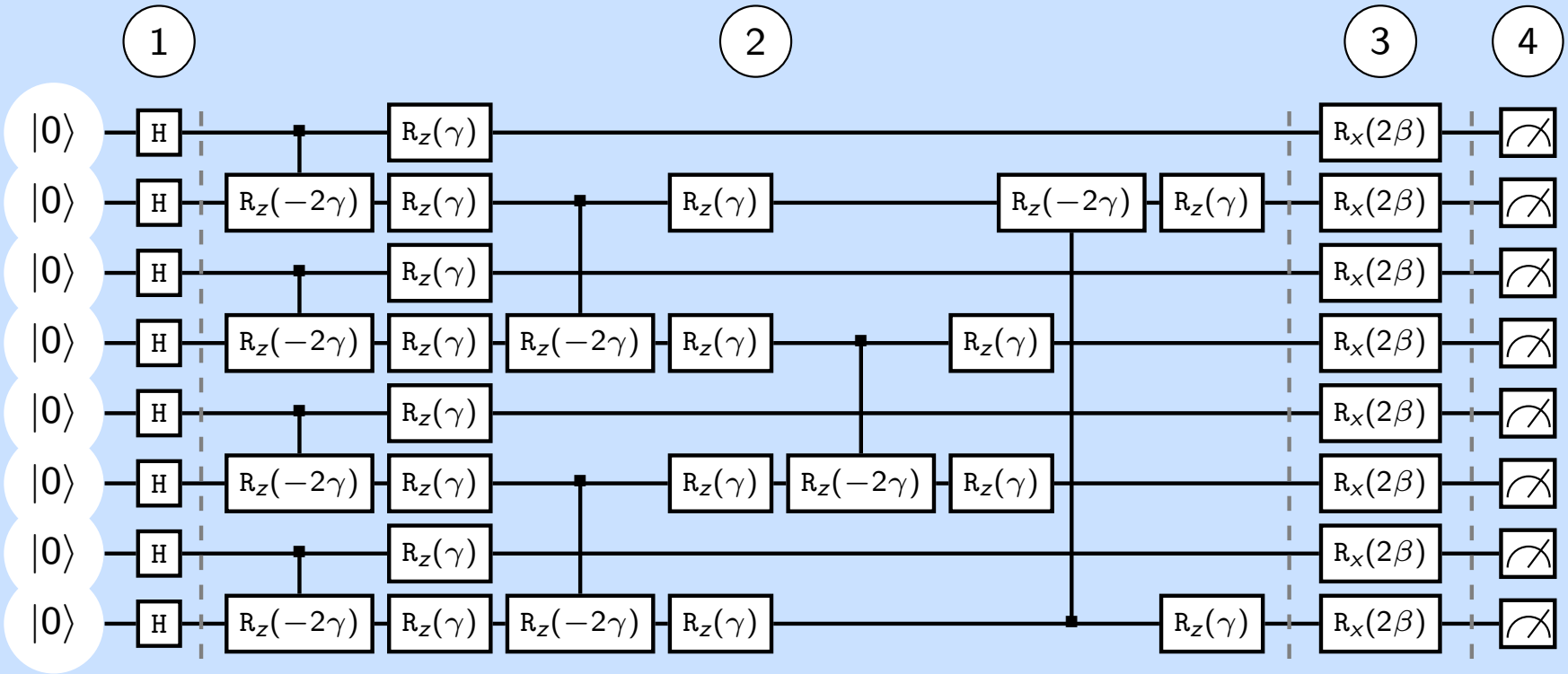
The Max-Cut graph consists of eight nodes (embedded in qubits $1, \ldots, 8$) and eight edges with equal weights.

Can we find a partitioning of this graph that would cut the maximal number of edges?



Embedding of Max-Cut optimisation problem on Rigetti's Aspen system.

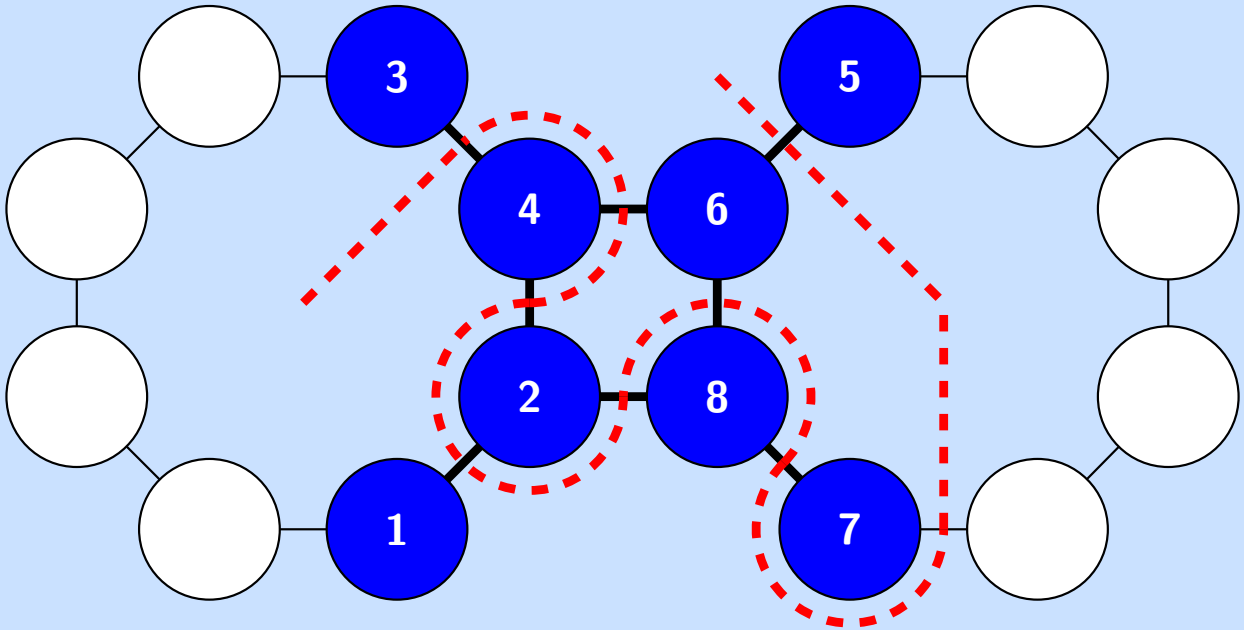
QAOA circuit



QAOA circuit for the Max-Cut problem.

Max-Cut solution

The optimal solution reads as 10011001 and is represented by the dashed curve that separates nodes into two equal subsets and cuts across all edges of the graph.



Visualisation of the Max-Cut problem solution.