A Deep BSDE approach for the simultaneous pricing and delta-gamma hedging of large portfolios consisting of high-dimensional multi-asset Bermudan options

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series of joint works with Kees Oosterlee

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Agenda

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Problem I

Possession of a portfolio of J options issued on a set of common risk factors (tradeable + non-tradeable) which form an $\mathbb{R}^{m+(d-m)}$ valued Itô process

$$X_t = x_0 + \int_0^t \mu(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s$$

Mixture of European/Bermudan(/American)contracts with $\mathcal{R}^j \subseteq [0, T]$ early exercise dates $j = 1, \dots, J$

Construct a (delta-)hedging portfolio

1. short position in the option: $-\sum_{j=1}^{J} v^j(t, X_t)\mathbb{1}_{\tau^j \geq t}$,
2. long position in the underlying assets: $+\sum_{i=1}^{m} \alpha_t^i X_t^i$,
3. risk-free bank account: $+B_t$,

Problem II

The value of the portfolio evolves according to

$$\mathrm{d}P_t^{\Delta} = -\sum_{j=1}^{J} v^j(t, X_t)\mathbb{1}_{\tau^j \geq t} + \sum_{i=1}^{m} \alpha_t^i \mathrm{d}X_t^i + \mathrm{d}B_t, \quad P_0^{\Delta} = 0$$

Rebalancing at a discrete set of points in time $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ according to the **first-order** constraint

$$\alpha_{t_n}^i = \sum_{j=1}^{J} \partial_i v^j(t_n, X_{t_n})\mathbb{1}_{\tau^j \geq t_n}, \quad i = 1, \ldots, m$$

and updating the position in the underlying by purchasing/selling $\alpha_{t_n}^i - \alpha_{t_{n-1}}^i$ of the i'th asset (borrow/deposit from/in bank account)

Risk

Since rebalancing only happens at discrete time intervals the corresponding strategy is not risk-free – only as $\sup_n |t_n - t_{n-1}| \to 0$

Quality assessed by the relative **profit-and-loss (PnL)**

$$\mathsf{PnL}_T^{\Delta} := \frac{e^{-rT} P_T^{\Delta}}{\sum_{j=1}^{J} v^j(0, X_0)},$$

which is an $\mathcal{F}_T$-measurable random variable

Statistics on its distribution then assess the quality of the hedging strategy

- $\mathbb{E}\left[\mathsf{PnL}_T^{\Delta}\right]$ – mean
- $\mathbb{V}ar\left[\mathsf{PnL}_T^{\Delta}\right]$ – variance
- $\mathsf{VaR}_\alpha := \inf\left\{x \in \mathbb{R} : \mathbb{P}\left[\mathsf{PnL}_T^{\Delta} < x\right] \leq \alpha\right\}$ – Value-at-Risk
- $\mathsf{ES}_\alpha := \mathbb{E}\left[\mathsf{PnL}_T^{\Delta} \middle| \mathsf{PnL}_T^{\Delta} \leq \mathsf{VaR}_\alpha\right]$ – expected shortfall
- semi-variance...

Gamma hedging

Itô's lemma: perfect replication in the continuous, complete framework
Sadly: the world is not continuous *(thanks Max Planck...)*
Mitigate finite hedging errors $\longrightarrow$ second-order hedging constraints (Gamma) Additional Gamma-hedging instruments needed with **non-vanishing Gammas**

$$\mathrm{d}P_t^\Gamma = -\sum_{j=1}^{J} \mathrm{d}v^j(t, X_t) + \sum_{i=1}^{m} \alpha_t^i \mathrm{d}X_t^i + \sum_{k=1}^{K} \beta_t^k u^k(t, X_t) + \mathrm{d}B_t, \quad P_0^\Gamma = 0.$$

Rebalancing according to **first-** and **second-order** constraints

$$\partial_{li}^2 P_{t_n}^\Gamma = 0 \implies \sum_{k=1}^{K} \beta_t^k \partial_{li}^2 u^k(t_n, X_{t_n}) = \sum_{j=1}^{J} \partial_{li}^2 v^j(t_n, X_{t_n}), \qquad 1 \le l, i \le d$$

$$\partial_i P_{t_n}^\Gamma = 0 \implies \alpha_{t_n}^i = \sum_{j=1}^{J} \partial_i v^j(t_n, X_{t_n}) - \sum_{k=1}^{K} \beta_{t_n}^k \partial_i u^k(t_n, X_{t_n}), \qquad 1 \le i \le m$$

✓Pros: **sharper PnLs** with **less frequent** rebalancing
✗Cons: more exposed to **model error** need to approximate $\mathrm{Hess}\, v^j$

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Backward Stochastic Differential Equations

$$X_t = x_0 + \int_0^t \mu(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s$$

$$Y_t^j = g^j(X_T) + \int_t^T f^j(s, X_s, Y_s^j, Z_s^j)\mathrm{d}s - \int_t^T Z_s^j \mathrm{d}W_s.$$

Semi-linear PDEs with terminal boundaries

$$\begin{aligned} \partial_t v^j + 1/2\,\mathrm{tr}\left(\sigma\sigma^T(t,x)\nabla^2 v^j\right) & \\ + \mu^T(t,x)\nabla v^j + f^j(t,x,v^j,\nabla v^j\sigma) = 0, & \qquad (t,x) \in [0,T]\times D, \\ v^j(T,x) = g^j(x), & \qquad x \in D. \end{aligned}$$

General Feynman–Kac relation

Under certain regularity conditions the solutions coincide $\mathbb{P}$-a.s.

$$Y_t^j = v^j(t, X_t), \quad Z_t^j = (\nabla v^j \sigma)(t, X_t).$$

Reflected BSDEs

Associated **reflected** BSDE – the solution "cannot go" below a certain *(Markovian)* **lower barrier process** $L_t^j := l^j(X_t)$

$$Y_t^j = g^j(X_T) + \int_t^T f^j(s, X_s, Y_s^j, Z_s^j)\mathrm{d}s - \int_t^T Z_s^j \mathrm{d}W_s + K_T^j - K_t^j,$$

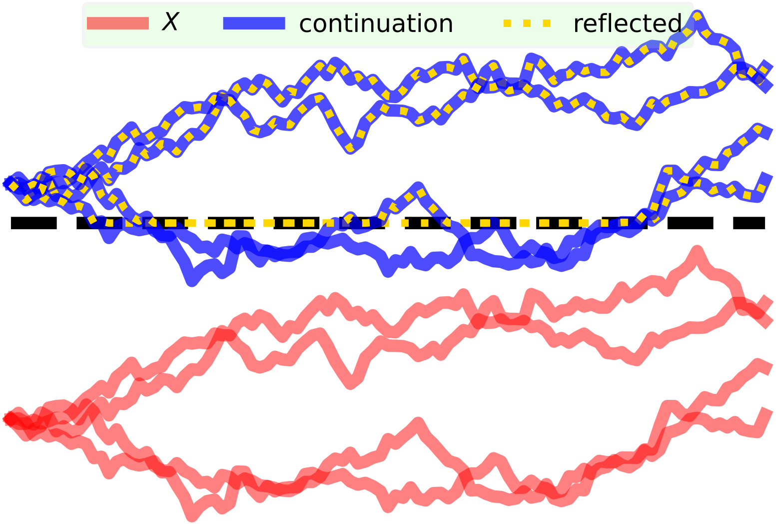
$$Y_t^j \geq l^j(X_t), \quad t \leq T \qquad \text{and} \qquad \int_0^T \left[Y_t^j - l^j(X_t)\right] \mathrm{d}K_t^j = 0.$$

Second-order semi-linear, **free-boundary** PDE

$$\begin{aligned} \min[v^j - l^j, \partial_t v^j + 1/2\,\mathrm{tr}\left(\sigma\sigma^T(t,x)\nabla^2 v^j\right) \\ + \mu^T(t,x)\nabla v^j + f(t,x,v^j,\nabla v^j\sigma)\Big] &= 0, \qquad (t,x) \in [0,T] \times D, \\ v^j(T,x) &= g^j(x), \qquad x \in D. \end{aligned}$$

discretely reflected BSDEs: reflections can only occur over a finite set of times $\{0 := r_0 < r_1 < \cdots < r_{R-1} < T =: r_R\}$, $R \to \infty \longrightarrow$ reflected BSDE

Illustration



Connections with Finance

The jth option with payoff g^j, instantaneous payoff $l^j \equiv g^j$ solves a BSDE

standard BSDE	reflected	discretely reflected
European	American	Bermudan

Simultaneous prices and **Deltas**

X	Y^j	Z^j	$\widetilde{Y}^j$
asset price	option price	delta	continuation value

Take-away

BSDE/reflected BSDE/discretely reflected BSDE $\implies$ **delta hedging** of European/American/Bermudan options

(Deep) BSDEs in the context of **single options**

- (delta-)hedging: Becker, Cheridito, and Jentzen 2020; Chen and Wan 2021
- incomplete markets: Gnoatto, Lavagnini, and Picarelli 2022

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One Step Malliavin (OSM) schemes

Back to FBSDE systems

$$X_t = x_0 + \int_0^t \mu(r, X_r)\mathrm{d}r + \int_0^t \sigma(r, X_r)\mathrm{d}W_r,$$
$$Y_t^j = g^j(X_T) + \int_t^T f^j(r, X_r, Y_r^j, Z_r^j)\mathrm{d}r - \int_t^T Z_r^j \mathrm{d}W_r.$$

Under suitable assumptions $X \in \mathbb{D}^{1,2}(\mathbb{R}^d)$, $Y^j \in \mathbb{D}^{1,2}(\mathbb{R})$, $Z^j \in \mathbb{D}^{1,2}(\mathbb{R}^{1\times d})$, for any $s \leq t$

$$D_s X_t = \sigma(s, X_s) + \int_s^t \nabla_x \mu(r, X_r) D_s X_r \mathrm{d}r + \int_s^t \nabla_x \sigma(r, X_r) D_s X_r \mathrm{d}W_r,$$
$$D_s Y_t^j = \nabla_x g^j(X_T) D_s X_T + \int_t^T \Big[\nabla_x f^j(r, \mathbf{X}_r^j) D_s X_r + \nabla_y f^j(r, \mathbf{X}_r^j) D_s Y_r^j + \nabla_z f^j(r, \mathbf{X}_r^j) D_s Z_r^j\Big]\mathrm{d}r - \int_t^T D_s Z_r^j \mathrm{d}W_r.$$

and $D_t Y_t^j = Z_t^j$ **and** $D_t Z_t^j \sim \mathrm{Hess}_x v^j \sim \Gamma^j$

One Step Malliavin (OSM) schemes

Simultaneous approximation of these **pairs of FBSDEs** $\to$ **One-Step-Malliavin** scheme *(Malliavin chain rule, Feynman-Kac)*

$$Y_N^{j,\pi} = g^j(X_N^\pi), \quad Z_N^{j,\pi} = \nabla_x g^j(X_N^\pi)\sigma(T, X_n^\pi),$$

$$\Gamma_n^{j,\pi} \sim \frac{1}{\Delta t_n}\mathbb{E}_n\Big[\dots\Big], \quad Z_n^{j,\pi} = \mathbb{E}_n\Big[\dots\Big], \qquad Y_n^{j,\pi} = \mathbb{E}_n\Big[\dots\Big]$$

Provides **second-order** Γ_n^π estimates. Sharper Monte Carlo $Z_n^{j,\pi}$. High-dimensional.

Convergence results

1. Standard BSDEs (European): Negyesi, Andersson, and Cornelis W Oosterlee 2024, IMA Jour. Num. Anal.
   ass.: C_b^2 *coefficients, analytical Mall. derivative* — error $\lesssim N^{-1/2}$
2. Extension to discretely reflected BSDEs (Bermudan): Negyesi and C. Oosterlee 2025.
   add. ass.: *risk neutral measure (no Z dependence in f)* — error $\lesssim R^{1/4}N^{-1/2}$
3. Reflected BSDEs (American): limit case *(only thing you can do in a computer)*
   error $\lesssim N^{-1/4}$

Portfolio gamma hedging

Treat the portfolio problem as a **collection** of discretely reflected BSDEs

$$X_t = x_0 + \int_0^t \mu(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s,$$

$$\widetilde{Y}_t := \begin{cases} \widetilde{Y}_t^1 \\ \vdots \\ \widetilde{Y}_t^J \end{cases}$$

$$Y_t := \begin{cases} \text{reflection}(t, X_t, \widetilde{Y}_t^1) \\ \vdots \\ \text{reflection}(t, X_t, \widetilde{Y}_t^J) \end{cases}$$

This results in a (huge) **system** of vector-valued, **discretely reflected BSDEs** where

$$\widetilde{Y}, Y \in \mathbb{R}^J, \quad Z \in \mathbb{R}^{J\times d}, \quad \Gamma \sim DZ \in \mathbb{R}^{J\times d\times d}$$

Deep BSDE – neural network Monte Carlo

Deep BSDE: neural network regression Monte Carlo – similar to Huré, Pham, and Warin 2020; Negyesi, Andersson, and Cornelis W Oosterlee 2024.

1. (Y, Z, Γ) are parametrized by (separate) DNNs at each time instance
2. A merged $L^2(\Omega, \mathbb{P}; \mathbb{R}^{J\times d})$ loss function is defined according to the **martingale representation theorem**

$$\mathcal{L}_n^{z,\gamma}(\theta^z, \theta^\gamma) := \mathbb{E}\Big[\Big|S^z_{t_n,t_{n+1}}(X) + \ldots - \psi(X_n^\pi|\theta^z) + (\chi(X_n^\pi|\theta^\gamma)\sigma(\ldots))^T \Delta W_n\Big|^2\Big] \longrightarrow \widehat{\theta}_n^z, \widehat{\theta}_n^\gamma,$$

$$\mathcal{L}_n^{y}(\theta^y) := \mathbb{E}\Big[\Big|S^y_{t_n,t_{n+1}}(X) + \cdots - \varphi(X_n^\pi|\theta^y) + \psi(X_n^\pi|\widehat{\theta}_n^z)\Delta W_n\Big|^2\Big] \longrightarrow \widehat{\theta}_n^y$$

3. Stochastic Gradient Descent (SGD) steps on finite Monte Carlo samples to approximate

$$(\theta_n^{z,*}, \theta_n^{\gamma,*}) \in \arg\inf_{\theta^z,\theta^\gamma} \mathcal{L}_n^{z,\gamma}(\theta^z, \theta^\gamma) \quad \theta_n^{y,*} \in \arg\inf_{\theta} \mathcal{L}_n^{y}(\theta^y)$$

4. solution into first- and second-order conditions to get $\alpha^i_{t_n}, \beta^k_{t_n}$,

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Single Bermudan(/American) option

$J = 1, m = d$ – European exchange options as Γ instruments (Margrabe)
Single, Black-Scholes (physical measure), Bermudan call $(r = 0, q > 0)$ – Chen and Wan 2021

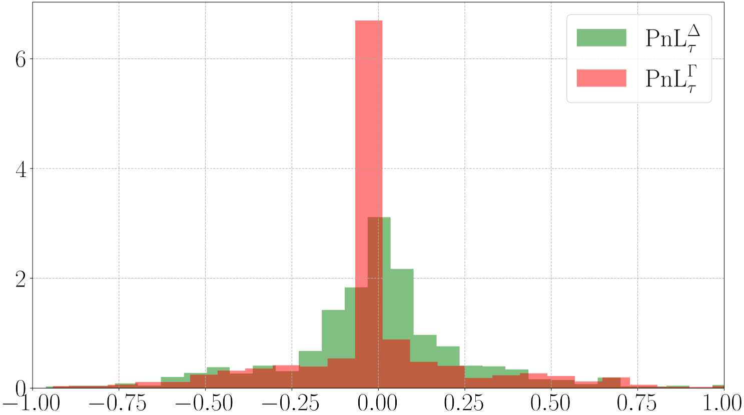

Figure: $d = 1$, $\mathrm{VaR}_{99}^{\Delta} = 27.1\%$ vs $\mathrm{VaR}_{99}^{\Gamma} = 3.3\%$

Single Bermudan(/American) option

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Single, Black-Scholes (physical measure), Bermudan call $(r = 0, q > 0)$ – Chen and Wan 2021

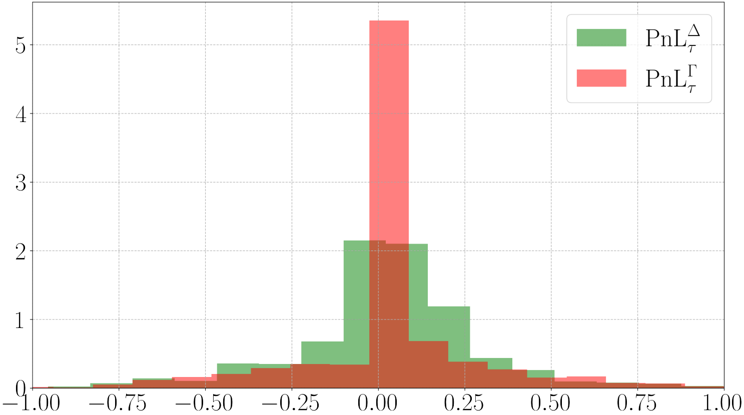

Figure: $d = 50$, $\text{VaR}_{99}^\Delta = 27.6\%$ vs $\text{VaR}_{99}^\Gamma = 5.0\%$

Single Bermudan(/American) option

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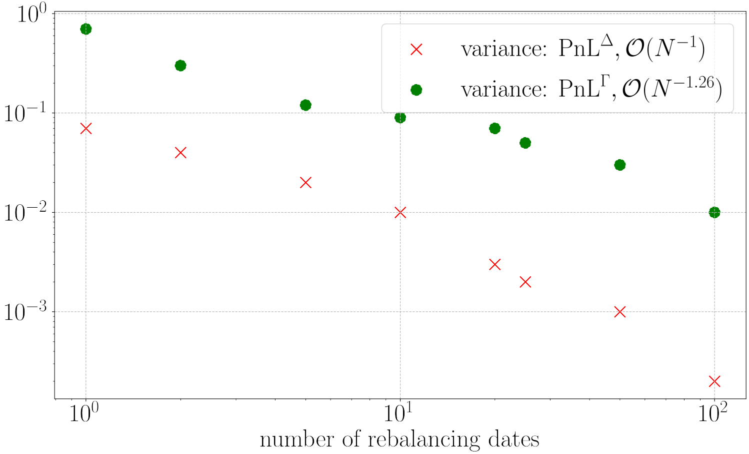
Single, Black-Scholes (physical measure), Bermudan call $(r = 0, q > 0)$ – Chen and Wan 2021



Portfolio

$J = 25, m = d = 20$

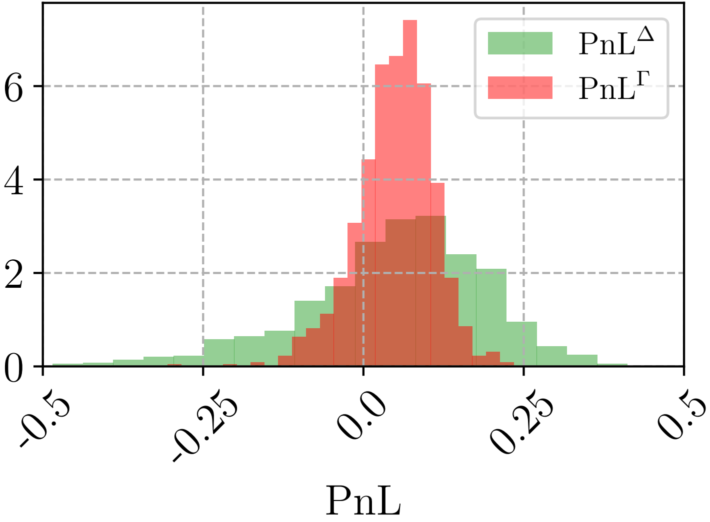
Black-Scholes, physical measure, $T = 1$ year, rebalance monthly, nonuniform pairwise correlation.
Different drift and diffusion coefficient for each asset.

	$\widetilde{Y}, Y$	$Z \sim \Delta$	$DZ \sim \Gamma$
dimensionality	$25 = J$	$500 = J \times d$	$10^4 = J \times d \times d$

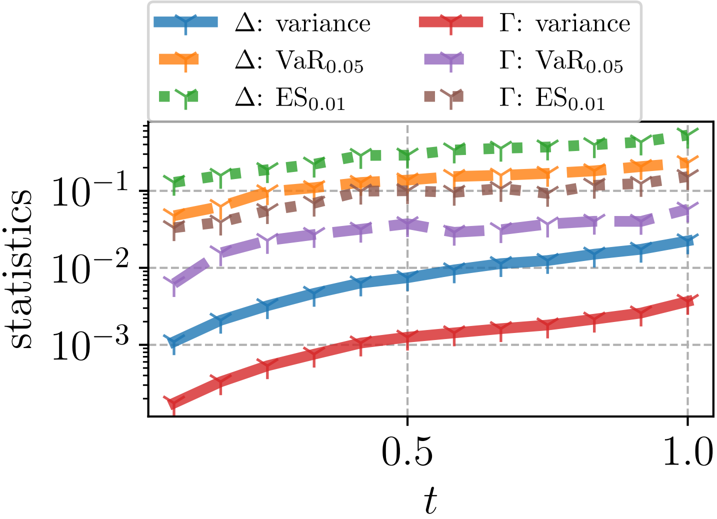
European exchange options as Γ instruments (Margrabe)
Mixture of European, Bermudan, American options including

	underlyings	position	early exercise dates
geometric put	all assets	ATM	monthly
maximum call	half of the assets	ITM	quarterly
cash or nothing	all assets	OTM	semi-anually
several vanilla calls	single assets	ATM/OTM/ITM	{none, any time}
⋮	⋮	⋮	⋮

Portfolio PnL



PnL statistics



PnL statistics

	Δ hedging	Γ hedging
variance	2.2×10^{-2}	3.7×10^{-3}
VaR95	-2.3×10^{-1}	-3.8×10^{-2}
VaR99	-4.0×10^{-1}	-7.2×10^{-2}
ES95	-3.4×10^{-1}	-6.3×10^{-2}
ES99	-5.3×10^{-1}	-8.4×10^{-2}
semivariance	1.5×10^{-2}	1.9×10^{-3}

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Summary

- **Gamma hedging** improves over standard delta hedging in exchange for additional model error
- BSDEs provide an elegant compact formulation to the simultaneous option pricing and delta-hedging problem of European/American/Bermudan options
- **OSM** schemes include second-order sensitivities, **Γs** and thus addresses the additional model error of Γ hedging
- A neural network regression approach yields robust estimates of high-accuracy in all **Greeks** up to Γs even in high-dimensional portfolios

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So Long, and Thanks for All the Fish!

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