Coherent Acceptability Measures in Multiperiod Models

Berend Roorda* Jacob Engwerda† Hans Schumacher‡

July, 2002

Abstract

The coherent risk framework has been introduced by Artzner *et al.* (1999) in a single-period setting. Here we investigate a similar framework in a multiperiod context. Acceptability measures are introduced as a function not only of a given position (payoff in each possible state of nature) but also of available information. A notion of time-consistency for acceptability measures is introduced, and conditions are given for this property to hold if the acceptability measure is expressed in terms of a family of test measures. We present sufficient conditions for the "no strictly acceptable opportunities" condition of Carr *et al.* (2001) to hold in the dynamic context. We show that the effect of hedging can be represented by a change in the set of test measures. Concerning the problem of computing hedges that optimize the degree of acceptability of a given position, we provide sufficient conditions under which an algorithm of dynamic programming type can be applied. For the special case of a derivative on a single underlying with convex payoff, and for a particular class of acceptability measures, we show that this algorithm simplifies considerably and we give explicit formulas for hedges that maximize the degree of acceptability.

Keywords: coherent risk; acceptability measures; robust hedging; bid-ask spread; option pricing; delta hedging; incomplete markets.

JEL Classification: G11; G13.

*B. Roorda, FELab and Dept. of Technology and Management, University of Twente, P.O. Box 217, 7500 AE, Enschede, the Netherlands. Phone: +3153-4894383. E-mail: b.roorda@sms.utwente.nl. This research was supported by the Netherlands Organization for Scientific Research (NWO - MaGW/ESR, project number 510-01-0025).

†J.C. Engwerda, Dept. of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, the Netherlands.

‡J.M. Schumacher, Dept. of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, the Netherlands.

1 Introduction

Many problems in finance come down to deciding the acceptability of a position that generates an uncertain stream of future revenues or losses. Problems of this type arise for instance when firms decide whether or not to undertake a given project, when regulators set limits for the institutions under their supervision, and when clearing house exchanges define margins for their members. Several recent papers have advocated the use of *collections of probability measures* to support acceptability decisions. In this approach, a position is deemed acceptable if, for each probability measure from the given collection, it passes a test based on the expected present value of the position under this measure. Typically a variable is available that can be used to move a position from unacceptable to acceptable, such as premium, collateral, capital reserve, or the amount held in a margin account; by considering the size of the shift in this variable that is needed to make a given position just acceptable, one then obtains a real-valued measure of the degree of acceptability rather than just a binary evaluation.

The recent interest in the approach based on collections of probability measures is to a considerable extent due to a paper by Artzner *et al.* (1999). In this paper, the authors introduce the class of "coherent" risk measures, and they show, assuming finiteness of the set of states of nature, that each coherent risk measure must in fact be obtained from an associated set of probability measures. Subsequently, Jaschke and Küchler (2001) have developed relations between coherent risk measures and valuation bounds in incomplete markets. Carr *et al.* (2001) have shown that collections of probability measures can play a role in financial analysis that is similar in several respects to the one usually associated with the set of states of nature.

The framework used in the cited papers is general in spirit and could be made to apply to multiperiod situations as well, as is done for instance in the paper of Jaschke and Küchler (2001). However, when it comes to actually computing acceptability measures in a multiperiod context, one expects to use backward recursions of the dynamic programming type. This requires the introduction of a notion of acceptability as a function of current state and time, akin to the value function of dynamic programming. It is the purpose of the present paper to provide a framework in which indeed the degree of acceptability is a function of the current state and time. There are computational advantages associated to such a framework, but also some issues arise of a conceptual nature; in particular we will discuss the notion of time-consistency for acceptability measures. In this early stage of development, we restrict ourselves to fully discrete models.

The paper is structured as follows. The next section introduces the basic framework that we use. In particular, we define multiperiod acceptability measures as functions of both position and current state and time, we introduce a notion of time-consistency for such measures, and we discuss the representation of acceptability measures by families of probability measures. Section 3 is concerned with conditions for absence of strictly acceptable opportunities (in the sense of Carr *et al.* (2001)) in the dynamic context. In Section 4, we note that the effect of optimal hedging can be represented by a change in the set of test measures, in analogy with the well known change of measure associated with hedging in complete markets. Section 5 presents sufficient conditions under which an algorithm of the dynamic programming type can be applied to compute the optimal degree of acceptability and the corresponding hedge strategy. Here we also interpret the result of Section 4 in terms of martingale measures. The dynamic programming algorithm is further developed for a particular situation in Section 6, where we consider a market with a single risky asset and a derivative that has a convex payoff. Under a suitable assumption on the type of acceptability measure used, we compute the change in the set of test measures corresponding to optimal hedging both of a short and of a long position in the derivative, and we provide explicit formulas for the optimal hedges. Conclusions follow in Section 7.

Throughout the paper we take the unit of accounting to be a suitably chosen numéraire. In this way the complexity of the notation is somewhat reduced; in particular there is no explicit mention of interest rates.

2 Basic framework

2.1 Single-period setting

Here we briefly review the axiomatic setting for risk measures that was proposed by Artzner *et al.* (1999). To emphasize the broad applicability of this framework, we use the term "acceptability measure" instead of "risk measure."

Let Ω be a finite set, say with n elements. The set of all functions from Ω to $\mathbb{R}$ will be denoted by $\mathcal{X}(\Omega)$ ($\simeq \mathbb{R}^n$). An element X of $\mathcal{X}(\Omega)$ is thought of as a representation of the position that generates outcome $X(\omega)$ when the state $\omega \in \Omega$ arises. An *acceptability measure defined on* Ω is a mapping from $\mathcal{X}(\Omega)$ to $\mathbb{R}$. The number $\rho(X)$ that is associated to the position $X \in \mathcal{X}(\Omega)$ by an acceptability measure ρ is interpreted as the "risk" or "degree of acceptability" of the position X. The formal setting of Artzner *et al.* (1999) furthermore includes a "reference instrument" $r : \Omega \to \mathbb{R}$. Depending on the context, the position r can be interpreted for instance as "amount of capital held in reserve" or "premium received"; in general the interpretation of r is such that the degree of acceptability of any position can be improved by adding a quantity of the position r. Since Artzner *et al.* use the risk measure interpretation of acceptability measures, their sign convention is such that positions that are less acceptable have larger values of the risk measure associated to them. Here we think in terms of acceptability, so that it is more natural to reverse the sign convention.

Our representation of the definitions and results from the paper by Artzner *et al.* has been adapted to this change of convention. Moreover, using the convention that our unit of accounting is a suitably chosen numéraire, we simply take $r = \underline{1}$ where $\underline{1} : \Omega \to \mathbb{R}$ is defined by $\underline{1}(\omega) = 1$ for all ω.

An acceptability measure is said to be *coherent* if it satisfies the following four axioms.

- *Translation property:* $\rho(X + \eta\underline{1}) = \rho(X) + \eta$ for all $\eta \in \mathbb{R}$
- *Superadditivity:* $\rho(X_1 + X_2) \geq \rho(X_1) + \rho(X_2)$
- *Positive homogeneity:* $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda \geq 0$
- *Monotonicity:* $X \geq Y$ implies $\rho(X) \geq \rho(Y)$.

Here we write $X \geq Y$ if $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$. A general motivation of the above principles is provided by Artzner *et al.* (1999).

The following result is fundamental.

Theorem 2.1 (Artzner *et al.* (1999), Prop. 4.1) *An acceptability measure ρ defined on a finite set Ω is coherent if and only if there exists a family $\mathcal{P}$ of probability measures on Ω such that, for all $X \in \mathcal{X}(\Omega)$,*

$$\rho(X) = \inf_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}} X. \tag{2.1}$$

If an acceptability measure ρ satisfies (2.1), it is said to be *represented* by the family $\mathcal{P}$ of probability measures, and the probability measures in the collection $\mathcal{P}$ are sometimes referred to as "test measures" for ρ. The above theorem can be generalized to the case of infinite sample spaces Ω if either the representation by probability measures is replaced by a representation in terms of finitely additive measures (Delbaen (2002)) or a continuity property is added to the coherence axioms (Föllmer and Schied (2002)).

If ρ is a coherent acceptability measure, then $\rho(\eta\underline{1}) = \eta$ for all $\eta \in \mathbb{R}$; this follows from the axiom of positive homogeneity (which implies $\rho(0) = 0$) and from the translation property. For convenience we introduce a separate term for this property.

Definition 2.2 An acceptability measure ρ on a sample space Ω is said to be *normalized* if $\rho(\eta\underline{1}) = \eta$ for all $\eta \in \mathbb{R}$.

In particular, if the set Ω consists of only one element ω, then the only normalized acceptability measure is $\rho(X) = X(\omega)$. The case in which all uncertainty has been resolved will be used below in the multiperiod context as a starting point for backward recursions.

2.2 Multiperiod setting

We now pass to a multiperiod setting. To keep the context as simple as possible we still work with a finite sample space Ω (following Artzner *et al.* (1999) and Carr *et al.* (2001)), but we consider each sample now as a discrete-time trajectory. We begin with introducing some notation and terminology that will be needed below.

2.2.1 Notation and conventions

Let T be a positive integer indicating the number of time periods over which we consider our economy. Let A be a finite set which we shall refer to as the "event set."[1] Define Ω as the set of all sequences $(\alpha_1, \ldots, \alpha_T)$ with $\alpha_i \in A$; we refer to such sequences as "full histories." The collection of sequences $(\alpha_1, \ldots, \alpha_\tau)$ of length τ $(1 \leq \tau \leq T)$ will be denoted by Ω_τ. We write $\Omega'_\tau := \cup_{1 \leq t \leq \tau} \Omega_\tau$ for the set of all sequences of length at most τ. The collection Ω'_T is also written simply as Ω', and we write $\Omega'' := \Omega'_{T-1}$. Elements of both Ω' and Ω'' will be referred to as "partial histories", where the term "partial" should be interpreted in a non-strict sense in the case of Ω' and in a strict sense in the case of Ω''.[2] The length of a sequence $\omega' \in \Omega'$ is denoted by $\tau(\omega')$. The set Ω_0 of sequences of zero length consists of a single element that we denote by 0; this element represents the initial state of the economy. For $\omega = (\alpha_1, \ldots, \alpha_T) \in \Omega$ and $1 \leq \tau \leq T$, define the *τ-restriction* $\omega|_\tau$ as $(\alpha_1, \ldots, \alpha_\tau)$. If a sequence $\omega' = (\alpha_1, \ldots, \alpha_\tau)$ is a *prefix* of $\omega \in \Omega$ (i.e. $\omega' = \omega|_\tau$), we write $\omega' \preceq \omega$. The collection of all sequences beginning with a given sequence ω' is denoted by $F(\omega') := \{\omega \in \Omega \,|\, \omega' \preceq \omega\}$. We denote by $\mathcal{F}_\tau$ the algebra generated by the sets $F(\omega')$ with ω' in the set Ω_τ of sequences of length exactly τ; in the present setting in which we have a finite sample space, this is of course the same as the σ-algebra generated by these sets. The collection $\mathcal{F}_T$ is the set of all subsets of Ω. The *concatenation* of a sequence $\omega' = (\alpha_1, \ldots, \alpha_\tau)$ with an element $\alpha \in A$ is the sequence $(\alpha_1, \ldots, \alpha_\tau, \alpha)$, which we write simply as $\omega'\alpha$. In this paper, we adopt the conventions $\inf \emptyset = \min \emptyset = \infty$ and $0 \cdot \infty = 0/0 = 0$.

2.2.2 Multiperiod acceptability measures

Consider a sequence space Ω as defined in the previous subsection. In the multiperiod setting, the acceptability of a given position should be considered not only as a function of the position itself, but also as a function of available information.

[1] This terminology is appropriate in particular for tree models; for instance in binomial models the event set consists of two elements ("up" and "down"). The framework that we use below applies equally well however to models obtained from discretization of a continuous state space, where A would rather be thought of as a representation of a grid in the state space.

[2] In the context of a non-recombining tree model, there is a one-one relation between the elements of Ω' and the nodes of the tree. The elements of Ω correspond to final nodes, and those of Ω'' to pre-final nodes.

We still define a "position" as a mapping from Ω to $\mathbb{R}$. Such a mapping may be restricted to the set $F(\omega')$ consisting of all sequences beginning with ω'. The restricted mapping $X|_{F(\omega')}$ defines a position on $F(\omega')$. We extend slightly the definition by acceptability measures that was given before by allowing that the degree of acceptability of a given position can be ∞. So, an acceptability measure on $F(\omega')$ is a mapping from $F(\omega')$ to the extended real line $\mathbb{R} \cup \{\infty\}$.

DEFINITION 2.3 A *multiperiod acceptability measure* on the sequence space Ω is a mapping that assigns to each partial history $\omega' \in \Omega'$ an acceptability measure on $F(\omega')$.

The acceptability measure on $F(\omega')$ that is provided by a multiperiod acceptability measure ρ will be denoted by $\rho(\cdot \,|\, \omega')$; the element of the extended real line associated by this mapping to a position X on $F(\omega')$ is denoted by $\rho(X \,|\, \omega')$. When X is a position on Ω, we also write $\rho(X \,|\, \omega')$ instead of $\rho(X|_{F(\omega')} \,|\, \omega')$. The situation at the initial time is represented by the sequence of zero length; instead of $\rho(X \,|\, 0)$, we write $\rho(X)$. Under the normalization condition (Def. 2.2), we have $\rho(X \,|\, \omega) = X(\omega)$ for all $\omega \in \Omega$.

We say that a multiperiod acceptability measure is *coherent* if all partial-information acceptability measures $\rho(\cdot \,|\, \omega')$ are coherent on $F(\omega')$. This implies in particular that, for all positions X and Y and for all partial histories ω', the following holds:

$$\text{if } \rho(X \,|\, \omega) \geq \rho(Y \,|\, \omega) \text{ for all } \omega \succeq \omega', \text{ then } \rho(X \,|\, \omega') \geq \rho(Y \,|\, \omega'). \tag{2.2}$$

We shall say that a multiperiod acceptability measure satisfies the *stepwise monotonicity condition* if the following condition holds for all positions X and Y and for all partial histories $\omega' \in \Omega''$:

$$\text{if } \rho(X \,|\, \omega'\alpha) \geq \rho(Y \,|\, \omega'\alpha) \text{ for all } \alpha \in A, \text{ then } \rho(X \,|\, \omega') \geq \rho(Y \,|\, \omega'). \tag{2.3}$$

The example below shows that there exists situations in which the monotonicity property (2.2) is satisfied but the stepwise monotonicity property (2.3) does not hold.

EXAMPLE 2.4 Consider a two-period binomial tree; that is, let $A = \{u, d\}$ and $\Omega = \{uu, ud, du, dd\}$. Specify an acceptability measure for products on Ω by

$$\rho(X \,|\, \omega') = \min_{i=1,2}(E_{\mathbb{P}_i}[X \,|\, \omega'])$$

where $\mathbb{P}_1$ is the probability measure that is obtained by assigning probability 0.6 to a u event and 0.4 to a d event, and $\mathbb{P}_2$ is obtained by reversing these probabilities. Clearly, ρ is a coherent multiperiod acceptability measure. Consider a position X that pays 100 if ud or du occurs, and that pays nothing otherwise (a "butterfly"). As is easily computed, we have $\rho(X) = 48$ whereas $\rho(X \,|\, u) = \rho(X \,|\, d) = 40$. Comparing the position X to the position Y that pays 44 in all states of nature, we see that ρ is not stepwise monotonic.

The phenomenon in the example may be illuminated somewhat further by considering in general a two-period binomial tree on which an acceptability measure is defined by a finite collection $\{\mathbb{P}_i\}_{i\in I}$ of probability measures. In self-explanatory notation, we can write for a general position X:

$$
\begin{aligned}
\rho(X) &= \min_i [p_i(uu)X(uu) + p_i(ud)X(ud) + p_i(du)X(du) + p_i(dd)X(dd)] \\
&= \min_i \Big[p_i(u)[p_i(u\,|\,u)X(uu) + p_i(d\,|\,u)X(ud)] \\
&\qquad\qquad + p_i(d)[p_i(u\,|\,d)X(du) + p_i(d\,|\,d)X(dd)] \Big]. \qquad (2.4)
\end{aligned}
$$

This expression may be compared to the worst-case expected degree of acceptability at time 1, which is

$$
\begin{aligned}
\min_i \Big[& p_i(u) \min_i [p_i(u\,|\,u)X(uu) + p_i(d\,|\,u)X(ud)] \\
& + p_i(d) \min_i [p_i(u\,|\,d)X(du) + p_i(d\,|\,d)X(dd)] \Big].
\end{aligned}
$$

This quantity is in general *not* equal to the one in (2.4), because the two expressions are related by an interchange of summation and minimization which is in general not without consequence.

A property that is implied by stepwise monotonicity is the following.

Definition 2.5 A multiperiod acceptability measure ρ defined on a sequence space Ω is said to be *time-consistent* if for all partial histories $\omega' \in \Omega''$ and all positions X and Y we have

$$
\text{if } \rho(X\,|\,\omega'\alpha) = \rho(Y\,|\,\omega'\alpha) \text{ for all } \alpha \in A, \text{ then } \rho(X\,|\,\omega') = \rho(Y\,|\,\omega'). \qquad (2.5)
$$

We introduce some further notation and terminology that will be needed below. To each $\omega' \in \Omega''$, one can associate a single-period economy in which the events that may occur (equivalently, the states of nature that may arise after one time step) are parametrized by the event set A. A *single-period position* is a mapping from A to $\mathbb{R}$. A *single-period acceptability measure* is a function that assigns a real number to single-period positions.

Let a multiperiod acceptability measure ρ be given. For any partial history $\omega' \in \Omega''$, one can generate a position X_Y on $F(\omega')$ from a given single-period position $Y : A \to \mathbb{R}$ by defining

$$
X_Y(\omega) = Y(\alpha) \quad \text{if } \omega \succeq \omega'\alpha. \qquad (2.6)
$$

In this way we can introduce for each $\omega' \in \Omega''$ a single-period acceptability measure denoted by $\rho_{\omega'}$:

$$
\rho_{\omega'} : Y \mapsto \rho(X_Y\,|\,\omega'). \qquad (2.7)
$$

The following lemma is easily verified directly from the coherence axioms.

LEMMA 2.6 *If ρ is a coherent multiperiod acceptability measure, then all single-period acceptability measures $\rho_{\omega'}$ derived from ρ are coherent as well.*

Given a product X on the sequence space Ω and a multiperiod acceptability measure ρ, we can define for each partial history a single-period position $\rho(X \,|\, \omega' \cdot)$ in the following way:

$$\rho(X \,|\, \omega' \cdot) : \alpha \mapsto \rho(X \,|\, \omega' \alpha). \tag{2.8}$$

Since this is a single-period position, its acceptability may be evaluated by means of the single-period acceptability measure $\rho_{\omega'}$. If ρ is time-consistent, we have

$$\rho(X \,|\, \omega') = \rho_{\omega'}(\rho(X \,|\, \omega' \cdot)). \tag{2.9}$$

An obvious way to construct a time-consistent acceptability measure is to start by assigning a single-period acceptability measure $\rho^{\omega'}$ to each partial history $\omega' \in \Omega''$ (for instance one may use the same acceptability measure for each ω') and then to define $\rho(\cdot \,|\, \omega')$ recursively by

$$\begin{aligned} \rho(X \,|\, \omega) &= X(\omega) && (\omega \in \Omega) \\ \rho(X \,|\, \omega') &= \rho^{\omega'}(\rho(X \,|\, \omega' \cdot)) && (\omega' \in \Omega''). \end{aligned} \tag{2.10}$$

In the following lemma we verify some properties of this scheme. Extending the terminology introduced in Def. 2.2, we say that a multiperiod acceptability measure ρ is *normalized* if each measure $\rho(\cdot \,|\, \omega')$ is a normalized acceptability measure on $F(\omega')$.

LEMMA 2.7 *The acceptability measure ρ that is defined on the sequence space Ω from a family $\{\rho^{\omega'}\}_{\omega' \in \Omega''}$ via the rule (2.10) is normalized if all measures $\rho^{\omega'}$ are normalized, and in this case we have $\rho_{\omega'} = \rho^{\omega'}$ for all ω'. Moreover, if all $\rho^{\omega'}$ are coherent, then ρ is coherent as well.*

PROOF The first claim follows easily by induction. To show that $\rho_{\omega'} = \rho^{\omega'}$, take a single-period position $Y : A \to \mathbb{R}$, and let X_Y be defined as in (2.6). Due to the normalization property we have $\rho(X_Y \,|\, \omega' \alpha) = Y(\alpha)$ for all $\alpha \in A$, and so

$$\rho_{\omega'}(Y) = \rho(X_Y \,|\, \omega') = \rho^{\omega'}(\rho(X_Y \,|\, \omega' \cdot)) = \rho^{\omega'}(Y).$$

Since this holds for all Y, it follows that $\rho_{\omega'} = \rho^{\omega'}$. Finally, the fact that coherence of ρ follows from coherence of all $\rho^{\omega'}$ can be verified by induction directly from the coherence axioms. $\square$

It follows from the lemma that any time-consistent acceptability measure can be thought of as having been constructed from single-period acceptability measures by means of the rule (2.10).

2.2.3 Representation by collections of probability measures

A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ can be defined in a straightforward way by assigning a probability to each trajectory ω. With slight abuse of notation, we denote the probability that ω will occur by $\mathbb{P}(\omega)$. The marginal probability of a sequence $\omega' \in \Omega'$ is given, with some further abuse of notation, by

$$\mathbb{P}(\omega') = \sum_{\omega' \preceq \omega} \mathbb{P}(\omega) = \mathbb{P}(F(\omega')). \tag{2.11}$$

The conditional probability given a sequence $\omega' \in \Omega'$ of a sequence $\omega \succeq \omega'$ is given by

$$\mathbb{P}(\omega \,|\, \omega') = \frac{\mathbb{P}(\omega)}{\mathbb{P}(\omega')}. \tag{2.12}$$

We will also need the "single-period conditional probabilities" defined by

$$\mathbb{P}^{\mathrm{s}}(\alpha \,|\, \omega') = \frac{\mathbb{P}(\omega'\alpha)}{\mathbb{P}(\omega')}. \tag{2.13}$$

For future reference we note the simple property

$$\mathbb{P}(\omega \,|\, \omega') = \mathbb{P}^{\mathrm{s}}(\alpha \,|\, \omega')\mathbb{P}(\omega \,|\, \omega'\alpha) \quad \text{if } \omega'\alpha \preceq \omega \tag{2.14}$$

and the law of iterated expectations

$$\begin{aligned} E[X \,|\, \omega'] &= \sum_{\omega \succeq \omega'} \mathbb{P}(\omega \,|\, \omega')X(\omega) \\ &= \sum_{\alpha \in A} \mathbb{P}^{\mathrm{s}}(\alpha \,|\, \omega') \sum_{\omega \succeq \omega'\alpha} \mathbb{P}(\omega \,|\, \omega'\alpha)X(\omega) \\ &= E_{\mathbb{P}^{\mathrm{s}}(\cdot \,|\, \omega')} E[X \,|\, \omega'\alpha]. \end{aligned} \tag{2.15}$$

Given a collection $\mathcal{P}$ of probability measures on Ω, we can define a multiperiod acceptability measure $\rho_{\mathcal{P}}$ by defining

$$\rho_{\mathcal{P}}(X \,|\, \omega') = \inf_{\mathbb{P} \in \mathcal{P},\, \mathbb{P}(\omega') > 0} E_{\mathbb{P}}[X \,|\, \omega'] \tag{2.16}$$

for positions X and partial histories ω'. A general multiperiod acceptability measure, even if it is coherent, is not necessarily of the above form; that is to say, the collection of probability measures on $F(\omega')$ that characterize the acceptability measure $\rho(\cdot \,|\, \omega')$ according to Thm. 2.1 need not coincide with the class of conditional probability measures induced on $F(\omega')$ by the probability measures in the family $\mathcal{P}$. Multiperiod acceptability measures of the form (2.16) might be called *completely coherent*. It is of interest to find a set of axioms that supports this notion; we shall not consider this problem here, however.

If a collection $\mathcal{P}$ of probability measures on Ω is given, one can define for each partial history $\omega' \in \Omega''$ a collection of single-period probability measures by

$$\mathcal{P}^{\mathrm{s}}(\omega') = \{\mathbb{P}^{\mathrm{s}}(\cdot \,|\, \omega') \,|\, \mathbb{P} \in \mathcal{P} \text{ with } \mathbb{P}(\omega') > 0\} \tag{2.17}$$

where $\mathbb{P}^{\mathrm{s}}(\cdot\,|\,\omega')$ is defined by (2.13). Conversely, if for each $\omega' \in \Omega''$ a collection of single-period measures $\mathcal{P}_{\mathrm{s}}(\omega')$ is given, then the family $\{\mathcal{P}_{\mathrm{s}}(\omega')\}_{\omega'\in\Omega''}$ defines a collection of probability measures on Ω by

$$\mathcal{P} = \{\mathbb{P}\,|\,\mathbb{P}^{\mathrm{s}}(\omega') \in \mathcal{P}_{\mathrm{s}}(\omega') \text{ for all } \omega' \in \Omega''\}. \tag{2.18}$$

If the above relation holds, we say that the collection $\mathcal{P}$ is *generated* by the family $\{\mathcal{P}_{\mathrm{s}}(\omega')\}_{\omega'\in\Omega''}$. In more concrete terms, the generated probability measures are of the form

$$\mathbb{P} : (\alpha_1, \dots, \alpha_T) \mapsto \prod_{t=1}^{\top} \mathbb{P}^{\mathrm{s}}_t(\alpha_t) \tag{2.19}$$

where for each t

$$\mathbb{P}^{\mathrm{s}}_t \in \mathcal{P}_{\mathrm{s}}((\alpha_1, \dots, \alpha_{t-1})).$$

Starting with a given collection of probability measures $\mathcal{P}$ we can first form its associated family of collections of single-period measures, and then form the collection $\mathcal{P}'$ of measures generated by this family. It may well happen that the collection $\mathcal{P}'$ obtained in this way is larger than the original collection $\mathcal{P}$; see for instance Example 2.4. In the following definition we introduce a term for collections of probability measures that do *not* change under the operation just described.

Definition 2.8 A collection of probability measures $\mathcal{P}$ on a sequence space Ω is said to be *of product type* if

$$\mathcal{P} = \{\mathbb{P}\,|\,\mathbb{P}^{\mathrm{s}}(\cdot\,|\,\omega') \in \mathcal{P}^{\mathrm{s}}(\omega') \text{ for all } \omega' \in \Omega'' \text{ s.t. } \mathbb{P}(\omega') > 0\}. \tag{2.20}$$

Example 2.4 has shown that a multiperiod acceptability measure specified by a collection of test measures via (2.16) is not necessarily time-consistent. The following lemma will be used below to prove that time-consistency does hold for acceptability measures obtained from product-type collections of probability measures.

Lemma 2.9 *Let $\mathcal{P}$ be a product-type collection of test measures, with generating family $\{\mathcal{P}^{\mathrm{s}}(\omega')\}_{\omega'\in\Omega''}$, and let the associated acceptability measure be denoted by ρ. For any product X and any partial history $\omega' \in \Omega''$, we have*

$$\rho(X\,|\,\omega') \;=\; \inf_{P\in\mathcal{P}^{\mathrm{s}}(\omega')} E_P\,\rho(X\,|\,\omega'\alpha). \tag{2.21}$$

Proof Take a partial history ω', a position X, and a test measure $\mathbb{P} \in \mathcal{P}$. By the law of iterated expectations, we have

$$\begin{aligned}
E_{\mathbb{P}}[X \,|\, \omega'] &= E_{\mathbb{P}^{\mathrm{s}}(\,\cdot\,|\,\omega')} E_{\mathbb{P}}[X \,|\, \omega'\alpha] \\
&\geq E_{\mathbb{P}^{\mathrm{s}}(\,\cdot\,|\,\omega')} \inf_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[X \,|\, \omega'\alpha] \\
&= E_{\mathbb{P}^{\mathrm{s}}(\,\cdot\,|\,\omega')} \rho(X \,|\, \omega'\alpha) \\
&\geq \inf_{P\in\mathcal{P}^{\mathrm{s}}(\omega')} E_P \rho(X \,|\, \omega'\alpha).
\end{aligned}$$

Since this holds for all $\mathbb{P} \in \mathcal{P}$, it follows that

$$\rho(X \,|\, \omega') = \inf_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[X \,|\, \omega'] \geq \inf_{P\in\mathcal{P}^{\mathrm{s}}(\omega')} E_P \rho(X \,|\, \omega'\alpha).$$

To show the reverse inequality, take $\varepsilon > 0$ and let $P \in \mathcal{P}^{\mathrm{s}}(\omega')$. Because $\rho(X \,|\, \omega'\alpha) = \inf_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[X \,|\, \omega'\alpha]$, we can choose for each α a probability measure $\mathbb{P}_\alpha \in \mathcal{P}$ such that

$$E_{\mathbb{P}_\alpha}[X \,|\, \omega'\alpha] \leq \rho(X \,|\, \omega'\alpha) + \varepsilon.$$

By the assumption that the collection $\mathcal{P}$ is of product type, there exists a probability measure $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}^{\mathrm{s}}(\cdot \,|\, \omega') = P$ and $\mathbb{P}(\cdot \,|\, \omega'\alpha) = \mathbb{P}_\alpha(\cdot \,|\, \omega'\alpha)$ for all $\alpha \in A$. Then

$$\begin{aligned}
E_{\mathbb{P}}[X \,|\, \omega'] &= E_{\mathbb{P}^{\mathrm{s}}(\,\cdot\,|\,\omega')} E_{\mathbb{P}}[X \,|\, \omega'\alpha] \\
&= E_P E_{\mathbb{P}_\alpha}[X \,|\, \omega'\alpha] \\
&\leq E_P(\rho(X \,|\, \omega'\alpha) + \varepsilon) \\
&= E_P \rho(X \,|\, \omega'\alpha) + \varepsilon.
\end{aligned}$$

It follows that

$$E_P \rho(X \,|\, \omega'\alpha) \geq E_{\mathbb{P}}[X \,|\, \omega'] - \varepsilon \geq \rho(X \,|\, \omega') - \varepsilon.$$

Since this holds for all positive ε, we obtain $E_P \rho(X \,|\, \omega'\alpha) \geq \rho(X \,|\, \omega')$, and since $P \in \mathcal{P}^{\mathrm{s}}(\omega')$ was arbitrary, it follows that

$$\inf_{P\in\mathcal{P}^{\mathrm{s}}(\omega')} E_P \rho(X \,|\, \omega'\alpha) \geq \rho(X \,|\, \omega').$$

This completes the proof. □

We now show that, for coherent multiperiod acceptability measures, the properties of time-consistency and of representability by a product-type collection are in fact equivalent.

Theorem 2.10 *A coherent multiperiod acceptability measure is time-consistent if and only if it can be represented, via relation (2.16), by a product-type collection of probability measures.*

Proof Let a multiperiod acceptability measure ρ be defined by the relation (2.16) where the collection $\mathcal{P}$ is of product type. The preceding lemma shows that, for each position X

and each $\omega' \in \Omega''$, the degree of acceptability $\rho(X \mid \omega')$ is determined in terms of the values of $\rho(X \mid \omega'\alpha)$, and it follows immediately that the acceptability measure ρ is time-consistent.

Conversely, let ρ be a time-consistent and coherent acceptability measure. As noted in Lemma 2.6, the single-period acceptability measures $\rho_{\omega'}$ derived from ρ are all coherent, and so by the representation result Thm. 2.1 there exists for each $\omega' \in \Omega''$ a collection $\mathcal{P}_{\mathrm{s}}(\omega')$ of probability measures on the event set A such that $\rho_{\omega'}(Y) = \inf_{P \in \mathcal{P}_{\mathrm{s}}(\omega')} E_P Y$ for any single-period position Y. Let $\mathcal{P}$ denote the product-type collection of probability measures on Ω that is generated by the family $\{\mathcal{P}_{\mathrm{s}}(\omega')\}_{\omega' \in \Omega''}$. For any position X, we have

$$\begin{aligned}\rho(X \mid \omega') &= \rho_{\omega'}(\rho(X \mid \omega' \cdot)) \\ &= \inf_{P \in \mathcal{P}_{\mathrm{s}}(\omega')} E_P \rho(X \mid \omega'\alpha) \\ &= \rho_{\mathcal{P}}(X \mid \omega')\end{aligned}$$

where we used the preceding lemma in the final equality. This shows that ρ is represented by the product-type collection $\mathcal{P}$. □

3 Absence of strictly acceptable opportunities

In order to discuss the effect of hedging on acceptability, we have to introduce tradables. We assume that n basic assets are present in the market, whose prices are described by a function $S : \Omega' \to \mathbb{R}^n$. For each $0 \le t \le T$, an $\mathcal{F}_t$-measurable function $S_t : \Omega \to \mathbb{R}^n$ is defined by $S_t(\omega) = S(\omega|_t)$. A *trading strategy* is a function from Ω' to $\mathbb{R}^n$, interpreted as a rule that assigns to each partial history $\omega' \in \Omega'$ a position in the basic instruments. Again, if $g : \Omega' \to \mathbb{R}^n$ is a strategy, we write $g_t(\omega) = g(\omega|_t)$. Each trading strategy defines a position, namely the total result of the strategy which is given by

$$H^g := \sum_{t=0}^{T-1} g_t^\top (S_{t+1} - S_t) \tag{3.1}$$

for a self-financing strategy with zero initial investment. Given a basic acceptability measure ρ, we define the acceptability measure of a position X subject to a strategy g by

$$\rho^g(X) := \rho(X + H^g). \tag{3.2}$$

Let us assume that a nonempty set $\mathbb{G}$ of allowed hedging strategies has been fixed. We can then define, for any position X, the optimal degree of acceptability taking hedging into account:

$$\rho^*(X) := \sup_{g \in \mathbb{G}} \rho^g(X). \tag{3.3}$$

In general the supremum need not be finite. If $\rho^*(X) = \infty$ then arbitrarily high degrees of acceptability can be achieved, which may not seem realistic at least in some interpretations of acceptability measures. Therefore we are interested in conditions that ensure finiteness of the supremum in (3.3).

Consider first, as in Carr *et al.* (2001), a single-period economy with traded assets $S^0, \ldots, S^n$ and with a collection $\mathcal{P}$ of probability measures on the finite set Ω of states of nature. The price of asset i at time t ($t = 0, 1$) is given by S_t^i; S^0 is the numéraire which always has price 1. The economy is said to allow *strictly acceptable opportunities* if it is possible to form a strictly acceptable portfolio at zero cost; that is, if there exist portfolio weights $a_0, \ldots, a_n$ such that

$$\begin{aligned} \textstyle\sum_{i=0}^n a_i S_0^i &= 0 & \\ E_{\mathbb{P}} \textstyle\sum_{i=0}^n a_i S_1^i &\geq 0 & \text{for all } \mathbb{P} \in \mathcal{P} \\ E_{\mathbb{P}} \textstyle\sum_{i=0}^n a_i S_1^i &> 0 & \text{for some } \mathbb{P} \in \mathcal{P}. \end{aligned}$$

Carr *et al.* (2001) have argued that if a collection of test measures is chosen sufficiently large so as to reflect a widely held market view, it can be assumed that there will be no strictly acceptable opportunities in the economy. The NSAO condition ("no strictly acceptable opportunities") is a stronger requirement than absence of arbitrage, and in incomplete markets it therefore leads in general to stronger bounds on prices of contingent claims than would be obtained by the no-arbitrage condition alone.[3]

In a multiperiod setting, we interpret the NSAO condition as the requirement that no self-financing investment strategy with zero initial cost should produce a strictly acceptable result. It is easily verified that a necessary condition for the NSAO condition to hold for a given multiperiod economy is that each of the associated single-period economies should be free of strictly acceptable opportunities. As can be seen from simple examples, however, this condition is not sufficient.

Example 3.1 Consider the two-period binomial tree of Example 2.4 again, with the same collection of two test measures. Suppose there are two assets S and B. The value of B is always 100, whereas for S we have

$$S(0) = 100, \quad S(u) = 110, \quad S(d) = 90,$$

$$S(uu) = 120, \quad S(ud) = S(du) = 100, \quad S(dd) = 80.$$

It is easily verified that none of the single-period economies derived from this model allows strictly acceptable opportunities. Now consider the dynamic strategy that is defined as follows. Take no position at the initial time; at time 1, take a position 1 in the asset S (and

[3]In the same spirit, various authors have suggested tightenings of the no-arbitrage bounds on the basis of an assumed measure of acceptability; see for instance Bernardo and Ledoit (1999), Cochrane and Saá Requejo (2000), Jaschke and Küchler (2001), Černý and Hodges (2002).

-1 in B) if an "up" movement occurs, and take the opposite position if a "down" step takes place. The expected result of this strategy under test measure $\mathbb{P}_1$ is

$$0.6 \cdot (0.6 \cdot 10 + 0.4 \cdot (-10)) + 0.4 \cdot (0.6 \cdot (-10) + 0.4 \cdot 10) \;=\; 0.4$$

while under $\mathbb{P}_2$ we find

$$0.4 \cdot (0.4 \cdot 10 + 0.6 \cdot (-10)) + 0.6 \cdot (0.4 \cdot (-10) + 0.6 \cdot 10) \;=\; 0.4.$$

So the expected result is positive in both cases; the "momentum" strategy creates a strictly acceptable opportunity.

In the example, the collection of test measures is not large enough to counterbalance the flexibility of dynamic strategies. An obvious way to extend the set of test measures is to form the product-type collection generated by the single-period probability measures that are implied by the two original test measures; this is the procedure already suggested just before Def. 2.8. For instance, this would generate a measure that assigns probability 0.6 to an "up" movement in the first step but probability 0.4 to the same movement at the second step, conditional on occurrence of an upward movement on the first step. It can easily be seen that the set of eight probability measures obtained in this way is sufficiently large to eliminate all strictly acceptable opportunities in the example economy.

A necessary condition for the multiperiod NSAO condition to hold is that each of the single-period economies satisfies the NSAO property. It will be shown below that this condition is also sufficient if the set of test measures is of product type. As a preparation, we need the following definition and lemma.

Definition 3.2 A set of vectors $\{x_1, \ldots, x_n\} \subset \mathbb{R}^n$ is said to be *positively complete* if there exists no vector $g \in \mathbb{R}^n$ such that $g^\top x_i \geq 0$ for all i and $g^\top x_i > 0$ for some i.

An equivalent formulation is that the nonnegative cone generated by the vectors x_i is the same as the linear span of these vectors; this motivates our terminology. If for each $i = 1, \ldots, N$ we have a set X_i of vectors in $\mathbb{R}^{n_i}$, we can form the *product set* $X = X_1 \times \cdots \times X_N$ which is defined by

$$X := \{ \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \mid x_1 \in X_1, \ldots, x_N \in X_N \} \subset \mathbb{R}^n \tag{3.4}$$

where $n = n_1 + \cdots + n_N$. For instance, the product of a set of three vectors in $\mathbb{R}^2$ and a set of four vectors in $\mathbb{R}^3$ is a set of twelve vectors in $\mathbb{R}^5$. We can now formulate the following lemma.

Lemma 3.3 *A product of positively complete sets is again positively complete.*

Proof For $i = 1, \ldots, N$, let X_i denote a set of k_i vectors in $\mathbb{R}^{n_i}$, and suppose that each set X_i is positively complete. Suppose $(g_1, \ldots, g_N) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$ is such that

$$\sum_{i=1}^{N} g_i^\top x_i \ \geq \ 0$$

for all $(x_1, \ldots, x_N) \in X_1 \times \cdots \times X_N$. We have to show that these conditions imply that

$$\sum_{i=1}^{N} g_i^\top x_i \ = \ 0$$

for all $(x_1, \ldots, x_N) \in X_1 \times \cdots \times X_N$. Take a fixed sequence $(x_2, \ldots, x_N)$ in $X_2 \times \cdots X_N$, and let X_1 consist of the vectors $x_1^1, \ldots, x_{k_1}^1$. By Stiemke's lemma (see for instance Mangasarian (1969), p. 32), there exist positive numbers $\alpha_1, \ldots, \alpha_{k_1}$ such that $\alpha_1 x_1^1 + \cdots + \alpha_{k_1} x_{k_1}^1 = 0$. It follows that

$$0 \leq \sum_{j=1}^{k_1} \alpha_j (g_1^\top x_j^1 + \sum_{i=2}^{N} g_i^\top x_i) = \sum_{j=1}^{k_1} \alpha_j \ \cdot \ \sum_{i=2}^{N} g_i^\top x_i$$

so that in particular

$$\sum_{i=2}^{N} g_i^\top x_i \geq 0.$$

Since $x_2, \ldots, x_N$ were chosen arbitrarily, it follows that the above relation holds for all collections $(x_2, \ldots, x_N)$. The reasoning can be repeated to show that $\sum_{i=3}^{N} g_i^\top x_i \geq 0$ for all collections $(x_3, \ldots, x_N)$. Continuing in this way, we finally find $g_N^\top x_N \geq 0$ for all $x_N \in X_N$. By the assumption in the lemma, it follows that actually we must have $g_N^\top x_N = 0$ for all x_N. Repeating the same exercise for a different order of the indices, we find for all i that $g_i^\top x_i = 0$ for all $x_i \in X_i$. The claim of the lemma follows. □

We can now show the announced result.

Theorem 3.4 *Consider a multiperiod economy with assets $S^1, \ldots, S^n$ and a collection $\mathcal{P}$ of test measures. If the collection $\mathcal{P}$ is of product type, and if no single-period economy allows a strictly acceptable opportunity, then the multiperiod economy allows no strictly acceptable opportunities.*

Proof Suppose that the condition of the theorem holds and let g be a trading strategy such that $E_{\mathbb{P}} H^g \geq 0$ for all $\mathbb{P} \in \mathcal{P}$; we then have to show that in fact $E_{\mathbb{P}} H^g = 0$ for all $\mathbb{P} \in \mathcal{P}$.

To see this, note that we may write

$$E_{\mathbb{P}} H^g = \sum_{\omega\in\Omega} \mathbb{P}(\omega) \left[\sum_{t=0}^{T-1} g_t^\top(\omega)(S_{t+1}(\omega) - S_t(\omega)) \right]$$

$$= \sum_{\omega\in\Omega} \mathbb{P}(\omega) \left[\sum_{\omega'\alpha\preceq\omega} g^\top(\omega')(S(\omega'\alpha) - S(\omega')) \right]$$

$$= \sum_{\omega'\in\Omega''} g^\top(\omega') \sum_{\alpha\in A} \mathbb{P}(\omega'\alpha)[S(\omega'\alpha) - S(\omega')]. \tag{3.5}$$

Lemma 3.3 applies with the collections of vectors $\{\sum_{\alpha\in A} \mathbb{P}(\omega'\alpha)[S(\omega'\alpha) - S(\omega')] \,|\, \mathbb{P} \in \mathcal{P}\}$ playing the role of the collections X_i, and the set of partial trajectories Ω'' playing the role of the index set $\{1, \dots, N\}$. The statement in the theorem follows in this way from the assumption on the single-period economies. □

4 Change of collection of test measures

Given a product X and a set of admissible hedging strategies $\mathbb{G}$, the degree of acceptability of X at time 0 under optimal hedging has been defined as

$$\rho^*(X) \;=\; \sup_{g\in\mathbb{G}} \rho(X + H^g). \tag{4.1}$$

In this way we define a new acceptability measure, and one may ask whether this measure is coherent. Under suitable assumptions, the answer is affirmative, as noted by Jaschke and Küchler (2001). For completeness we provide a brief direct proof of this fact.

Proposition 4.1 *Assume that ρ is a coherent acceptability measure and that the set of admissible strategies $\mathbb{G}$ is a cone, that is to say, all linear combinations of admissible strategies with positive coefficients are themselves admissible strategies. Then the acceptability measure ρ^* defined in (4.1) is coherent.*

Proof If the NSAO property is not satisfied, then $\rho^*(X) = \infty$ for all products X, and the coherence axioms are trivially satisfied. Consider now the case where the NSAO property does hold so that $\rho^*(X)$ is finite for all products X. To prove superadditivity, take $\varepsilon > 0$ and let g_1 and g_2 be such that $\rho(X + H^{g_1}) \geq \rho^*(X) - \varepsilon$ and $\rho(Y + H^{g_2}) \geq \rho^*(Y) - \varepsilon$. Note that $g_1 + g_2$ is an admissible strategy and that $H^{g_1+g_2} = H^{g_1} + H^{g_2}$. We have

$$\begin{aligned}
\rho^*(X+Y) \;&\geq\; \rho(X + Y + H^{g_1+g_2}) \\
&=\; \rho((X + H^{g_1}) + (Y + H^{g_2})) \\
&\geq\; \rho(X + H^{g_1}) + \rho(Y + H^{g_2}) \\
&\geq\; \rho^*(X) + \rho^*(Y) - 2\varepsilon.
\end{aligned}$$

Since this holds for any positive ε, we obtain $\rho^*(X+Y) \geq \rho^*(X) + \rho^*(Y)$ as claimed. The remaining three axioms (translation property, monotonicity, and positive homogeneity) are similarly verified in a straightforward way. $\square$

From the representation result Thm. 2.1, we have the following immediate corollary.

COROLLARY 4.2 *Suppose that the set of admissible strategies is a cone. For every family of test measures $\mathcal{P}$, there exists a family of measures $\mathcal{P}^*$ such that*

$$\rho^*_{\mathcal{P}} = \rho_{\mathcal{P}^*}.$$

The corollary shows that the acceptability of a position under optimal hedging can be computed as an acceptability measure *without* hedging under a *transformed* family of test measures. Note that the transformation from $\mathcal{P}$ to $\mathcal{P}^*$ depends on the chosen set of admissible strategies. It will be shown below that, under suitable circumstances, the class $\mathcal{P}^*$ can be much smaller than the class $\mathcal{P}$.

5 Recursive method

In this section we discuss a recursive method similar to the well known method of dynamic programming. This approach is applicable if our set of test measures is of product type. We also assume that there are no intertemporal constraints on the hedging strategy set $\mathbb{G}$, such as constraints concerning the change of a hedging position from one period to the next. For the purposes of the recursion, we define for a given hedging strategy g:

$$H^g_\tau(\omega) = \sum_{t=\tau}^{T-1} g_t^\top(\omega)(S_{t+1}(\omega) - S_t(\omega)) \quad (\omega \in \Omega). \tag{5.1}$$

Note that the lower limit of the summation in (5.1) is $t = \tau$. In keeping with the standard convention that assigns the value 0 to a sum with no terms, we set

$$H^g_T(\omega) = 0$$

for all histories ω and all strategies g. For a given product X we define

$$\rho^g(X \,|\, \omega') = \rho(X + H^g_{\tau(\omega')} \,|\, \omega') \tag{5.2}$$

and

$$\rho^*(X \,|\, \omega') = \sup_{g \in \mathbb{G}} \rho^g(X \,|\, \omega'). \tag{5.3}$$

THEOREM 5.1 *Consider an economy with assets $S^i : \Omega' \to \mathbb{R}$ $(i = 1, \dots, n)$ and with a collection $\mathcal{P}$ of test measures that is of product type, with generating family $\{\mathbb{P}^{\mathrm{s}}(\omega')\}_{\omega' \in \Omega''}$.*

Assume that a set of admissible hedge strategies $\mathbb{G}$ is given consisting of all functions from Ω' to G where $G \subset \mathbb{R}^n$. For any product X and any partial history $\omega' \in \Omega''$, we have

$$\rho^*(X \mid \omega') \;=\; \sup_{\gamma \in G} \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^*(X \mid \omega'\alpha) + \gamma^\top (S(\omega'\alpha) - S(\omega'))]. \tag{5.4}$$

Proof Take a product X and a partial history ω'; let $\tau = \tau(\omega')$. To show that the left hand side of (5.4) is at most equal to the right hand side, take a strategy $g \in \mathbb{G}$. We have:

$$\begin{aligned}
\rho^g(X \mid \omega') &= \rho(X + H^g_\tau \mid \omega') \\
&= \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P \rho(X + H^g_\tau \mid \omega'\alpha) \\
&= \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P \rho(X + H^g_{\tau+1} + g(\omega')^\top (S(\omega'\alpha) - S(\omega')) \mid \omega'\alpha) \\
&= \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho(X + H^g_{\tau+1} \mid \omega'\alpha) + g(\omega')^\top (S(\omega'\alpha) - S(\omega'))] \\
&\leq \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^*(X \mid \omega'\alpha) + g(\omega')^\top (S(\omega'\alpha) - S(\omega'))] \\
&\leq \sup_{\gamma \in G} \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^*(X \mid \omega'\alpha) + \gamma^\top (S(\omega'\alpha) - S(\omega'))].
\end{aligned}$$

Since this holds for every $g \in \mathbb{G}$, we find

$$\rho^g(X \mid \omega') \leq \sup_{\gamma \in G} \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^*(X \mid \omega'\alpha) + \gamma^\top (S(\omega'\alpha) - S(\omega'))]$$

which completes the first part of the proof. To show the reverse inequality, take $\gamma \in G$ and $\varepsilon > 0$, and let g be a strategy such that

$$\rho^g(X \mid \omega'\alpha) \geq \rho^*(X \mid \omega'\alpha) - \varepsilon$$

and $g(\omega') = \gamma$. We have

$$\begin{aligned}
\rho^*(X \mid \omega') &\geq \rho^g(X \mid \omega') \\
&= \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^g(X \mid \omega'\alpha) + \gamma^\top (S(\omega'\alpha) - S(\omega'))] \\
&\geq \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^*(X \mid \omega'\alpha) + \gamma^\top (S(\omega'\alpha) - S(\omega'))] - \varepsilon.
\end{aligned}$$

Since this holds for every $\gamma \in G$ and every $\varepsilon > 0$, we can take supremum with respect to both γ and ε and obtain

$$\rho^*(X \mid \omega') \geq \sup_{\gamma \in G} \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^*(X \mid \omega'\alpha) + \gamma^\top (S(\omega'\alpha) - S(\omega'))]$$

as required. □

This result also allows us to compute optimal hedge strategies.

Theorem 5.2 *In the setting of Thm. 5.1, assume that for all $\omega' \in \Omega'$ the supremum in (5.4) is achieved, say in $\gamma(\omega')$. Define the strategy g^* by*

$$g^* : \omega' \mapsto \gamma(\omega') \quad (\omega' \in \Omega'').$$

Then we have, for all $\omega' \in \Omega'$,

$$\rho^*(X \mid \omega') = \rho^{g^*}(X \mid \omega'). \tag{5.5}$$

In other words, the strategy g^* optimizes the degree acceptability of the position X.

Proof Take a strategy $g \in \mathbb{G}$ and a product $X : \Omega \to \mathbb{R}$. The proof proceeds by induction with respect to the difference of final time and current time. For complete histories $\omega \in \Omega$, the normalization property implies that $\rho^g(X \mid \omega) = X(\omega)$ for all strategies g and so the condition (5.5) is trivially satisfied. Now assume that (5.5) holds for all partial histories of length $\tau + 1$, and let ω' be a sequence of length τ. Then we can write:

$$\begin{aligned}
\rho^g(X \mid \omega') &= \rho(X + H^g_\tau \mid \omega') \\
&= \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P \rho(X + H^g_\tau \mid \omega'\alpha) \\
&= \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho(X + H^g_{\tau+1} \mid \omega'\alpha) + g(\omega')^\top (S(\omega'\alpha) - S(\omega'))] \\
&\leq \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^*(X \mid \omega'\alpha) + g(\omega')^\top (S(\omega'\alpha) - S(\omega'))] \\
&\leq \sup_{\gamma \in \mathbb{G}} \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^*(X \mid \omega'\alpha) + \gamma^\top (S(\omega'\alpha) - S(\omega'))] \\
&= \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P[\rho^{g^*}(X \mid \omega'\alpha) + (g^*(\omega'))^\top (S(\omega'\alpha) - S(\omega'))] \\
&= \inf_{P \in \mathcal{P}^{\mathrm{s}}(\omega')} E_P \rho(X + H^{g^*}_\tau \mid \omega'\alpha) \\
&= \rho^{g^*}(X \mid \omega').
\end{aligned}$$

This completes the induction step. The statement of the theorem follows. □

The theorem does not claim that there is a unique strategy that optimizes acceptability. It may well happen that at one or more partial histories ω' there are several different values of γ that achieve the supremum in (5.4). In this case there are several different strategies that reach the same optimal level of acceptability.

So far we have excluded intertemporal constraints on hedging strategies in this section, but we have allowed intraperiod restrictions so that for instance short-selling constraints can be accommodated. If hedging positions are unrestricted, it can be shown that the worst-case test measure is always a martingale measure. This is a consequence of the following lemma, of which we provide a proof for completeness.

Before stating the lemma we recall a few conventions and definitions from convex analysis (Rockafellar (1970, §12)). The minimum of an empty subset of $\mathbb{R}$ is taken to be ∞; the same convention applies to the infimum. For a subset C of $\mathbb{R}^n \times \mathbb{R}$, the *lower-bound function* of C is defined by

$$\ell_C(x) = \inf\{y \in \mathbb{R} \mid (x, y) \in C\}. \tag{5.6}$$

This function is (closed) convex when C is (closed) convex (Hiriart-Urruty and Lemaréchal (1991), Thm. IV.1.3.1). The closure of a subset C of $\mathbb{R}^n$ is denoted by $\bar{C}$. The *conjugate* of a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(\gamma) = \sup_{x \in \mathbb{R}^n} \gamma^\top x - f(x).$$

The conjugate of the conjugate of f, denoted by f^{**}, is the convex closure of f.

Lemma 5.3 *Let C be a convex set in $\mathbb{R}^n \times \mathbb{R}$. Then*

$$\sup_{\gamma \in \mathbb{R}^n} \inf_{(x,y) \in C} y - \gamma^\top x = \ell_{\bar{C}}(0) \tag{5.7}$$

and the supremum in (5.7) is achieved by any element of the set of subgradients of $\ell_{\bar{C}}(x)$ at $x = 0$.

Proof We have:

$$\begin{aligned}
\sup_{\gamma \in \mathbb{R}^n} \inf_{(x,y) \in C} y - \gamma^\top x &= \sup_{\gamma \in \mathbb{R}^n} \inf_{x \in \mathbb{R}^n} \inf_{\{y \,|\, (x,y) \in C\}} y - \gamma^\top x \\
&= \sup_{\gamma \in \mathbb{R}^n} \inf_{x \in \mathbb{R}^n} \ell_C(x) - \gamma^\top x \\
&= \sup_{\gamma \in \mathbb{R}^n} -\ell_C^*(\gamma) \\
&= \ell_C^{**}(0) \\
&= \ell_{\bar{C}}(0).
\end{aligned}$$

Furthermore, $\gamma \in \mathbb{R}^n$ is a maximizer in (5.7) if and only if γ defines a supporting hyperplane to the epigraph of ℓ_C at $(0, \ell_C(0))$. Therefore, the set of maximizers of (5.7) coincides with the set of subgradients of $\ell_{\bar{C}}$ at 0. □

The characterization of the optimal hedge that is provided by the lemma will be used in the next section in a more concrete case.

Corollary 5.4 *Assume the setting of Thm. 5.1 with unrestricted hedging positions (i. e. $G = \mathbb{R}^n$); let the collection $\mathcal{P}$ of test measures be convex. Denote by $\mathcal{P}_0^{\mathrm{s}}(\omega')$ the collection of single-step martingale measures in $\mathcal{P}^{\mathrm{s}}(\omega')$; that is to say, $\mathcal{P}_0^{\mathrm{s}}(\omega')$ consists of the measures in $\mathcal{P}^{\mathrm{s}}$ satisfying*

$$E_P S(\omega'\alpha) = S(\omega').$$

If $\mathcal{P}_0^{\mathrm{s}}(\omega')$ is not empty, then

$$\rho^*(X \,|\, \omega') = \inf_{P \in \mathcal{P}_0^{\mathrm{s}}(\omega')} E_P \rho^*(X \,|\, \omega'\alpha). \tag{5.8}$$

Proof The statement follows from (5.4), by applying the lemma above to the set C defined by

$$C = \{(-E_P(S(\omega'\alpha) - S(\omega')), E_P \rho^*(X \,|\, \omega'\alpha)) \,|\, P \in \mathcal{P}^{\mathrm{s}}(\omega')\}.$$

□

Since a discrete-time process S is a martingale if and only if $E[S(\tau+1)\,|\,\mathcal{F}_\tau] = S(\tau)$ for all τ (assuming integrability), we can also write, instead of (5.8),

$$\rho^*(X\,|\,\omega') = \inf_{\mathbb{P}\in\mathcal{P}_0} E_{\mathbb{P}}[X\,|\,\omega'] \tag{5.9}$$

where $\mathcal{P}_0$ denotes the class of measures in $\mathcal{P}$ subject to which the price process S is a martingale. In terms of Cor. 4.2, this means that the class $\mathcal{P}^*$ of that corollary can be taken to be the class of martingale measures in $\mathcal{P}$. The relation (5.9) may also be seen as an extension of Thm. 1 in Carr *et al.* (2001) to the multiperiod setting. Since Carr *et al.* work with a finite collection of test measures, the set C (as in the proof above) generated by the convex hull of these measures is closed. In this case the infimum in (5.8) and (5.9) can be replaced by a minimum. The measure at which the minimum is reached serves in the single-period case as the representative state pricing density of the cited theorem. It may be noted that when the set over which minimization takes place is empty, the relations (5.8) and (5.9) still hold on the basis of the convention that the minimum over an empty set is ∞; in this case, the NSAO condition is not satisfied.

6 Options on a single underlying

In the foregoing sections, we have obtained general expressions for optimal acceptability under hedging. We now apply these results in a special situation. We consider a European option on a single underlying. The option is assumed to have a convex payoff function. The event set A is a finite subset of $\mathbb{R}$; the elements of A are interpreted as the possible relative returns that may be realized after one period. The price of the underlying at the initial time is S_0, and further asset prices are determined in a forward recursion by the rule

$$S(\omega'\alpha) = (1+\alpha)S(\omega') \qquad (\alpha \in A). \tag{6.1}$$

For each single-period economy we use the same acceptability measure, which is derived from a class $\mathcal{P}^{\mathrm{s}}$ of probability measures on A. The overall acceptability measure is the one that is obtained from these single-period acceptability measures. The associated class of overall probability measures on Ω will be denoted by $\mathcal{P}$. We assume that the family $\mathcal{P}^{\mathrm{s}}$ is convex (this is not an essential restriction); a further assumption on $\mathcal{P}^{\mathrm{s}}$ is introduced below. We consider unrestricted hedging strategies.

It is immediate from the results of Carr *et al.* (2001) that the single-period acceptability measure defined by the family $\mathcal{P}^{\mathrm{s}}$ allows no strictly acceptable opportunities if and only if there is a measure $P \in \mathcal{P}^{\mathrm{s}}$ such that $E_P\alpha = 0$. Alternatively, we may say that the NSAO condition holds if and only if the class $\mathcal{P}$ of overall probability measures contains a measure

$\mathbb{P}$ with respect to which the process defined by (6.1) is a martingale. We shall henceforth assume that this condition holds.

To phrase a further assumption on the class $\mathcal{P}^{\mathrm{s}}$ of single-period acceptability measures, we introduce the following definition.

DEFINITION 6.1 Let $\mathcal{P}$ be a collection of zero-expectation probability distributions on $\mathbb{R}$. The collection $\mathcal{P}$ is said to be *pointed below* if there exists a distribution $P_{\min} \in \mathcal{P}$ such that, for all $P \in \mathcal{P}$ and for all convex functions $f(z)$ that are integrable with respect to all $P \in \mathcal{P}$, we have

$$E_{P_{\min}} f(Z) \le E_P f(Z). \tag{6.2}$$

The distribution $P_{\min}$ is then called the *minimal distribution* in $\mathcal{P}$. The collection $\mathcal{P}$ is said to be *pointed above* if there exists a distribution $P_{\max} \in \mathcal{P}$ such that, for all $P \in \mathcal{P}$ and for all convex functions f that are integrable with respect to all $P \in \mathcal{P}$, we have

$$E_P f(Z) \le E_{P_{\max}} f(Z). \tag{6.3}$$

In this case the distribution $P_{\max}$ is called the *maximal distribution* in $\mathcal{P}$. The collection $\mathcal{P}$ is said to be *pointed* if it is both pointed below and pointed above.

The term "pointed" refers to the observation that, if (6.2) holds, the distribution $P_{\min}$ is an extreme element of the set $\mathcal{P}$ with respect to all of the functionals $P \mapsto E_P f$. A slightly more general notion of pointedness below is obtained if the convex functions appearing in (6.2) are required to be positive (which under the zero-expectation requirement does not detract from the strength of the condition) and the integrability condition is dropped. Examples of pointed collections are given below. We first prove a theorem that uses the pointedness property.

THEOREM 6.2 *Consider a multiperiod economy as described in the first paragraph of this section. Assume that the collection $\mathcal{P}_0^{\mathrm{s}}$ defined by*

$$\mathcal{P}_0^{\mathrm{s}} = \{P \in \mathcal{P}^{\mathrm{s}} \mid E_P \alpha = 0\}$$

is pointed with minimal element $P_{\min}$ and maximal element $P_{\max}$. Denote by $\mathbb{P}_{\min}$ and $\mathbb{P}_{\max}$ the probability measures induced on Ω by $P_{\min}$ and $P_{\max}$ respectively. Let a product X be defined by $X(\omega) = f(S(\omega))$ where f is a convex function. Under these conditions, we have for all $\omega' \in \Omega'$:

$$\rho^*(X \mid \omega') = E_{\mathbb{P}_{\min}}[X \mid \omega'] \tag{6.4}$$

and

$$\rho^*(-X \mid \omega') = -E_{\mathbb{P}_{\max}}[X \mid \omega']. \tag{6.5}$$

Proof We prove the validity of (6.4); the proof of (6.5) is similar. Define a sequence of functions f_τ ($\tau = T, T-1, \ldots, 0$) by $f_T(S) = f(S)$ and

$$f_\tau(S) = E_{P_{\min}} f_{\tau+1}((1+\alpha)S). \tag{6.6}$$

By induction with respect to the difference of final time and current time, we will show that the following holds for all times τ and for all partial histories ω':

(i) the function $f_\tau(\cdot)$ is convex

(ii) $\rho^*(X \mid \omega') = f_{\tau(\omega')}(S(\omega'))$

(iii) equation (6.4) is satisfied.

For full histories $\omega \in \Omega$, these statements are valid by definition. Now assume that statements (i)–(iii) hold for all $t \geq \tau + 1$ and for all partial histories of length $\tau + 1$ or more. Item (i) follows from the fact that a linear combination, with positive coefficients, of convex functions is again convex. To show (ii), take $\omega' \in \Omega'_\tau$. From Cor. 5.4, we have

$$\rho^*(X \mid \omega') = \inf_{P \in \mathcal{P}_0^{\mathrm{s}}} E_P[f_{\tau+1}((1+\alpha)S(\omega'))].$$

By the induction assumption, the function $f_{\tau+1}$ is convex; since the convexity of a function is not affected by an affine transformation of the argument, the function $z \mapsto f_{\tau+1}((z+1)S(\omega'))$ is convex as well. Because the family $\mathcal{P}_0^{\mathrm{s}}$ has been assumed to be pointed, we find that in fact

$$\rho^*(X \mid \omega') = E_{P_{\min}}[f_{\tau+1}((1+\alpha)S(\omega'))]. \tag{6.7}$$

By definition, the right hand side is equal to $f_\tau(S(\omega'))$ and so item (ii) is proved. Finally, note that from (6.7) and the induction assumption we have

$$\rho^*(X \mid \omega') = E_{P_{\min}} \rho^*(X \mid \omega'\alpha) = E_{P_{\min}} E_{\mathbb{P}_{\min}}[X \mid \omega'\alpha] = E_{\mathbb{P}_{\min}}[X \mid \omega'].$$

This completes the proof. □

Note that X and $-X$ correspond to a long and a short position in the option, respectively. The quantities $\rho^*(X)$ and $-\rho^*(-X)$ can be seen as a lower and an upper bound for *bid price* and *ask price* respectively. The theorem may be compared to Cor. 4.2. In terms of that corollary, the family $\mathcal{P}^*$ that determines the acceptability measure under optimal hedging consists, in the situation considered in the theorem, only of a single element.

We now consider examples of pointed collections of probability measures. A simple observation is the following.

Proposition 6.3 *If a family $\mathcal{P}$ of zero-expectation probability distributions on $\mathbb{R}$ contains the distribution P_0 that assigns probability 1 to outcome 0, then $\mathcal{P}$ is pointed below and P_0 is its minimal distribution.*

PROOF The claim is just a reformulation of Jensen's inequality. □

Somewhat less restrictive sufficient conditions for pointedness are given in the next proposition. For any Borel subset A of $\mathbb{R}$ such that 0 is in the convex hull of A, let $\mathcal{P}_A$ denote the family of probability distributions P that satisfy $P(A) = 1$, and let $\mathcal{P}_A^0$ denote the subfamily of distributions in $\mathcal{P}_A$ that have zero expectation.

PROPOSITION 6.4 *If $A \subset \mathbb{R}$ is finite, then the family $\mathcal{P}_A^0$ is pointed.*

PROOF We first prove that $\mathcal{P}_A^0$ is pointed below. This follows from the previous proposition if $0 \in A$. Suppose now that the set A does not contain 0; then A is of the form $A = \{\alpha_1, \ldots, \alpha_n\}$ with

$$\alpha_1 < \cdots < \alpha_k < 0 < \alpha_{k+1} < \cdots < \alpha_n$$

and $1 \leq k \leq n-1$. We claim that the minimal distribution $P_{\min}$ for the associated family $\mathcal{P}_A^0$ is the one that assigns mass p to α_k and $1-p$ to α_{k+1}, where p is chosen such that the zero-expectation requirement is satisfied:

$$p = \frac{\alpha_{k+1}}{\alpha_{k+1} - \alpha_k}.$$

We prove this by induction with respect to n. For $n = 2$ the indicated measure is in fact the only element of $\mathcal{P}_A^0$ and so the claim is trivially true. Assume now that the claim is valid for all sets A with at most n points. Let f be a convex function, and let A be a set having $n+1$ points. We parametrize the family $\mathcal{P}_A^0$ by the weights p_i assigned to the points α_i. Our optimization problem may then be formulated as follows:

$$\begin{aligned} \text{minimize} \qquad & \sum_{i=1}^{n+1} p_i f(\alpha_i) \\ \text{subject to} \qquad & p_i \geq 0 \quad (i = 1, \ldots, n+1) \\ & \sum_{i=1}^{n+1} p_i = 1 \\ & \sum_{i=1}^{n+1} p_i \alpha_i = 0. \end{aligned}$$

Without loss of generality, we may assume that $k + 1 \leq n$. For a fixed choice of p_{n+1}, the optimal choice of the weights $p_1, \ldots, p_n$ is determined by the optimization problem

$$\begin{aligned} \text{minimize} \qquad & \sum_{i=1}^{n} p_i f(\alpha_i) \\ \text{subject to} \qquad & p_i \geq 0 \quad (i = 1, \ldots, n) \\ & \sum_{i=1}^{n} p_i = 1 - p_{n+1} \\ & \sum_{i=1}^{n} p_i \alpha_i = 0. \end{aligned}$$

An optimal solution $(p_1, \ldots, p_n)$ of this problem can be transformed to a minimal distribution for the family associated with $A' := \{\alpha_1, \ldots, \alpha_n\}$ by multiplying all p_i's by $1/(1-p_{n+1})$. Therefore, it follows from the induction assumption that a minimal distribution measure for A must be present in the one-parameter family of measures $(0, \ldots, 0, p_k, p_{k+1}, 0, \ldots, 0, p_{n+1})$, where p_k and p_{k+1} are determined from p_{n+1} by the constraints

$$p_k + p_{k+1} + p_{n+1} = 1, \qquad p_k\alpha_k + p_{k+1}\alpha_{k+1} + p_{n+1}\alpha_{n+1} = 0.$$

Taking these two constraints into account, we can think of $E_P f(\alpha) = p_k f(\alpha_k) + p_{k+1} f(\alpha_{k+1}) + p_{n+1} f(\alpha_{n+1})$ as a function of p_{n+1} only. Note that this function is linear. Direct computation shows the derivative of $E_P f(\alpha)$ with respect to p_{n+1} to be

$$\frac{d}{dp_{n+1}} E_P f(\alpha) = \frac{\alpha_{n+1} - \alpha_k}{\alpha_{k+1} - \alpha_k} \left[\frac{\alpha_{n+1} - \alpha_{k+1}}{\alpha_{n+1} - \alpha_k} f(\alpha_k) + \frac{\alpha_{k+1} - \alpha_k}{\alpha_{n+1} - \alpha_k} f(\alpha_{n+1}) - f(\alpha_{k+1}) \right]. \tag{6.8}$$

The numbers $(\alpha_{n+1} - \alpha_{k+1})/(\alpha_{n+1} - \alpha_k)$ and $(\alpha_{k+1} - \alpha_k)/(\alpha_{n+1} - \alpha_k)$ are nonnegative, sum to 1, and satisfy

$$\frac{\alpha_{n+1} - \alpha_{k+1}}{\alpha_{n+1} - \alpha_k} \alpha_k + \frac{\alpha_{k+1} - \alpha_k}{\alpha_{n+1} - \alpha_k} \alpha_{n+1} = \alpha_{k+1}.$$

Therefore it follows from the convexity of f that the right hand side of (6.8) is nonnegative. Also note that $p_{n+1} = 0$ is feasible, in the sense that under this choice of p_{n+1} the inequality constraints $p_i \geq 0$ for $i = k, k+1, n+1$ are satisfied. It follows that it is optimal to take $p_{n+1} = 0$.

In the same way, it can be shown that there is a maximal distribution $P_{\max}$ in $\mathcal{P}_A^0$. This distribution is obtained by concentrating all mass at the two outermost points of A. □

For an example of a family of probability distributions that is *not* pointed, take the standard normal distribution and a uniform distribution concentrated on $[-a, a]$ where $\sqrt{3} < a < \sqrt[4]{15}$. The second moment of the uniform distribution is larger than 1 but its fourth moment is less than 3, which shows that the two distributions are ordered differently by the functionals $P \mapsto E_P f(\alpha)$ induced by the convex functions $f_1(\alpha) = \alpha^2$ and $f_2(\alpha) = \alpha^4$.

In concrete cases such as the one of the family $\mathcal{P}_A$ associated to a finite set A, the second claim in Lemma 5.3 can be used to find the hedging strategy that optimizes acceptability. This is shown in the following proposition.

Proposition 6.5 *Consider the situation of Thm. 6.2, and assume that the family of test measures is given by the collection $\mathcal{P}_A$ where A is the finite event set. Assume that 0 is an element of the convex closure of A but not of A itself, and let α_l and α_r be the largest negative and the smallest positive element of A, respectively. Under these assumptions, the strategy that optimizes acceptability of the position X is given by*

$$g^*(\omega') = -\frac{\rho^*(X \mid \omega'\alpha_r) - \rho^*(X \mid \omega'\alpha_l)}{S(\omega'\alpha_r) - S(\omega'\alpha_l)}. \tag{6.9}$$

Proof According to Lemma 5.3, any subgradient of the function

$$x \mapsto \min\{E_P f_{\tau+1}((1+\alpha)S(\omega')) \mid P \in \mathcal{P},\ S(\omega')E_P\alpha = -x\} \tag{6.10}$$

at $x = 0$ is an optimal hedge at the node ω'. It follows from reasoning as in the proof of Prop. 6.4 that for sufficiently small $|x|$ the distribution that achieves the minimum in the above expression is the one that concentrates all mass at the points α_l and α_r. Therefore the right hand side in (6.10) is, for sufficiently small $|x|$, equal to

$$p_l(x)f_{\tau+1}((1+\alpha_l)S(\omega')) + p_r(x)f_{\tau+1}((1+\alpha_r)S(\omega'))$$

where p_l and p_r are determined as functions of x by

$$p_l + p_r = 1, \quad p_l\alpha_l + p_r\alpha_r = -\frac{x}{S(\omega')}.$$

It follows that p_l and p_r are in fact linear functions of x, and their derivatives satisfy

$$p_l' + p_r' = 0, \quad p_l'\alpha_l + p_r'\alpha_r = -\frac{1}{S(\omega')}.$$

Consequently, the function (6.10) has a unique subgradient at $x = 0$ which is given by

$$-\frac{f_{\tau+1}((1+\alpha_r)S(\omega')) - f_{\tau+1}((1+\alpha_l)S(\omega'))}{(\alpha_r - \alpha_l)S(\omega')}.$$

In view of (6.1) and relation (ii) in the proof of Thm. 6.2, the above expression may also be written as the right hand side of (6.9). □

The strategy that is found to be optimal is exactly the hedge that should be applied to replicate the option in a binomial model based on the returns α_l and α_r (Cox *et al.* (1979)). A similar result can be obtained for the strategy that optimizes the acceptability of a short position in an option with a convex payoff; in this case the optimal strategy uses the replicating hedge in the binomial model corresponding to the extreme returns allowed by the set A. So the long position is hedged based on the model that has minimal volatility in the model class considered, and the hedge of a short position is based on the maximum-volatility model. These results are in line what has been found in the continuous-time literature on uncertain volatility models (see for instance Lyons (1995) and Avellaneda and Parás (1996)), and in the discrete-time literature on interval models (Kolokoltsov (1998), Roorda *et al.* (1999)). Of course, the assumption of a convex payoff is crucial in obtaining such a simple characterization of the optimal hedge. For options with non-convex payoffs the optimal hedge can still be computed by the general method outlined in Section 5, although the computational cost may be high.

Families of the type $\mathcal{P}_A$ where A is a finite set allow limited scope for modeling. More flexibility can be obtained, still assuming a finite event set A, by placing constraints on the way that probability mass may be distributed across the points of A. Such constraints can be formulated in many ways; one possibility is the following.

Definition 6.6 Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a collection of Borel subsets of $\mathbb{R}$ which are ordered in the sense that $\alpha < \alpha'$ whenever $\alpha \in A_i$, $\alpha' \in A_j$, and $i < j$. Let $p = (p_1, \ldots, p_m)$ be a vector of nonnegative numbers summing to 1. The *interval probability model* associated to the collection $\mathcal{A}$ and the vector p is the class of probability distributions on $\mathbb{R}$ given by

$$\mathcal{P}_{\mathcal{A}}^{p} = \{P \mid P(A_i) = p_i\}. \tag{6.11}$$

To obtain an interval probability model that fits into the framework of this paper, the subsets A_i should be finite. If a given interval probability model contains zero-expectation distributions, then the family of such distributions can be shown to be pointed and so one obtains results similar to the ones above. The maximum-volatility and minimum-volatility distributions now correspond to multinomial models however, rather than to binomial models, and so there is more room for calibration.

Another way of defining a class of probability distributions on $\mathbb{R}$ associated to a finite collection of Borel subsets, not necessarily ordered as in the above definition, is to consider all distributions such that $P(A_i) \leq P(A_j)$ for $i \leq j$. A model of this type has been used by Bühler (1981) in the context of capital budgeting. This model shares with the interval probability model the desirable property that the infimization problem in (2.21) takes the form of a linear program.

7 Conclusions

The development of multiperiod acceptability measures has, under various headings and in several different contexts, been a long-standing subject of interest in finance. The idea of characterizing acceptability in terms of a suitable collection of probability measures can be applied in a multiperiod context, and this paper has contributed some notions that may be relevant in that setting. Working within a discrete-state discrete-time context, we have defined a multiperiod acceptability measure as a function of both the position held and the current state of nature. We extended the coherence concept to multiperiod acceptability measures. A notion of time-consistency for such measures was introduced, and we found that in the case of coherent measures time-consistency is equivalent to representability by a product-type collection of test measures. The product property was shown to allow a convenient characterization of absence of strictly acceptable opportunities in the multiperiod setting. Turning to dynamic hedging, we noted that acceptability under optimal hedging is equivalent to acceptability under a transformed collection of test measures which under suitable assumptions consists of the martingale measures in the original collection. Assuming again the product property as well as absence of intertemporal constraints on hedging strategies, we have presented an algorithm of the dynamic programming type to compute the optimal degree of acceptability as well as the corresponding optimal hedge strategy. In an economy with a single risky asset, and under a further assumption on the class of test measures used, the dynamic programming algorithm was shown to take on a particularly simple form for options with convex payoffs.

We have worked with the simplest framework that allows the introduction of partially revealed information and dynamic hedging. Obviously it would be of interest to extend the theory developed here to continuous-state and continuous-time models. In this paper we have admitted the axiom of positive homogeneity following Artzner *et al.* (1999); the development of a theory of multiperiod acceptability measures that do not satisfy this axiom is another topic of interest. For practical purposes, classes of collections of test measures are needed that are flexible enough to express actual preferences seen in the market and that are at the same time computationally feasible for purposes of calibration, pricing, and hedging. Some possible choices have been mentioned in this paper, such as the interval probability model; clearly, however, there is considerable scope for further analytical as well as empirical research in this area.

References

Artzner Ph., F. Delbaen, J.-M. Eber, and D. Heath, 1999. Coherent measures of risk. *Mathematical Finance*, **9**, 203–228.

Avellaneda M. and A. Parás, 1996. Managing the volatility risk of portfolios of derivative securities: the Langrangian uncertain volatility model. *Applied Mathematical Finance*, **3**, 21–52.

Bernardo A. and O. Ledoit, 1999. Gain-loss and asset pricing. *Journal of Political Economy*, **108**, 144–172.

Carr P., H. Geman, and D. B. Madan, 2001. Pricing and hedging in incomplete markets. *Journal of Financial Economics*, **32**, 131–167.

Černý A. and S. Hodges, 2002. The theory of good-deal pricing in financial markets. In: H. Geman, D. Madan, S.R. Pliska, T. Vorst (eds.), *Mathematical Finance — Bachelier Congress 2000*, Springer, Berlin, pp. 175–202.

Cochrane J. and J. Saá Requejo, 2000. Beyond arbitrage: good-deal asset price bounds in incomplete markets. *Journal of Political Economy*, **108**, 79–119.

Cox J., S. Ross, and M. Rubinstein, 1979. Option pricing: a simplified approach. *Journal of Financial Economics*, **7**, 229–263.

Delbaen F., 2002. Coherent risk measures on general probability spaces. In: K. Sandmann, P. J. Schönbucher (eds.), *Advances in Finance and Stochastics. Essays in Honour of Dieter Sondermann*, Springer, Berlin, pp. 1–38.

Föllmer H. and A. Schied, 2002. Robust preferences and convex measures of risk. In: K. Sandmann, P. J. Schönbucher (eds.), *Advances in Finance and Stochastics. Essays in Honour of Dieter Sondermann*, Springer, Berlin, pp. 39–56.

Hiriart-Urruty J.-B. and C. Lemaréchal, 1993. *Convex Analysis and Minimization Algorithms I.* Springer, Berlin.

Jaschke S. and U. Küchler, 2001. Coherent risk measures and good-deal bounds. *Finance and Stochastics*, **5**, 181–200.

Kolokoltsov V. N., 1998. Nonexpansive maps and option pricing theory. *Kybernetika*, **34**, 713–724.

Lyons T. J., 1995. Uncertain volatility and the risk-free synthesis of derivatives. *Applied Mathematical Finance*, **2**, 117–133.

Mangasarian O. L., 1969. *Nonlinear Programming.* McGraw-Hill, New York.

Rockafellar R. T., 1970. *Convex Analysis.* Princeton University Press, Princeton, NJ.

Roorda B., J. C. Engwerda, and J. M. Schumacher, 1999. Performance of delta-hedging strategies in interval models — a robustness study. CentER Discussion Paper 9905, Tilburg University.