# Martingale characterizations of coherent acceptability measures

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Submitted August 9, 2002

**Abstract**

The coherent risk framework is linked to martingale valuation by adding hedge-invariance as a fifth axiom, motivated by the concept of consistent hedging. The resulting subclass, called coherent pre-hedge (CoPr) measures, is characterized by a martingale condition on the test set that underlies a coherent measure. It is also made explicit how consistent hedging, optimal as well as non-optimal, transforms the test set of a given coherent measure into a martingale test set. These results are put in perspective of the fundamental theorems of asset pricing and the concept of valuation bounds.

*Keywords: coherent risk measures; acceptability measures; martingale measures; incomplete markets; asset pricing; valuation bounds*

*JEL Classification:* G12;G13;D52.

## 1 Introduction

According to the fundamental theorem of asset pricing, arbitrage-free prices of assets in ideal markets can be expressed as the expected value under a martingale measure of their random future worth, in units of a numeraire. In this paper we extend this martingale characterization to the class of coherent acceptability measures that are hedge-invariant: they can be expressed as the worst expected value over a set of martingale measures. Reflecting the perspective, linear pricing functions are those coherent acceptability measures for which the underlying martingale set is a singleton. Whereas linear pricing functions ignore transaction costs in the entire market, hedge-invariant coherent acceptability measures only ignore these for hedging instruments, providing only price limits for positions in general.

The key idea behind our approach is that *before a hedging decision is made* out of a given set of hedging possibilities, called the hedge set, there is no need to discriminate between the acceptability of two positions that admit exactly the same set of hedged positions: under 'consistent' hedging, that consistently favors one out of a given set of feasible hedged positions, both position will transform to the same. From this perspective, it makes sense to impose that acceptability is invariant under hedging, in particular if the hedge set is linear.

In order to stress that this acceptability concept anticipates an appropriate hedging action, we use the term pre-hedge acceptability measures, in contrast to what we sometimes will call post-hedge or final acceptability measures, for which hedge-invariance would be an absurd requirement. Pre-hedge measures are introduced in Section 3.

An acceptability measure is called coherent if it satisfies the four coherence axioms, with riskiness defined as acceptability with a minus sign; we refer to Artzner et al. 1999 (ADEH for short), for an extensive description of and motivation for the coherent framework. For

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completeness, we include the axioms in terms of acceptability in the next section, together with some basic facts on coherent measures. Of particular interest in our setup is the representation result (ADEH, Proposition 4.1) stating that coherent measures amount to taking worst expected value over a set of probability measures, called the test set.[1]

In view of the considerations above, hedge-invariant coherent acceptability measures are called coherent pre-hedge (CoPr) measures; we often omit the term acceptability as the term also applies to risk measures in the obvious way. The class of CoPr measures is the central object in this paper. It is introduced in Section 4, where it is also shown that for linear hedge sets, CoPr measures amount to taking worst expected value over a martingale test set.

We then focus on the relation between pre- and post-hedge acceptability measures, at the level of their test sets. First, in Section 5, we concentrate on hedging that maximizes acceptability under a given measure. We derive that the corresponding 'maximum acceptability measure' is the CoPr measure with test set consisting of the martingale measures in the closed convex hull of the test set of the given post-hedge measure. To complete the picture, also non-optimal hedging is considered, and the martingale inclusion theorem in Section 6 characterizes *all* CoPr measures that can be obtained by consistent hedging.

Finally, we put our results in perspective of the fundamental theorems of asset pricing and the concept of valuation bounds, and reformulate several results in Jaschke and Küchler (2001) and Carr et al. (2001) at the relatively concrete level of martingale test sets.

A simple, stylized example illustrates the exposition throughout.

# 2 Preliminaries

## 2.1 Notation

$\Omega$ is a finite set of $n$ elements, each $\omega \in \Omega$ representing an 'outcome' or future 'state of nature'. $\Pr(\Omega)$ denotes the set of all probability measures on $\Omega$, which we identify with the unit simplex in $\mathbb{R}^n$. Further, $\mathcal{X}(\Omega)$ is the class of all real valued functionals on $\Omega$, which is equivalent to $\mathbb{R}^n$; in ADEH this space is denoted as $\mathcal{G}$. $E_P[\cdot]$ denotes the expected value under a probability measure $P \in \Pr(\Omega)$. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, so $E_P X = \langle P, X \rangle$. $\mathcal{S}^{\perp}$ denotes the orthogonal complement of a set $\mathcal{S} \subset \mathbb{R}^n$; $\mathrm{ch}(\mathcal{S})$ its convex hull, $\mathrm{cch}(\mathcal{S})$ the closure of its convex hull, and $\mathrm{ext}(\mathcal{S})$ the exterior of $\mathrm{cch}(S)$.

## 2.2 Coherent Framework

In this section we summarize the coherent risk framework, in as far we use it in our exposition. The main differences with the setup in ADEH is that our notation is in relative prices, and that our formulation is in terms of acceptability, which is, mathematically spoken, just riskiness with a minus sign, but economically may refer to a somewhat broader interpretation.

An acceptability measure assigns a level of acceptability to a financial position. This is formalized as follows. Let $X : \Omega \to \mathbb{R}$ denote the future net worth of a position as a function of a finite set of 'states of nature' $\Omega$; so $X \in \mathcal{X}(\Omega)(\simeq \mathbb{R}^n)$. Values are expressed in units of a reference instrument, which itself then corresponds to $X(\omega) = 1$ for all $\omega \in \Omega$. In principle, any tradable instrument with a well defined payoff on $\Omega$ may be chosen as numeraire, but in view of the interpretation in terms of acceptability it should have relative secure cash flows, as argued in Jaschke and Küchler (2001).

An acceptability measure is defined as a function $\rho : \mathcal{X}(\Omega) \to \mathbb{R}$, which quantifies the acceptability of $X$ by $\rho(X)$. A position is considered as unacceptable if $\rho(X) < 0$, and as acceptable if $\rho(X) \geq 0$. Unacceptable deals are also called bad deals; good deals are strictly acceptable, and OK-deals have zero acceptability.

[1]The terminology is inspired by Carr et al. (2001), where elements of the test set are called (valuation) test measures. In ADEH $\mathcal{P}$ is called a generalized scenario set.

An acceptability measure is called coherent if it satisfies the following axioms.[2]

- *Translation invariance:* $\rho(X+\alpha) = \rho(X) + \alpha$
- *Superadditivity:* $\rho(X_1 + X_2) \geq \rho(X_1) + \rho(X_2)$
- *Positive Homogeneity:* $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda \geq 0$
- *Monotonicity:* $X \leq Y$ implies $\rho(X) \leq \rho(Y)$

Here $X \leq Y$ means that $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$.

The value $\rho(X)$ can be interpreted as the maximum acceptable price for $X$, as from translation invariance it follows that $\rho(X - \rho(X)) = 0$. Similarly, $-\rho(-X)$ is the minimum acceptable premium for selling $X$.

Although the setup suggests a one step economy, multiperiod models can be dealt with by an outcome space $\Omega$ consisting of sequences of events. For explicit dynamical aspects, such as the evolution of acceptability over time, and recursive evaluation of acceptability measures, we refer to Roorda et al. (2002).

## 2.3 Representation by Test Sets

Central in our exposition is the following representation result in ADEH, the credit for the proof being given to Huber (1981). We formulate it in the version of acceptability measures, in relative prices.

Proposition 2.1 (ADEH, prop 4.1)
*An acceptability measure $\rho : \mathcal{X}(\Omega) \to \mathbb{R}$, with $\Omega$ finite,[3] is coherent if and only if there exists a family of probability measures $\mathcal{P} \subset \Pr(\Omega)$ such that for all $X \in \mathcal{X}(\Omega)$*

$$\rho(X) = \inf_{P \in \mathcal{P}} E_P\, X \tag{1}$$

The notion of coherent acceptability is hence equivalent to worst expected value over a set of probabilities, called the test set. The coherent acceptability measure corresponding to $\mathcal{P}$ is referred to as $\rho_{\mathcal{P}}$. In order to streamline the exposition, also empty test sets are allowed, corresponding to $\rho(\cdot) = \infty$ by convention.

## 2.4 Convexity Results

We conclude the preliminaries with some elementary convexity results, mostly remaining just below the surface in ADEH. They partly rely on the well-known *hyperplane separation theorem*,[4] which we will exploit via the following lemma. Recall that $\Pr(\Omega)$ denotes the set of all probability measures on a finite outcome space $\Omega$.

Lemma 2.2
*For $\mathcal{C}, \mathcal{D} \subset \Pr(\Omega)$ non-empty, disjoint, closed convex sets, there exists $X' \in \mathcal{X}(\Omega)$ such that*

$$\rho_{\mathcal{C}}(X') := \inf_{C \in \mathcal{C}} E_C[X'] > 0 \geq \sup_{D \in \mathcal{D}} E_D[X'] =: -\rho_{\mathcal{D}}(-X'). \tag{2}$$

*Moreover, if $\mathcal{D} = \mathcal{L} \cap \Pr(\Omega)$ with $\mathcal{L}$ a linear subspace of $\mathbb{R}^n$, $X'$ can be chosen in $\mathcal{L}^{\perp}$, hence with $E_D[X'] = 0$ for all $D \in \mathcal{D}$.*

[2] These axioms relate to what now is also called strong coherence; weak coherence has positive homogeneity and superadditivity together replaced by one concavity axiom (hence convexity in terms of risk), cf. e.g. Carr et al. (2001). Extension of our results to weakly coherent acceptability measures is a topic for future research.

[3] We remark that for infinite $\Omega$, the representation results holds under a continuity condition (Föllmer and Schied, 2002), or, alternatively, in terms of finitely additive measures (Delbaen, 2002).

[4] see e.g. Debreu (1959), Duffie (1992); for an instantly accessible introduction to its use in economics we refer to the 'history of economic thought' web site (2001).

Proof. By the separating hyperplane theorem, there exists a $X \in \mathbb{R}^n$, and $c \in \mathbb{R}$, such that the hyperplane $\mathcal{S} := \{M \in \mathbb{R}^n \mid \langle M, X \rangle = c\}$ separates $\mathcal{C}$ and $\mathcal{D}$, and has no point in common with $\mathcal{C}$. Replacing $X$ with $X - c$ yields $c = 0$. Hence either $\sup_{C \in \mathcal{C}} \langle C, X \rangle < 0 \leq \inf_{D \in \mathcal{D}} \langle D, X \rangle$ or $\inf_{C \in \mathcal{C}} \langle C, X \rangle > 0 \geq \sup_{D \in \mathcal{D}} \langle D, X \rangle$. Hence (2) holds for either $X' = -X$ or $X' = X$.

If $\mathcal{D} = \mathcal{L} \cap \Pr(\Omega)$, with $\mathcal{L}$ a linear subspace, choose $X'$ parallel to the shortest connection between $\mathcal{L}$ and $\mathcal{C}$. Then $X' \perp \mathcal{L}$, and hence $\mathcal{D}$ is contained in the corresponding separating hyperplane, so $\langle D, X' \rangle = 0$ for all $D \in \mathcal{D}$. $\square$

A one-to-one relationship between coherent measures and closed convex test sets can be derived from this. Recall that $\mathrm{ch}(\mathcal{S})$ denotes the convex hull of a set $S$, $\mathrm{cch}(\mathcal{S})$ the closure of its convex hull, and $\mathrm{ext}(\mathcal{S})$ the exterior of $\mathrm{cch}(S)$.

Lemma 2.3

1. $\rho_{\mathcal{P}} = \rho_{\mathrm{ch}(\mathcal{P})} = \rho_{\mathrm{cch}(\mathcal{P})} = \max_{P \in \mathrm{cch}(\mathcal{P})} E_P X = \max_{P \in \mathrm{ext}(\mathcal{P})} E_P X$
2. $\rho_{\mathcal{P}} \geq \rho_{\mathcal{P}'} \Leftrightarrow \mathrm{cch}(\mathcal{P}) \subset \mathrm{cch}(\mathcal{P}')$
3. $\rho_{\mathcal{P}} = \rho_{\mathcal{P}'} \Leftrightarrow \mathrm{cch}(\mathcal{P}) = \mathrm{cch}(\mathcal{P}')$

Proof.

1: Convex combinations of probability measures do not contribute to the supremum, as $E_{\lambda P + (1-\lambda)P'}[\cdot] = \lambda E_P[\cdot] + (1-\lambda) E_{P'}[\cdot]$. Neither does closure, as the operator $P \mapsto E_P[X]$ is continuous in $P$. The fact that the supremum is achieved for some $P \in \mathrm{cch}(\mathcal{P})$ is a consequence of the Bolzano-Weierstrass theorem, stating that every infinite bounded sequence in $\mathbb{R}^n$ must have an accumulation point $P^*$. The supremum is attained in $P^*$, and, as $\mathcal{P}$ is closed, $P^* \in \mathcal{P}$. If $P^* \notin \mathrm{ext}(\mathcal{P})$, it must be a convex combination of two exterior points in which the supremum also is reached.

2, $\Rightarrow$: Suppose there exists a $P \in \mathrm{cch}(\mathcal{P})$ that lies outside $\mathcal{P}'$. Then Lemma 2.2 with $\mathcal{C} = \{P\}$ and $\mathcal{D} = \mathrm{cch}(\mathcal{P}')$ implies that for some $X' \in \mathcal{X}(\Omega)$, $E_P[X'] > \rho_{\mathcal{P}'}(X')$, hence $\rho_{\mathcal{P}} \not\leq \rho_{\mathcal{P}'}$.

2, $\Leftarrow$: Obvious.

3: From 2. $\square$

Concerning the interpretation of the lemmas, notice that in (2) the position $X'$ is acceptable, both to a buyer with acceptability measure $\rho_{\mathcal{C}}$ and to a seller with measure $\rho_{\mathcal{D}}$. Any zero-sum good deal generates a strictly separating hyperplane between test sets in this way, having the traded position as normal vector. Lemma 2.3 now implies that such a deal is possible if and only if the closed convex hulls of $\mathcal{C}$ and $\mathcal{D}$ have no point in common. In Section 7 we further discuss this interpretation in the context of hedging.

Example 2.4

We illustrate the exposition by a simple example. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$. $\mathcal{X}(\Omega) (\simeq \mathbb{R}^3)$ is the set of all future future net worths $X : \Omega \to \mathbb{R}$; we write $X = (a, b, c)$ for $X \in \mathcal{X}(\Omega)$ with $X(\omega_1) = a, X(\omega_2) = b, X(\omega_3) = c$.

An assets future net worth $X_S$ is given by $X_S := (1, 0, -1)$, the net payoff of a derivative by $X_D := (1, -2, 1)$, and $X_C := (1, 1, 1)$ is the unit certain net worth.

We write $P = [a, b, c]$ for $P \in \Pr(\Omega)$ with $P(\omega_1) = a$, $P(\omega_1) = b$, $P(\omega_1) = c$. We consider acceptability measure $\rho_{\mathcal{P}}$ with test set $\mathcal{P} = \{P_1, P_2\}$, where $P_i = [\frac{1}{2} p_i, \frac{1}{2}, \frac{1}{2}(1 - p_i)]$ and $p_1 < \frac{1}{2} < p_2$. From the definition of $\rho_{\mathcal{P}}$ it follows that for $X = (a, b, c)$,

$$\rho_{\mathcal{P}}(X) = \frac{1}{2}(p_i a + b + (1 - p_i) c) \text{ with } i = 1 \text{ if } a \geq c, i = 2 \text{ if } a \leq c. \tag{3}$$

Note that according to lemma Lemma 2.3, $\rho_{\mathcal{P}} = \rho_{\mathrm{cch}(\mathcal{P})}$, with $\mathrm{cch}(\mathcal{P}) = \{\lambda P_1 + (1 - \lambda) P_2\}_{\lambda \in [0,1]}$.

# 3 Pre-hedge acceptability measures

A hedge opportunity $H : \Omega \to \mathbb{R}$ represents the future net worth of hedging as function of future states. As net worths are considered, $H$ represents a costless hedge. The set of all hedge opportunities is denoted as $\mathcal{H} \subset \mathcal{X}(\Omega)$. Typically $\mathcal{H}$ is generated by a set of hedge instruments, in which positions can be taken according to a class of self-financing hedging strategies. As mentioned earlier, dynamical aspects are left implicit in this paper; $\mathcal{H}$ just represents net hedging effects, without making explicit how these effects can be obtained.

We most often assume that the hedge set is linear, not in the last place because this reduces the mathematical complexity considerably. Economically, it means that we assume an ideal market on hedge instruments, and no limitations on hedge positions, which is motivated by the fact that hedge instruments are often very liquid, and hedging transaction costs are often relatively small as compared to the uncertainty involved in the to-be-hedged position. Linearity of $\mathcal{H}$ may even be defended, at least in principle, when transaction costs on hedging do matter, by interpreting ‘costless’ as ‘without value’, rather than as ‘literally without any costs’.

For reasons explained below, we call $\rho$ a pre-hedge acceptability measure if it satisfies the following axiom.

- *Hedge-invariance:* $\rho(X) = \rho(X + H)$ for all $H \in \mathcal{H}$.

The justification of this axiom lies in the interpretation of $\rho$ as the acceptability of position $X$ under some *not yet determined* hedge. For a linear hedge set, positions $X$ and $X + H$ are equivalent in the sense that their corresponding set of hedged positions coincides: $\{X + H'\}_{H' \in \mathcal{H}} = \{X + H + H'\}_{H' \in \mathcal{H}}$. Hence, in presence of hedging possibilities $\mathcal{H}$, but before choosing a specific hedge $H \in \mathcal{H}$, there is no need to discern between the acceptability of $X$ and $X + H$. This explains the term pre-hedge acceptability, which also is a warning that the concept anticipates a proper hedging action; without such action the concept of pre-hedge acceptability measure makes hardly any sense.

Notice that the argument is also valid under the weaker assumption that the hedge set is closed under addition and subtraction. In order to put pre-and post hedge acceptability in sharper contrast, observe that if $\mathcal{H}$ is only closed under addition, like cones, $\{X+H'\}_{H' \in \mathcal{H}} \supset \{X+H+H'\}_{H' \in \mathcal{H}}$, which would justify to adopt $\rho(X) \geq \rho(X+H)$ as a hedge-monotonicity axiom. At face value, this seems to be the absurd requirement that hedging should have an adverse effect on acceptability. In pre-hedge sense, however, this should be read as $\rho(\mathit{hedged} X) \geq \rho(\mathit{hedged}(X + H))$, which is justified by the inclusion above.

Any acceptability measure $\rho$ can be transformed into a pre-hedge measure $\rho'$ on the basis of a hedging scheme, i.e., a mapping $h : \mathcal{X}(\Omega) \to \mathcal{H}$, that is consistent, i.e, with

$$h(X + H) = h(X) - H \text{ for all } H \in \mathcal{H}, \tag{4}$$

by the rule

$$\rho'(X) := \rho(X + h(X)). \tag{5}$$

Notice that consistency implies that all positions $\{X + H\}_{H \in \mathcal{H}}$ result in the same hedged position $X + h(X)$, so that hedge-invariance of $\rho'$ is automatically satisfied. Any hedge-invariant $\rho'$ that satisfies (5) for some hedge scheme, and hence for some consistent hedge scheme $h$, is called an $\mathcal{H}$-transform of $\rho$. Conversely, then $\rho$ is called a final (or post-hedge) measure for $\rho'$.

As a further explanation, it might be observed that consistent hedging induces a fixed point for $X + h(X)$, as (4) implies $h(X + h(X)) = 0$. A pre-hedge measure hence may be considered as an acceptability measure on equivalence classes of positions $[X] := \{X + H\}_{H \in \mathcal{H}}$, with acceptability derived from the acceptability under $\rho$ of the unique fixed point $X + h(X)$ in this class that remains unhedged.

To summarize, pre-hedge acceptability measures express the acceptability of positions that will be hedged consistently.

# 4 Coherent Pre-hedge (CoPR) Acceptability Measures

Combining hedge-invariance with the coherence axioms results in the following class of acceptability measures.

Definition 4.1 (Coherent Pre-Hedge (CoPr) Acceptability)
*An acceptability measure that satisfies the coherence axioms together with the axiom of hedge invariance, with respect to a given hedge set $\mathcal{H}$, is called a coherent pre-hedge (CoPr) measure with respect to $\mathcal{H}$, or an $\mathcal{H}$-invariant coherent risk measure.*

In view of Proposition 2.1 and the fundamental theorems of asset pricing, it is not surprising that CoPr measures can be characterized in terms of martingales. A probability measure $Q$ is called a martingale with respect to a hedge set $\mathcal{H}$ if all $H \in \mathcal{H}$ have zero expected value under $Q$. The set of all martingales with respect to $\mathcal{H}$ is denoted as

$$\mathbb{M}(\mathcal{H}) := \{Q \in \Pr(\Omega) \mid E_Q[H] = 0 \text{ for all } H \in \mathcal{H}\}. \tag{6}$$

Theorem 4.2 (Martingale Representation)
*For a linear hedge set $\mathcal{H}$ the following equivalences hold true:*

1. *$\rho$ is $\mathcal{H}$-invariant and coherent $\Leftrightarrow \rho = \rho_{\mathcal{Q}}$ for some $\mathcal{Q} \subset \mathbb{M}(\mathcal{H})$*
2. *$\rho_{\mathcal{P}}$ is $\mathcal{H}$-invariant $\Leftrightarrow \mathcal{P} \subset \mathbb{M}(\mathcal{H})$*

Proof.
**1, $\Rightarrow$:** As $\rho$ is coherent, $\rho = \rho_{\mathcal{P}}$ for some test set $\mathcal{P}$, see Proposition 2.1, and hence $\rho(0) = \rho_{\mathcal{P}}(0) = 0$. As $\rho$ is $\mathcal{H}$-invariant, it holds that for all $X \in \mathcal{X}(\Omega)$ and all $H \in \mathcal{H}$, $\rho(X) = \rho(X + H)$. With $X = 0$ this implies

$$\rho(H) = \rho_{\mathcal{P}}(H) = \inf_{P \in \mathcal{P}} E_P[H] = 0 \text{ for all } H \in \mathcal{H}. \tag{7}$$

Now suppose that $\mathcal{P} \not\subset \mathbb{M}(\mathcal{H})$, hence there exists a $P \in \mathcal{P}$ with $E_P[H] = \mu \neq 0$ for some $H \in \mathcal{H}$. From (7) it follows that $\mu > 0$, but this leads to the contradiction $E_P[-H)] < 0$ with $-H \in \mathcal{H}$. Conclude that $\mathcal{P} \subset \mathbb{M}(\mathcal{H})$, hence $\rho = \rho_{\mathcal{Q}}$ for some martingale set $\mathcal{Q} \subset \mathbb{M}(\mathcal{H})$.
1, $\Leftarrow$: As $\rho(X) = \inf_{Q \in \mathcal{Q}} E_Q[X]$, Proposition 2.1 implies that $\rho$ is coherent, and hedge-invariance follows directly from (6).
2 From 1. and Lemma 5.3. $\square$

We remark that linearity of $\mathcal{H}$ may be weakened to the property that $0 \in \mathcal{H}$ and that for all $H \in \mathcal{H}$, $-\lambda H \in \mathcal{H}$ for some $\lambda \in \mathbb{R}^+$. However, hedge-invariance of pre-hedge measures is then less convincing.

Example 4.3
We consider unrestricted hedging by assets, wich corresponds to a linear hedge set $\mathcal{H} := \{\gamma X_S\}_{\gamma \in \mathbb{R}}$. The set of martingale measures is given by $\mathbb{M}(\mathcal{H}) = \{[\frac{1}{2}(1-q), q, \frac{1}{2}(1-q)\}_{q \in [0,1]}$. A CoPr measure amounts to taking worst case expected loss over a subset of $\mathbb{M}(\mathcal{H})$. Lemma 2.3 implies that every CoPr measure takes the form $\rho_{\{Q_1, Q_2\}}$ with $Q1, Q2 \in \mathbb{M}(\mathcal{H})$.

# 5 Maximum Acceptability Hedging

In this section we analyze the maximum possible effect of hedging on acceptability, at the level of test sets. Given a measure $\rho$ and a hedge set $\mathcal{H}$, we define

$$\rho^{\mathcal{H}}(X) := \sup_{H \in \mathcal{H}} \rho(X + H), \tag{8}$$

which is the outcome of risk according to $\rho$ under *maximum acceptability hedging.* The next lemma addresses a well-posedness condition.

Lemma 5.1

*Suppose $\mathcal{H} \subset \mathcal{X}(\Omega)$ is linear. If $\rho(H) \leq 0$ for all $H \in \mathcal{H}$, $\rho^{\mathcal{H}}$ is the $\mathcal{H}$-transform of $\rho$ corresponding to a (finite) consistent hedging scheme $h^*$. If $\rho(H) > 0$ for some $H \in \mathcal{H}$, $\rho^{\mathcal{H}}(X) = \infty$ for all $X \in \mathcal{X}(\Omega)$.*

Proof. As $\mathcal{H}$ is linear, the supremum $S$ in (8) is finite if and only if $\rho(H) \leq 0$ for all $H \in \mathcal{H}$. Obviously a bounded supremum for given $X$ in (8) can always be achieved by a bounded sequence of hedges, which must contain an accumulation point $H^* \in \mathcal{H}$, for which the supremum is attained (cf. the proof of Lemma 2.3). Define $h^*(X) := H^*$, and $h^*(X + H) := H^* - H$ for all $H \in \mathcal{H}$. Observe that indeed $H^* - H$ must be optimal for $X + H$, so $\rho(X + h^*(X)) = \rho^{\mathcal{H}}(X)$, as required by (5), with $h^*$ consistent, cf. (4).

The last claim follows from scaling up any $H \in \mathcal{H}$ with $\rho(H) > 0$ and superadditivity of $\rho$. □

The following theorem states that acceptability maximization preserves coherence, and characterizes its effect in terms of underlying test sets.

Theorem 5.2 (Maximum Acceptability Measures)

*Let be given a coherent acceptability measure $\rho = \rho_{\mathcal{P}}$, and a linear hedge set $\mathcal{H} \subset \mathcal{X}(\Omega)$. Then*

$$\rho^{\mathcal{H}}_{\mathcal{P}} = \rho_{\mathcal{Q}} \text{ with } \mathcal{Q} := \mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H}) \tag{9}$$

Proof. First suppose $\mathcal{Q} = \emptyset$. From the next lemma it then follows that for some $H' \in \mathcal{H}$, $\rho_{\mathcal{P}}(H') = \inf_{P \in \mathcal{P}} E_P[H'] > 0$. Linearity of $\mathcal{H}$ then implies that $\sup_{H\in\mathcal{H}} \rho_{\mathcal{P}}(X + H) \geq \sup_{\lambda\in\mathbb{R}} E_P[\lambda H'] + \min_{\omega\in\Omega} X(\omega) = \infty$ for all $X \in \mathcal{X}(\Omega)$. This proves the theorem for $\mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H}) = \emptyset$.

Lemma 5.3

*For linear $\mathcal{H} \subset \mathcal{X}(\Omega)$, $\mathrm{cch}(P) \cap \mathbb{M}(\mathcal{H}) = \emptyset \Leftrightarrow \exists\, H \in \mathcal{H}$ such that $\rho_{\mathcal{P}}(H) > 0$.*

*Proof of the lemma.*

$\Rightarrow$: Applying Lemma 2.2 with $\mathcal{C} = \mathrm{cch}(\mathcal{P})$ and $\mathcal{D} = \mathbb{M}(\mathcal{H}) = \mathcal{L} \cap \mathrm{Pr}(\Omega)$, with $\mathcal{L} := \mathcal{H}^{\perp}$, yields $\rho_{\mathcal{P}}(X') > 0 = \rho_{\mathbb{M}(\mathcal{H})}(X')$ for some $X'$ in $\mathcal{L}^{\perp}$, hence in $\mathcal{H}$.

$\Leftarrow$: $\rho_{\mathcal{P}}(H) = \inf_{P\in\mathcal{P}} E_P[H] > 0$ implies $\mathcal{P} \cap \mathbb{M}(H) = \emptyset$, hence $\mathcal{P} \cap \mathbb{M}(\mathcal{H}) = \emptyset$.

*end of the proof of the lemma.*

Next suppose $\mathcal{Q} \neq \emptyset$. It follows from Lemma 5.1 that $\rho^{\mathcal{H}}_{\mathcal{P}}$ is a finite $\mathcal{H}$-transform of $\rho_{\mathcal{P}}$. It is also a coherent measure, which is proved by a straightforward check of the four axioms; in fact the proof only relies on the fact that the hedge set is a cone; we refer to Jaschke and Küchler (2001) for an alternative proof, and to Roorda et al. (2002) for a similar result in a multiperiod setting.

- *Translation invariance:* $\rho^{\mathcal{H}}_{\mathcal{P}}(X+\alpha) = \sup_{H\in\mathcal{H}} \rho_{\mathcal{P}}(X+\alpha+H) = \sup_{H\in\mathcal{H}} \rho_{\mathcal{P}}(X+H) + \alpha = \rho^{\mathcal{H}}_{\mathcal{P}}(X) + \alpha$
- *Superadditivity:* $\rho^{\mathcal{H}}_{\mathcal{P}}(X_1 + X_2) = \sup_{H\in\mathcal{H}} \rho_{\mathcal{P}}(X_1 + X_2 + H) = \sup_{H_1,H_2\in\mathcal{H}} \rho_{\mathcal{P}}(X_1 + H_1 + X_2 + H_2) \geq \sup_{H_1,H_2\in\mathcal{H}} \rho_{\mathcal{P}}(X_1 + H_1) + \rho_{\mathcal{P}}(X_2 + H_2) = \rho^{\mathcal{H}}_{\mathcal{P}}(X_1) + \rho^{\mathcal{H}}_{\mathcal{P}}(X_2)$
- *Positive Homogeneity:* $\rho^{\mathcal{H}}_{\mathcal{P}}(\lambda X) = \sup_{H\in\mathcal{H}} \rho_{\mathcal{P}}(\lambda X + H) = \sup_{H\in\mathcal{H}} \lambda\rho_{\mathcal{P}}(X + H/\lambda) = \sup_{H\in\mathcal{H}} \lambda\rho_{\mathcal{P}}(X + H) = \lambda\rho^{\mathcal{H}}_{\mathcal{P}}(X)$ for all $\lambda > 0$; trivial for $\lambda = 0$.
- *Monotonicity:* $X \leq Y$ implies $\rho^{\mathcal{H}}_{\mathcal{P}}(X) = \sup_{H\in\mathcal{H}} \rho_{\mathcal{P}}(X + H) \leq \sup_{H\in\mathcal{H}} \rho_{\mathcal{P}}(Y + H) = \rho^{\mathcal{H}}_{\mathcal{P}}(Y)$

The martingale theorem now implies that $\rho^{\mathcal{H}}_{\mathcal{P}} = \rho_{\mathcal{Q}'}$ for some $\mathcal{Q}' \subset \mathbb{M}(\mathcal{H})$. As $\mathcal{Q} \subset \mathcal{P}$, cf. (9), for all $X \in \mathcal{X}(\Omega)$, $\rho^{\mathcal{H}}_{\mathcal{P}}(X) = \sup_{H\in\mathcal{H}} \rho_{\mathcal{P}}(X+H) \leq \sup_{H\in\mathcal{H}} \rho_{\mathcal{Q}}(X+H) = \rho_{\mathcal{Q}}(X)$, hence from Lemma 2.3, $\mathcal{Q} \subset \mathrm{cch}(\mathcal{Q}')$. On the other hand, $\rho_{\mathcal{Q}'} \geq \rho_{\mathcal{P}}$, hence $\mathrm{cch}(\mathcal{Q}') \subset \mathrm{cch}(\mathcal{P})$, hence $\mathrm{cch}(\mathcal{Q}') \subset \mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H}) = \mathcal{Q}$. Conclude that $\mathrm{cch}(\mathcal{Q}') = \mathcal{Q}$ and hence $\rho^{\mathcal{H}}_{\mathcal{P}} = \rho_{\mathcal{Q}}$. □

It is interesting to point here at the fact that in ADEH (Condition 4.3) it is suggested to let the closed convex hull of the test set contain a pricing martingale, in order to avoid unwanted concentration of risk. This precisely rules out the case $\mathcal{Q} = \emptyset$.

EXAMPLE 5.4
For $\mathcal{P} = \{P_1, P_2\}$ as specified in Example 2.4, the corresponding minimum risk measure is given by $\rho_{\mathcal{P}}^{\mathcal{H}} = \rho_{\mathcal{Q}}$ with $\mathcal{Q} = \mathrm{cch}(P) \cap \mathbb{M}(\mathcal{H}) = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$. The corresponding optimal hedging scheme $h^*$ maps $X = (a, b, c)$ to $h^*(X) = ((c-a)/2, 0, (a-c)/2)$, resulting in the optimally hedged position $X + h^*(X) = ((a+c)/2, b, (a+c)/2)$. Notice that $h^*$ is indeed consistent. The optimal hedge for the derivative turns out to be zero.

# 6 Coherence preserving hedging

There is a very practical reason to consider other hedging schemes than maximum acceptability hedging alone: it is not to be expected that in practice a hedging strategy will be based entirely on a single coherent acceptability measure proposed by someone that turns out to be well informed about some of the latest developments in risk measurement. In addition, it is both of theoretical and practical interest to compare the effect of different hedging strategies.

The next theorem characterizes the effect on test sets of any consistent hedging scheme, provided that it is coherence preserving.

THEOREM 6.1 (MARTINGALE INCLUSION)
*For a linear hedge set $\mathcal{H} \in \mathcal{X}(\Omega)$ and $\mathcal{P} \not\subset \mathbb{M}(\mathcal{H})$, the following equivalence holds:*

$$\rho_{\mathcal{Q}} \textit{ is a } \mathcal{H}\textit{-transform of } \rho_{\mathcal{P}} \tag{10}$$

*if and only if $\mathcal{Q} \neq \emptyset$ and*

$$\mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H}) \subset \mathrm{cch}(\mathcal{Q}) \subset \mathbb{M}(\mathcal{H}) \tag{11}$$

PROOF. $\Rightarrow$: We prove that the negation of (11) implies the negation of (10). First notice that $\mathrm{cch}(\mathcal{Q}) \not\subset \mathbb{M}(\mathcal{H})$ implies that $\rho$ is not $\mathcal{H}$-invariant, cf. Theorem 4.2. Secondly, if $\mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H}) \not\subset \mathrm{cch}(\mathcal{Q})$, then there exists a $Q' \in \mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H})$ that lies outside $\mathcal{Q}$. The separating hyperplane theorem (take Lemma 2.2 with $\mathcal{C} = \mathcal{Q}$ and $\mathcal{D} = \{Q'\}$) implies that there exists an $X' \in \mathcal{X}(\Omega)$ for which $\rho_{\mathcal{Q}}(X') > 0 \geq E_{Q'}[X'] = E_{Q'}[X' + H] \geq \rho_{\mathcal{P}}(X' + H)$ for all $H \in \mathcal{H}$, contradicting (10).

$\Leftarrow$: If $\mathcal{Q} = \emptyset$, this follows easily from Lemma 5.3. For $\mathcal{Q} \neq \emptyset$, we prove that the negation of (10) implies the negation of (11). If $\mathcal{Q} \not\subset \mathbb{M}(\mathcal{H})$, this is trivial, so suppose $\mathcal{Q} \subset \mathbb{M}(\mathcal{H})$, hence $\mathrm{cch}(\mathcal{Q}) \subset \mathbb{M}(\mathcal{H})$.

By assumption $\mathcal{P} \not\subset \mathbb{M}(\mathcal{H})$, so there exists a $P \in \mathcal{P}$ with $E_P[H] \neq 0$ for some $H \in \mathcal{H}$. As $\mathcal{H}$ is linear, there must also exist a $P^- \in \mathcal{P}$ with $E_{P^-}[H^+] < 0$ for some $H^- \in \mathcal{H}$. Also by assumption, $\rho_{\mathcal{Q}}$ is not an $\mathcal{H}$-transform of $\mathcal{P}$, so there exists an $X' \in \mathcal{X}(\Omega)$ such that $\rho_{\mathcal{Q}}(X') \neq \rho_{\mathcal{P}}(X' + H)$ for all $H \in \mathcal{H}$. From linearity of $\mathcal{H}$ and the fact that $E_{P^-}[H^-] < 0$ it follows that $\{\rho_{\mathcal{P}}(X + H)\}_{H \in \mathcal{H}}$ is a connected set in $\mathbb{R}$ without (finite) lower bound, which implies that $\rho_{\mathcal{Q}}(X') > \rho_{\mathcal{P}}(X' + H)$ for all $H \in \mathcal{H}$. So $\rho_{\mathcal{P}}^{\mathcal{H}}(X') \leq \rho_{\mathcal{Q}}(X')$, and Lemma 5.1 implies that this inequality is in fact strict, so $\rho_{\mathcal{P}}^{\mathcal{H}} \not\geq \rho_{\mathcal{Q}}$. According to Theorem 5.2, $\rho_{\mathcal{P}}^{\mathcal{H}} = \rho_{\mathrm{cch}\,\mathcal{P} \cap \mathbb{M}(\mathcal{H})}$. From Lemma 2.3 it follows that $\mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H}) \not\subset \mathrm{cch}(\mathcal{Q})$. □

If $\mathcal{P} \subset \mathbb{M}(\mathcal{H})$, in which case the theorem does not apply, $\rho_{\mathcal{Q}}$ is a $\mathcal{H}$-transformation of $\rho_{\mathcal{P}}$ if and only if $\mathrm{cch}(\mathcal{P}) = \mathrm{cch}(\mathcal{Q})$. On the other hand, if $\mathrm{cch}(\mathcal{P})$ does not contain any martingale, $\mathcal{Q}$ can be any martingale set in (10), and hence no feature of $\rho_{\mathcal{P}}$ is preserved under hedging.

The theorem underlines the special role of optimal hedging: given a coherent measure $\rho_{\mathcal{P}}$, maximum acceptability hedging $h^*$ is the only consistent hedging scheme for which the corresponding martingale test set remains within $\mathrm{cch}(\mathcal{P})$. From Lemma 2.3 it follows that any non-optimal scheme $h \neq h^*$ induces some adverse hedging, i.e, $\rho(X + h(X)) < \rho(X)$ for some $X$. Notice, however, that this is only a true anomaly if $\rho$ is absolutely convincing as an acceptability measure on the whole space $\mathcal{X}(\Omega)$, and if $\mathcal{H}$ indeed accurately reflects hedging opportunities.

The theorem may also be used for determining which final measures $\rho_{\mathcal{P}}$ can be obtained from a given CoPr measure $\rho_{\mathcal{Q}}$: $\mathcal{P}$ may be any set of probability measures for which the closed convex hull does not contain 'new' martingales outside $\mathrm{cch}(\mathcal{Q})$; if $\mathrm{cch}(\mathcal{P})$ contains all 'old' martingales in $\mathrm{cch}(Q)$, it requires optimal hedging to really attain level $\rho_{\mathcal{Q}}$ of final acceptability specified by $\rho_{\mathcal{P}}$. Example 5.4 showed that there may be many extensions $\mathcal{P}$ with this property, even for the same optimal hedging scheme.

A characterization of coherence preserving hedging schemes is left as a topic for future research. In the example below the set of all such hedging schemes is determined, but it is not obvious how this can be generalized.

Example 6.2
The martingale inclusion theorem states that any coherent preserving hedge transforms $\rho_{\mathcal{P}}$ to $\rho_{\mathcal{Q}}$ with $\mathrm{cch}(\mathcal{Q})$ containing $\rho_{\mathcal{P}} \cap \mathbb{M}(\mathcal{H}) = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$, cf. Example 5.4. From the expression for $\mathbb{M}(\mathcal{H})$ in Example 4.3 it follows that, without loss of generality, $\mathcal{Q}$ may be chosen of the form $\mathcal{Q} = \{[\frac{1}{2}(1-q_i), q_i, \frac{1}{2}(1-q_i)]\}_{i=1,2}$, with $q_1 \leq \frac{1}{2} \leq q_2$. The corresponding hedge for a position with $X = (a, b, c)$, is $\gamma^* + \bar{\gamma}$ assets with $\gamma^* = (c-a)/2$ the optimal hedge, and $\bar{\gamma} = \frac{\frac{1}{2}-q_1}{\frac{1}{2}-p_2}(b - (a+c)/2)$ if $b \geq (a+c)/2$ and $\bar{\gamma} = \frac{q_2-\frac{1}{2}}{p_1-\frac{1}{2}}((a+c)/2 - b)$ if $b \leq (a+c)/2$. All these hedges, parameterized by $q_1 \in [0, \frac{1}{2}]$ and $q_2 \in [\frac{1}{2}, 1]$ for given $p_1 < \frac{1}{2} < p_2$ are coherence preserving, and no other consistent hedging schemes are.

# 7 Comparison with linear asset pricing

In this section we interpret our results from the perspective of linear asset pricing, which corresponds to taking expected value over a *single* martingale. In the next section we discuss valuation bounds, which relate to *sets* of martingales.

## 7.1 The fundamental theorems of asset pricing

We indicate how the fundamental theorems of asset pricing, for finite $\Omega$, can be derived from the martingale representation theorem for CoPr measures, as a special case.

In our notation, a pricing measure takes the form $\pi : \mathcal{X}(\Omega) \to \mathbb{R}$ that assigns price $\pi(X)$ to a future net worth $X$. It is obvious that, under the standard assumption of idealized, arbitrage-free markets, pricing measures should satisfy the coherence axioms; in fact the axioms of superadditivity and positive homogeneity can be strengthened to the axiom of linearity,

- *Linearity:* $\pi(\lambda X_1 + \mu X_2) = \lambda\pi(X_1) + \mu\pi(X_2)$ for all $\lambda, \mu \in \mathbb{R}$

In a pricing context it is most appropriate to let the hedge set $\mathcal{H}$ coincide with all market opportunities, i.e., all final net worths available at the market. It is assumed that these 'marketed cash flows' are already well-priced by the market, so a pricing measure should simply reprice them: $\pi(H) = 0$ for all $H \in \mathcal{H}$. Consequently, pricing measures belong to the class of CoPr measures, hence can be represented by a martingale test set. Linearity implies that the test set should be a singleton, i.e., $\pi = \rho_{\mathcal{Q}}$ for some $\mathcal{Q} = \{Q\}$ with $Q \in \mathbb{M}(\mathcal{H})$. This shows that a pricing measure, if it exists, amounts to expected values under a martingale measure.

The fundamental asset pricing theorems relate existence and uniqueness of pricing measures to arbitrage opportunities and completeness of markets. In our setup, these relationships can be formulated and derived as follows.

An arbitrage opportunity is a market opportunity $H \in \mathcal{H}$ with $H(\omega) \geq 0$ for all $\omega \in \Omega$ and $H(\omega) > 0$ for at least one $\omega \in \Omega$. A strict arbitrage opportunity has $H(\omega) > 0$ for all

$\omega \in \Omega$. Applying Lemma 5.3 with unrestricted test set $\mathcal{P} = \Pr(\Omega)$ implies that $\mathbb{M}(\mathcal{H}) = \emptyset$ if and only if there exists a strict arbitrage opportunity $H \in \mathcal{H}$. Also, $\mathbb{M}(\mathcal{H})$ does not contain any measure with full support $\Omega$ if and only if $\mathcal{H}$ contains an arbitrage opportunity, which can be derived as follows. Lemma 5.3 with $\mathcal{P}$ any closed convex set consisting only of full support test measures, implies the existence of $H \in \mathcal{H}$ with $E_P[H] > 0$ for all $P \in \mathcal{P}$. Hence $E_P[H] \geq 0$ for all $P \in \Pr(\Omega)$, so this $H$ must be an arbitrage opportunity. Now if the outcome space $\Omega$ is defined as the support of some physical measure, full support martingales are precisely those martingales that are equivalent to that measure, and the first fundamental theorem follows.

Concerning the second, the market is called complete if $\{H - c\}_{H \in \mathcal{H}, c \in \mathbb{R}} = \mathcal{X}(\Omega)$, expressing that all positions are available at the market, at some price. By assumption, $\mathcal{H}$ is a linear subspace of $\mathcal{X}(\Omega)$, hence must have dimension $n-1$ in complete markets, so $\mathbb{M}(\mathcal{H}) \subset \mathcal{H}^{\perp} \cap \Pr(\Omega)$ contains at most one element. Hence complete markets have a unique pricing measure.

Notice that the class of CoPr measures reduce to this single pricing measure, if all opportunities in a complete market indeed are considered as hedge opportunities.

## 7.2 Pricing by tests

In Carr et al. (2001) the notion of a representative state pricing function is introduced. These are price functions that can be expressed as a strictly positive linear combination of the outcome of tests in a finite test set $\mathcal{P} = \{P_1, \ldots, P_M\}$, i.e.,

$$\pi(X) = \Sigma_{m=1}^{M} w_m E_{P_m} X, \text{ with } w_m \in \mathbb{R}^+ \text{ for } m = 1, \ldots, M. \tag{12}$$

We refer to this type of pricing functions as ‘pricing by tests’. According to their first theorem, pricing by tests is equivalent to the condition of no strictly acceptable opportunities (NSAO), which, for the purpose of our exposition, we formulate as

$$\forall\, H \in \mathcal{H} : \rho_{\mathcal{P}}(H) < 0 \text{ or } E_P H = 0\; \forall\, P \in \mathcal{P}, \tag{13}$$

where $\mathcal{H}$ now has the interpretation of all opportunities under consideration. So this says that opportunities are either bad deals, or ‘zero-like’ positions, as seen through test set $\mathcal{P}$. This assumption reduces the space of no-arbitrage pricing measures in incomplete markets considerably, if the number $M$ of tests in (12) is small relative to the co-dimension of $\mathbb{M}(\mathcal{H})$.

According to the authors, their first theorem can be interpreted as a version of the first fundamental theorem of asset pricing, with the stronger assumption of NSAO replacing no-arbitrage, and, in effect, pricing by equivalent martingales sharpened to pricing by tests.

For a linear set of opportunities $\mathcal{H}$, it is easily derived that the NSAO condition is equivalent to the condition that the interior[5] of $\mathrm{cch}(\mathcal{P})$ intersects $\mathbb{M}(\mathcal{H})$. Martingales in this intersection precisely correspond to pricing by tests: they reprice $\mathcal{H}$ and are of the form (12). Notice that the slightly weaker condition of no-good-deal opportunities, i.e., $\rho_{\mathcal{P}}(H) \leq 0$ for all $H \in \mathcal{H}$, is equivalent to $\mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H}) \neq \emptyset$, which directly follows from Lemma 5.3.

Concerning the interpretation, notice that with $\mathcal{P} = \Pr(\Omega)$, with $\Omega$ again the support of some physical measure, (12) amounts to pricing by equivalent martingales. The step from the first fundamental theorem to the first theorem in Carr et al. (2001) hence for linear hedge sets comes down to replacing $\Pr(\Omega)$ with a finite test set $\mathcal{P}$; the notion of ‘equivalence’ translates to ‘being inside the closed convex hull’.

In order to arrive at a similar counterpart of the second fundamental theorem, they introduce the notion of acceptable completeness. In our terminology, this is defined by the condition

$$\forall\, X \in \mathcal{X}(\Omega),\ \exists H \in \mathcal{H}, c \in \mathbb{R} \text{ such that } E_P[X + H - c] = 0\; \forall\, P \in \mathcal{P}. \tag{14}$$

This condition states that any position can be turned into a zero-like position by hedging at some price, but not necessarily to the zero position itself, as is required for classical completeness. Their second theorem states, slightly reformulated, that acceptable completeness

[5] for a singleton, the set itself

is equivalent to the uniqueness of pricing by tests, which in turn is equivalent to uniqueness of weights $w_i$ in (12) if there is no 'overtesting', i.e., if the functional that maps positions into the vector of outcomes of tests in $\mathbb{R}^M$ is surjective. In an NSAO market hence prices are unique if and only if the market is acceptably complete.

In our setting, with $\mathcal{H}$ a linear set, acceptable completeness amounts to the geometrical condition

$$\mathcal{P}^\perp + \mathcal{H} + C = \mathcal{X}(\Omega) \tag{15}$$

where $C$ denotes the set of all constants on $\Omega$. This implies that $\mathcal{P}^\perp + \mathcal{H}$ is a $n-1$ dimensional space; its one-dimensional orthogonal complement, $(\mathcal{P}^\perp)^\perp \cap \mathcal{H}^\perp$, contains $\mathrm{cch}(\mathcal{P}) \cap \mathbb{M}(\mathcal{H})$, which hence must consist of just one martingale, thus implying uniqueness of prices by Theorem 5.2. The corresponding optimal hedging scheme $h^*$ is the one that brings a position $X \in \mathcal{X}(\Omega)$ into zero-like positions $\mathcal{P}^\perp$ at some price, whose existence is guaranteed by (15).

### 7.3 Comparison with valuation bounds

There are various reasons to concentrate on price limits rather than on single linear pricing functions, varying from modest to ambitious. One might resign oneself to indeterministic results, for lack of arguments to arrive at a unique price in incomplete markets, or, in the contrary, try to go beyond linear pricing and attempt to explain bid-ask spreads in terms of price limits.

The most 'solid' but often very conservative price limits are based on arbitrage arguments, cf. Merton (1973) for an early work in this direction. Recall that, according to the first fundamental theorem, no-arbitrage price bounds correspond to extreme prices under equivalent martingales. In our notation, take $\Omega$ the support of a given physical measure, $\mathcal{P} = \Pr(\Omega)$, and $\mathcal{H}$ the set of all (already well-priced) market opportunities. The no-arbitrage price limits for a position $X$ are given by

$$\inf_{Q \in \mathcal{Q}} E_Q X, \sup_{Q \in \mathcal{Q}} E_Q X \tag{16}$$

with $\mathcal{Q} := \mathcal{P} \cap \mathbb{M}(\mathcal{H}) = \mathbb{M}(\mathcal{H})$. Notice that a restriction to 'equivalent' martingales, i.e., of full support $\Omega$, would not change the outcome.

The search for less conservative price limits has raised an extensive literature on so-called (no-)good-deal bounds (see e.g. Černý and Hodges, 2002 and the references therein). These are based on the stronger assumption that the market does not offer good deals, which is motivated by an equilibrium argument: such opportunities would disappear immediately. The no strictly acceptable opportunity (NSAO) condition, discussed in the previous section, is in fact nothing else than such a no-good-deal condition. Its implication of pricing by tests illustrates that narrower price limits can be obtained: in (16), $\mathcal{Q}$ reduces to $\mathbb{M}(\mathcal{H}) \cap \mathrm{cch}(\mathcal{P})$ for linear test sets.

It is out of our scope to discuss the wide variety of good-deal concepts proposed in the literature; we confine ourselves to those induced by coherent measures. We remark that this does not include bounds in terms of the Sharpe ratio, as introduced in Cochrane and Saá Requejo (2000),[6] but does include, interestingly enough, valuation bounds based on *generalized* Sharpe ratios, as described in Hodges (1998).

Coherent acceptability measures induce valuation bounds in a mathematically obvious way. As already stated in Section 2, $\rho(X)$ and $-\rho(-X)$ may be interpreted as resp. upper and lower price limits for a position $X$, if no hedging is taken into account. According to Proposition 2.1 this amounts to considering extreme expected values over a set $\mathcal{P}$ of probability measures.

In presence of a linear set $\mathcal{H}$ of hedging opportunities, these bounds tighten to $\rho_{\mathcal{P}}^{\mathcal{H}}(X)$, $-\rho_{\mathcal{P}}^{\mathcal{H}}(-X)$, cf. (8). A price for buying $X$ beyond $\rho_{\mathcal{P}}^{\mathcal{H}}(X)$ is unacceptable under any hedge, and a seller of $X$ will not be satisfied with a premium below $-\rho_{\mathcal{P}}^{\mathcal{H}}(-X)$, no matter how $X$

[6]In fact this was already available in 1996 as working paper of Graduate School of Business, University of Chicago.

is hedged. For linear hedge sets, this amounts to extreme expected value over the set of martingales in $\mathrm{cch}(\mathcal{P})$, as shown in Theorem 5.2. Non-optimal hedging would yield price limits in between these bounds and no-arbitrage bounds, according to Theorem 6.1.

In Jaschke and Küchler (2001), coherent valuation bounds are studied for the more general case of hedge sets consisting of a cone. As mentioned in Section 5, they proved that the measure obtained from optimal hedging, i.e., $\rho^{\mathcal{H}}$ defined according to (8), is again coherent. In addition, they observed that the set of acceptable positions $\mathcal{A} := \{X \in \mathcal{X}(\Omega) \mid \rho(X) \geq 0\}$ transforms to $\mathcal{A} - \mathcal{H}$ under optimal hedging, i.e., $\{X \in \mathcal{X}(\Omega) \mid \rho^{\mathcal{H}}(X) \geq 0\} = \mathcal{A} - \mathcal{H}$. On an abstract level, this identifies the test set of $\rho^{\mathcal{H}}$ as the right polar cone of $\mathcal{A} - \mathcal{H}$, cf. Corollary 8 in Jaschke and Küchler (2001). Our results may be interpreted as a more concrete version of this result for linear hedge sets, with an extension to non-optimal hedging.

### 7.4 On subjective valuation bounds

A more realistic, but also more complex setting for market equilibrium theory involves 'subjective' opinions of agents about what is acceptable. It is out of scope to sketch an overview of the vast literature on economic equilibrium theory that addresses this kind of issues; we only present a concrete result on the level of test sets that may serve as an illustration of our framework. For simplicity we consider a market with two agents (1 and 2), who may have different opinions about acceptability, and not necessarily the same hedging opportunities. Let $\mathcal{P}_i$ and $\mathcal{H}_i$ denote respectively the test and hedge set of agent $i$. From the previous analysis it is obvious that if $\mathcal{Q}_i := \mathrm{cch}(\mathcal{P}_i) \cap \mathbb{M}(\mathcal{H})_i$ would be empty, this would cause a good deal for agent $i$. Notice that $\rho_{\mathcal{Q}_i}$ measures the agent $i$'s acceptability under optimal hedging.

However, a good deal now also may consist of a transaction between agents, in which agent 1 buys a position $X$ from agent 2 at such a price that it is a good deal for both. In order to rule out this type of good deals, it should also hold that

$$\mathcal{Q}_1 \cap \mathcal{Q}_2 \neq \emptyset. \tag{17}$$

This follows from Lemma 2.2 and its interpretation at the end of Section 2: any hyperplane that would strictly separate $\mathcal{Q}_1$ and $\mathcal{Q}_2$ has a normal vector $X$, for which it would be a good deal to both agents to trade it mutually, provided it is optimally hedged. Obviously (17) is also sufficient for excluding good deals in the two-agents market.

This suggests the generalization that in a market with subjective (but coherent) acceptability and heterogeneous (but linear) hedging opportunities, the subjective test sets of agents will have a joint intersection $\mathcal{Q}^*$ of tests that are martingale measures with respect to the union of all hedging opportunities of all agents. This $Q^*$ may be interpreted as the 'objective' intersection of all opinions that everyone agrees upon, and that are consistent with all 'marketed cash flows'. The no-good-deal condition now rules out subjective optimism, as it only permits the subjective test sets to extend $\mathcal{Q}^*$. This justifies, to some extend, the use of one martingale test set, even in the context of subjective valuation bounds. In the more ambitious interpretation of valuation bounds, even bid and ask prices could be explained in terms of this $\mathcal{Q}^*$. In fact, it is only the exterior of $\mathcal{Q}^*$ that matters, which consists of those tests that are at the boundary of at least one agents test set. This reflects the intuitive idea that prices are formed at the edge of common sense.

## 8 Summary and Conclusions

We bridged the gap between the coherent risk framework and standard asset pricing by adding hedge-invariance as a fifth axiom: the corresponding sub-class of coherent pre-hedge (CoPr) measures amounts to taking worst expected value over a set of martingale measures, which is nothing else than considering the worst price of a position over a set of linear pricing functions that reprice a given set of linear hedge opportunities.

CoPr measures were motivated by their interpretation as acceptability measures for positions that will be hedged consistently. We made transparent how the presence of a linear set of hedging opportunities transforms the test set of a given coherent measure to a martingale test set, under optimal hedging (Theorem 5.2), as well as under any consistent hedge that preserves coherence (Theorem 6.1).

We interpreted these results as an extension of the fundamental theorems of asset pricing, that mathematically amounts to replacing the set of all probability measures $\mathrm{Pr}(\Omega)$ by a given test set $\mathcal{P} \subset \mathrm{Pr}(\Omega)$. By assuming linear hedge sets, we obtained somewhat sharper formulations of the results in Carr et al. (2001) on the acceptable completeness, and in Jaschke and Küchler (2001) on valuation bounds. We also indicated how the framework could be applied for analyzing subjective valuation bounds.

Our analysis raised some specific mathematical questions, e.g. the extension of the framework to weak coherence, and formulation of explicit criteria for hedging schemes to be coherence preserving. At the more practical side, CoPr measures might be used to simplify the design and calibration of test sets in applications, by disconnecting hedging from acceptability assessment in the spirit of risk-neutral valuation.

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