Option Pricing, Lévy Processes, Stochastic Volatility, Stochastic Lévy Volatility, VG Markov Chains and Derivative Investment.

Dilip B. Madan
Robert H. Smith School of Business
University of Maryland

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EMPIRICAL OBSERVATIONS I

- FROM TIME SERIES DATA.

  1. IT HAS BEEN KNOWN SINCE EARLY WORK BY FAMA THAT DAILY RETURNS ARE MORE LONG-TAILED RELATIVE TO THE NORMAL DENSITY, WITH AN APPROACH TO NORMALITY AS WE CONSIDER MONTHLY RETURNS.

  2. MORE RECENTLY WE HAVE EVALUATIONS OF ONE MINUTE, 15 MINUTE, HOURLY, AND DAILY RETURN DATA ON S&P 500 FUTURES RETURN DATA THAT CONFIRMS AND EXAGGERATES THIS PICTURE.

S&P 500 FUTURES RETURNS

NOV. 1992-FEB. 1993

	1 Min.	15 Min.	Hourly	Daily
Kurtosis	58.59	13.85	5.97	10.31
χ^2 test statistic	437.12	931.85	98.323	123.84
χ^2 critical value 5%	9.26	5.7	3.57	0.989

Source: Doctoral Dissertation of Theiry Ané, Université de Paris IX Dauphine-ESSEC.

EMPIRICAL OBSERVATIONS II

- FROM OPTIONS DATA.
  - BLACK-MERTON-SCHOLES IMPLIED VOLATILITY SMILES FROM OPTIONS DATA ALSO SUGGEST LONGER THAN NORMAL RISK NEUTRAL TAILS FOR RETURN DATA
  - SKEWNESS PREMIA DOCUMENTED BY BATES SUGGEST LONGER LEFT TAILS THAN RIGHT TAILS FOR THE RISK NEUTRAL PROCESS.

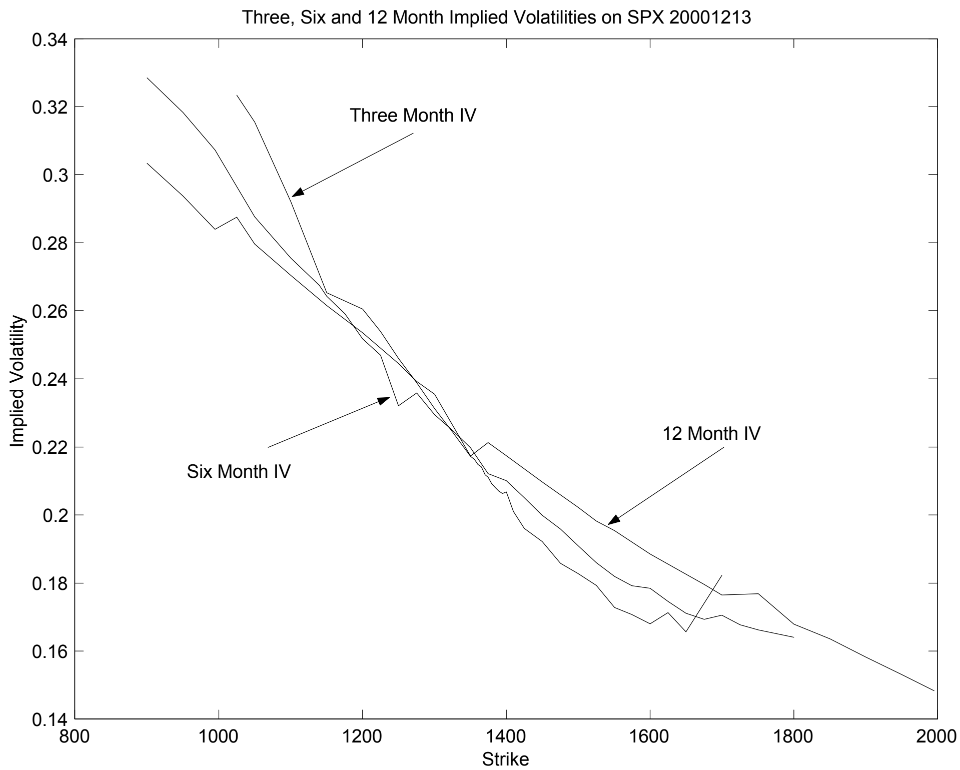

Three, Six and 12 Month Implied Volatilities on SPX 20001213
Implied Volatility
Strike
Three Month IV
Six Month IV
12 Month IV
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
800
1000
1200
1400
1600
1800
2000

Figure 1:

EMPIRICAL OBSERVATIONS III

- FROM THE ANALYSIS OF EXTREMES.

  1. THE LIMITING DISTRIBUTION OF THE MAXIMA OR MINIMA OF INDEPENDENTLY SAMPLED OBSERVATIONS FROM A DISTRIBUTION IS KNOWN TO BELONG UP TO A SCALE AND SHIFT CONSTANT TO EITHER THE

     GUMBEL, WEIBULL OR FRECHET

FAMILIES OF DISTRIBUTIONS

- 1. THE NORMAL OR LOGNORMAL IS IN THE DOMAIN OF ATTRACTION OF THE GUMBEL

  2. THE LOG GAMMA OR THE VG MODEL DISCUSSED LATER IS IN THE DOMAIN OF ATTRACTION OF THE WEIBULL OR FRECHET.

- FOR DAILY RETURN DATA ON THE DJIA FOR 100 YEARS WE ESTIMATED BY MAXIMUM LIKELIHOOD THE GUMBEL AND WEIBULL OR FRECHET DENSITIES FOR THE MAXIMUM AND MINIMUM RETURN OVER 100 DAYS

DISTRIBUTION OF EXTREMES

	MIN DAILY DROP100 DAY		
	GUMBELL LL	WEIBULL LL	P VALUE
1897-1997	768.37	808.58	0.00
1897-1945	380.22	389.98	0.01
1946-1997	409.93	434.74	0.00
	MAX DAILY RISE 100 DAY		
	GUMBELL LL	FRECHET LL	P VALUE
1897-1997	811.66	833.77	0.01
1897-1945	395.79	408.92	0.01
1946-1997	358.33	432.95	0.01

EMPIRICAL OBSERVATIONS IV

- FROM THE HISTOGRAM OF UP AND DOWN MOVES.

  1. TREATING 100 YEAR DJIA DAILY RETURN AS ARRIVAL RATE OF JUMPS IN A LÉVY MEASURE WE ESTIMATE BY REGRESSIONS A COMMON JUMP DIFFUSION FORM THAT IS NOT COMPLETELY MONOTONE (MORE ON THIS LATER) AND A GENERALIZED VG FORM THAT IS COMPLETELY MONOTONE

  2. THE JUMP DIFFUSION ASSERTS LOG ARRIVAL RATE LINEAR IN JUMP SIZE AND ITS SQUARE

3. THE GENERALIZED VG ASSERTS LOG ARRIVAL RATE LINEAR IN JUMP SIZE AND LOG JUMP SIZE

REGRESSION OF LOG ARRIVAL RATES ON JUMP SIZES

	LOG ARRIVAL OF DOWN MOVE			
	CONST	SIZE	LOG SIZE	RSQ
1897-1997	-9.88(1.44)	-31.6(8.36)	-1.92(0.32)	0.97
1897-1945	-8.51(1.45)	-33.0(8.53)	-1.65(0.32)	0.97
1946-1997	-12.35(2.22)	-32.0(17.78)	-2.41(0.45)	0.95
	LOG ARRIVAL OF UP MOVES			
	CONST	SIZE	LOG SIZE	RSQ
1897-1997	-11.55(1.71)	-24.5(9.10)	-2.25(0.38)	0.96
1897-1945	-10.29(1.65)	-25.4(8.97)	-1.99(0.37)	0.97
1946-1997	-13.66(3.23)	-25.8(24.45)	-2.67(0.65)	0.93
	LOG ARRIVAL FOR JUMP DIFFUSION			
	CONST	SIZE	SIZE^2	RSQ
1897-1997	-3.66(0.53)	-1.73(3.86)	-447(66)	0.70
1897-1945	-3.36(0.48)	-1.77(3.66)	-421(62)	0.71
1946-1997	-3.17(0.65)	1.54(8.98)	-928(191)	0.64

EMPIRICAL OBSERVATIONS V

- For multivariate Gaussian returns one may deduce independence from zero correlations. Hence zero correlations in return levels would imply zero correlations among the squares.

- From 4 years of daily data on the SPX returns we observe the following results for regressions of returns on their lagged values and the regressions of squared returns on their lagged values.

Return Dependencies					
	Slope	SE	R^2	F	pvalue
Return	-.0093	.03	.0000086	.117	.7321
Level					
Squared	.2517	.1021	.0633	91.55	0
Returns					

- The presence of strong correlation at the squared level also argues for the absence of joint normality of returns.

7 No Arbitrage and Asset Returns

1. The implications of no arbitrage

(a) Discounted Prices are Martingales under a change of measure and hence by Girsanov's Theorem, they are semimartingales under the original statistical measure.

(b) Semimartingales can be written as time changed Brownian motion and hence if $X(t)$ is the log price process we may write

$$X(t) = W(T(t))$$

for an increasing random process $T(t)$ that is a process for the time change.

(c) $X(t)$ is a continuous process essentially only if $T(t)$ is continuous, but then we must have

$$T(t) = \int_0^t a(u)du + \int_0^t b(u)dZ(u)$$

for a Brownian motion $Z(t)$.

It follows from the fact that T is increasing that $b = 0$, and hence that the time change is locally deterministic.

- In fact $a(t)$ is then the local variance and local volatility is all that we need to be concerned about in describing risk exposures.

(d) Supposing the time change to be related to locally random economic activity like the arrival of orders or information we conclude that the time change and the price process is discontinuous with possibly no continuous martingale component.

8 The Economic Foundations

1. The time interval of our economy is $[0, \Upsilon]$.

2. Our fundamental departure from traditional modeling assumptions is in the filtration describing the evolution of the underlying uncertainty.

   - Traditionally these are modeled by continuous martingales or stochastic integrals with respect to Brownian motion.

   - We consider instead increasing random processes that are of necessity pure jump processes representing cummulated demand and supply shocks for the asset or commodity under consideration.

3. Let $U(t)$ be the process of cummulated demand shocks. $U(t)$ is a strictly increasing pure jump process and

$$u(t) = \Delta U(t) = U(t) - U(t_{-}) \geq 0$$

represents the number of units of the asset demanded by some economic agent at the prevailing market price of $p(t_{-})$.

$U(t)$ models the arrival of orders to buy at market.

4. Analogously let $V(t)$ represent the cummulated level of supply shocks with

$$v(t) = \Delta V(t) = V(t) - V(t_{-}) \geq 0$$

being the number of units of the asset that some economic agent wishes to sell at the prevailing market price of $p(t_{-})$.

5. ASSUMPTION

We suppose that at any instant of continuous time the market processes either a market buy or market sell order. These two types of orders do not coincide in their arrival time on the time continuum.

6. QUALIFICATION

We do not model the determination of $u(t), v(t)$ as the outcome of optimizing behavior on the part of economic agents. The motivations for such orders may well include in addition to liquidity or information based trades, the demand and supply generated by chartists for example.

The processes $U(t), V(t)$ are the primitives of our model.

8.0.1 Modeling the process of price increases

1. Economic agents realize that buy orders in execution may face an adverse price response and effectively communicate a curtailment of demand in response to such price increases. They in fact supply a demand function

   $$q_t^{du} = q^{du}(p(t)/p(t_-), u_t, t)$$

   where $q^{du}(1, u_t, t) = u_t$ and

   $$\frac{\partial q^{du}}{\partial p(t)} < 0.$$

2. Market buy orders are cleared through meeting or being crossed with limit sell orders. We suppose the existence of supply at a positive price response and all market buy orders are cleared through such matching with the price response being determined in the process. The supply function of the limit sell side is

$$q_t^{su} = q^{su}(p(t)/p(t_-), u_t, t)$$

where we suppose no supply without a price response as markets are always already cleared so

$$q^{su}(1, u_t, t) = 0,$$

and in addition

$$\frac{\partial q^{su}}{\partial p(t)} > 0.$$

3. Market prices and transacted quantities are simultaneously determined by the market clearing condition

$$q_t^{du} = q_t^{su} = q_t^u.$$

We suppose that these equations are solved to determine $p(t)$ and q_t^u in response to a demand shock as

$$\begin{aligned} \ln\left(\frac{p(t)}{p(t_-)}\right) &= \Phi^u(u_t, t) > 0 \\ q_t^u &= \Psi^u(u_t, t) > 0. \end{aligned}$$

8.0.2 Modeling the Price Decrease

1. Similar to the modeling of price responses to a demand shock we suppose that the price response to a supply shock $v(t)$ is given by

$$\begin{aligned} \ln\left(\frac{p(t)}{p(t_-)}\right) &= -\Phi^v(v_t, t) < 0 \\ q_t^v &= \Psi^v(v_t, t) > 0. \end{aligned}$$

8.0.3 The Price Process

1. Putting together the processes for price increases and decreases we obtain the price process as

$$\ln(p(t)) = \ln(p(0)) + \sum_{s\leq t} \Phi^u(\Delta U(s), s) - \sum_{s\leq t} \Phi^v(\Delta V(s), s).$$

2. It follows that the resulting price process stands in sharp contrast to traditional assumptions about such processes in the finance literature.

   - Traditional process assumptions yield continuous price processes: Ours is a pure jump process.

   - Traditional process assumptions yield processes of infinite variation: Ours is a finite variation process as it is by construction a difference of two increasing processes.

9 The Variance Gamma model as the core example

- This is the Variance Gamma process defined by Brownian motion with drift θ and volatility σ, time changed by an increasing Gamma process with unit mean rate and variance rate ν resulting in the three parameter process

$$X(t;\sigma,\nu,\theta) = \theta G(t;\nu) + \sigma W(G(t;\nu))$$

  where $G(t;\nu)$ is the Gamma process and $W(t)$ is a standard Brownian motion.

- The Variance Gamma process has a particularly simple characteristic function given by evaluating the Gamma Characteristic function at $i\theta u - \sigma^2 u^2/2$ the log of the Gaussian characteristic function. It is

$$\phi_{VG}(u) = \left(\frac{1}{1 - i\theta\nu u + \frac{\sigma^2\nu}{2}u^2}\right)^{t/\nu}.$$

- The moment equations can be uniquely solved for the parameters provided skewness satisfies an upper bound given in terms of kurtosis.

  - The moments are given by

$$
\begin{aligned}
Variance &= \theta^2\nu + \sigma^2 \\
Central\ \text{3rd}\ moment &= 2\theta^3\nu^2 + 3\sigma^2\theta\nu \\
Central\ \text{4th}\ moment &= 3Variance^2 + 3\sigma^4\nu \\
&\quad +12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3
\end{aligned}
$$

  - The bound on skewness is

$$1.5 * skewness^2 < (kurtosis - 3).$$

- We may also write $X(t)$ as the difference of two gamma processes

$$X(t) = G_p(t) - G_n(t)$$

on writing

$$\frac{1}{1 - i\theta\nu u + \sigma^2 \nu u^2/2} = \left(\frac{1}{1 - i\eta_p u}\right)\left(\frac{1}{1 + i\eta_n u}\right)$$

whereby

$$\begin{aligned} \eta_p - \eta_n &= \theta\nu \\ \eta_p \eta_n &= \frac{\sigma^2 \nu}{2} \end{aligned}$$

1. (a) and hence

$$\begin{aligned} \eta_p &= \left(\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}\right)^{1/2} + \frac{\theta\nu}{2} \\ \eta_n &= \left(\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}\right)^{1/2} - \frac{\theta\nu}{2} \end{aligned}$$

with Lévy density

$$k_{VG}(x) = \begin{cases} C\frac{\exp(-Mx)}{x} & x > 0 \\ C\frac{\exp(-G|x|)}{|x|} & x < 0 \end{cases}$$

for

$$\begin{aligned} C &= \frac{1}{\nu} \\ G &= \frac{1}{\eta_n} \\ M &= \frac{1}{\eta_p}. \end{aligned}$$

- The parameter θ provides skewness to the distribution as it enhances the left tail when negative by both decreasing G and simultaneously increasing M. The parameter ν provides kurtosis which in the absence of skew ($\theta = 0$) is $3(1 + \nu)$.

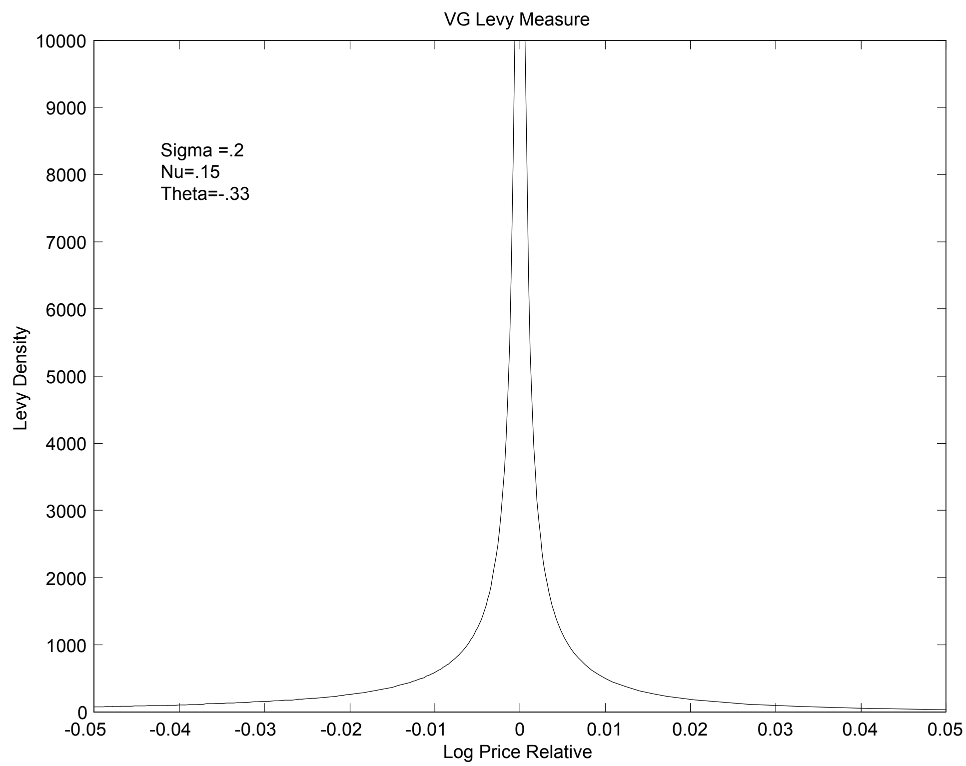


Figure 2:

Graph of Levy Density

10 Economic interpretation of the Parameters.

- The rates of decay on the left G and the right M may be reparameterized in terms of directional and size premia. The parameter C captures quadratic variation.
  - The percentage excess price for a 2% down move relative to a similar up move we call a dircetional premium and for the parameter values of the above Lévy density this is 39.097%.
  - The geometric average of a 2% percent move independent of direction relative to a 4% move we call a size premium and for the Lévy density shown this is 49.2866%.
  - The parameter $C = 6.67$ for the displayed Lévy density is a measure of the speed of the economy as it essentially measures the rate at which time or quadratic variation is changing.

- One may recover the exponential rates of decay on the left and the right from a specification of directional and size premia.

11 Contrast with Jump Diffusion

- We note by way of contrast that Jump Diffusion models typically have finite activity Lévy densities that are not completely monotone for their jump components.
  - These processes model the behavior of frequent small moves using a diffusion.
  - They model rare large moves by an unconnected and orthogonal jump process.
  - The jump component can induce blips in the Lévy density on the right or the left.
- We speculate that perhaps
  - an infinite activity Lévy process,
  - with a monotone density that links small and large behavior,

– adequately dispenses with the need to consider an additional, orthogonal and unrelated diffusion component.

12 Option Pricing with Lévy process models using the FFT

- We typically model the stock price risk neutrally by

$$S(t) = S(0)\exp{(r - q + \omega)\,t + X(t)}$$

where $X(t)$ is a process with a known characteristic function

$$\begin{aligned}\phi(u) &= E\left[\exp(iuX(t))\right] \\ &= \exp(t\psi(u))\end{aligned}$$

- To organize the forward price at $S(0)\exp(r-q)t$ we take the value of ω as defined by

$$\omega = -\psi(-i).$$

- We employ the fast Fourier transform as developed in Carr and Madan (1998). If we define the transform of the modified call price in log strike by

$$\gamma(u) = \int_{-\infty}^{\infty} e^{iuk} e^{\alpha k} C(k) dk$$

where k is the log of the strike and $C(k)$ is the price of a European call of maturity T and strike e^k, then

$$\gamma(u) = \frac{e^{-rt}\phi_{\ln S}(\alpha + 1 + iu)}{(\alpha + iu)(\alpha + 1 + iu)}$$

Call prices may be recovered easily on inversion.

$$C(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \gamma(u) du$$

- The method is valid and applicable once we have analytical expressions for the characteristic functions of the log price. It may be applied uniformly across all strikes to provide us with very fast algorithms for surface calibration.

12.1 Calculation of Modified Call transform from the characteristic function

- We have noting $x = \ln(S(T)$ that

$$
\begin{aligned}
&\gamma(u) \\
= &\int_{-\infty}^{\infty} e^{iuk} \int_{k}^{\infty} e^{-rt} e^{\alpha k}(e^{x} - e^{k}) q(x) dx dk \\
= &e^{-rt} \int_{-\infty}^{\infty} q(x) \int_{-\infty}^{x} \\
&(e^{x+(\alpha+iu)k} - e^{(\alpha+1+iu)k}) dk dx \\
= &e^{-rt} \int_{-\infty}^{\infty} q(x) e^{(\alpha+1+iu)x} \times \\
&\left[\frac{1}{\alpha + iu} - \frac{1}{\alpha + 1 + iu}\right] dx \\
= &e^{-rt} \frac{\phi(u - i(\alpha + 1))}{(\alpha + iu)(\alpha + 1 + iu)}
\end{aligned}
$$

13 The CGMY Process, measure changes and fine structure questions

- The CGMY process is obtained on generalizing the VG Lévy density to

$$k_{CGMY}(x) = \begin{cases} C\frac{\exp(-Mx)}{x^{1+Y}} & x > 0 \\ C\frac{\exp(-G|x|)}{|x|^{1+Y}} & x < 0 \end{cases}$$

- The parameter Y captures the fine structure of the process in the following way

$$\begin{array}{cc} Y < -1 & FA \text{ and } not\ CM \\ -1 \leq Y < 0 & FA \text{ and } CM \\ 0 \leq Y < 1 & IA \text{ and } FV \\ 1 \leq Y < 2 & IV \end{array}$$

- The $CGMY$ characteristic function is obtained on integration as

$$\log\left[\phi_{CGMY}(u)\right] =$$
$$tC\Gamma(-Y)\left\{ \begin{array}{c} M - iu)^Y - M^Y + \\ (G + iu)^Y - G^Y \end{array} \right\}$$

- The density is quite robust and we illustrate a few parameter settings

- The $CGMYe$ is the process

$$X_{CGMYe}(t) = X_{CGMY}(t) + \eta W(t)$$

- The $CGMYe$ characteristic function is given by

$$\log\left[\phi_{CGMYe}(u)\right] = \log\left[\phi_{CGMY}(u)\right] - \eta^2 u^2 t/2$$

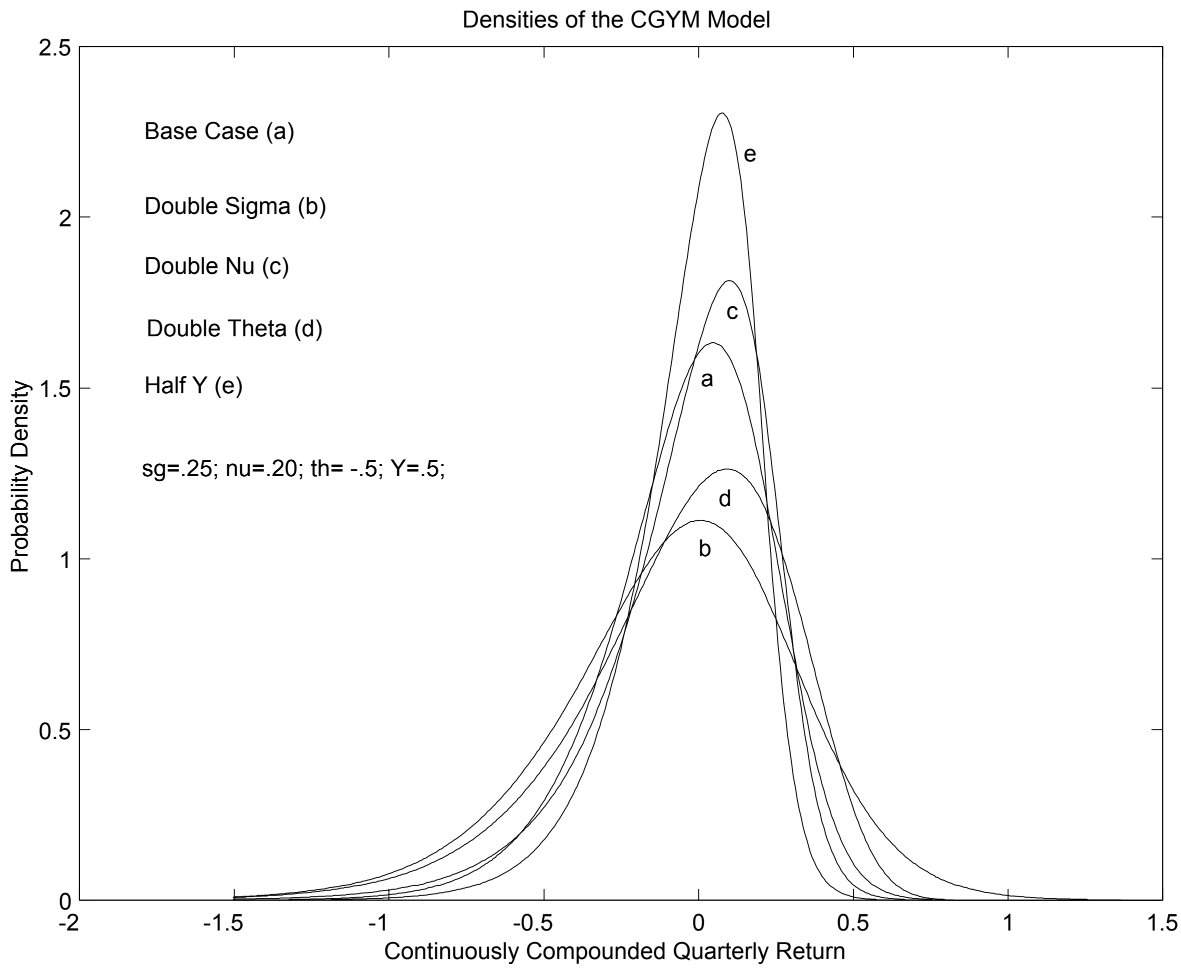

Densities of the CGYM Model
Base Case (a)
Double Sigma (b)
Double Nu (c)
Double Theta (d)
Half Y (e)
sg=.25; nu=.20; th= -.5; Y=.5;
Probability Density
Continuously Compounded Quarterly Return
a
b
c
d
e
0
0.5
1
1.5
2
2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5

Figure 3:

13.1 The CGMYe Stock Price Process

- The $CGMYe$ stock price process is defined by

$$S(t) = S(0)\exp\left(\left(\mu + \omega - \eta^2/2\right)t + X_{CGMYe}(t)\right)$$

where

$$\omega = -\frac{1}{t}\log\left(\phi_{CGYM}(-i)\right)$$

and ensures that the mean rate of return is μ.

- The log characteristic function of the log Stock price is

$$\log\left(\phi_{\ln S}(u)\right) =$$
$$iu(\ln(S(0)) + (\mu + \omega - \eta^2/2)t) +$$
$$\log\left[\phi_{CGMYe}(u)\right]$$

- For the risk neutral process we employ the same process with the mean rate set to equal the interest rate and the other parameters determined by matching option prices.

14 Analyzing the Results

- Higher Moments of the CGMYe Process

$$
\begin{aligned}
E\left[X - E[X]\right]^2 &= \eta^2 + \int_{-\infty}^{\infty} x^2 k(x) dx \\
E\left[X - E[X]\right]^3 &= \int_{-\infty}^{\infty} x^3 k(x) dx \\
E\left[X - E[X]\right]^4 &= 3\left(Variance\right)^2 + \int_{-\infty}^{\infty} x^4 k(x) dx
\end{aligned}
$$

- Decomposition of Quadratic Variation

The quadratic variation contributed by the diffusion component is

$$\eta^2 t$$

The contribution of the $CGMY$ jump component is

$$C\Gamma(2-Y)\left[\frac{1}{M^{2-Y}} + \frac{1}{G^{2-Y}}\right].$$

obtained on integrating x^2 against the Lévy measure.

- Explicit Measure Change.

Let the statistical Lévy measure have parameters

$$C, G, M, Y$$

and suppose the risk neutral Lévy measure is estimated in the $CGMY$ class with parameters

$$\widetilde{C}, \widetilde{G}, \widetilde{M}, \widetilde{Y}$$

then

$$\frac{dQ}{dP} = \mathcal{E}(Y - 1)$$

where

$$k_Q(x) = Y(x) k_P(x).$$

More explicitly we have

$$\left[\frac{dQ}{dP}\right]_t = \exp\left(-t\int_{-\infty}^{\infty}(Y(x)-1)k_P(x)dx\right) \prod_{s<t} Y(\Delta X_{CGMY}(s))$$

where

$$Y(x) = \begin{cases} \frac{\widetilde{C}}{C} x^{Y-\widetilde{Y}} \exp\left(-(\widetilde{M}-M)x\right) & x > \varepsilon \\ \frac{\widetilde{C}}{C} |x|^{Y-\widetilde{Y}} \exp\left(-(\widetilde{G}-G)|x|\right) & x < -\varepsilon \end{cases}$$

15 Estimation Methodology and Results

- For the statistical estimation we invert using the fast Fourier transform the log characteristic function for daily returns at obtain the density at a prespecified grid of points.

- We also bin the data into this grid and maximize the likelihood of the binned data for our parameter estimates. We employ $N = 16384 = 2^{14}$ which gives a spacing of .00154.

- For the statistical estimation we employ time series data on 13 stocks and 8 market indices.

- For the risk neutral process we fit the implied option prices to market data by non-linear least squares using option prices of a maturity between one and two months.

- For the risk neutral process we use data on five names including SPX index and estimate the parameters for five Wednesdays from October 14 1998 to February 10 1999.

Statistical Results

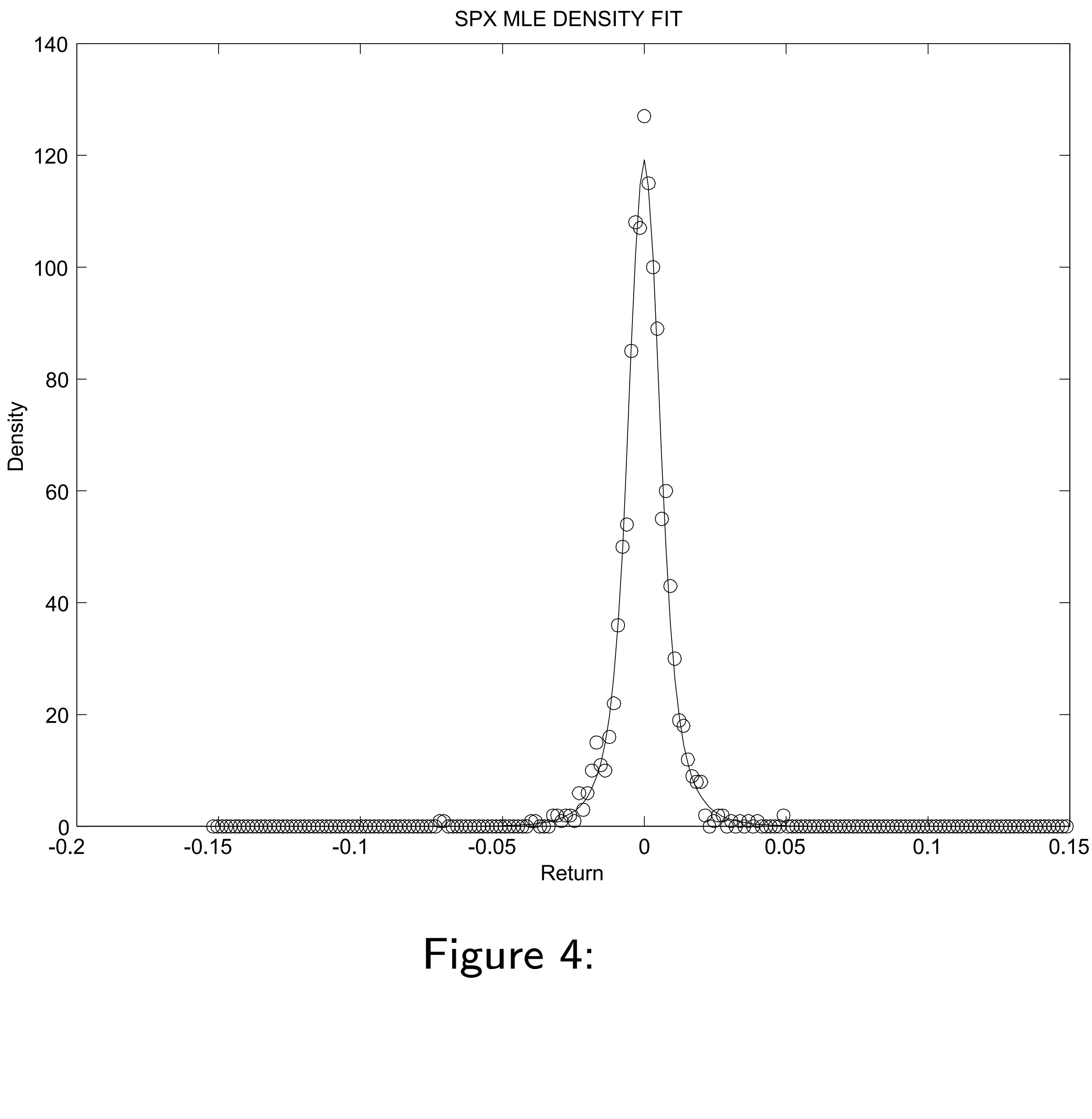

SPX MLE DENSITY FIT
Density
Return
140
120
100
80
60
40
20
0
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15

Figure 4:

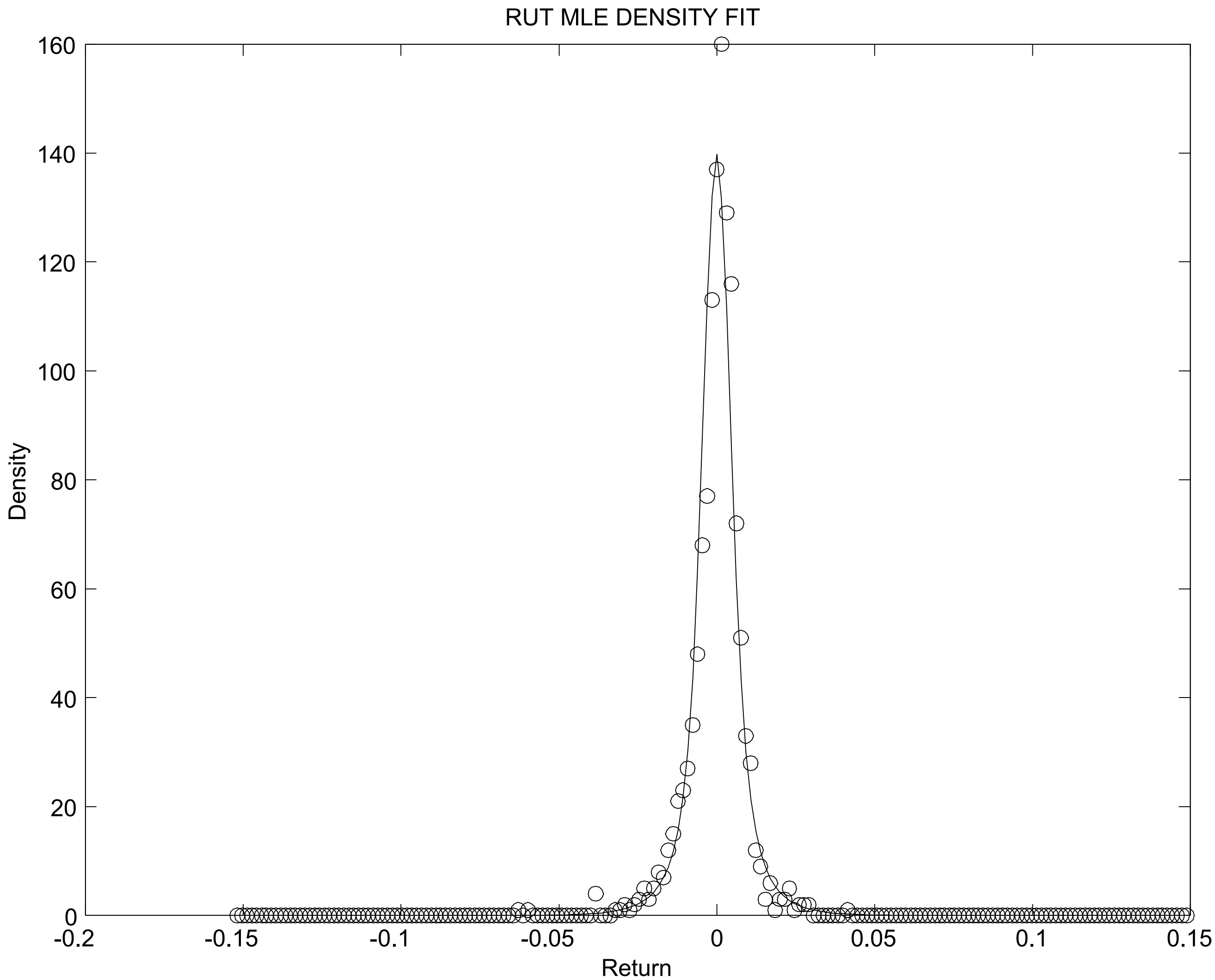

RUT MLE DENSITY FIT
Density
Return
160
140
120
100
80
60
40
20
0
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15

Figure 5:

16 Risk Neutral Results

1. Discussion of Results

   (a) Skewness and Kurtosis

   Statistical:

      i. Skewness is small generally.

      ii. Often the estimated skewness is positive.

      iii. Kurtosis is generally present but is marginally above 3 when annualized.

         - For the excess daily kurtosis one has to multiply the excess over 3 by 365, and this is substantial.

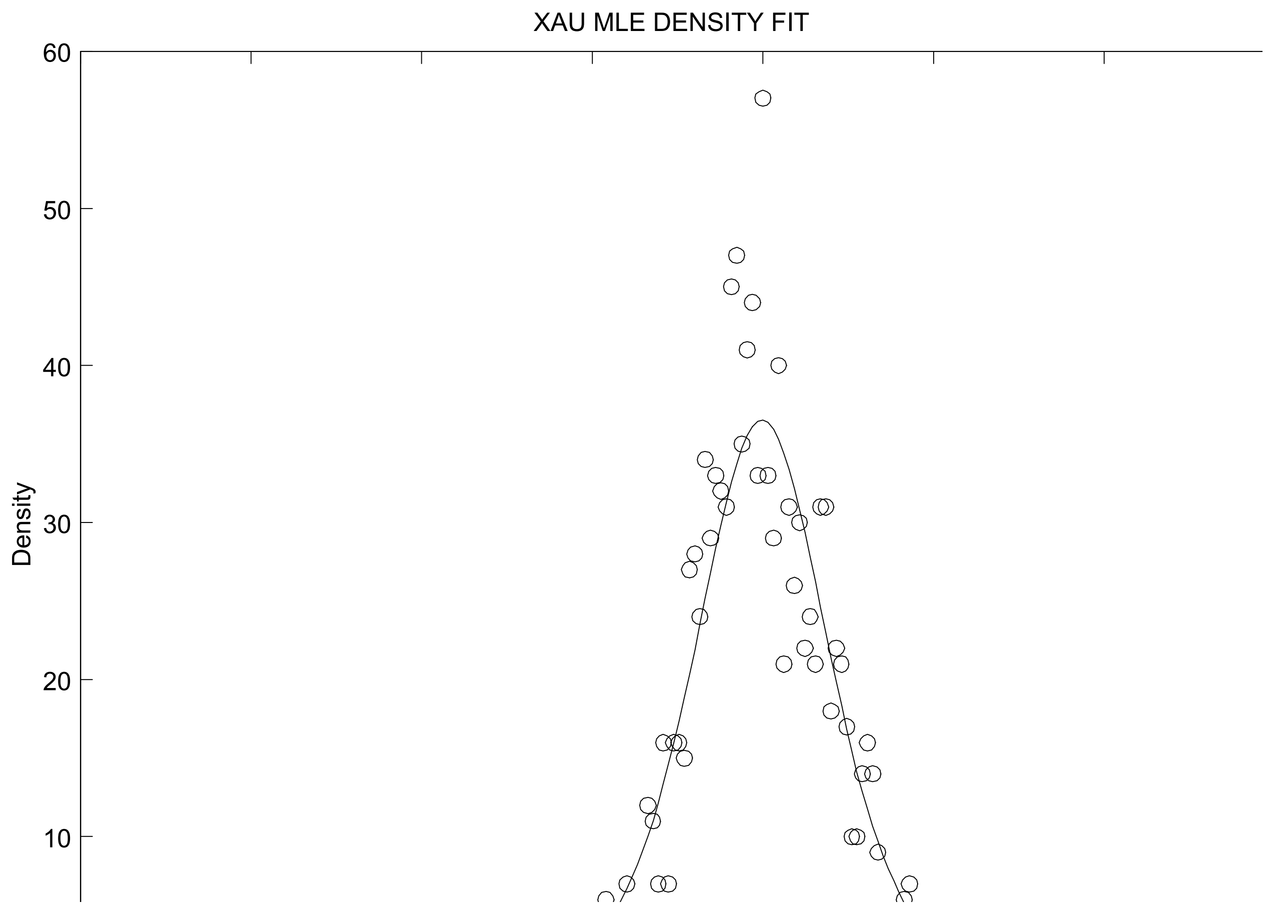


Figure 6:

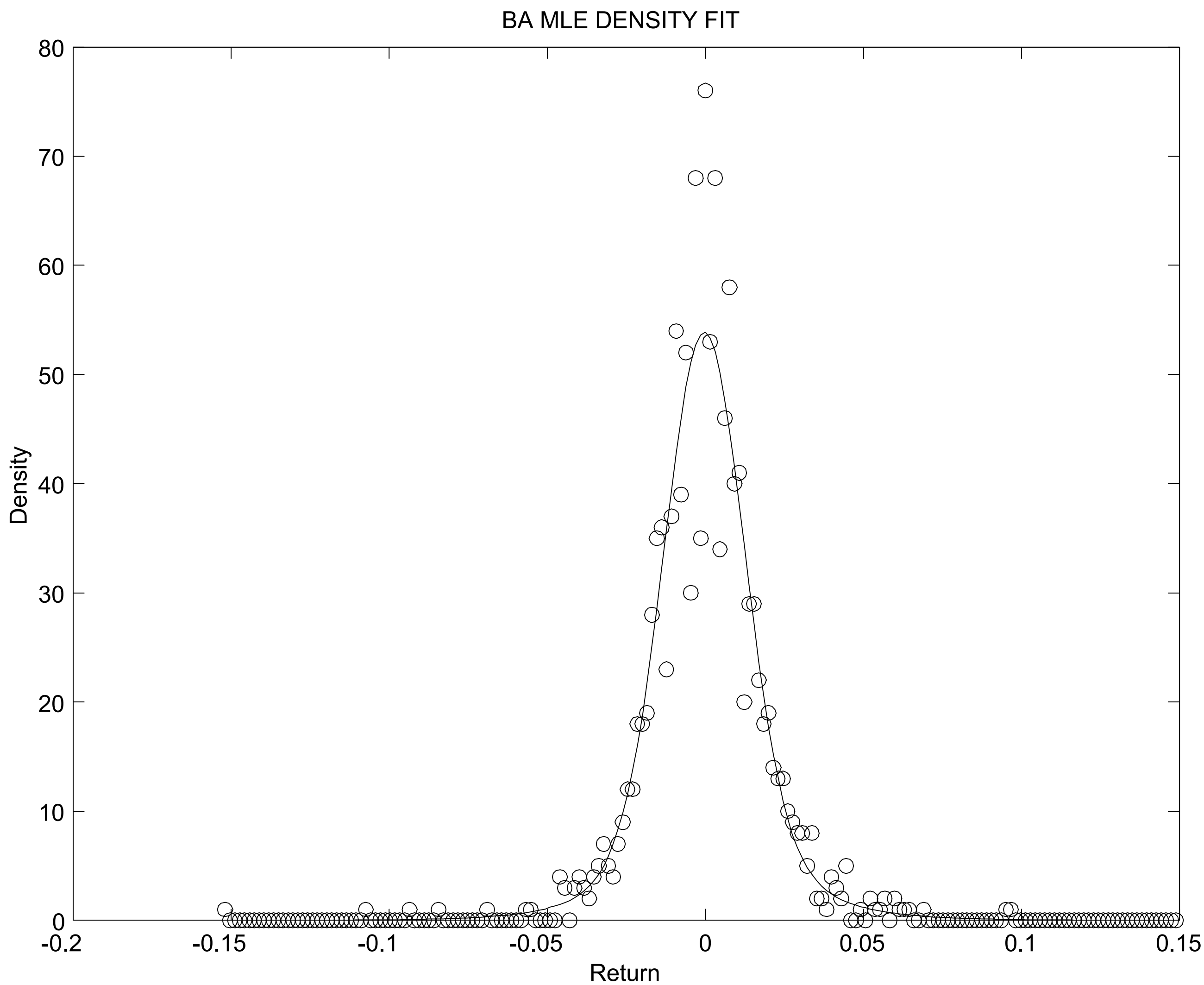

BA MLE DENSITY FIT
Density
Return
80
70
60
50
40
30
20
10
0
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15

Figure 7:

(b) Risk Neutral:

   i. Skewness is substantial.

   ii. Skewness is consistently negative.

   iii. Kurtosis generally much larger than it is statistically.

(c) Decomposition of Predictable Quadratic Variation

i. The Statistical Process for the indices has no diffusion component.

ii. Some Single names, 7 out of 13, do have a diffusion component. Though they are statistically insignificant in all cases.

BA	15.32
GE	1.48
HWP	12.60
IBM	0.71
JNJ	0.23
$MSFT$	62.29
WMT	2.61

iii. This suggests that the diffusion component is the diversifiable noise component while the correlated information component is pure jump.

iv. The risk neutral process has no significant diffusion component in all cases.

(d) The Fine Structure of Returns

Statistical

i. FA: BA, INTC,WMT.

ii. IA, FV: All the rest

iii. IV: MCD, BIX, SOX.

Risk Neutral

i. FA: SPX1014, MSFT1111

ii. IA, FV: All the rest.

iii. IV: IBM1111

(e) Explicit Measure Changes

SPX

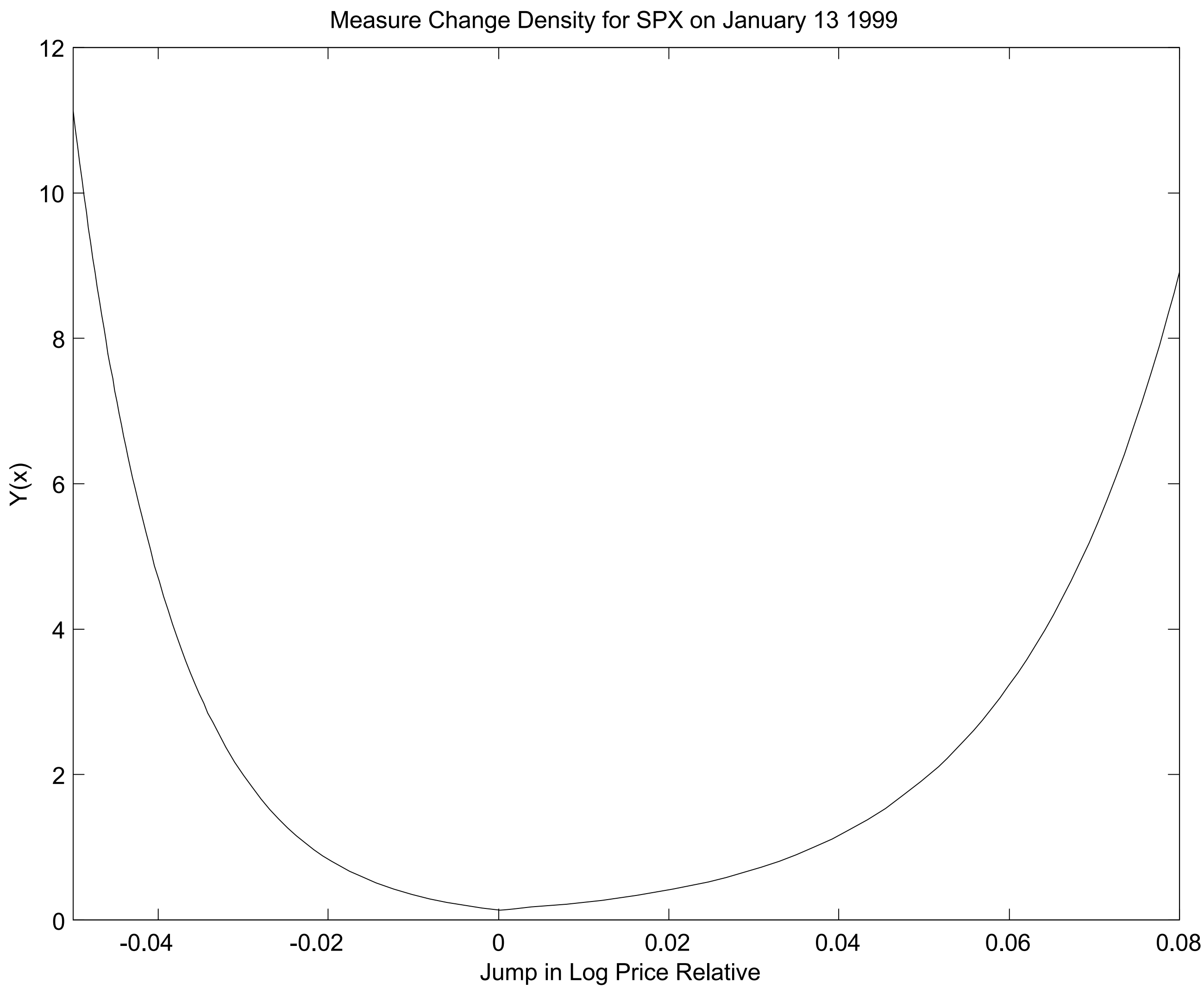

Measure Change Density for SPX on January 13 1999
Y(x)
Jump in Log Price Relative
0
2
4
6
8
10
12
-0.04
-0.02
0
0.02
0.04
0.06
0.08

Figure 8:

DISCUSSION OF MEASURE CHANGE

- Lévy densities are limits of probability densities

- The measure change function is the ratio of Lévy densities and we may build some intuition by considering ratios of probability densities.

- Economic theory for probability densities suggests that

$$
\begin{aligned}
Y(x) &= \frac{U'(c(S_{-}e^{x}))p_S(x)}{U'(c(S_{-}))p_O(x)} \\
& \quad U \text{ is the utility function} \\
S &= S_{-}e^{x} \text{ is post jump stock price} \\
& \quad c(S) \text{ is the investor's position} \\
& \quad p_S(x) \text{ is the subjective probability} \\
& \quad p_O(x) \text{ is the objective probability}
\end{aligned}
$$

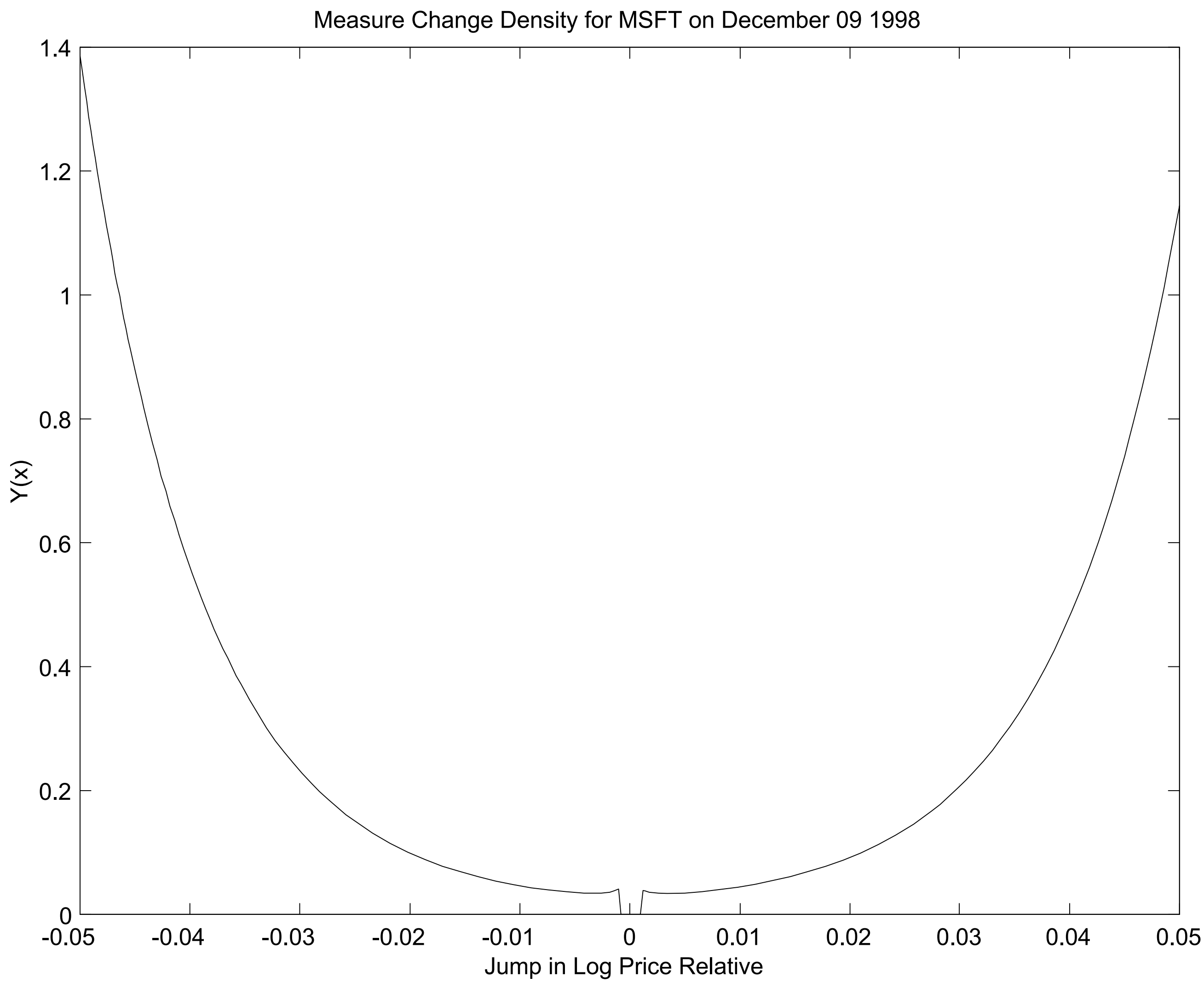
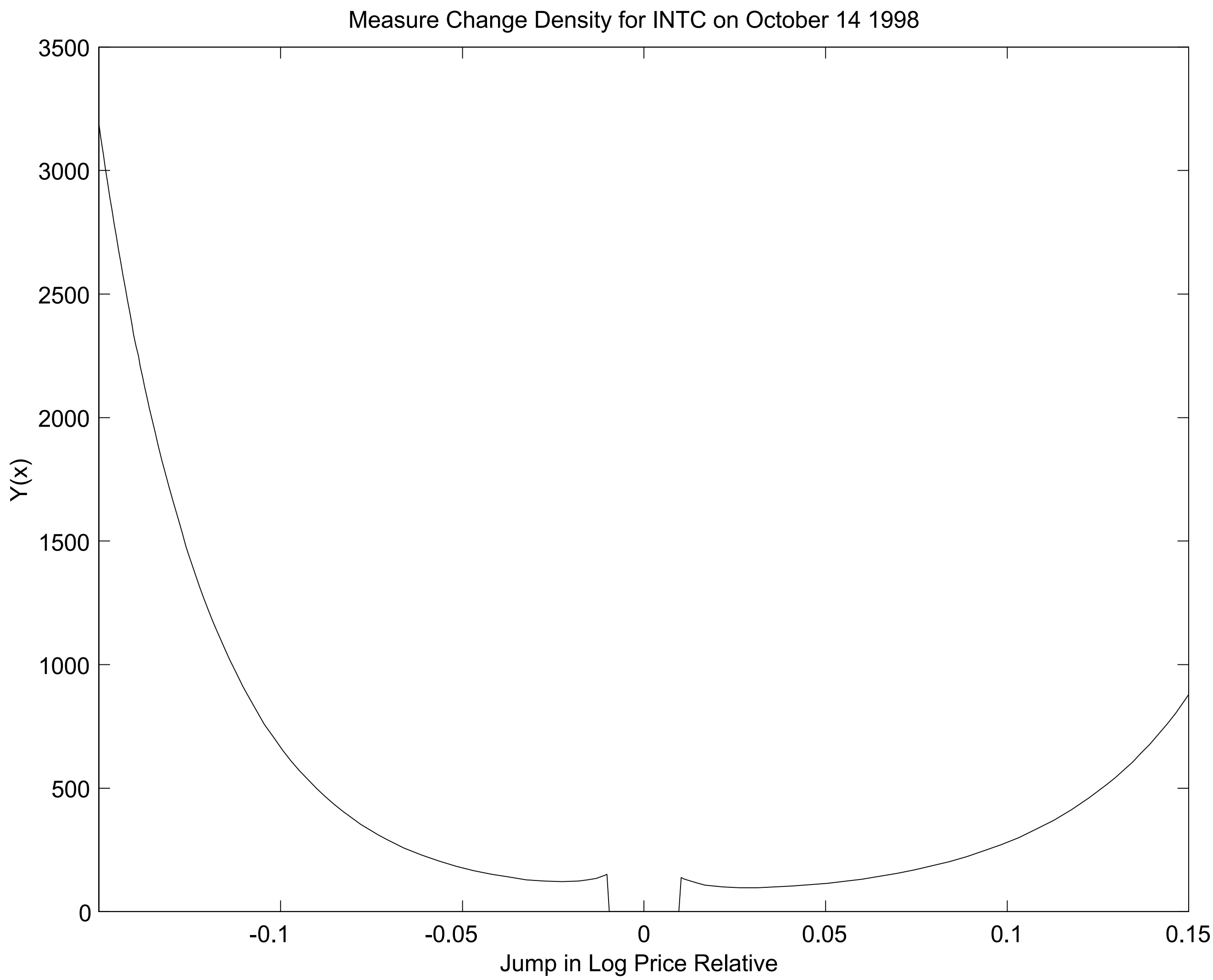

Measure Change Density for MSFT on December 09 1998
Y(x)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.05
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04
0.05
Jump in Log Price Relative

Figure 9:



Measure Change Density for INTC on October 14 1998
3500
3000
2500
2000
1500
1000
500
0
Y(x)
-0.1
-0.05
0
0.05
0.1
0.15
Jump in Log Price Relative

Figure 10:

- Under Rational Expectations

$$p_S(x) = p_O(x)$$

- With a Lucas representative agent

$$c(S) = S$$

- Adding constant relative risk aversion we obtain

$$Y(x) = e^{-\alpha x}$$

where α is the coefficient of relative risk aversion.

- – This is a decreasing function of x with no room for the increase observed with respect to positive values of x.

RESOLUTION 1.

- Failure of rational expectations:
  - Investors do not know the mean of the statistical distribution and the need to mix over this parameter gives the subjective probability greater spread relative to the objective probability.

RESOLUTION 2

- There are heteregeneous beliefs

- Subjective probabilities are closest to objective beliefs for delta hedged option writers who closely monitor movements in this probability.

- These writers delta hedge the position and hence

$$c(S_{\_}e^{x}) \approx a - x^2$$

- Marginal utility applied to a delta hedged option write is $U - shaped$ and losses are experienced with large market moves on either side.

RESOLUTION 3

- The measure change reflects weighted individual personalized state price densities

- Positive weights are given to persons both long the market and short the market

- This leads to measure changes shaped like a hyperbolic cosine function