Non-parametric calibration of jump-diffusion models.

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References:

R Cont , P Tankov (2002) Calibration of jump-diffusion option pricing models: a robust non-parametric approach, Rapport Interne CMAP No. 490

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Outline:

- Overview of exponential Lévy models.
- The calibration problem for exp Lévy models.
- Ill posedness and instability of least squares calibration.
- Regularization using convex penalization.
- Relative entropy for Lévy processes.
- Numerical implementation.
- Tests on simulated data.
- Empirical results for DAX options.
- Conclusion and perspectives.

Jump diffusion models for option pricing

Time homogeneous jumps with finite intensity:

$$S_t = S_0 \ \exp(rt + \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i)$$

σ : Volatility coefficient

N_t : Number of jumps : Poisson process with intensity λ

Y_i : Jump sizes : IID random variables with density $f(.)$

Parameters: $\sigma > 0$, frequency of jumps λ, probability density of jump sizes $f(.)$

Definition: $\nu = \lambda f$ is called the Lévy density.

Extensions: infinite jump rates, time inhomogeneity.

Exponential Lévy models for option pricing

Assumption: dynamics of log price under risk-neutral measure Q is a Lévy process

$$S_t = S_0 \ \exp(rt + X_t)$$

$$E^Q[\exp(iuX_t)] = \exp t\phi(u)$$

$$\phi(u) = iu\gamma - \frac{\sigma^2 u^2}{2} + \int \nu(dx)(e^{iux} - 1 - iux1_{|x|\leq 1})$$

$$\int_{-1}^{1} |x|^2 \nu(dx) < \infty \qquad \int_{|x|>1} \nu(dx) < \infty$$

Martingale condition: Se^{-rt} is a martingale iff

$$\int_{|y|>1} \nu(dy)e^y < \infty \qquad \text{and} \qquad \gamma = -\frac{\sigma^2}{2} - \int (e^y - 1 - y1_{|y|\leq 1})\nu(dy)$$

Different parametrizations of the Lévy measure

- Compound Poisson models: $X_t = \sum_{i=1}^{N^\lambda(t)} Y_i$, $Y_i \sim f$ IID)

Merton model: $f = N(0, \sigma^2)$ Poisson : $f = \sum_{k=1}^{n} p_k \delta_{y_k}$.

- Double exponential (Kou) :

$\nu_0(dx) = [1_{x>0} p \alpha_1 e^{-\alpha_2 x} + (1-p)\alpha_2 e^{-\alpha_2 x} 1_{x<0}]dx$

- Variance Gamma (Madan Seneta) $\nu(dx) = A|x|^{-1} \exp(-\eta_\pm |x|)$
- Tempered stable processes : $\nu(dx) = A_\pm |x|^{-(1+\alpha)} \exp(-\eta_\pm |x|)$
- Normal inverse gaussian process (Barndorff-Nielsen)
- Hyperbolic and generalized hyperbolic processes (Eberlein et al):
- Meixner process : $\nu(dx) = \frac{Ae^{-ax}}{sinh(x)} dx$

Call options in exponential Lévy models

Call option: a security that pays $(S-K)^+$ at date T.

$\rightarrow$ Pricing using Fourier transform (Carr & Madan)

$$C_T^0(k) = e^{-rT}E^Q[(S_T-K)^+] = e^{-rT}E^Q[(e^{s_T}-e^k)^+]$$

$$= e^{-rT}\int_{-\infty}^{\infty}(e^s-e^k)^+ q_T(s)ds$$

$$z_T(k) = e^{-rT}E[(e^{s_T}-e^k)^+] - (1-e^{k-rT})^+$$

$$\zeta_T(v) = \int_{-\infty}^{+\infty} e^{ivk} z_T(k)dk = \frac{e^{-rT}\phi_T(v-i)-e^{ivrT}}{iv(1+iv)}$$

where

$$\phi_T(u) \equiv \int_{-\infty}^{\infty} e^{ius} q_T(s)ds.$$

Non-parametric identification of exponential Lévy models

• Historical point of view: given a Lévy process estimated from underlying time series, how to pick among the infinite number of equivalent martingale measures for pricing options? An ad-hoc choice (ex: Esscher transform) will not give option prices consistent with market prices of options.

• Risk neutral point of view: there are many choices for the form / parametrization of the Lévy measure : which one to choose? $\rightarrow$ a non-parametric analysis can be of guidance.

Goal: non-parametric identification of an exponential Lévy process:

- compatible with a given prior family of equivalent measures, for ex specified from historical data

- compatible with observed market prices of options

Calibration of exp-Lévy models

Model: $S_t = \exp X_t$ where X_t is a Lévy process defined by the characteristic function (σ, ν)

Problem 1: Given the (observed) market prices $C^*(T_i, K_i), i = 1..n$ for a set of liquid call options, find a constant $\sigma > 0$ and a Lévy measure ν such that

$$C^{\sigma,\nu}(T_i, K_i) = C^*(T_i, K_i) \tag{1}$$

where $C^{\sigma,\nu}$ is the option price computed for the Lévy process with triplet $(\sigma, \nu, \gamma(\sigma, \nu))$.

Difficulties:

- The parameter-to-price map $(\sigma, \nu) \rightarrow C^{\sigma,\nu}(T_i, K_i)$ is not explicit: it must be computed using Fourier transform (Carr & Madan).
- Typically this equation may have many or no solutions.

A popular solution: non-linear least squares

$$(\sigma^*, \nu^*) = \arg \inf_{\sigma,\nu} \sum_{i=1}^{N} \omega_i |C^{\sigma,\nu}(t_0, S_0, T_i, K_i) - C^*_{t_0}(T_i, K_i)|^2 \qquad (2)$$

This is still an ill-posed problem

- There may be many Lévy triplets which reproduce call prices with equal precision (pricing error can have many local minima).
- The calibrated Lévy measure is very sensitive to the input prices and to the numerical intialization value in the minimization algorithm.

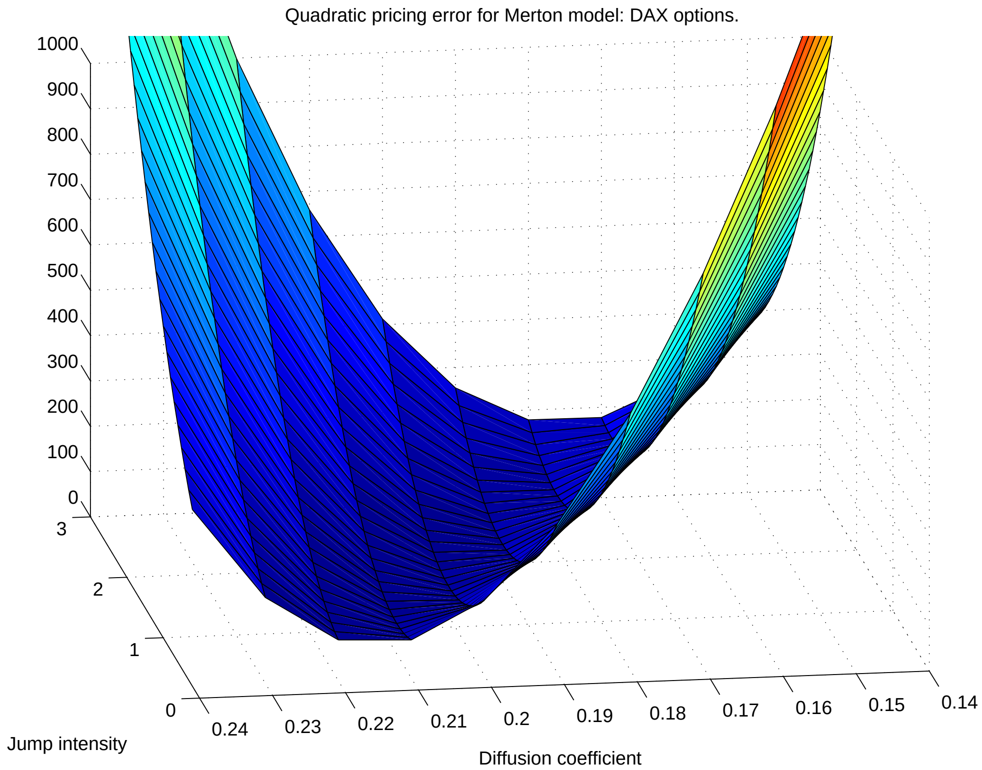

Figure 1: This figure illustrates the difficulty of calibrating even a simple parametric model to real data (we see a line of local minima)

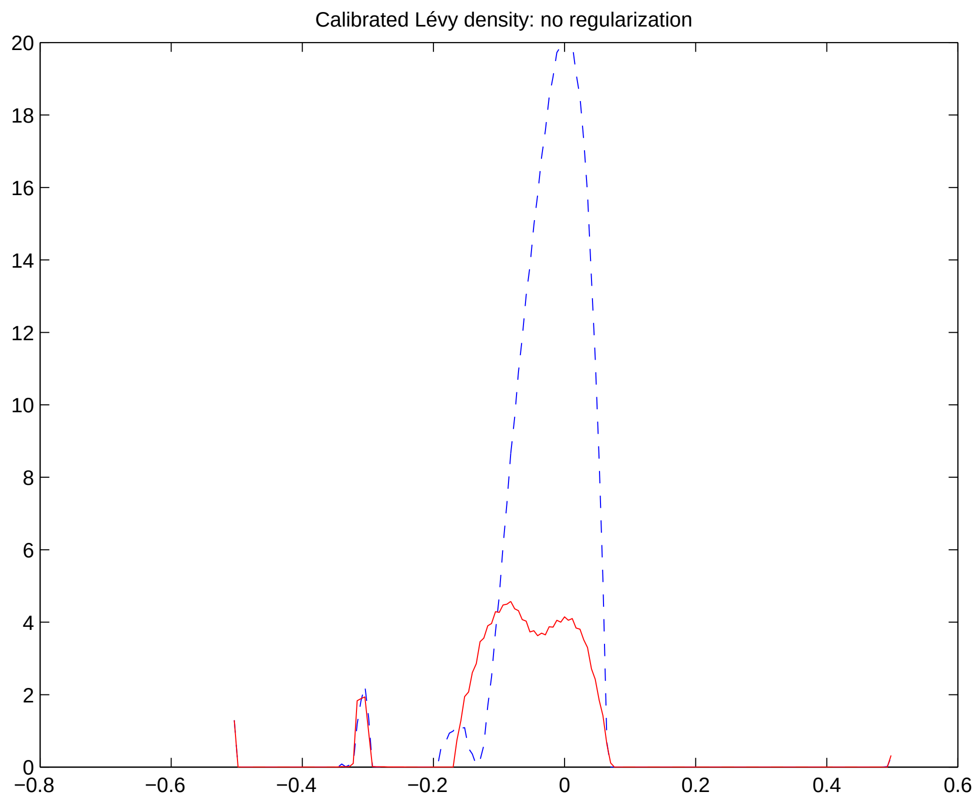

Figure 2: Lévy measures calibrated by least squares. DAX options, 10 May 2001. Maturity 1 month. Least squares, using Merton models with different intensities $\lambda_1 = 1$ and $\lambda_2 = 5$ as initializers.

Regularization using convex penalization term

$$(\sigma^*, \nu^*) = \arg\inf \sum_{i=1}^{N} \omega_i |C^{\sigma,\nu}(T_i, K_i) - C_i^*|^2 + \alpha F(\mathbb{Q}, \mathbb{Q}_0) \qquad (3)$$

When α is small, the solution is close to the least-squares solution (precision).

When α is large, the functional (3) is convex and the solution is close to the prior (stability).

Here we take $F(\mathbb{Q}, \mathbb{Q}_0) = H(\nu, \nu_0)$ (relative entropy)

Relative entropy for Lévy processes

Relative entropy of measure $\mathbb{Q}$ with respect to $\mathbb{Q}_0$ on $\mathcal{F}_{\mathcal{T}}$:

$$H_T(\mathbb{Q}|\mathbb{Q}_0) = E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}}{d\mathbb{Q}_0} \ln(\frac{d\mathbb{Q}}{d\mathbb{Q}_0})]$$

• Convex non-negative functional of $\mathbb{Q}$ for fixed $\mathbb{Q}_0$, equal to zero if and only if $\frac{d\mathbb{Q}}{d\mathbb{Q}_0} = 1$ a.s.

In case of two Lévy processes:

Condition of absolute continuity (Sato) for $\sigma > 0$

$$\sigma = \sigma_0$$

$$\nu \sim \nu_0$$

$$\int_{-\infty}^{+\infty} \left(1 - \sqrt{\frac{d\nu}{d\nu_0}}\right)^2 d\nu_0 < \infty$$

$$H_T(\mathbb{Q}|\mathbb{Q}_0) = \frac{T}{2\sigma^2}\left\{\gamma - \gamma_0 - \int_{-1}^{1} x(\nu - \nu_0)(dx)\right\}^2 + T\int_{-\infty}^{\infty}\left(\frac{d\nu}{d\nu_0}\log\left(\frac{d\nu}{d\nu_0}\right) + 1 - \frac{d\nu}{d\nu_0}\right)\nu_0(dx)$$

Here the first term penalizes the difference of drifts and the second one penalizes the difference of Lévy measures.

If $\mathbb{Q}$ and $\mathbb{Q}_0$ are martingale measures, the first term becomes

$$\frac{T}{2\sigma^2}\left\{\int_{-\infty}^{\infty}(e^x - 1)(\nu - \nu_0)(dx)\right\}^2$$

and the relative entropy only depends on ν and ν_0, i.e.

$$H(\mathbb{Q}|\mathbb{Q}_0) = H(\nu, \nu_0)$$

Properties of relative entropy

- Preserves absolute continuity
- $H(\nu, \nu_0)$ is a convex non-negative functional of ν for fixed ν_0, equal to zero iff $\nu = \nu_0$ almost everywhere
- Easy to compute
- Corresponds to adding the least possible amount of information to the prior
- Widely used in the literature

Relation to other entropy-based calibration algorithms

- Weighted Monte Carlo (WMC) method by Avellaneda & al.

$$\mathbb{Q} = \arg \min_{Q \sim Q_0} \mathcal{E}(Q, Q_0) + \sum_{i=1}^{n} |C^*(T_i, K_i) - E^Q(S(T_i) - K_i)^+|^2$$

where Q and Q_0 are probability measures on a finite set of trajectories, simulated from the $\mathbb{Q}_0$ by Monte Carlo.

Principal differences:

- In the WMC method the optimization is done over the measure $\mathbb{Q}$. Here it is done over the parameters σ, ν of the infinitesimal generator.
- The result of WMC is a set of weights $Q(\omega)$ over a (finite) set of paths. In our case the result is a *process*, defined by its local characteristics $\gamma(\sigma, \nu), \sigma, \nu$.
- Consequence of 1): In our approach the calibrated measure

belongs to the class of risk-neutral measures, corresponding to Lévy processes/ jump diffusions.

- In WMC discretization is essential to make the problem meaningful: the continuous problem does not make sense. Here the limit is well defined and discretization is only used in the numerical implementation. In particular the continuum limit is not singular.
- Under this approach other options can only be priced by Monte Carlo using the *same* sample paths, while our method allows using PIDE methods or Monte Carlo methods for pricing. In particular Monte Carlo pricing can be done with an arbitrary number of paths.

Regularization using relative entropy

$$\nu^* = \arg\inf \alpha H(\nu, \nu_0) + \sum_{i=1}^{N} \omega_i (C^\nu(T_i, K_i) - C^*(T_i, K_i))^2 \qquad (4)$$

Properties of solution:

- Depends continuously on the input prices
- Does not depend on the initial measure (when α is large enough)
- The entropic regularization makes the calibrated measure more smooth

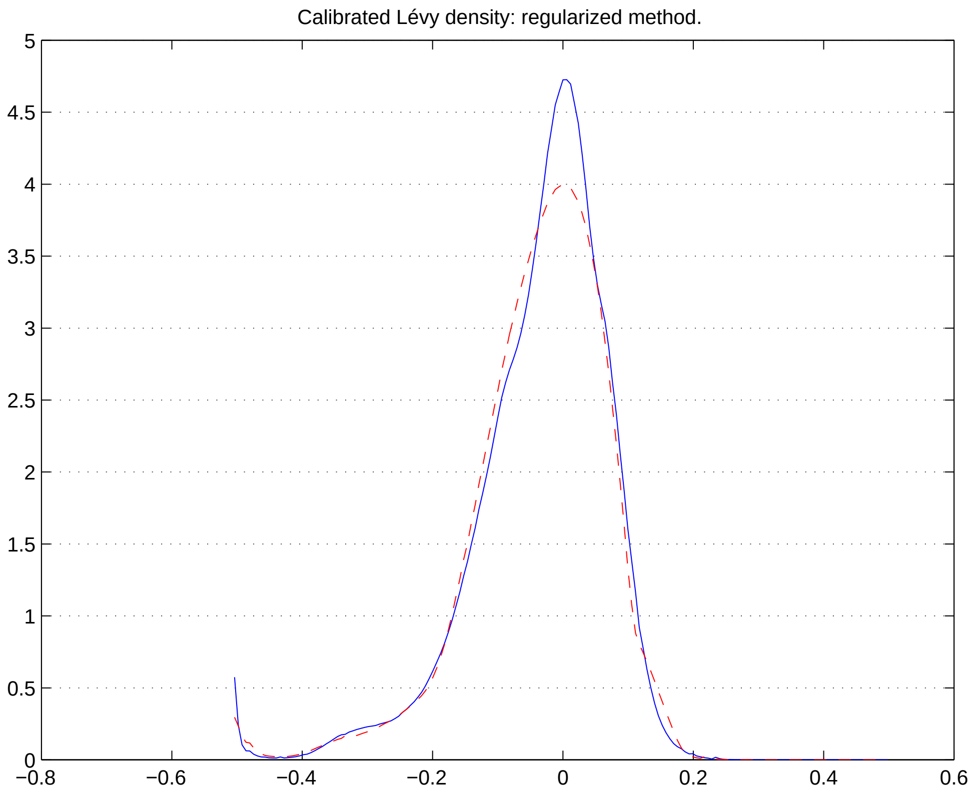

Figure 3: Lévy measure calibrated by entropic regularization. DAX options, 10 May 2001. Maturity 1 month. Again, Merton models with different intensities $\lambda_1 = 1$ and $\lambda_2 = 5$ are used as initializers

Numerical implementation: choice of the prior measure $\mathbb{Q}_0$

- Based on historical estimation (in this case we obtain the risk-neutral measure, closest to the historical one)
- Based on the calibrated measure of the day before. This ensures smooth variation with calendar time.
- From the same dataset, using pre-calibration. In this case we first calibrate a simple parametric model (i.e. Merton's model) using least squares and then use it as prior. Here, the prior does not contain any additional information and is only used to regularize the problem.

Numerical implementation: choice of the regularization parameter

$$\nu^* = \arg\inf \alpha H(\nu, \nu_0) + \sum_{i=1}^{N} \omega_i (C^\nu(T_i, K_i) - C^*(T_i, K_i))^2 \qquad (5)$$

Small α: high precision in calibration, low stability (non convex).

High α: low precision, high stability.

Typically the a posteriori error level $\epsilon(\alpha)$ increases with α.

Idea: choose α such that the a posteriori error (calibration error) has the same level as the a priori error (error on input prices).

Morozov discrepancy principle : given the "noise" level ϵ_0 on the input prices, choose $\alpha > 0$ such that $\epsilon(\alpha) \simeq \epsilon_0$.

Typically ϵ_0 is due to bid/ask spreads.

Numerical implementation: other issues

- The weights ω_i of different prices must reflect relative liquidity of these options: a simple solution is to take $\omega_i = \frac{1}{Vega_i^2}$
- The Lévy measure is discretized on a uniform grid in order to use FFT.
- An explicit representation of the gradient of the minimization functional allows to use a gradient based optimization method to solve the minimization problem.

Overview of the algorithm: bid ask prices

Define $C_i^* = (C_i^{bid} + C_i^{ask})/2$

1. Calibrate a Merton model (with Gaussian jumps) to obtain an estimate of volatility σ_0.

2. Compute uncertainty on prices as $\epsilon_0^2 = \sum_{i=1}^N \omega_i |C_i^{bid} - C_i^{ask}|^2$.

3. Use several BFGS runs with low precision to compute optimal regularization parameter α^* acheiving tradeoff between precision and stability:

$$\varepsilon(\alpha^*) = \sum_{i=1}^{N} \omega_i |C^{\sigma,\nu}{}_i - C_i^*|^2 \simeq \epsilon_0^2$$

4. Solve variational problem for $\mathcal{J}(\nu)$ with α^* by BFGS with high precision using prespecified prior or result of 1) as prior.

Overview of the algorithm: transaction prices

1. Calibrate a Merton model (with Gaussian jumps) to obtain an estimate of volatility σ_0.

2. Fix $\sigma = \sigma_0$ and run least squares ($\alpha = 0$) to get estimate of "distance to model" $\epsilon_0^2 = \inf_\nu \sum_{i=1}^N \omega_i |C^{\sigma_0,\nu}{}_i - C_i^*|^2$.

3. Use several BFGS runs with low precision to compute optimal regularization parameter α^* acheiving tradeoff between precision and stability:

$$\varepsilon(\alpha^*) = \sum_{i=1}^N \omega_i |C^{\sigma,\nu}{}_i - C_i^*|^2 \simeq \epsilon_0^2$$

4. Solve variational problem for $\mathcal{J}(\nu)$ with α^* by BFGS with high precision using prespecified prior or result of 1) as prior.

Tests on simulated data

Model 1: Kou's model (compound Poisson)

$$\nu(x) = \lambda[1_{x>0} p\alpha_1 e^{-\alpha_2 x} + (1-p)\alpha_2 e^{-\alpha_2 x} 1_{x<0}]$$

Option prices were computed for 21 equidistant strikes, ranging from 6 to 14 (the money being at 10).

Model 2: Variance Gamma model (infinite activity, no diffusion component)

$$\nu(x) = A|x|^{-1} \exp(-\eta_{\pm}|x|)$$

Option prices were computed for 45 equidistant strikes ranging from 7.5 to 12.

The Merton's model (with symmetric Gaussian jumps) was used as prior in both cases.

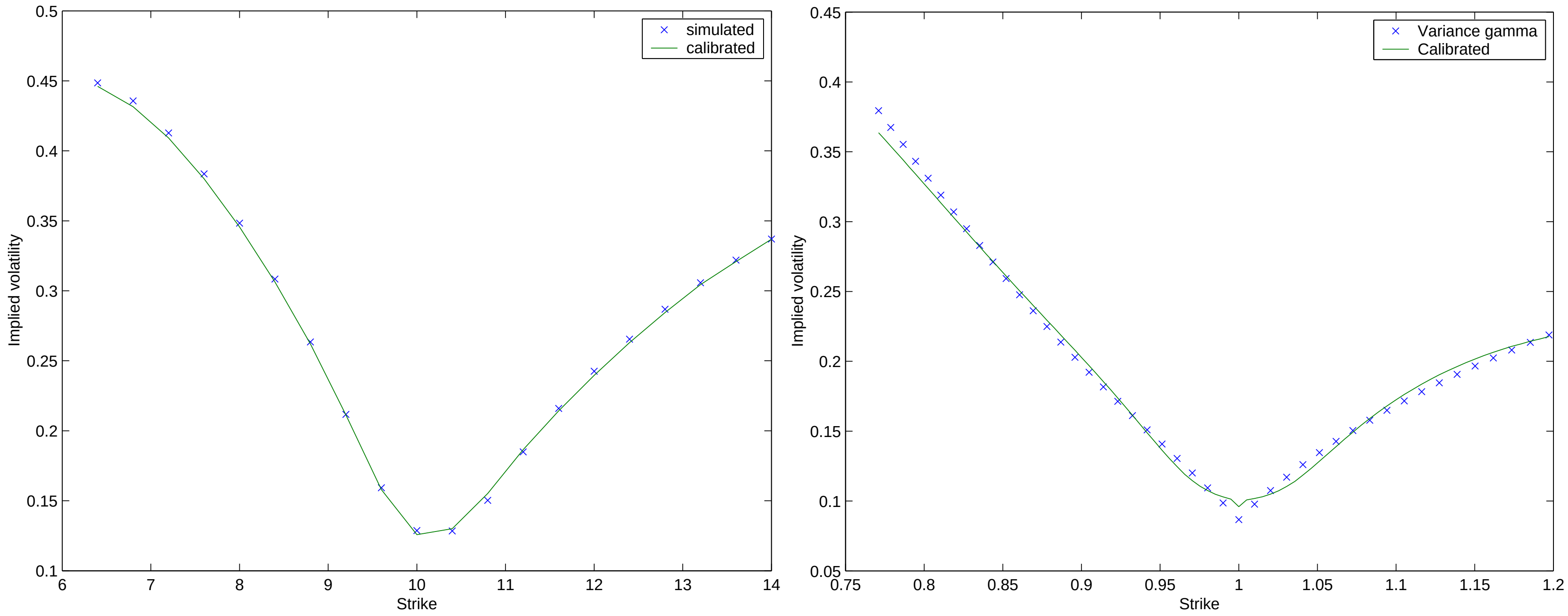

Figure 4: Calibration quality for Kou's jump diffusion model (left) and VG model (right)

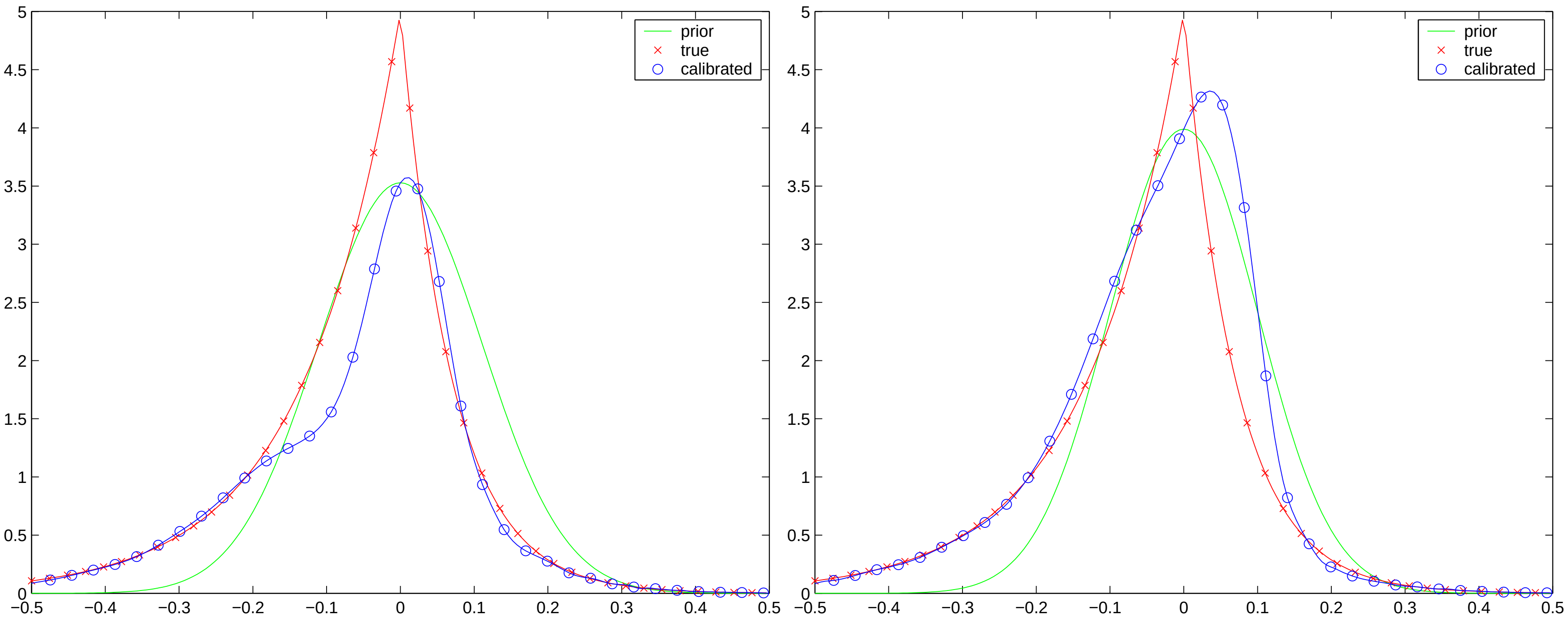

Figure 5: Lévy measure calibrated to option prices simulated from Kou's jump diffusion model with $\sigma_0 = 10\%$. Left: σ has been calibrated in a separate step ($\sigma = 10.5\%$). Right: σ was fixed to $9.5\% < \sigma_0$.

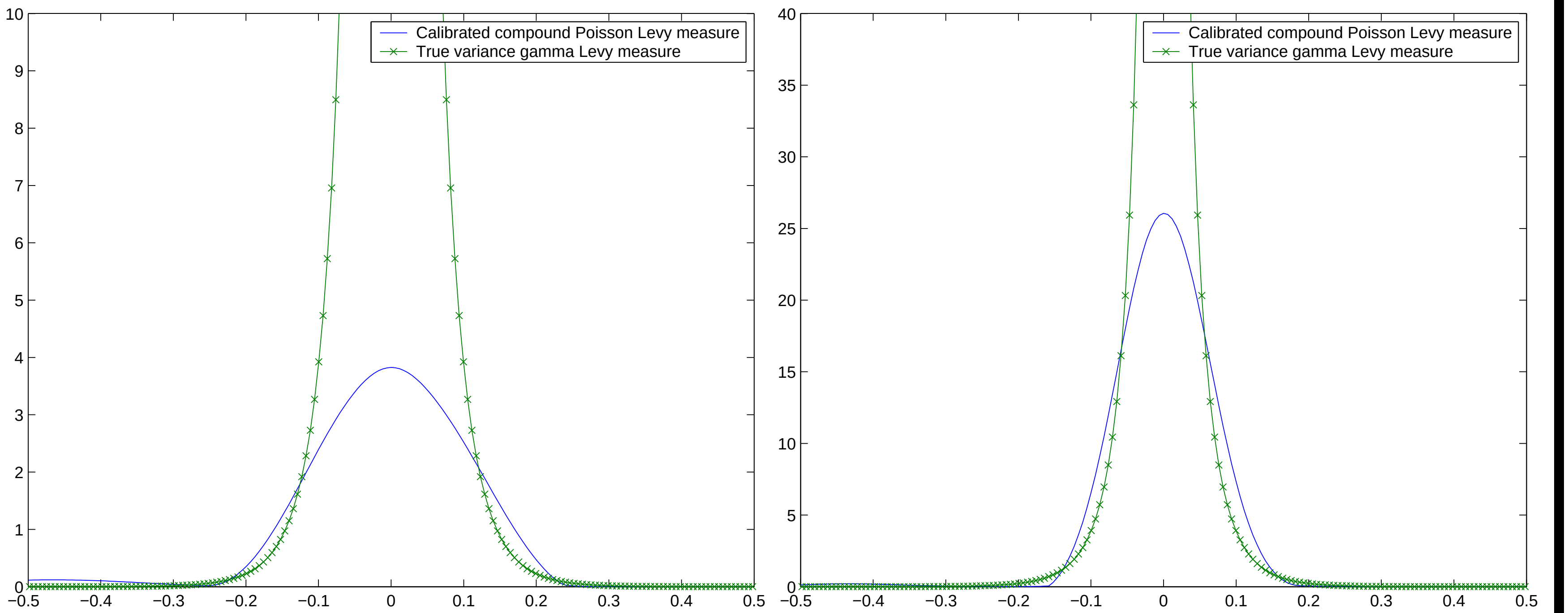

Figure 6: Lévy measure calibrated to variance gamma option prices with $\sigma = 0$ using a compound Poisson prior with $\sigma = 10\%$ (left) and $\sigma = 7.5\%$ (right). Increasing the diffusion coefficient decreases the intensity of small jumps in the calibrated measure.

Summary of empirical results

The calibrated Lévy measures we obtain are strongly asymmetric: the distribution of jump sizes is highly skewed towards negative values.

A small intensity of jumps λ can be sufficient for explaining the shape of the implied volatility for small maturities: empirically $\lambda \simeq 1$

Regularization by entropy strongly reduces sensitivity of results to the initialization: stable numerical results.

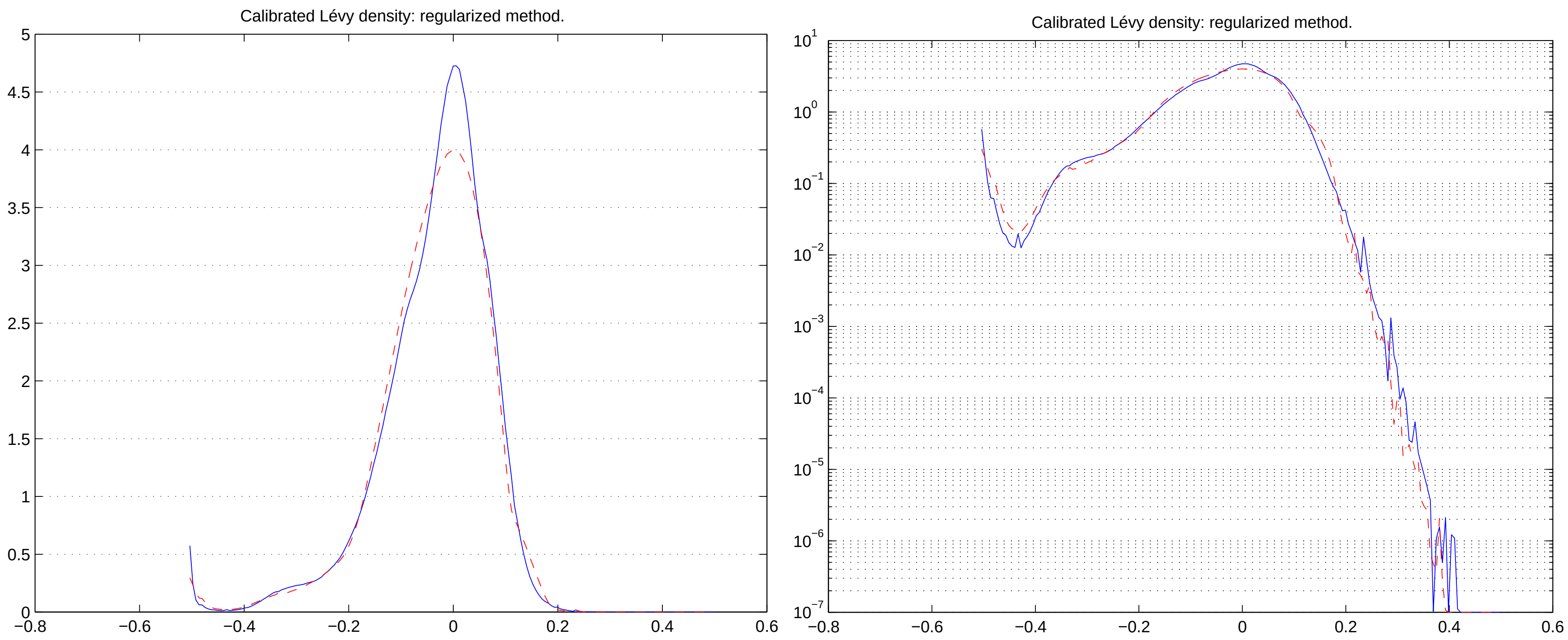

Figure 7: Lévy density calibrated to DAX option prices, maturity 3 months, linear and logarithmic scale.

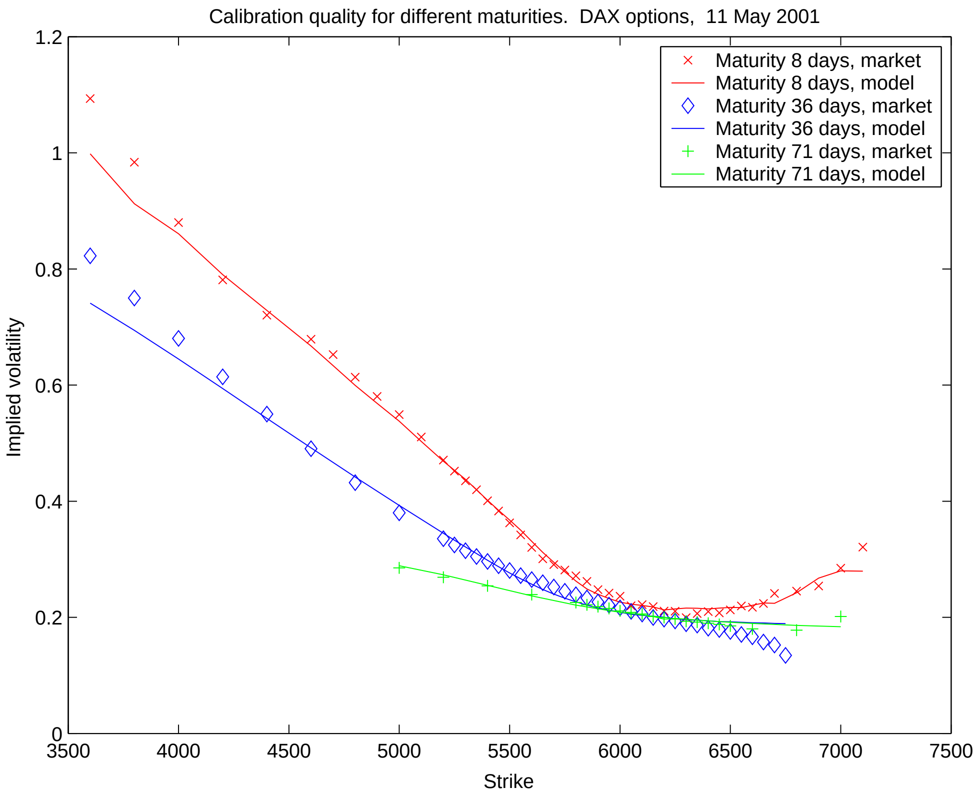

Figure 8: Calibration quality for different maturities: market implied volatilities for DAX options against model implied volatilities. Each maturity has been calibrated separately.

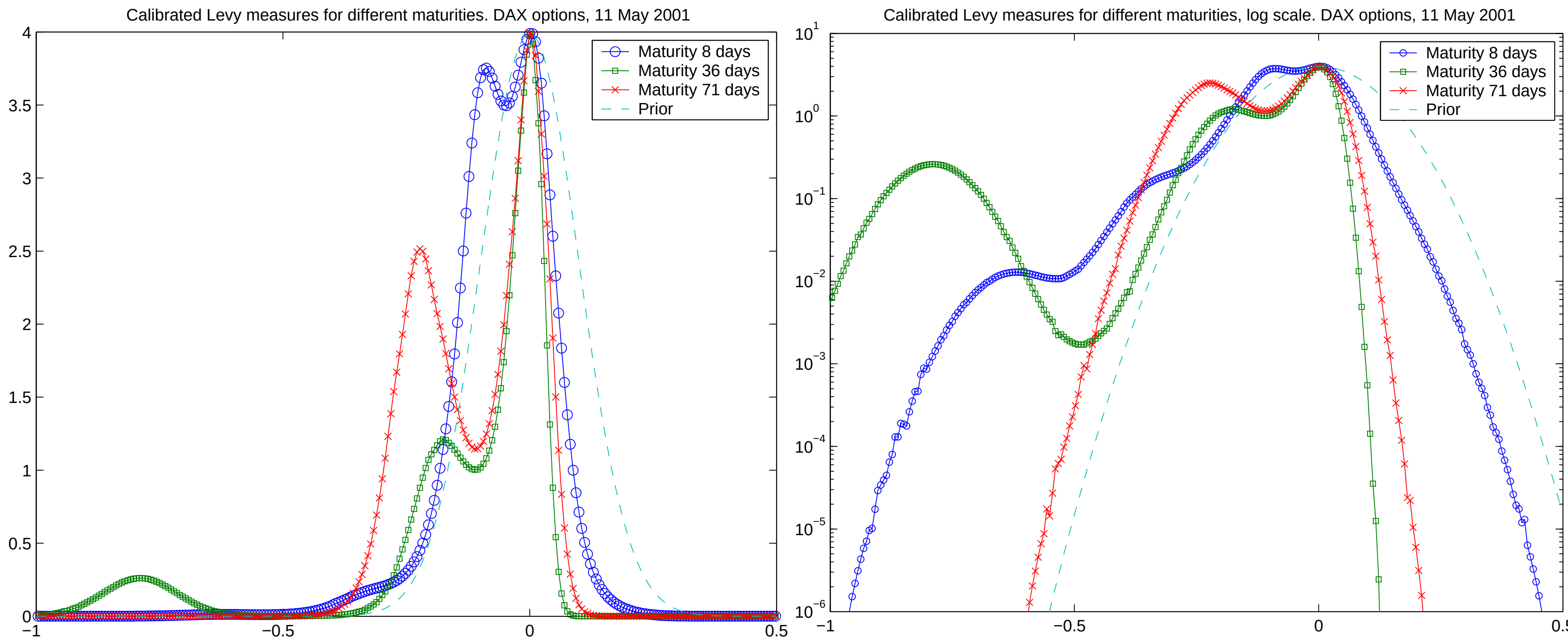

Figure 9: Lévy measures calibrated to DAX option prices for three different maturieis, linear and logarithmic scale.

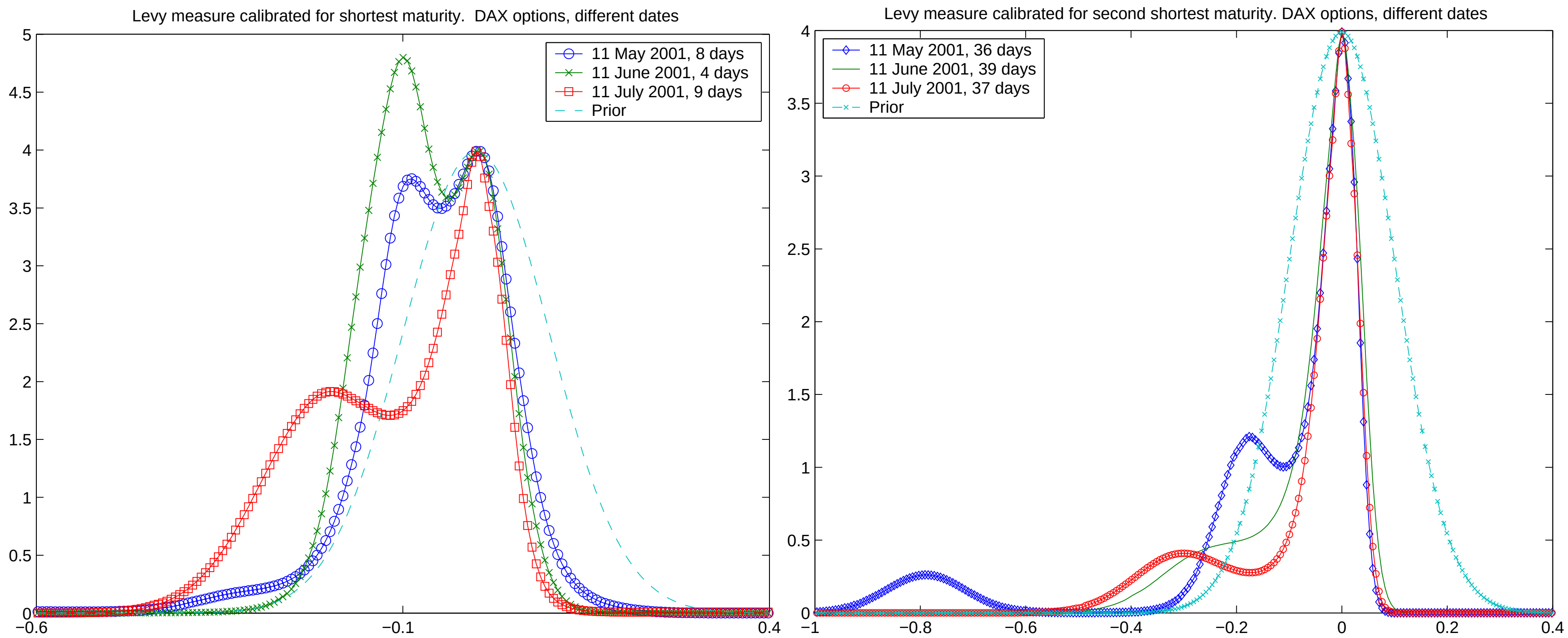

Figure 10: Stability of calibration over calendar dates. Lévy measures have been calibrated at different dates for shortest (left) and second shortest (right) maturity.

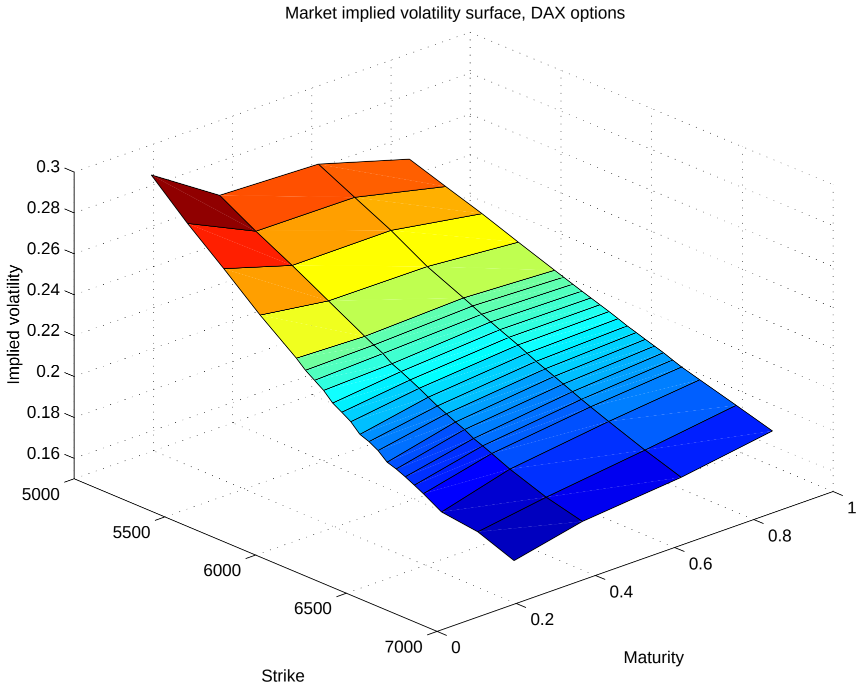

Figure 11: Market implied volatility surface

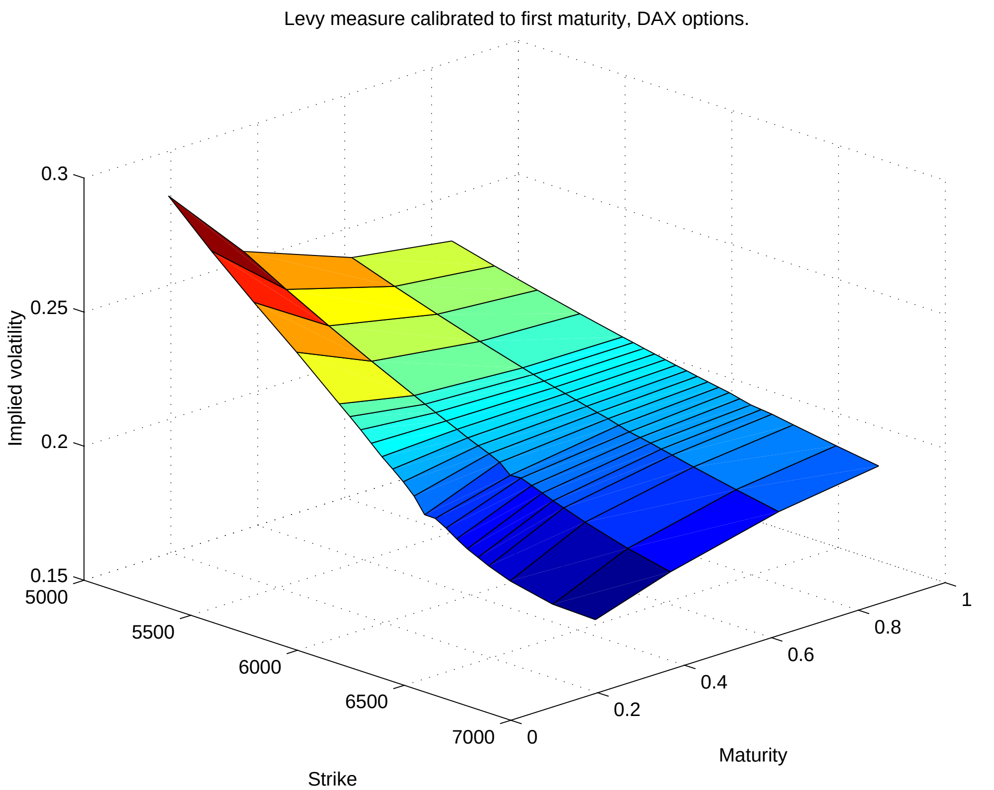

Figure 12: Implied volatilities for all maturities were computed, using the Lévy measure, calibrated to the first maturity

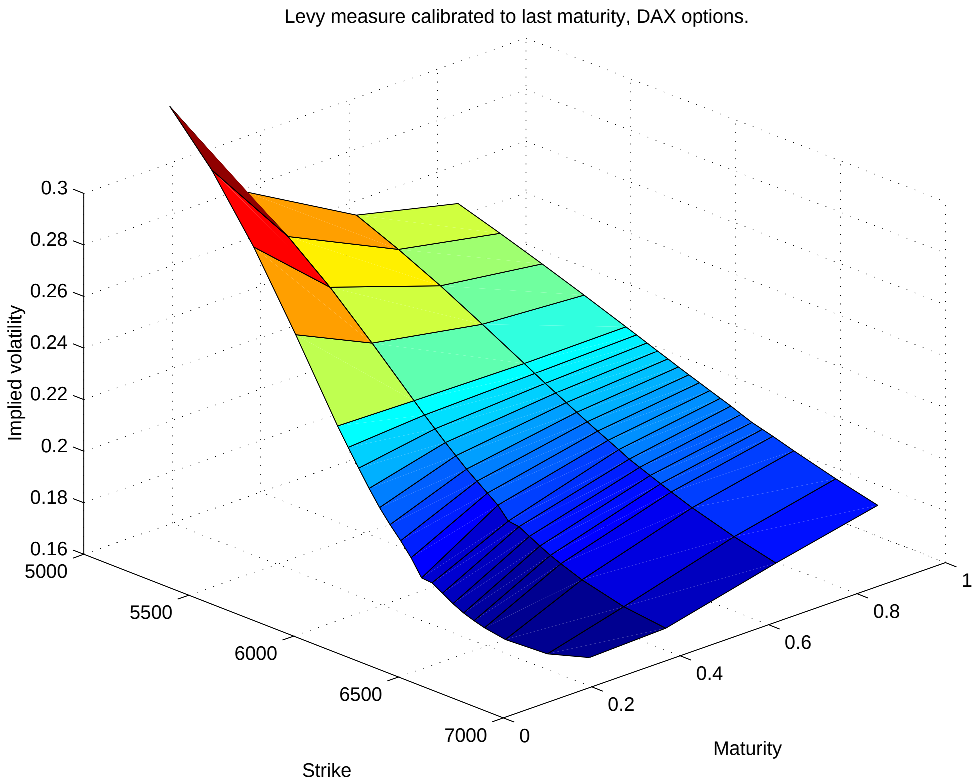

Figure 13: Implied volatilities for all maturities were computed, using the Lévy measure, calibrated to the last maturity

Conclusion

We have proposed a non-parametric method for identifying risk neutral jump-diffusion models consistent with market prices of options and equivalent to a prespecified prior + a stable numerical algorithm for computing it.

Theoretically : an extension of pricing using minimal entropy martingale measure made consistent with observed market prices of options.

Computationally, it is a stable version of current least squares calibration methods for Lévy models which does not assume shape restrictions on the Lévy measure.

Time-inhomogeneities can be easily incorporated into the framework.

Applications and Extensions

- Specification tests for parametric exp Lévy models.
- Identification of interesting parametric classes of Lévy measures from options data.
- Investigation of appropriate time-inhomogeneous extensions.
- Calibration of mixed jump diffusion/ stochastic volatility models.
- Calibration of reduced form/ hybrid credit risk models.
- Multivariate jump diffusion models.