CREDIT RISK MODELLING
COURSE NOTES

Marek Rutkowski
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland
markrut@mini.pw.edu.pl

Contents

Textbook: Tomasz Bielecki and Marek Rutkowski: *Credit Risk: Modeling, Valuation and Hedging.* Springer-Verlag, Berlin Heidelberg New York, 2002.

WARSAW UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 1

1 Hazard Function of a Random Time

Let $\tau : \Omega \to \mathbb{R}_+$ be a non-negative random variable, henceforth referred to as the *random time,* which is defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. For convenience, we assume that $\mathbb{P}\{\tau = 0\} = 0$ and $\mathbb{P}\{\tau > t\} > 0$ for any $t \in \mathbb{R}_+$. The last condition means that τ is assumed to be unbounded; more precisely, it is not dominated with probability 1 by a constant. A bounded random time can also be studied using techniques presented in what follows, though. Let F stand for the (right-continuous) cumulative distribution function of τ, i.e., $F(t) = \mathbb{P}\{\tau \le t\}$ for every $t \in \mathbb{R}_+$. The *survival function* G of τ is defined by the formula: $G(t) := 1 - F(t) = \mathbb{P}\{\tau > t\}$ for every $t \in \mathbb{R}_+$.

Example 1.1 *If τ is exponentially distributed under $\mathbb{P}$ with parameter λ, then $F(t) = 1 - e^{-\lambda t}$ and thus the survival function equals $G(t) = e^{-\lambda t}$.*

We define the *jump process* H associated with the random time τ by setting $H_t = \mathbb{1}_{\{\tau \le t\}}$ for $t \in \mathbb{R}_+$. It is obvious that the process H has right-continuous sample paths, specifically, each sample paths is equal to 0 before random time τ, and it equals 1 for $t \ge \tau$.

1.1 Conditional Expectations

Let $\mathbb{H} = (\mathcal{H}_t)_{t \ge 0}$ stand for the filtration generated by H, specifically, for any $t \in \mathbb{R}_+$ we set $\mathcal{H}_t = \sigma(H_u : u \le t)$. The filtration $\mathbb{H}$ is assumed to be $(\mathbb{P}, \mathcal{G})$-completed. Finally, we set $\mathcal{H}_\infty = \sigma(H_u : u \in \mathbb{R}_+)$. The σ-field $\mathcal{H}_t$ represents the information generated by the observations of the occurrence of the random time τ up to time t – that is, on the time interval $[0, t]$.

We use the commonly standard notation $\sigma(\eta)$ for the σ-field generated by a random variable η. We also assume that Y is an integrable random variable on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ – that is, $\mathbb{E}_{\mathbb{P}}|Y| < \infty$.

Let us first enumerate a few basic properties of the filtration $\mathbb{H}$:

(H.1) $\mathcal{H}_t = \sigma(\{\tau \le u\} : u \le t)$,
(H.2) $\mathcal{H}_t = \sigma(\sigma(\tau) \cap \{\tau \le t\})$,
(H.3) $\mathcal{H}_t = \sigma(\tau \wedge t) \vee (\{\tau > t\})$,
(H.4) $\mathcal{H}_t = \mathcal{H}_{t+}$,
(H.5) $\mathcal{H}_\infty = \sigma(\tau)$,
(H.6) for any $A \in \mathcal{H}_\infty$ we have: $A \cap \{\tau \le t\} \in \mathcal{H}_t$.

All properties above are easy to check; let us only mention that in order to establish (H.6), it is enough to consider an arbitrary event A of the following form: $A = \{\tau \le s\}$ for some $s \in \mathbb{R}_+$.

Lemma 1.1 *Let Y be a $\mathcal{G}$-measurable random variable. Then*

$$\mathbb{1}_{\{\tau \le t\}} \mathbb{E}_{\mathbb{P}}(Y \mid \mathcal{H}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \le t\}} Y \mid \mathcal{H}_\infty) = \mathbb{1}_{\{\tau \le t\}} \mathbb{E}_{\mathbb{P}}(Y \mid \tau) \tag{1.1}$$

and

$$\mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{P}}(Y \mid \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y)}{\mathbb{P}\{\tau > t\}} .$$

Proof. We shall first check that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}Y\,|\,\mathcal{H}_\infty)=\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}Y\,|\,\mathcal{H}_t).$$

In view of (H.6), we have $A\cap\{\tau\le t\}\in\mathcal{H}_t$ for any $A\in\mathcal{H}_\infty$. Consequently,

$$\begin{aligned}\int_A \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}Y\,|\,\mathcal{H}_\infty)\,d\mathbb{P} &= \int_A \mathbb{1}_{\{\tau\le t\}}Y\,d\mathbb{P}=\int_{A\cap\{\tau\le t\}} Y\,d\mathbb{P}\\ &= \int_{A\cap\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(Y\,|\,\mathcal{H}_t)\,d\mathbb{P}=\int_A \mathbb{1}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(Y\,|\,\mathcal{H}_t)\,d\mathbb{P}\\ &= \int_A \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}Y\,|\,\mathcal{H}_t)\,d\mathbb{P}\end{aligned}$$

since the event $\{\tau\le t\}$ belongs to $\mathcal{H}_t$. To establish the second formula, we need to show that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y\,|\,\mathcal{H}_t)=c\mathbb{1}_{\{\tau>t\}},\quad\text{where}\quad c=\frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y)}{\mathbb{P}\{\tau>t\}}.$$

Equivalently, we need to check that for any $A\in\mathcal{H}_t$

$$\int_A \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y\,|\,\mathcal{H}_t)\,d\mathbb{P}=\int_A c\mathbb{1}_{\{\tau>t\}}\,d\mathbb{P}.$$

In this case, it is enough to consider events of the form: $A=\{\tau\le s\}$ for $s\le t$, as well as the event $A=\{\tau>t\}$. In the former case, both sides of the last equality are equal to 0. Furthermore, since $A=\{\tau>t\}\in\mathcal{H}_t$ we obtain

$$\int_A \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A Y\,|\,\mathcal{H}_t)\,d\mathbb{P}=\int_A \mathbb{1}_A Y\,d\mathbb{P}=\int_\Omega \mathbb{1}_A Y\,d\mathbb{P}=c\,\mathbb{P}\{A\}=\int_A c\mathbb{1}_A\,d\mathbb{P}.$$

This completes the proof of the lemma. □

Corollary 1.1 *For any $\mathcal{G}$-measurable random variable Y we have*

$$\mathbb{E}_{\mathbb{P}}(Y\,|\,\mathcal{H}_t)=\mathbb{1}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(Y\,|\,\tau)+\mathbb{1}_{\{\tau>t\}}\,\frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y)}{\mathbb{P}\{\tau>t\}}\,. \tag{1.2}$$

For any $\mathcal{H}_t$-measurable random variable Y we have

$$Y=\mathbb{1}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(Y\,|\,\tau)+\mathbb{1}_{\{\tau>t\}}\,\frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y)}{\mathbb{P}\{\tau>t\}}\,, \tag{1.3}$$

that is, $Y=h(\tau)$ for a Borel measurable $h:\mathbb{R}\to\mathbb{R}$, which is constant on the open interval $]\,t,\infty[$.

The basic formula (1.2), though simple, appears to be quite useful. Let us state some special cases of this result. For any $t<s$ we have

$$\mathbb{P}\{\tau\ge s\,|\,\mathcal{H}_t\}=\mathbb{1}_{\{\tau>t\}}\mathbb{P}\{\tau\ge s\,|\,\tau>t\}$$

and

$$\mathbb{P}\{\tau>s\,|\,\mathcal{H}_t\}=\mathbb{1}_{\{\tau>t\}}\mathbb{P}\{\tau>s\,|\,\tau>t\}. \tag{1.4}$$

The following result is a straightforward consequence of (1.4).

Corollary 1.2 *The process M given by the formula*

$$M_t = \frac{1 - H_t}{1 - F(t)}, \quad \forall\, t \in \mathbb{R}_+, \tag{1.5}$$

follows an $\mathbb{H}$-martingale. Equivalently, for every $0 \le t \le s$,

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} \, \frac{F(s) - F(t)}{1 - F(t)}. \tag{1.6}$$

Proof. Equality (1.4) can be rewritten as follows:

$$\mathbb{E}_{\mathbb{P}}(1 - H_s \,|\, \mathcal{H}_t) = (1 - H_t) \, \frac{1 - F(s)}{1 - F(t)}.$$

This immediately yields the martingale property of M. The second formula is also clear. □

Definition 1.1 *An increasing function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula*

$$\Gamma(t) := -\ln G(t) = -\ln\left(1 - F(t)\right), \quad \forall\, t \in \mathbb{R}_+,$$

is called the hazard function of τ. If the cumulative distribution function F is absolutely continuous with respect to the Lebesgue measure – that is, when $F(t) = \int_0^t f(u)\, du$, for a Lebesgue integrable function $f : \mathbb{R}_+ \to \mathbb{R}_+$, then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\, du},$$

where $\gamma(t) = f(t)(1 - F(t))^{-1}$. The function γ is called the intensity function (or the hazard rate) of the random time τ.

Notice that $\Gamma(t)$ is well defined for any $t \in \mathbb{R}_+$, since by assumption $F(t) < 1$ for every $t \in \mathbb{R}_+$. Furthermore, we have

$$\Gamma(\infty) := \lim_{t \to \infty} \Gamma(t) = \infty,$$

since clearly $\lim_{t\to\infty}(1 - F(t)) = 0$. It is also obvious that the intensity function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ (if it exists) is a non-negative function. Finally, γ is Lebesgue integrable on any bounded interval $[0, t]$ and $\int_0^\infty \gamma(u)\, du = \infty$.

Example 1.2 *If τ is exponentially distributed with parameter λ under $\mathbb{P}$ the hazard rate of τ is constant: $\gamma(t) = \lambda$ for every $t \in \mathbb{R}_+$.*

Using the hazard function Γ, we may rewrite (1.2) as follows:

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} \mathbb{E}_{\mathbb{P}}(Y \,|\, \tau) + \mathbb{1}_{\{\tau > t\}}\, e^{\Gamma(t)}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y). \tag{1.7}$$

In particular, for any $t \le s$ equality (1.4) takes the following form:

$$\mathbb{P}\{\tau > s \,|\, \mathcal{H}_t\} = \mathbb{1}_{\{\tau > t\}}\, e^{\Gamma(t) - \Gamma(s)} = \mathbb{1}_{\{\tau > t\}}\, e^{-\int_t^s \gamma(u)\, du},$$

where the second equality holds, provided that τ admits the hazard rate γ.

Corollary 1.3 *Let Y be $\mathcal{H}_\infty$-measurable so that $Y = h(\tau)$ for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$. Then the following statements are true.*
(i) *If the hazard function Γ of τ is continuous, then we have*

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau \le t\}} h(\tau) + 1\!\!1_{\{\tau > t\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)} \, d\Gamma(u). \tag{1.8}$$

(ii) *If τ admits the intensity function γ, then we have*

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau \le t\}} h(\tau) + 1\!\!1_{\{\tau > t\}} \int_t^\infty h(u) \gamma(u) e^{-\int_t^u \gamma(v)\, dv} \, du.$$

In particular, for any $t \le s$,

$$\mathbb{P}\{\tau > s \,|\, \mathcal{H}_t\} = 1\!\!1_{\{\tau > t\}} e^{-\int_t^s \gamma(v)\, dv}$$

and

$$\mathbb{P}\{t < \tau < s \,|\, \mathcal{H}_t\} = 1\!\!1_{\{\tau > t\}} \Big(1 - e^{-\int_t^s \gamma(v)\, dv} \Big).$$

Lemma 1.2 *The process L, given by the formula*

$$L_t := 1\!\!1_{\{\tau > t\}} e^{\Gamma(t)} = (1 - H_t) e^{\Gamma(t)}, \quad \forall \, t \in \mathbb{R}_+, \tag{1.9}$$

follows an $\mathbb{H}$-martingale.

Proof. It suffices to observe that the process L coincides with the process M introduced in Corollary 1.2. □

1.2 Martingales Associated with a Continuous Hazard Function

We already know that the $\mathbb{H}$-adapted process of finite variation L given by formula (1.9) is an $\mathbb{H}$-martingale (no matter whether Γ is a continuous or a discontinuous function). In this section, we will examine further important examples of martingales associated with the hazard function. We make throughout an additional assumption that the hazard function Γ of a random time τ is continuous.

We shall first assume that the cumulative distribution function F is an absolutely continuous function, so that the random time τ admits the intensity function γ. Our goal is to establish a martingale characterization of γ. More specifically, we shall check directly that the process $\hat{M}$, defined as:

$$\hat{M}_t := H_t - \int_0^t \gamma(u) 1\!\!1_{\{u \le \tau\}} \, du = H_t - \int_0^{t \wedge \tau} \gamma(u) \, du = H_t - \Gamma(t \wedge \tau),$$

follows an $\mathbb{H}$-martingale. To this end, recall that by virtue of (1.6) we have

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{H}_t) = 1\!\!1_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)}.$$

On the other hand, if we denote

$$Y = \int_t^s \gamma(u) 1\!\!1_{\{u \le \tau\}} \, du = \int_{t \wedge \tau}^{s \wedge \tau} \frac{f(u)}{1 - F(u)} \, du = \ln \frac{1 - F(t \wedge \tau)}{1 - F(s \wedge \tau)} \,,$$

then obviously $Y = \mathbb{1}_{\{\tau > t\}} Y$. Let us set $A = \{\tau > t\}$. Using first (1.2) and then Fubini's theorem, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(Y \mid \mathcal{H}_t) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A Y \mid \mathcal{H}_t) = \mathbb{1}_A \frac{\mathbb{E}_{\mathbb{P}}(Y)}{\mathbb{P}\{A\}} = \mathbb{1}_A \frac{\mathbb{E}_{\mathbb{P}}\Big(\int_t^s \gamma(u) \mathbb{1}_{\{u \le \tau\}}\, du\Big)}{1 - F(t)} \\ &= \mathbb{1}_A \frac{\int_t^s \gamma(u)(1 - F(u))\, du}{1 - F(t)} = \mathbb{1}_A \frac{F(s) - F(t)}{1 - F(t)} = \mathbb{E}_{\mathbb{P}}(H_s - H_t \mid \mathcal{H}_t).\end{aligned}$$

This shows that the process $\hat{M}$ follows an $\mathbb{H}$-martingale. We have thus established the following simple, but remarkable, result.

Lemma 1.3 *Assume that*

$$F(t) = 1 - e^{-\int_0^t \gamma(u)\, du}, \quad \forall\, t \in \mathbb{R}_+,$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is the hazard rate of τ. Then the process $\hat{M}$

$$\hat{M}_t = H_t - \int_0^{t \wedge \tau} \gamma(u)\, du, \quad \forall\, t \in \mathbb{R}_+, \tag{1.10}$$

follows an $\mathbb{H}$-martingale.

It appears that a counterpart of Lemma 1.3 can be established when F is merely continuous. Before examining this extension, we recall an auxiliary result. For the proof of Lemma 1.4, the interested reader is referred, for instance, to Brémaud (1981) or Revuz and Yor (1999).

Lemma 1.4 *Let g and h be two right-continuous functions with left-hand limits. If g and h are of finite variation on $[0, t]$ then we have*

$$\begin{aligned} g(t)h(t) &= g(0)h(0) + \int_{]0,t]} g(u-)\, dh(u) + \int_{]0,t]} h(u)\, dg(u) \\ &= g(0)h(0) + \int_{]0,t]} g(u)\, dh(u) + \int_{]0,t]} h(u-)\, dg(u) \\ &= g(0)h(0) + \int_{]0,t]} g(u-)\, dh(u) + \int_{]0,t]} h(u-)\, dg(u) \\ &\quad + \sum_{u \le t} \Delta g(u) \Delta h(u), \end{aligned}$$

where $\Delta g(u) = g(u) - g(u-)$ and $\Delta h(u) = h(u) - h(u-)$.

Any of the equalities of Lemma 1.4 will be referred to as the *integration by parts formula* (or the *product rule*) for functions of finite variation. We shall frequently apply this formula to stochastic processes of finite variation. In such a case, the integrals should be understood as the path-wise integrals, defined with probability 1.

Proposition 1.1 *Assume that the hazard function Γ is continuous. Then the process of finite variation $\hat{M}_t = H_t - \Gamma(t \wedge \tau)$ follows an $\mathbb{H}$-martingale. Furthermore, for every $t \in \mathbb{R}_+$ we have*

$$L_t = 1 - \int_{]0,t]} L_{u-}\, d\hat{M}_u. \tag{1.11}$$

Proof. For the sake of brevity, we shall make use of Lemma 1.2 (the direct calculations also give, of course, the required result). It is clear that $\hat{M}$ follows an $\mathbb{H}$-adapted integrable process. Using the integration by parts formula for functions of finite variation, we obtain

$$L_t = (1 - H_t)e^{\Gamma(t)} = 1 + \int_{]0,t]} e^{\Gamma(u)}\big((1 - H_u)\, d\Gamma(u) - dH_u\big) \tag{1.12}$$

since Γ is a continuous increasing function. This in turn yields

$$\hat{M}_t = H_t - \Gamma(t \wedge \tau) = \int_{]0,t]} \big(dH_u - (1 - H_u)\, d\Gamma(u)\big) = -\int_{]0,t]} e^{-\Gamma(u)}\, dL_u,$$

so that $\hat{M}$ is manifestly an $\mathbb{H}$-martingale. Since (1.12) may be rewritten as follows:

$$L_t = 1 + \int_{]0,t]} e^{\Gamma(u)}(1 - H_{u-})(d\Gamma(u \wedge \tau) - dH_u) = 1 - \int_{]0,t]} L_{u-}\, d\hat{M}_u,$$

it is clear that (1.11) is valid. □

Proposition 1.2 *Assume that the hazard function Γ of τ is continuous. Then for any Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ such that the random variable $h(\tau)$ is integrable, the process M^h, given by the formula*

$$\hat{M}^h_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) - \int_0^{t\wedge\tau} h(u)\, d\Gamma(u), \quad \forall\, t \in \mathbb{R}_+,$$

is an $\mathbb{H}$-martingale.

Proof. We shall directly verify the martingale property of $\hat{M}^h$. Therefore, the demonstration given below provides also an alternative proof of Proposition 1.1. On one hand, formula (1.8) in Corollary 1.3 yields

$$I := \mathbb{E}_{\mathbb{P}}\big(h(\tau)\mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{H}_t\big) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma(t)} \int_t^s h(u)e^{-\Gamma(u)}\, d\Gamma(u).$$

On the other hand, it is clear that

$$J := \mathbb{E}_{\mathbb{P}}\Big(\int_{t\wedge\tau}^{s\wedge\tau} h(u)\, d\Gamma(u) \,\Big|\, \mathcal{H}_t\Big) = \mathbb{E}_{\mathbb{P}}\big(\tilde{h}(\tau)\mathbb{1}_{\{t<\tau\le s\}} + \tilde{h}(s)\mathbb{1}_{\{\tau>s\}} \,|\, \mathcal{H}_t\big),$$

where we set $\tilde{h}(s) = \int_t^s h(u)\, d\Gamma(u)$. Consequently, using again formula (1.8), we get

$$J = \mathbb{1}_{\{\tau>t\}} e^{\Gamma(t)} \Big(\int_t^s \tilde{h}(u)e^{-\Gamma(u)}\, d\Gamma(u) + e^{-\Gamma(s)}\tilde{h}(s)\Big).$$

To conclude the proof, it is enough to observe that Fubini's theorem yields

$$\begin{aligned}
&\int_t^s e^{-\Gamma(u)} \int_t^u h(v)\, d\Gamma(v)\, d\Gamma(u) + e^{-\Gamma(s)}\tilde{h}(s) \\
&\quad = \int_t^s h(u) \int_u^s e^{-\Gamma(v)}\, d\Gamma(v)\, d\Gamma(u) + e^{-\Gamma(s)} \int_t^s h(u)\, d\Gamma(u) \\
&\quad = \int_t^s h(u)e^{-\Gamma(u)}\, d\Gamma(u),
\end{aligned}$$

as expected. □

Remarks. It is apparent that $\hat{M}^h$ admits the following integral representation

$$\hat{M}^h_t = \int_{]0,t]} h(u)\, d\hat{M}_u.$$

This equality shows that the martingale property of $\hat{M}^h$ is also a straightforward consequence of Proposition 1.1.

Corollary 1.4 *Assume that the hazard function Γ of τ is continuous. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a Borel measurable function such that the random variable $Y = e^{h(\tau)}$ is integrable. Then the process*

$$\tilde{M}^h_t = \exp\big(1\!\!1_{\{\tau \le t\}} h(\tau)\big) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\, d\Gamma(u)$$

is an $\mathbb{H}$-martingale.

Proof. Notice that

$$\exp\big(1\!\!1_{\{\tau \le t\}} h(\tau)\big) - 1 = 1\!\!1_{\{\tau \le t\}} e^{h(\tau)} + 1\!\!1_{\{\tau > t\}} - 1 = 1\!\!1_{\{\tau \le t\}} e^{h(\tau)} - H_t,$$

so that

$$\tilde{M}^h_t = 1\!\!1_{\{\tau \le t\}} e^{h(\tau)} - \int_0^{t\wedge\tau} e^{h(u)}\, d\Gamma(u) - \hat{M}_t.$$

To complete the proof of the corollary, it is thus enough to make use of Proposition 1.2. □

The next result offers a still another example of an $\mathbb{H}$-martingale associated with a random time τ.

Corollary 1.5 *Assume that the hazard function Γ of τ is continuous. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a Borel measurable function such that the random variable $h(\tau)$ is integrable. Then the process*

$$\bar{M}^h_t = (1 + 1\!\!1_{\{\tau \le t\}} h(\tau)) \exp\Big(- \int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big)$$

is an $\mathbb{H}$-martingale.

Proof. Let us denote by U the following decreasing continuous process:

$$U_t = \exp\Big(- \int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big).$$

Notice that

$$1 + 1\!\!1_{\{\tau \le t\}} h(\tau) = 1 + \int_{]0,t]} h(u)\, dH_u =: H^h_t.$$

An application of the product rule yields

$$d\bar{M}^h_t = d(H^h_t U_t) = U_t h(t)\, dH_t - (1 + 1\!\!1_{\{\tau \le t\}} h(\tau)) U_t h(t)\, d\Gamma(t\wedge\tau).$$

Consequently, we have

$$d\bar{M}^h_t = U_t h(t)\, d(H_t - \Gamma(t\wedge\tau)) = U_t h(t)\, d\hat{M}_t.$$

The last equality makes it clear that the process $\bar{M}^h$ indeed follows an $\mathbb{H}$-martingale. □

1.3 Martingale Representation Theorem

The following elementary version of the martingale representation theorem is commonly known (see, for instance, Brémaud (1981)).

Proposition 1.3 *Assume that F is an absolutely continuous function. Let $M^h_t := \mathbb{E}_{\mathbb{P}}(h(\tau)\,|\,\mathcal{H}_t)$ for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ such that the random variable $h(\tau)$ is integrable. Then*

$$M^h_t = M^h_0 + \int_{]0,t]} \hat{h}(u)\, d\hat{M}_u, \tag{1.13}$$

where $\hat{M}_t = H_t - \int_0^{t\wedge\tau} \gamma_u \, du$ and the function $\hat{h} : \mathbb{R}_+ \to \mathbb{R}$ is given by the formula

$$\hat{h}(t) = h(t) - e^{\Gamma(t)}\, \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>t\}} h(\tau)\big). \tag{1.14}$$

Proof. Observe first that $M^h_0 = \mathbb{E}_{\mathbb{P}}(h(\tau))$. Recall also that the random variable M^h_t admits the following representation (cf. (1.8))

$$M^h_t = \mathbb{E}_{\mathbb{P}}(h(\tau)\,|\,\mathcal{H}_t) = \mathbb{1}_{\{\tau\le t\}} h(\tau) + \mathbb{1}_{\{\tau>t\}} g(t), \tag{1.15}$$

where the function $g : \mathbb{R} \to \mathbb{R}$ equals

$$g(t) := e^{\Gamma(t)} \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>t\}} h(\tau)\big) = e^{\Gamma(t)} \int_t^\infty h(u) f(u)\, du. \tag{1.16}$$

If representation (1.13) is valid for some function $\hat{h}$, then we have, on the set $\{\tau > t\}$,

$$M^h_t = \mathbb{E}_{\mathbb{P}}(h(\tau)) - \int_0^t \hat{h}(s)\gamma(s)\, ds = \mathbb{E}_{\mathbb{P}}(h(\tau)) - \int_0^t \hat{h}(s) e^{\Gamma(s)} f(s)\, ds.$$

On the other hand, by virtue of (1.15), equality $M^h_t = g(t)$ holds on this set. Differentiating both sides with respect to t, and taking into account the equality $\gamma(t) = e^{\Gamma(t)} f(t)$, we obtain

$$-e^{\Gamma(t)} f(t)\hat{h}(t) = g'(t) = e^{\Gamma(t)} f(t)(g(t) - h(t)).$$

The equality $\hat{h}(t) = h(t) - g(t)$ is thus straightforward on the set $\{t < \tau\}$. Since the process M^h is manifestly continuous on this set, we also have

$$\hat{h}(t) = h(t) - M^h_t = h(t) - M^h_{t-}$$

on the set $\{t < \tau\}$. In view of the last equality, it is clear that, on the event $\{\tau \le t\}$, the right-hand side of (1.13) gives $h(\tau)$, as expected. □

Proposition 1.3 remains valid when the hazard function Γ is merely continuous, as the next result shows.

Proposition 1.4 *Assume that F is a continuous function. Let $M^h_t := \mathbb{E}_{\mathbb{P}}(h(\tau)\,|\,\mathcal{H}_t)$ for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ such that the random variable $h(\tau)$ is integrable. Then*

$$M^h_t = M^h_0 + \int_{]0,t]} \hat{h}(u)\, d\hat{M}_u, \tag{1.17}$$

where $\hat{M}_t = H_t - \Gamma(t \wedge \tau)$ and $\hat{h}$ satisfies (1.14), i.e., $\hat{h} = h - g$, where g is given by (1.16).

Proof. By virtue of (1.8), the left-hand side of formula (1.17) equals (see also (1.15))

$$I = \mathbb{E}_{\mathbb{P}}(h(\tau) \,|\, \mathcal{H}_t) = H_t h(\tau) + (1 - H_t) g(t).$$

On the other hand, the right-hand side of (1.17) can be rewritten as follows:

$$\begin{aligned}
J &= g(0) + \int_{]0,t]} \hat{h}(u)\, d\hat{M}_u \\
&= g(0) + \int_{]0,t]} (h(u) - g(u))\, d(H_u - \Gamma(u \wedge \tau)) \\
&= g(0) + H_t(h(\tau) - g(\tau)) + \int_0^{t\wedge\tau} (g(u) - h(u))\, d\Gamma(u) \\
&= g(0) + H_t h(\tau) + (1 - H_t) g(t) - g(t \wedge \tau) + \int_0^{t\wedge\tau} (g(u) - h(u))\, d\Gamma(u).
\end{aligned}$$

To check that $I = J$, it suffices to show that

$$g(t \wedge \tau) = g(0) + \int_0^{t\wedge\tau} (g(u) - h(u))\, d\Gamma(u)$$

or, equivalently, that for any $t \in \mathbb{R}_+$ we have

$$g(t) = g(0) + \int_0^t (g(u) - h(u))\, d\Gamma(u).$$

Put another way, we need to verify that the following equality holds:

$$e^{\Gamma(t)} \int_t^\infty h(u)\, dF(u) = \int_0^\infty h(u)\, dF(u) + \int_0^t e^{\Gamma(u)} (g(u) - h(u))\, dF(u).$$

By applying Fubini's theorem, we get (recall that $e^{\Gamma(u)} dF(u) = d\Gamma(u)$)

$$\begin{aligned}
\int_0^t e^{\Gamma(u)} g(u)\, dF(u) &= \int_0^t e^{2\Gamma(u)} \int_u^\infty h(v)\, dF(v)\, dF(u) \\
&= \int_0^t h(v) \int_0^v e^{\Gamma(u)}\, d\Gamma(u)\, dF(v) + \int_t^\infty h(v) \int_0^t e^{\Gamma(u)}\, d\Gamma(u)\, dF(v) \\
&= \int_0^t h(u)\big(e^{\Gamma(u)} - 1\big)\, dF(u) + \big(e^{\Gamma(t)} - 1\big) \int_t^\infty h(u)\, dF(u).
\end{aligned}$$

This completes the proof. □

Notice that representation (1.17) can also be rewritten as follows:

$$M_t^h = M_0^h + \int_{]0,t]} (h(u) - M_{u-}^h)\, d\hat{M}_u. \tag{1.18}$$

Remarks. Since an arbitrary $\mathcal{H}_\infty$-measurable random variable X has the form $X = h(\tau)$, we may also deduce from Proposition 1.4 that any $\mathbb{H}$-martingale admits the representation (1.17). Hence, any $\mathbb{H}$-martingale is a purely discontinuous martingale, as it follows a process of finite variation. Put another way, any continuous $\mathbb{H}$-martingale necessarily follows a constant process.

1.4 Change of a Probability Measure

Let $\mathbb{P}^*$ be an arbitrary probability measure on $(\Omega, \mathcal{H}_\infty)$. Assume that $\mathbb{P}^*$ is absolutely continuous with respect to $\mathbb{P}$, i.e., $\mathbb{P}^*\{A\} = 0$ for any event $A \in \mathcal{H}_\infty$ such that $\mathbb{P}\{A\} = 0$. Then there exists a Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}_+$, which satisfies

$$\mathbb{E}_{\mathbb{P}}(h(\tau)) = \int_{]0,\infty[} h(u)\, dF(u) = 1,$$

and such that the Radon-Nikodým density of $\mathbb{P}^*$ with respect to $\mathbb{P}$ equals

$$\eta := \frac{d\mathbb{P}^*}{d\mathbb{P}} = h(\tau) \geq 0, \quad \mathbb{P}\text{-a.s.}, \tag{1.19}$$

We shall henceforth write $\mathbb{E}_{\mathbb{P}}$ ($\mathbb{E}_{\mathbb{P}^*}$, resp.) to denote the expected value with respect to the probability measure $\mathbb{P}$ ($\mathbb{P}^*$, resp.) Probability measure $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ if and only if the inequality in (1.19) is strict, $\mathbb{P}$-a.s.

Furthermore, we shall assume that $\mathbb{P}^*\{\tau = 0\} = 0$ and $\mathbb{P}^*\{\tau > t\} > 0$ for every $t \in \mathbb{R}_+$. The first condition is in fact satisfied for an arbitrary probability measure $\mathbb{P}^*$ absolutely continuous with respect to $\mathbb{P}$. For the second condition to hold, it is sufficient and necessary to postulate that for every $t \in \mathbb{R}_+$

$$\mathbb{P}^*\{\tau > t\} = 1 - F^*(t) = \int_{]t,\infty[} h(u)\, dF(u) > 0, \tag{1.20}$$

where F^* is the cumulative distribution function of τ under $\mathbb{P}^*$, specifically,

$$F^*(t) := \mathbb{P}^*\{\tau \leq t\} = \int_{]0,t]} h(u)\, dF(u).$$

Condition (1.20) is equivalent to the following one (cf. (1.16))

$$g(t) = e^{\Gamma(t)} \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau > t\}} h(\tau)\big) = e^{\Gamma(t)} \int_{]t,\infty[} h(u)\, dF(u) = e^{\Gamma(t)}\, \mathbb{P}^*\{\tau > t\} > 0.$$

From now on, we assume that this is indeed the case, so that the hazard function Γ^* of τ with respect to $\mathbb{P}^*$ is well defined.

Is not difficult to establish the relationship between the hazard functions Γ^* and Γ. Indeed, we have

$$\frac{\Gamma^*(t)}{\Gamma(t)} = \frac{\ln\Big(\int_{]t,\infty[} h(u)\, dF(u)\Big)}{\ln(1 - F(t))} =: g^*(t),$$

since, by the definition of the hazard function, $\Gamma^*(t) = -\ln(1 - F^*(t))$. Let us now analyze some special cases of the last relationship.

In the first step, we will assume that F is an absolutely continuous function, so that the intensity function γ of τ under $\mathbb{P}$ is well defined. Recall that γ is given by the following formula:

$$\gamma(t) = f(t)(1 - F(t))^{-1}, \quad \forall\, t \in \mathbb{R}_+.$$

Under the present assumptions, the c.d.f. F^* of τ under $\mathbb{P}^*$ equals

$$F^*(t) := \mathbb{P}^*\{\tau \leq t\} = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \leq t\}} h(\tau)) = \int_0^t h(u) f(u)\, du = \int_0^t f^*(u)\, du,$$

where $f^*(u) = h(u)f(u)$, and thus F^* is an absolutely continuous function. Thus, the intensity function γ^* of the random time τ under $\mathbb{P}^*$ exists, and is given by the formula

$$\gamma^*(t) = \frac{f^*(t)}{1 - F^*(t)} = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\,du}\,.$$

To derive a more explicit relationship between the intensities γ and γ^*, we define an auxiliary function $h^* : \mathbb{R}_+ \to \mathbb{R}$ by setting $h^*(t) = h(t)g^{-1}(t)$. Notice that

$$\begin{aligned}\gamma^*(t) &= \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\,du} = \frac{h(t)f(t)}{\int_t^\infty h(u)f(u)\,du} \\ &= \frac{h(t)f(t)}{e^{-\Gamma(t)}g(t)} = \frac{h^*(t)f(t)}{1 - F(t)} = h^*(t)\gamma(t).\end{aligned}$$

This also means that $d\Gamma^*(t) = h^*(t)\,d\Gamma(t)$. It appears that the last equality holds true if F is merely a continuous function. Indeed, if F (and thus F^*) is continuous, we get

$$d\Gamma^*(t) = \frac{dF^*(t)}{1 - F^*(t)} = \frac{d(1 - e^{-\Gamma(t)}g(t))}{e^{-\Gamma(t)}g(t)} = \frac{g(t)d\Gamma(t) - dg(t)}{g(t)} = h^*(t)\,d\Gamma(t).$$

We have thus established the following partial result in which, for the sake of convenience, we denote $\kappa(t) = h^*(t) - 1 = h(t)g^{-1}(t) - 1$.

Proposition 1.5 *Let the two probability measures $\mathbb{P}^*$ and $\mathbb{P}$ be related to each other by means of* (1.19)*. If the hazard function Γ of τ under $\mathbb{P}$ is continuous, then the hazard function Γ^* of τ under $\mathbb{P}^*$ is also continuous and $d\Gamma^*(t) = (1 + \kappa(t))\,d\Gamma(t)$, where $\kappa(t) = h(t)g^{-1}(t) - 1$ and the functions h and g are given by formulae* (1.19) *and* (1.16)*, respectively.*

Let us now take a closer look at the auxiliary function κ. To this end, we introduce the following non-negative $\mathbb{P}$-martingale η:

$$\eta_t := \frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{H}_t} = \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{H}_t) = \mathbb{E}_{\mathbb{P}}(h(\tau) \,|\, \mathcal{H}_t). \tag{1.21}$$

It is clear that $\eta_t = M_t^h$. We shall refer to the process η as the *Radon-Nikodým density process* of $\mathbb{P}^*$ with respect to $\mathbb{P}$. In view of (1.7), we have

$$\eta_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}}\, e^{\Gamma(t)} \int_{]t,\infty[} h(u)\,dF(u) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} g(t).$$

If, in addition, F is a continuous function then (cf. (1.8))

$$\eta_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)}\,d\Gamma(u).$$

On the other hand, using (1.17) and (1.18), we obtain

$$M_t^h = M_0^h + \int_{]0,t]} M_{u-}^h (h^*(u) - 1)\,d\hat{M}_u = M_0^h + \int_{]0,t]} M_{u-}^h \kappa(u)\,d\hat{M}_u,$$

which shows that η solves the following SDE:

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \kappa(u)\,d\hat{M}_u. \tag{1.22}$$

It is not difficult to find an explicit solution to this equation, namely,

$$\eta_t = \big(1 + 1\!\!1_{\{\tau \le t\}}\kappa(\tau)\big) \exp\Big(- \int_0^{t\wedge\tau} \kappa(u)\, d\Gamma(u)\Big). \tag{1.23}$$

In view of the last formula, the martingale property of the process η – that is apparent from (1.21) – is thus also a simple consequence of Corollary 1.5. The proof of the following classic result is left to the reader.

Lemma 1.5 *Let Y follow a process of finite variation. Consider the following linear stochastic differential equation*

$$Z_t = 1 + \int_{]0,t]} Z_{u-}\, dY_u. \tag{1.24}$$

The unique solution $Z_t = \mathcal{E}_t(Y)$ to (1.24), referred to as the Doléans exponential of Y, equals

$$\mathcal{E}_t(Y) = e^{Y_t} \prod_{0<u\le t} (1 + \Delta Y_u) e^{-\Delta Y_u} = e^{Y_t^c} \prod_{0<u\le t} (1 + \Delta Y_u), \tag{1.25}$$

where Y^c is the continuous part of Y, i.e., $Y_t^c = Y_t - \sum_{0<u\le t} \Delta Y_u$.

Since the process η satisfies (1.22), it is clear that it can be represented as follows:

$$\eta_t = \mathcal{E}_t\Big(\int_{]0,\,\cdot\,]} \kappa(u)\, d\hat{M}_u \Big).$$

Expression (1.23) for the random variable η_t can thus also be obtained from (1.25), upon setting $dY_u = \kappa(u)\, d\hat{M}_u$. Let us stress that (1.25) is merely a special case of the well known general formula for the Doléans exponential (see, e.g., Elliott (1982), Protter (1990), or Revuz and Yor (1999)). We are in a position to formulate the following result (all statements in Proposition 1.6 were already established above).

Proposition 1.6 *Assume that F is a continuous function. Let $\mathbb{P}^*$ be any probability measure on $(\Omega, \mathcal{H}_\infty)$ absolutely continuous with respect to $\mathbb{P}$, so that (1.19) holds for some function h. Assume that $\mathbb{P}^*\{\tau > t\} > 0$ for $t \in \mathbb{R}_+$. Then the Radon-Nikodým density process η of $\mathbb{P}^*$ with respect to $\mathbb{P}$ satisfies*

$$\eta_t := \frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{H}_t} = \mathcal{E}_t\Big(\int_{]0,\,\cdot\,]} \kappa(u)\, d\hat{M}_u \Big),$$

where $\kappa(t) = h(t)g^{-1}(t) - 1$ and

$$g(t) = e^{\Gamma(t)} \int_t^\infty h(u)\, dF(u).$$

Moreover, the hazard function of τ under $\mathbb{P}^*$ equals $\Gamma^*(t) = g^*(t)\Gamma(t)$ with

$$g^*(t) = \frac{\ln\Big(\int_{]t,\infty[} h(u)\, dF(u) \Big)}{\ln(1 - F(t))}.$$

1.5 Martingale Characterization of the Hazard Function

Proposition 1.1 raises the natural question whether the martingale property of the process $H_t - \Gamma(t \wedge \tau)$ with respect to the filtration $\mathbb{H}$ uniquely characterizes the hazard function of a random time τ? Our goal is to show that the answer to this question is positive, provided that the hazard function Γ is continuous. Notice that for a discontinuous hazard function Γ, equality (1.12) takes the following form:

$$L_t = L_0 + \int_{]0,t]} (1 - H_u)\, de^{\Gamma(u)} - \int_{]0,t]} e^{\Gamma(u-)}\, dH_u$$

or, equivalently,

$$L_t = 1 + \int_{]0,t]} e^{\Gamma(u-)} \big((1 - H_u)\, d\Gamma(u) - dH_u \big) + \sum_{s \le t,\, s < \tau} \big(\Delta e^{\Gamma(s)} - e^{\Gamma(s-)}\, \Delta\Gamma(s) \big),$$

where

$$\Delta e^{\Gamma(s)} = e^{\Gamma(s)} - e^{\Gamma(s-)}, \quad \Delta\Gamma(s) = \Gamma(s) - \Gamma(s-).$$

The last formula makes in clear that in the case of a discontinuous hazard function Γ, the process $H_t - \Gamma(t \wedge \tau)$ is not an $\mathbb{H}$-martingale.

Let us recall that $H_t = H_{t\wedge\tau}$; that is, the process H is stopped at time τ. We find it convenient to introduce the notion of a martingale hazard function of a random time.

Definition 1.2 *A function $\Lambda : \mathbb{R}_+ \to \mathbb{R}$ is called a martingale hazard function of a random time τ with respect to its natural filtration $\mathbb{H}$ if and only if the process $H_t - \Lambda(t \wedge \tau)$ follows an $\mathbb{H}$-martingale.*

The function Λ may also be seen as an $\mathbb{F}^0$-adapted right-continuous stochastic process, where $\mathbb{F}^0$ is the trivial filtration, i.e., $\mathcal{F}^0_t = \mathcal{F}^0_0 = \{\emptyset, \Omega\}$ for every $t \in \mathbb{R}_+$. We shall sometimes find it useful to refer to the martingale hazard function as the $(\mathbb{F}^0, \mathbb{H})$-*martingale hazard process* of τ.

Proposition 1.7 (i) *The unique martingale hazard function of τ with respect to $\mathbb{H}$ is the right-continuous increasing function Λ given by the formula*

$$\Lambda(t) = \int_{]0,t\,]} \frac{dF(u)}{1 - F(u-)} = \int_{]0,t\,]} \frac{d\mathbb{P}\{\tau \le u\}}{1 - \mathbb{P}\{\tau < u\}}, \quad \forall\, t \in \mathbb{R}_+. \tag{1.26}$$

(ii) *The martingale hazard function Λ coincides with the hazard function Γ if and only if F is a continuous function. In general, for every $t \in \mathbb{R}_+$ we have*

$$e^{-\Gamma(t)} = e^{-\Lambda^c(t)} \prod_{0<u\le t} (1 - \Delta\Lambda(u)), \tag{1.27}$$

where $\Lambda^c(t) = \Lambda(t) - \sum_{0\le u\le t} \Delta\Lambda(u)$ and $\Delta\Lambda(u) = \Lambda(u) - \Lambda(u-)$.
(iii) *The martingale hazard function Λ is continuous if and only if the cumulative distribution function F of τ is continuous. In this case, Λ satisfies $\Lambda(t) = -\ln(1 - F(t)) = \Gamma(t)$ for every $t \in \mathbb{R}_+$.*

Proof. Let us first examine the uniqueness. The definition of Λ implies that $\mathbb{E}_{\mathbb{P}}(H_t) = \mathbb{E}_{\mathbb{P}}(\Lambda(t\wedge\tau))$. Put more explicitly (recall that $F(0) = 0$),

$$F(t) = \int_{]0,t\,]} \Lambda(u)\, dF(u) + \Lambda(t)(1 - F(t)), \tag{1.28}$$

so that Λ is necessarily a right-continuous function. Furthermore, if Λ_1 and Λ_2 are the two right-continuous functions, which satisfy (1.28), then for every $t \in \mathbb{R}_+$ we have

$$\int_{]0,t\,]} (\Lambda_1(u) - \Lambda_2(u))\, dF(u) + (\Lambda_1(t) - \Lambda_2(t))(1 - F(t)) = 0.$$

Using the last equality, one can show – by making use of rather standard contraction arguments – that the martingale hazard function Λ is unique.

To complete the proof of part (i), we need to establish the martingale property of the process $H_t - \Lambda(t \wedge \tau)$. It is enough to check that for any $t \le s$ we have

$$\mathbb{E}_{\mathbb{P}}(H_s - H_t \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)} = \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_t),$$

where the first equality is a consequence of (1.6), and where we have set

$$Y := \Lambda(s \wedge \tau) - \Lambda(t \wedge \tau) = \int_{]t\wedge\tau, s\wedge\tau]} \frac{dF(u)}{1 - F(u-)}\,.$$

Since $Y = \mathbb{1}_{\{\tau > t\}}\, Y$, using (1.2), we obtain

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}}\, Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(Y)}{1 - F(t)}\,.$$

Furthermore, we have

$$\mathbb{E}_{\mathbb{P}}(Y) = \mathbb{P}\{\tau > s\} \int_{]t,s]} \frac{dF(u)}{1 - F(u-)} + \int_{]t,s]} \int_{]t,u]} \frac{dF(v)}{1 - F(v-)}\, dF(u).$$

Consequently,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(Y) &= (\Lambda(s) - \Lambda(t))(1 - F(s)) + \int_{]t,s]} (\Lambda(u) - \Lambda(t))\, dF(u) \\ &= (\Lambda(s) - \Lambda(t))(1 - F(s)) - \Lambda(t)(F(s) - F(t)) + \int_{]t,s]} \Lambda(u)\, dF(u). \end{aligned}$$

The product rule yields

$$\int_{]t,s]} \Lambda(u)\, dF(u) = \Lambda(s)F(s) - \Lambda(t)F(t) - \int_{]t,s]} F(u-)\, d\Lambda(u). \tag{1.29}$$

Finally, it is clear from (1.26) that

$$\int_{]t,s]} F(u-)\, d\Lambda(u) = \Lambda(s) - \Lambda(t) - F(s) + F(t).$$

Combining the above equalities, we find that $\mathbb{E}_{\mathbb{P}}(Y) = F(s) - F(t)$ for every $t \le s$. This completes the proof of (i). To establish (ii), notice that by virtue of (1.26), the survival function $G(t) = 1 - F(t)$ satisfies

$$G(t) = -\int_{]0,t\,]} G(u-)\, d\Lambda(u).$$

Therefore (cf. (1.24)–(1.25)),

$$e^{-\Gamma(t)} = G(t) = e^{-\Lambda^c(t)} \prod_{0<u\leq t} (1 - \Delta\Lambda(u)).$$

This completes the proof of (1.27). In particular, the martingale hazard function Λ and the hazard function Γ are not equal to each other, when the function F is discontinuous. All statements of part (iii) are immediate consequences of part (ii). □

Remarks. Assume that the cumulative distribution function F is absolutely continuous, with the probability density function f. Then necessarily

$$\Lambda(t) = \Gamma(t) = \int_0^t f(u)(1 - F(u))^{-1}\, du$$

and thus the martingale hazard function Λ is absolutely continuous as well. Specifically, $\Lambda(t) = \int_0^t \lambda(u)\, du$, where $\lambda(u) = \gamma(u) = f(u)(1 - F(u))^{-1}$ for every $u \in \mathbb{R}_+$.

1.6 Compensator of a Random Time

By virtue of the properties of the martingale hazard function, the process $C_t := \Lambda(t\wedge\tau)$ satisfies: (i) C is an increasing, right-continuous, $\mathbb{H}$-adapted process, and (ii) the compensated process $H - C$ follows an $\mathbb{H}$-martingale. This shows that the notion of the martingale hazard function is closely related to the concept of the $\mathbb{H}$-compensator of τ or, more precisely, to the concept of the $\mathbb{H}$-compensator of the associated jump process H.

We adopt here the standard convention, which stipulates that B is an *increasing process* if B is an adapted process with non-decreasing, right-continuous sample paths. The process H is, of course, a bounded increasing process, and so also a bounded $\mathbb{H}$-submartingale.

Let us first recall the definition of the compensator of an increasing process (the compensator of an increasing process is also known as its *dual predictable projection*; see, e.g., Dellacherie (1972) or Jacod (1979)). When specified to our situation, it can be stated as follows.

Definition 1.3 *A process A is called the $\mathbb{H}$-compensator of the process H if and only if the following conditions are satisfied: (i) A is an $\mathbb{H}$-predictable increasing process, with $A_0 = 0$, (ii) the compensated process $H - A$ follows an $\mathbb{H}$-martingale.*

Existence and uniqueness of the Doob-Meyer decomposition for a bounded submartingale[1] imply that the process H admits a unique $\mathbb{H}$-compensator. We are in a position to prove the following result.

Lemma 1.6 *Assume that the cumulative distribution function F of τ is continuous. Then the unique $\mathbb{H}$-compensator A of τ equals, for every $t \in \mathbb{R}_+$,*

$$A_t = \Lambda(t \wedge \tau) = \Gamma(t \wedge \tau) = -\ln(1 - F(t \wedge \tau)).$$

Proof. In view of the definition of the martingale hazard function, part (ii) in Proposition 1.7, and Lemma 1.6, it is enough to check that the process $A_t = \Lambda(t \wedge \tau)$, $t \in \mathbb{R}_+$, is $\mathbb{H}$-predictable. But this is clear, since the mapping $t \to t \wedge \tau$ defines a continuous, $\mathbb{H}$-adapted process, so that it is an $\mathbb{H}$-predictable process. In view of the continuity of Λ, we conclude that A is an $\mathbb{H}$-predictable process. □

[1]See Theorem 4.10 in Sect. 1.4 of Karatzas and Shreve (1991).

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WARSAW UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 2

2 Hazard Process of a Random Time

The concepts introduced in the previous lecture will now be extended to a more general set-up, when allowance for a larger flow of information − formally represented by some reference filtration $\mathbb{F}$ − is made.

We denote by τ a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, satisfying: $\mathbb{P}\{\tau = 0\} = 0$ and $\mathbb{P}\{\tau > t\} > 0$ for any $t \in \mathbb{R}_+$. We introduce a right-continuous process H by setting $H_t = \mathbb{1}_{\{\tau \le t\}}$ and we denote by $\mathbb{H}$ the associated filtration: $\mathcal{H}_t = \sigma(H_u : u \le t)$. Let $\mathbb{G} = (\mathcal{G}_t)_{t \ge 0}$ be an arbitrary filtration on $(\Omega, \mathcal{G}, \mathbb{P})$. All filtrations are assumed to satisfy the 'usual conditions' of right-continuity and completeness. For each $t \in \mathbb{R}_+$, the information available at time t is captured by the σ-field $\mathcal{G}_t$. We shall focus on the case described in the following assumption.

Condition (G.1) We assume that we are given an auxiliary filtration $\mathbb{F}$ such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$; i.e., $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in \mathbb{R}_+$.

For the sake of simplicity, we assume that the σ-field $\mathcal{F}_0$ is trivial (so that $\mathcal{G}_0$ is a trivial σ-field as well). For given filtrations $\mathbb{H} \subseteq \mathbb{G}$, the equality $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ does not specify uniquely an auxiliary filtration $\mathbb{F}$. For instance, when $\mathcal{G}_t = \mathcal{H}_t$, we may take $\mathbb{F} = \mathbb{F}^0$, but also $\mathbb{F} = \mathbb{H}$ (or indeed any other sub-filtration of $\mathbb{H}$). In most applications, $\mathbb{F}$ will appear in a natural way as the filtration generated by a certain stochastic process.

Condition (G.1a) For every $t \in \mathbb{R}_+$, the event $\{\tau \le t\}$ belongs to the σ-field $\mathcal{F}_t$ (and thus τ is an $\mathbb{F}$-stopping time).

Under (G.1a), we have $\mathbb{G} = \mathbb{F}$, and thus τ also is a $\mathbb{G}$-stopping time. In some models, only a partial observation of the random time τ is postulated. Such a case corresponds to the following condition.

Condition (G.1b) For some dates $t \in \mathbb{R}_+$, the event $\{\tau \le t\}$ does not belong to the σ-field $\mathcal{G}_t$.

Let $\hat{\mathbb{H}} \subset \mathbb{H}$ stand for the filtration associated with the partial observations of τ. Then the enlarged filtration $\mathbb{G}$ equals $\mathbb{G} = \hat{\mathbb{H}} \vee \mathbb{F}$.

Under (G.1), the process H is obviously $\mathbb{G}$-adapted, but it is not necessarily $\mathbb{F}$-adapted. In other words, the random time τ is a $\mathbb{G}$-stopping time, but it may fail to be an $\mathbb{F}$-stopping time. Under (G.1b), the process H is not $\mathbb{G}$-adapted, i.e., τ is not a $\mathbb{G}$-stopping time. However, in both cases the following condition is satisfied.

Condition (G.2) For every $t \in \mathbb{R}_+$ we have $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t \vee \mathcal{F}_t$.

Lemma 2.1 *Assume that the filtration $\mathbb{G}$ satisfies $\mathbb{G} \subseteq \mathbb{H} \vee \mathbb{F}$, that is, $\mathcal{G}_t \subseteq \mathcal{H}_t \vee \mathcal{F}_t$ for every $t \in \mathbb{R}_+$. Then $\mathbb{G} \subseteq \mathbb{G}^*$, where $\mathbb{G}^* = (\mathcal{G}^*_t)_{t \ge 0}$ with*

$$\mathcal{G}^*_t := \big\{ A \in \mathcal{G} \,:\, \exists\, B \in \mathcal{F}_t,\ \ A \cap \{\tau > t\} = B \cap \{\tau > t\} \big\}.$$

Proof. It is rather clear that the class $\mathcal{G}^*_t$ is a sub-σ-field of $\mathcal{G}$. Therefore, it is enough to check that $\mathcal{H}_t \subseteq \mathcal{G}^*_t$ and $\mathcal{F}_t \subseteq \mathcal{G}^*_t$ for every $t \in \mathbb{R}_+$. Put another way, we need to verify that if either $A = \{\tau \le u\}$ for some $u \le t$ or $A \in \mathcal{F}_t$, then there exists an event $B \in \mathcal{F}_t$ such that $A \cap \{\tau > t\} = B \cap \{\tau > t\}$. In the former case we may take $B = \emptyset$, and in the latter $B = A$. □

Remarks. By a suitable modification of arguments used in the proof of Lemma 2.1, one can show that under (G.2) for any $\mathcal{G}_t$-measurable random variable Y there exists an $\mathcal{F}_t$-measurable random variable $\tilde{Y}$ such that $Y = \tilde{Y}$ on the set $\{\tau > t\}$. Under (G.1), this remarkable property is also a straightforward consequence of part (ii) in Lemma 2.2.

For any $t \in \mathbb{R}_+$, we write $F_t = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\}$, and we denote by G the $\mathbb{F}$-*survival process* of τ with respect to the filtration $\mathbb{F}$, given as:

$$G_t := 1 - F_t = \mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}, \quad \forall\, t \in \mathbb{R}_+.$$

Notice that for any $0 \le t \le s$ we have $\{\tau \le t\} \subseteq \{\tau \le s\}$, and so

$$\mathbb{E}_{\mathbb{P}}(F_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{P}\{\tau \le s \,|\, \mathcal{F}_s\} \,|\, \mathcal{F}_t) = \mathbb{P}\{\tau \le s \,|\, \mathcal{F}_t\} \ge \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\} = F_t.$$

This shows that the process F (G, resp.) follows a bounded, non-negative $\mathbb{F}$-submartingale ($\mathbb{F}$-supermartingale, resp.) under $\mathbb{P}$. We may thus deal with the right-continuous modification of F (of G) with finite left-hand limits. The next definition is a straightforward generalization of Definition 1.1.

Definition 2.1 *Assume that $F_t < 1$ for $t \in \mathbb{R}_+$. The $\mathbb{F}$-hazard process of τ under $\mathbb{P}$, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$. Equivalently, $\Gamma_t = -\ln G_t = -\ln(1 - F_t)$ for every $t \in \mathbb{R}_+$.*

Since $G_0 = 1$, it is clear that $\Gamma_0 = 0$. For the sake of conciseness, we shall refer briefly to Γ as the $\mathbb{F}$-hazard process, rather than the $\mathbb{F}$-hazard process under $\mathbb{P}$, unless there is a danger of confusion.

In this chapter, we assume that the inequality $F_t < 1$ holds for every $t \in \mathbb{R}_+$, so that the $\mathbb{F}$-hazard process Γ is well defined. It should be stressed that the case when τ is an $\mathbb{F}$-stopping time (i.e., the case when $\mathbb{F} = \mathbb{G}$) is not dealt with here.

2.1 Conditional Expectations

We shall first focus on the conditional expectation $\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{G}_t)$, where Y is a $\mathbb{P}$-integrable random variable. We start by the following result, which is a direct counterpart of Lemma 1.1. Unless explicitly stated otherwise, we assume that Condition (G.2) is valid, and thus the filtration $\mathbb{G}$ is the sub-filtration of $\mathbb{G}^*$.

Lemma 2.2 (i) *Assume that* (G.2) *holds. Then for any $\mathcal{G}$-measurable random variable Y and any $t \in \mathbb{R}_+$ we have*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{G}_t) = \mathbb{P}\{\tau > t \,|\, \mathcal{G}_t\} \, \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{F}_t)}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}}. \tag{2.1}$$

(ii) *If, in addition, $\mathcal{H}_t \subseteq \mathcal{G}_t$ (so that* (G.1) *holds) then*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \, \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{F}_t)}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}}. \tag{2.2}$$

In particular, for any $t \le s$

$$\mathbb{P}\{t < \tau \le s \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \, \frac{\mathbb{P}\{t < \tau \le s \,|\, \mathcal{F}_t\}}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}}. \tag{2.3}$$

Proof. Since (ii) is a straightforward consequence of (i), it is enough to establish the first statement. Let us denote $C = \{\tau > t\}$. To prove (i), we need to verify that (recall that $\mathcal{F}_t \subseteq \mathcal{G}_t$)

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_C Y \mathbb{P}(C \,|\, \mathcal{F}_t) \,\big|\, \mathcal{G}_t\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_C \mathbb{E}_{\mathbb{P}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t) \,\big|\, \mathcal{G}_t\big).$$

Put another way, we need to show that for any $A \in \mathcal{G}_t$ we have

$$\int_A \mathbb{1}_C Y \mathbb{P}(C \,|\, \mathcal{F}_t)\, d\mathbb{P} = \int_A \mathbb{1}_C \mathbb{E}_{\mathbb{P}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\, d\mathbb{P}.$$

In view of Lemma 2.1, for any $A \in \mathcal{G}_t$ we have $A \cap C = B \cap C$ for some event $B \in \mathcal{F}_t$, and so

$$\begin{aligned}
\int_A \mathbb{1}_C Y \mathbb{P}(C \,|\, \mathcal{F}_t)\, d\mathbb{P} &= \int_{A\cap C} Y \mathbb{P}(C \,|\, \mathcal{F}_t)\, d\mathbb{P} = \int_{B\cap C} Y \mathbb{P}(C \,|\, \mathcal{F}_t)\, d\mathbb{P} \\
&= \int_B \mathbb{1}_C Y \mathbb{P}(C \,|\, \mathcal{F}_t)\, d\mathbb{P} = \int_B \mathbb{E}_{\mathbb{P}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t) \mathbb{P}(C \,|\, \mathcal{F}_t)\, d\mathbb{P} \\
&= \int_B \mathbb{E}_{\mathbb{P}}(\mathbb{1}_C \mathbb{E}_{\mathbb{P}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t) \,|\, \mathcal{F}_t)\, d\mathbb{P} = \int_{B\cap C} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\, d\mathbb{P} \\
&= \int_{A\cap C} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\, d\mathbb{P} = \int_A \mathbb{1}_C \mathbb{E}_{\mathbb{P}}(\mathbb{1}_C Y \,|\, \mathcal{F}_t)\, d\mathbb{P}.
\end{aligned}$$

This ends the proof. □

Assume that (G.1) holds. By virtue of part (ii) in Lemma 2.2, for any $\mathcal{G}_t$-measurable random variable Y, there exists an $\mathcal{F}_t$-measurable random variable $\tilde{Y}$ such that $\mathbb{1}_{\{\tau>t\}} Y = \mathbb{1}_{\{\tau>t\}} \tilde{Y}$. As already mentioned (see remarks after Lemma 2.1), this property can also be derived by approximation arguments. If it is taken for granted, the derivation of (10.6) can be substantially simplified. Indeed, suppose that we know that (the first equality below is obvious)

$$\mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \zeta \tag{2.4}$$

for some integrable $\mathcal{F}_t$-measurable random variable ζ such that $\zeta = \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t)$ on $\{\tau > t\}$. By taking the conditional expectation with respect to $\mathcal{F}_t$ of both terms of the second equality in (2.4), we obtain

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{G}_t) \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{F}_t) = \zeta\, \mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\},$$

and this immediately yields (10.6). However, it does not seem to be possible to derive (2.1) using this argument. Since (recall that $\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\} > 0$)

$$\tilde{Y} = \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{F}_t)}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}}\,, \tag{2.5}$$

we have, as expected, $\tilde{Y} = Y$ when Y is an $\mathcal{F}_t$-measurable random variable.

Before we state the next lemma, let us introduce another auxiliary random variable by setting $\hat{Y} = \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_{\tau-})$, where $\mathcal{F}_{\tau-}$ stands for the σ-field generated by all events that strictly precede the random time τ (let us stress that τ is not necessarily an $\mathbb{F}$-stopping time). Since $\mathcal{F}_0$ is trivial, by definition we have (see, e.g., Dellacherie [2])

$$\mathcal{F}_{\tau-} = \sigma\big(B \cap \{\tau > t\} : B \in \mathcal{F}_t,\, t \in \mathbb{R}_+\big). \tag{2.6}$$

In particular, the inclusion $\sigma(\tau) \subseteq \mathcal{F}_{\tau-}$ is always valid, and $\sigma(\tau) = \mathcal{F}_{\tau-}$ when $\mathbb{F} = \mathbb{F}^0$ is the trivial filtration. It is also not difficult to check that $\mathcal{G}_{\tau-} = \mathcal{F}_{\tau-}$. Consequently, the equality $\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_{\tau-}) = \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_{\tau-})$ is valid for any $\mathcal{G}$-measurable integrable random variable Y.

Lemma 2.3 *Let Y be an integrable $\mathcal{G}$-measurable random variable and let $\hat{Y} = \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_{\tau-})$. For any $0 \le t \le s$ we have*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} \hat{Y} \,|\, \mathcal{G}_t), \tag{2.7}$$

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} Y \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} \hat{Y} \,|\, \mathcal{G}_t). \tag{2.8}$$

Proof. Consider an arbitrary event $A \in \mathcal{G}_t$. By virtue of Lemma 2.1, we may, and do, assume that $A \cap C = B \cap C$, where we write $C = \{\tau > t\}$. Since $B \cap C$ is manifestly in $\mathcal{F}_{\tau-}$, we have

$$\begin{aligned}\int_A \mathbb{1}_C Y \, d\mathbb{P} &= \int_{A\cap C} Y \, d\mathbb{P} = \int_{B\cap C} Y \, d\mathbb{P} = \int_{B\cap C} \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_{\tau-}) \, d\mathbb{P} \\ &= \int_{A\cap C} \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_{\tau-}) \, d\mathbb{P} = \int_A \mathbb{1}_C \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_{\tau-}) \, d\mathbb{P} = \int_A \mathbb{1}_C \hat{Y} \, d\mathbb{P}.\end{aligned}$$

This gives (2.7). For (2.8), notice that the event $\{\tau > s\}$ is in $\mathcal{F}_{\tau-}$. □

It is apparent that formulae (2.1)–(2.3) can be rewritten as follows:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{G}_t) &= \mathbb{P}\{\tau > t \,|\, \mathcal{G}_t\} \, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} \, e^{\Gamma_t} Y \,|\, \mathcal{F}_t), \\ \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} Y \,|\, \mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}} \, e^{\Gamma_t} Y \,|\, \mathcal{F}_t)\end{aligned} \tag{2.9}$$

and

$$\mathbb{P}\{t < \tau \le s \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}\big(1 - e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t\big).$$

The next corollary deals with some simple, but useful, modifications of these expressions.

Corollary 2.1 *Let Y be a $\mathcal{G}$-measurable random variable and let $t \le s$.* (i) *Assume that* (G.2) *holds. Then*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{P}\{\tau > t \,|\, \mathcal{G}_t\} \, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} \, e^{\Gamma_t} Y \,|\, \mathcal{F}_t). \tag{2.10}$$

(ii) *If* (G.1) *is valid then*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} \, e^{\Gamma_t} Y \,|\, \mathcal{F}_t) \tag{2.11}$$

and

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} \, e^{\Gamma_t} Y \,|\, \mathcal{F}_t). \tag{2.12}$$

If Y is $\mathcal{F}_s$-measurable, then

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}\big(e^{\Gamma_t - \Gamma_s} Y \,|\, \mathcal{F}_t\big) \tag{2.13}$$

and

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}\big((\mathbb{1}_{\{\tau>t\}} - e^{-\Gamma_s}) e^{\Gamma_t} Y \,\big|\, \mathcal{F}_t\big).$$

Proof. In view of (2.1), to show that (2.10) holds, it is enough to observe that $\mathbb{1}_{\{\tau>t\}} \mathbb{1}_{\{\tau>s\}} = \mathbb{1}_{\{\tau>s\}}$. Equalities (10.7)–(10.8) are immediate consequences of (2.10). For (10.9), notice that, by virtue of (10.7), we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}} Y \,|\, \mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau>s\}} \, e^{\Gamma_t} Y \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}\big(\mathbb{P}\{t > s \,|\, \mathcal{F}_s\} \, e^{\Gamma_t} Y \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}\big((1 - F_s) \, e^{\Gamma_t} Y \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{P}}\big(e^{\Gamma_t - \Gamma_s} Y \,\big|\, \mathcal{F}_t\big).\end{aligned}$$

To derive the last formula, it suffices to combine (2.9) with (10.9). □

It is worth noticing that equality (10.9) remains valid when the random variable Y is merely $\mathcal{G}$-measurable, rather than $\mathcal{F}_s$-measurable, provided that, on the right-hand side of (10.9), we substitute Y with the $\mathcal{F}_s$-measurable random variable $\tilde{Y}$ for which $\mathbb{1}_{\{\tau>s\}}Y = \mathbb{1}_{\{\tau>s\}}\tilde{Y}$. More explicitly, we need to replace Y by $\tilde{Y}$ given by the following expression (cf. (2.5)):

$$\tilde{Y} = \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>s\}}Y \,|\, \mathcal{F}_s)}{\mathbb{P}\{\tau > s \,|\, \mathcal{F}_s\}}\,.$$

The proof of the next auxiliary result is essentially the same as the proof of part (i) in Lemma 2.2.

Lemma 2.4 *For any $\mathcal{G}$-measurable random variable Y and any sub-σ-field $\mathcal{F}$ of $\mathcal{G}$ we have*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y \,|\, \mathcal{H}_t \vee \mathcal{F}) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}Y \,|\, \mathcal{F})}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}\}}. \tag{2.14}$$

For any $t \le s$ we have

$$\mathbb{P}\{\tau > s \,|\, \mathcal{H}_t \vee \mathcal{F}\} = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{P}\{\tau > s \,|\, \mathcal{F}\}}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}\}}. \tag{2.15}$$

Our next goal is to examine the conditional expectation $\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}Y \,|\, \mathcal{G}_t)$. Its evaluation under (G.2) is rather difficult, and thus we shall introduce an alternative condition.

Condition (G.3) For any $t \in \mathbb{R}_+$ and arbitrary event $A \in \mathcal{H}_\infty \vee \mathcal{F}_t$ we have $A \cap \{\tau \le t\} \in \mathcal{G}_t$.

Under (G.3), for every $t \in \mathbb{R}_+$ we have $\mathcal{H}_t \subseteq \mathcal{G}_t$. It is easy to see that (G.1) is sufficient for (G.3) to hold; however, (G.2) does not imply (G.3). Finally, conditions (G.2) and (G.3), taken together, imply (G.1).

Lemma 2.5 *Assume that* (G.3) *holds. For any $\mathcal{G}$-measurable random variable Y we have*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau\le t\}}Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_\infty \vee \mathcal{F}_t). \tag{2.16}$$

Proof. Let us denote $D = \{\tau \le t\}$. For any $A \in \mathcal{H}_\infty \vee \mathcal{F}_t$ we have (notice that $D \in \mathcal{G}_t$)

$$\begin{aligned}\int_A \mathbb{E}_{\mathbb{P}}(\mathbb{1}_D Y \,|\, \mathcal{H}_\infty \vee \mathcal{F}_t)\, d\mathbb{P} &= \int_A \mathbb{1}_D Y\, d\mathbb{P} = \int_{A\cap D} Y\, d\mathbb{P} \\ &= \int_{A\cap D} \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t)\, d\mathbb{P} = \int_A \mathbb{1}_D \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t)\, d\mathbb{P}.\end{aligned}$$

The random variable $\mathbb{1}_D\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t)$ is manifestly $\mathcal{H}_t \vee \mathcal{G}_t$-measurable, so that it is also $\mathcal{H}_\infty \vee \mathcal{F}_t$-measurable. We conclude that (2.16) holds. □

Unless explicitly stated otherwise, we assume from now on that Condition (G.1) holds, i.e., we consider the case when $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. By combining (2.16) with (10.6), we obtain the following well known result, which is a straightforward generalization of equality (1.2).

Corollary 2.2 *For any $\mathcal{G}$-measurable random variable Y we have*

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau\le t\}}\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_\infty \vee \mathcal{F}_t) + \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}Y \,|\, \mathcal{F}_t).$$

Any $\mathcal{G}_t$-measurable random variable Y admits the following representation

$$Y = \mathbb{1}_{\{\tau\le t\}}\, \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{H}_\infty \vee \mathcal{F}_t) + \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau>t\}}\, e^{\Gamma_t}Y \,|\, \mathcal{F}_t).$$

Proposition 2.1 (i) *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a bounded, continuous function. Then for any $t < s \le \infty$*

$$\mathbb{E}_{\mathbb{P}}(1\!\!1_{\{t<\tau\le s\}} h(\tau) \mid \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} h(u)\, dF_u \,\Big|\, \mathcal{F}_t \Big). \tag{2.17}$$

(ii) *Let Z be a bounded, $\mathbb{F}$-predictable process. Then for any $t < s \le \infty$*

$$\mathbb{E}_{\mathbb{P}}(1\!\!1_{\{t<\tau\le s\}} Z_\tau \mid \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} Z_u\, dF_u \,\Big|\, \mathcal{F}_t \Big). \tag{2.18}$$

Proof. In view of (10.8), to establish (2.17), it is enough to check that

$$\mathbb{E}_{\mathbb{P}}(1\!\!1_{\{t<\tau\le s\}}\, h(\tau) \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} h(u)\, dF_u \,\Big|\, \mathcal{F}_t \Big).$$

We first consider a piecewise constant function $h(u) = \sum_{i=0}^n h_i 1\!\!1_{\{t_i<u\le t_{i+1}\}}$, where, without loss of generality, we take $t_0 = t < \cdots < t_{n+1} = s$. Then

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(1\!\!1_{\{t<\tau\le s\}} h(\tau) \,|\, \mathcal{F}_t) &= \sum_{i=0}^n \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(h_i 1\!\!1_{]t_i,t_{i+1}]}(\tau) \,|\, \mathcal{F}_{t_{i+1}}) \,\big|\, \mathcal{F}_t\big) \\
&= \mathbb{E}_{\mathbb{P}}\Big(\sum_{i=0}^n h_i (F_{t_{i+1}} - F_{t_i}) \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{P}}\Big(\sum_{i=0}^n \int_{]t_i,t_{i+1}]} h(u)\, dF_u \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} h(u)\, dF_u \,\Big|\, \mathcal{F}_t \Big).
\end{aligned}$$

To complete the proof of part (i), it suffices to approximate an arbitrary continuous function h by a suitable sequence of piecewise constant functions.

The proof of (2.18) relies on similar arguments. We begin by assuming that Z is a stepwise $\mathbb{F}$-predictable process; that is, $Z_u = \sum_{i=0}^n Z_{t_i} 1\!\!1_{\{t_i<u\le t_{i+1}\}}$ for $t < u \le s$, where $t_0 = t < \cdots < t_{n+1} = s$, and Z_{t_i} is $\mathcal{F}_{t_i}$-measurable random variable for $i = 0, \dots, n$. We have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big(1\!\!1_{\{t<\tau\le s\}}\, Z_\tau \,|\, \mathcal{F}_t\big) &= \mathbb{E}_{\mathbb{P}}\big(1\!\!1_{\{t<\tau\le s\}} Z_\tau \mid \mathcal{F}_t\big) \\
&= \mathbb{E}_{\mathbb{P}}\Big(\sum_{i=0}^n 1\!\!1_{\{t_i<\tau\le t_{i+1}\}} Z_{t_i} \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{P}}\Big(\sum_{i=0}^n Z_{t_i} (F_{t_{i+1}} - F_{t_i}) \,\Big|\, \mathcal{F}_t \Big).
\end{aligned}$$

Consequently, for any stepwise, bounded, $\mathbb{F}$-predictable process Z, we obtain

$$\mathbb{E}_{\mathbb{P}}\big(1\!\!1_{\{t<\tau\le s\}}\, Z_\tau \,|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} Z_u\, dF_u \,\Big|\, \mathcal{F}_t \Big). \tag{2.19}$$

In the second step, Z is approximated by a suitable sequence of bounded, stepwise, $\mathbb{F}$-predictable processes. The sum under the sign of the conditional expectation converges to the Itô integral (or to the Lebesque-Stieltjes integral if F is of finite variation). The boundedness of Z and F is a sufficient condition for the convergence of the sequence of conditional expectations. □

For the validity of (2.17), it suffices to assume that the function h is piecewise continuous. Also, the boundedness of the function h (the process Z, resp.) is not a necessary condition for (2.17) ((2.18), resp.) to hold; we have imposed this rather restrictive condition for the sake of convenience. On the other hand, in general the $\mathbb{F}$-predictability of Z cannot be replaced by the weaker condition of the $\mathbb{G}$-predictability of Z in Proposition 10.2.

Corollary 2.3 *Under the assumptions of Proposition* 10.2, *if, in addition, the hazard process* Γ *of* τ *is continuous, then*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\leq s\}}h(\tau)\,|\,\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^s h(u)e^{\Gamma_t-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big) \tag{2.20}$$

and

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\leq s\}}Z_\tau\,|\,\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^s Z_u e^{\Gamma_t-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big). \tag{2.21}$$

Proof. Under the present assumptions, $dF_u = e^{-\Gamma_u}\,d\Gamma_u$, and thus equality (2.20) ((2.21), resp.) is an immediate consequence of (2.17) ((2.18), resp.) □

2.1.1 Case of a $\mathcal{G}$-measurable random variable

Let us return to the general case of a $\mathcal{G}$-measurable (bounded) random variable. The following natural and practically important question arises: is it possible to derive an expression similar to (2.21), when Z_τ is replaced by a $\mathcal{G}$-measurable random variable. We claim that the answer to this question is positive. To show this, we proceed as follows. First, we associate with Y the conditional expectation $\hat{Y} = \mathbb{E}_{\mathbb{P}}(Y\,|\,\mathcal{F}_{\tau-}) = \mathbb{E}_{\mathbb{P}}(Y\,|\,\mathcal{G}_{\tau-})$, where the σ-field $\mathcal{F}_{\tau-} = \mathcal{G}_{\tau-}$ of all events strictly preceding τ is formally defined by (2.6). It is known that there exists an $\mathbb{F}$-predictable process $\hat{Z}$ such that $\hat{Z}_\tau = \hat{Y}$ (see p. 126 in Dellacherie and Meyer (1978a)). The following chain of equalities is thus valid (cf. (2.8))

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\leq s\}}\,Y\,|\,\mathcal{G}_t) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\leq s\}}\,\hat{Y}\,|\,\mathcal{G}_t)\\ &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\leq s\}}\,\hat{Z}_\tau\,|\,\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\,\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}\hat{Z}_u\,dF_u\,\Big|\,\mathcal{F}_t\Big)\\ &= \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^s \hat{Z}_u e^{\Gamma_t-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big),\end{aligned} \tag{2.22}$$

where the last equality holds, provided that the hazard process Γ is continuous. It is noteworthy that the uniqueness of the process $\hat{Z}$ is neither claimed, nor required here. In the case when several bounded, $\mathbb{F}$-predictable processes $\hat{Z}$ satisfying the equality $\hat{Z}_\tau = \hat{Y}$ exist, they all yield the same result for the conditional expectation we are interested in.

The next result appears to be useful for the valuation of a defaultable security that promises to pay dividends prior to the default time.

Proposition 2.2 *Assume that* A *is a bounded,* $\mathbb{F}$*-predictable process of finite variation. Then for every* $t \leq s$

$$\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}(1-H_u)\,dA_u\,\Big|\,\mathcal{G}_t\Big) = \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}(1-F_u)\,dA_u\,\Big|\,\mathcal{F}_t\Big)$$

or, equivalently,

$$\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}(1-H_u)\,dA_u\,\Big|\,\mathcal{G}_t\Big) = 1\!\!1_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} e^{\Gamma_t-\Gamma_u}\,dA_u\,\Big|\,\mathcal{F}_t\Big).$$

Proof. For a fixed, but arbitrary, $t \le s$, we introduce an auxiliary process $\tilde{A}$ by setting: $\tilde{A}_u = A_u - A_t$ for $u \in [t,s]$. It is clear that $\tilde{A}$ is a bounded, $\mathbb{F}$-predictable process of finite variation; the same remark applies to the process of left-hand limits: $\tilde{A}_{t-}$. Therefore,

$$\begin{aligned}
J_t := {} & \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}(1-H_u)\,dA_u\,\Big|\,\mathcal{G}_t\Big)\\
= {} & \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} 1\!\!1_{\{\tau>u\}}\,d\tilde{A}_u\,\Big|\,\mathcal{G}_t\Big)\\
= {} & \mathbb{E}_{\mathbb{P}}\Big(\tilde{A}_{\tau-}1\!\!1_{\{t<\tau\le s\}} + \tilde{A}_s 1\!\!1_{\{\tau>s\}}\,\Big|\,\mathcal{G}_t\Big)\\
= {} & 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}\tilde{A}_{u-}\,dF_u + \tilde{A}_s(1-F_s)\,\Big|\,\mathcal{F}_t\Big),
\end{aligned}$$

where the last equality follows from formulae (10.9) and (2.18). Using an obvious equality $G_t = 1 - F_t$, we obtain

$$\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}\tilde{A}_{u-}\,dF_u + \tilde{A}_s(1-F_s)\,\Big|\,\mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{P}}\Big(-\int_{]t,s]}\tilde{A}_{u-}\,dG_u + \tilde{A}_s G_s\,\Big|\,\mathcal{F}_t\Big).$$

Since $\tilde{A}$ follows a process of finite variation (so that its continuous martingale part vanishes), the following version of Itô's product rule is in force

$$\tilde{A}_s G_s = \tilde{A}_t G_t + \int_{]t,s]}\tilde{A}_{u-}\,dG_u + \int_{]t,s]} G_u\,d\tilde{A}_u.$$

But $\tilde{A}_t = 0$, and so

$$\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}\tilde{A}_{u-}\,dF_u + \tilde{A}_s(1-F_s)\,\Big|\,\mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]}(1-F_u)\,dA_u\,\Big|\,\mathcal{F}_t\Big).$$

This proves the first asserted formula. The second equality is a simple reformulation of the first. □

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WARSAW UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 3

3 Poisson Process and Conditional Poisson Process

In Appendix 1, we have focused our attention on the case of a single random time and the associated jump process. In some financial applications, we need to model a sequence of successive random times. Almost invariably, this is done by making use of the so-called $\mathbb{F}$-*conditional Poisson process,* also known as the *doubly stochastic Poisson process.* The general idea is quite similar to the canonical construction of a single random time, which was examined in Appendix 1. We start by assuming that we are given a stochastic process Φ, to be interpreted as the *hazard process,* and we construct a jump process, with unit jump size, such that the probabilistic features of consecutive jump times are governed by the hazard process Φ.

3.1 Poisson Process with Constant Intensity

Let us first recall the definition and the basic properties of the (time-homogeneous) Poisson process N with constant intensity $\lambda > 0$.

Definition 3.1 A process N defined on a probability space $(\Omega, \mathbb{G}, \mathbb{P})$ is called the *Poisson process* with intensity λ with respect to $\mathbb{G}$ if $N_0 = 0$ and for any $0 \leq s < t$ the following two conditions are satisfied:
(i) the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$,
(ii) the increment $N_t - N_s$ has the Poisson law with parameter $\lambda(t-s)$; specifically, for any $k = 0, 1, \ldots$ we have:

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s\} = \mathbb{P}\{N_t - N_s = k\} = \frac{\lambda^k (t-s)^k}{k!}\, e^{-\lambda(t-s)}.$$

The Poisson process of Definition 3.1 is termed *time-homogeneous,* since the probability law of the increment $N_{t+h} - N_{s+h}$ is invariant with respect to the shift $h \geq -s$. In particular, for arbitrary $s < t$ the probability law of the increment $N_t - N_s$ coincides with the law of the random variable N_{t-s}. Let us finally observe that, for every $0 \leq s < t$,

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_t - N_s) = \lambda(t-s). \tag{3.1}$$

We take a version of the Poisson process whose sample paths are, with probability 1, right-continuous stepwise functions with all jumps of size 1. Let us set $\tau_0 = 0$, and let us denote by $\tau_1, \tau_2, \ldots$ the $\mathbb{G}$-stopping times given as the random moments of the successive jumps of N. For any $k = 0, 1, \ldots$

$$\tau_{k+1} = \inf\{t > \tau_k : N_t \neq N_{\tau_k}\} = \inf\{t > \tau_k : N_t - N_{\tau_k} = 1\}.$$

One shows without difficulties that $\mathbb{P}\{\lim_{k\to\infty} \tau_k = \infty\} = 1$. It is convenient to introduce the sequence ξ_k, $k \in \mathbb{N}$ of non-negative random variables, where $\xi_k = \tau_k - \tau_{k-1}$ for every $k \in \mathbb{N}$. Let us quote the following well known result.

Proposition 3.1 *The random variables ξ_k, $k \in \mathbb{N}$ are mutually independent and identically distributed, with the exponential law with parameter λ, that is, for every $k \in \mathbb{N}$ we have*

$$\mathbb{P}\{\xi_k \leq t\} = \mathbb{P}\{\tau_k - \tau_k \leq t\} = 1 - e^{-\lambda t}, \quad \forall\, t \in \mathbb{R}_+.$$

Proposition 3.1 suggests a simple construction of a process N, which follows a time-homogeneous Poisson process with respect to its natural filtration $\mathbb{F}^N$. Suppose that the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is large enough to support a family of mutually independent random variables ξ_k, $k \in \mathbb{N}$ with the common exponential law with parameter $\lambda > 0$. We define the process N on $(\Omega, \mathcal{G}, \mathbb{P})$ by setting: $N_t = 0$ if $\{t < \xi_1\}$ and, for any natural k,

$$N_t = k \quad \text{if and only if} \quad \sum_{i=1}^{k} \xi_i \le t < \sum_{i=1}^{k+1} \xi_i.$$

It can checked that the process N defined in this way is indeed a Poisson process with parameter λ, with respect to its natural filtration $\mathbb{F}^N$. The jump times of N are, of course, the random times $\tau_k = \sum_{i=1}^{k} \xi_i$, $k \in \mathbb{N}$.

Let us recall some useful equalities that are not hard to establish through elementary calculations involving the Poisson law. For any $a \in \mathbb{R}$ and $0 \le s < t$ we have

$$\mathbb{E}_{\mathbb{P}}\big(e^{ia(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = \mathbb{E}_{\mathbb{P}}\big(e^{ia(N_t - N_s)}\big) = e^{\lambda(t-s)(e^{ia}-1)},$$

and

$$\mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = \mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)}\big) = e^{\lambda(t-s)(e^{a}-1)}.$$

The next result is an easy consequence of (3.1) and the above formulae. The proof of the proposition is thus left to the reader.

Proposition 3.2 *The following stochastic processes follow $\mathbb{G}$-martingales.* (i) *The compensated Poisson process $\hat{N}$ defined as*

$$\hat{N}_t := N_t - \lambda t.$$

(ii) *For any $k \in \mathbb{N}$, the compensated Poisson process stopped at τ_k*

$$\hat{M}^k_t := N_{t \wedge \tau_k} - \lambda(t \wedge \tau_k).$$

(iii) *For any $a \in \mathbb{R}$, the exponential martingale M^a given by the formula*

$$M^a_t := e^{aN_t - \lambda t(e^a - 1)} = e^{a\hat{N}_t - \lambda t(e^a - a - 1)}.$$

(iv) *For any fixed $a \in \mathbb{R}$, the exponential martingale K^a given by the formula*

$$K^a_t := e^{iaN_t - \lambda t(e^{ia} - 1)} = e^{ia\hat{N}_t - \lambda t(e^{ia} - ia - 1)}.$$

Remarks. (i) For any $\mathbb{G}$-martingale M, defined on some filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, and an arbitrary $\mathbb{G}$-stopping time τ, the stopped process $M^\tau_t = M_{t\wedge\tau}$ necessarily follows a $\mathbb{G}$-martingale. Thus, the second statement of the proposition is an immediate consequence of the first, combined with the simple observation that each jump time τ_k is a $\mathbb{G}$-stopping time.
(ii) Consider the random time $\tau = \tau_1$, where τ_1 is the time of the first jump of the Poisson process N. Then $N_{t\wedge\tau} = N_{t\wedge\tau_1} = H_t$, so that the process $\hat{M}^1$ introduced in part (ii) of the proposition coincides with the martingale $\hat{M}$ associated with τ.
(iii) The property described in part (iii) of Proposition 3.2 characterizes the Poisson process in the following sense: if $N_0 = 0$ and for every $a \in \mathbb{R}$ the process M^a is a $\mathbb{G}$-martingale, then N follows the Poisson process with parameter λ. Indeed, the martingale property of M^a yields

$$\mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = e^{\lambda(t-s)(e^{a}-1)}, \quad \forall\, 0 \le s < t.$$

By standard arguments, this implies that the random variable $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$, and has the Poisson law with parameter $\lambda(t-s)$. A similar remark applies to property (iv) in Proposition 3.2.

Let us consider the case of a Brownian motion W and a Poisson process N that are defined on a common filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$. In particular, for every $0 \le s < t$, the increment $W_t - W_s$ is independent of the σ-field $\mathcal{G}_s$, and has the Gaussian law $N(0, t-s)$. It might be useful to recall that for any real number b the following processes follow martingales with respect to $\mathbb{G}$:

$$\hat{W}_t = W_t - t, \quad m_t^b = e^{bW_t - \frac{1}{2}b^2 t}, \quad k_t^b = e^{ibW_t + \frac{1}{2}b^2 t}.$$

The next result shows that a Brownian motion W and a Poisson process N, with respect to a common filtration $\mathbb{G}$, are necessarily mutually independent.

Proposition 3.3 *Let a Brownian motion W and a Poisson process N be defined on a common filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$. Then the two processes W and N are mutually independent.*

Proof. Let us sketch the proof. For a fixed $a \in \mathbb{R}$ and any $t > 0$, we have

$$\begin{aligned} e^{iaN_t} &= 1 + \sum_{0<u\le t} (e^{iaN_t} - e^{iaN_{t-}}) = 1 + \int_{]0,t]} (e^{ia} - 1) e^{iaN_{u-}} \, dN_u, \\ &= 1 + \int_{]0,t]} (e^{ia} - 1) e^{iaN_{u-}} \, d\hat{N}_u + \lambda \int_0^t (e^{ia} - 1) e^{iaN_{u-}} \, du. \end{aligned}$$

On the other hand, for any $b \in \mathbb{R}$, the Itô formula yields

$$e^{ibW_t} = 1 + ib \int_0^t e^{ibW_u} \, dW_u - \frac{1}{2} b^2 \int_0^t e^{ibW_u} \, du.$$

The continuous martingale part of the compensated Poisson process $\hat{N}$ is identically equal to 0 (since $\hat{N}$ is a process of finite variation), and obviously the processes $\hat{N}$ and W have no common jumps. Thus, using the Itô product rule for semimartingales, we obtain

$$\begin{aligned} e^{i(aN_t + bW_t)} &= 1 + ib \int_0^t e^{i(aN_u + bW_u)} \, dW_u - \frac{1}{2} b^2 \int_0^t e^{i(aN_u + bW_u)} \, du \\ &+ \int_{]0,t]} (e^{ia} - 1) e^{i(aN_{u-} + bW_u)} \, d\hat{N}_u + \lambda \int_0^t (e^{ia} - 1) e^{i(aN_u + bW_u)} \, du. \end{aligned}$$

Let us denote $f_{a,b}(t) = \mathbb{E}_{\mathbb{P}}(e^{i(aN_t + bW_t)})$. By taking the expectations of both sides of the last equality, we get

$$f_{a,b}(t) = 1 + \lambda \int_0^t (e^{ia} - 1) f_{a,b}(u) \, du - \frac{1}{2} b^2 \int_0^t f_{a,b}(u) \, du.$$

By solving the last equation, we obtain, for arbitrary $a, b \in \mathbb{R}$,

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(aN_t + bW_t)}\big) = f_{a,b}(t) = e^{\lambda t (e^{ia} - 1)} e^{-\frac{1}{2} b^2 t} = \mathbb{E}_{\mathbb{P}}\big(e^{iaN_t}\big) \mathbb{E}_{\mathbb{P}}\big(e^{ibW_t}\big).$$

Thus, for any $t \in \mathbb{R}_+$ the random variables W_t and N_t are mutually independent under $\mathbb{P}$.

In the second step, we fix $0 < t < s$, and we consider the following expectation, for arbitrary real numbers a_1, a_2, b_1 and b_2,

$$f(t, s) := \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s + b_1 W_t + b_2 W_s)}\big).$$

Let us denote $\tilde{a}_1 = a_1 + a_2$ and $\tilde{b}_1 = b_1 + b_2$. Then

$$\begin{aligned} f(t,s) &= \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1N_t + a_2N_s + b_1W_t + b_2W_s)}\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1N_t + a_2(N_s - N_t) + \tilde{b}_1W_t + b_2(W_s - W_t))} \,\big|\, \mathcal{G}_t\big)\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1N_t + \tilde{b}_1W_t)}\mathbb{E}_{\mathbb{P}}\big(e^{i(a_2(N_s-N_t) + b_2(W_s - W_t))} \,\big|\, \mathcal{G}_t\big)\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1N_t + \tilde{b}_1W_t)}\mathbb{E}_{\mathbb{P}}\big(e^{i(a_2N_{t-s} + b_2W_{t-s})}\big)\big) \\ &= f_{a_1,b_1}(t-s)\,\mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1N_t + \tilde{b}_1W_t)}\big) \\ &= f_{a_1,b_1}(t-s)f_{\tilde{a}_1,\tilde{b}_1}(t), \end{aligned}$$

where we have used, in particular, the independence of the increment $N_t - N_s$ (and $W_t - W_s$) of the σ-field $\mathcal{G}_t$, and the time-homogeneity of N and W. By setting $b_1 = b_2 = 0$ in the last formula, we obtain

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(a_1N_t + a_2N_s)}\big) = f_{a_1,0}(t-s)f_{\tilde{a}_1,0}(t),$$

while the choice of $a_1 = a_2 = 0$ yields

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(b_1W_t + b_2W_s)}\big) = f_{0,b_1}(t-s)f_{0,\tilde{b}_1}(t).$$

It is not difficult to check that

$$f_{a_1,b_1}(t-s)f_{\tilde{a}_1,\tilde{b}_1}(t) = f_{a_1,0}(t-s)f_{\tilde{a}_1,0}(t)f_{0,b_1}(t-s)f_{0,\tilde{b}_1}(t).$$

We conclude that for any $0 \le t < s$ and arbitrary $a_1, a_2, b_1, b_2 \in \mathbb{R}$:

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(a_1N_t + a_2N_s + b_1W_t + b_2W_s)}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1N_t + a_2N_s)}\big)\mathbb{E}_{\mathbb{P}}\big(e^{i(b_1W_t + b_2W_s)}\big).$$

This means that the random variables (N_t, N_s) and (W_t, W_s) are mutually independent. By proceeding along the same lines, one may check that the random variables $(N_{t_1}, \dots, N_{t_n})$ and $(W_{t_1}, \dots, W_{t_n})$ are mutually independent for any $n \in \mathbb{N}$ and for any choice of $0 \le t_1 < \dots < t_n$. □

Let us now examine the behavior of the Poisson process under a specific equivalent change of the underlying probability measure. For a fixed $T > 0$, we introduce a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{G}_T)$ by setting

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \eta_T, \quad \mathbb{P}\text{-a.s.}, \tag{3.2}$$

where the Radon-Nikodým density process η_t, $t \in [0,T]$, satisfies

$$d\eta_t = \eta_{t-}\kappa\, d\hat{N}_t, \quad \eta_0 = 1, \tag{3.3}$$

for some constant $\kappa > -1$. Since $Y := \kappa\hat{N}$ is a process of finite variation, (3.3) admits a unique solution, denoted as $\mathcal{E}_t(Y)$ or $\mathcal{E}_t(\kappa\hat{N})$; it can be seen as a special case of the Doléans (or stochastic) exponential. By solving (3.3) path-by-path, we obtain

$$\eta_t = \mathcal{E}_t(\kappa\hat{N}) = e^{Y_t}\prod_{0<u\le t}(1 + \Delta Y_u)e^{-\Delta Y_u} = e^{Y_t^c}\prod_{0<u\le t}(1 + \Delta Y_u),$$

where $Y_t^c := Y_t - \sum_{0<u\le t}\Delta Y_u$ is the path-by-path continuous part of Y. Direct calculations show that

$$\eta_t = e^{-\kappa\lambda t}\prod_{0<u\le t}(1 + \kappa\Delta N_u) = e^{-\kappa\lambda t}(1+\kappa)^{N_t} = e^{N_t\ln(1+\kappa) - \kappa\lambda t},$$

where the last equality holds if $\kappa > -1$. Upon setting $a = \ln(1+\kappa)$ in part (iii) of Proposition 3.2, we get $M^a = \eta$; this confirms that the process η follows a $\mathbb{G}$-martingale under $\mathbb{P}$. We have thus proved the following result.

Lemma 3.1 *Assume that $\kappa > -1$. The unique solution η to the SDE* (3.3) *follows an exponential $\mathbb{G}$-martingale under $\mathbb{P}$. Specifically,*

$$\eta_t = e^{N_t \ln(1+\kappa) - \kappa\lambda t} = e^{\hat{N}_t \ln(1+\kappa) - \lambda t(\kappa - \ln(1+\kappa))} = M_t^a, \tag{3.4}$$

where $a = \ln(1+\kappa)$. In particular, the random variable η_T is strictly positive, $\mathbb{P}$a.s. *and* $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$. *Furthermore, the process M^a solves the following SDE:*

$$dM_t^a = M_{t-}^a (e^a - 1)\, d\hat{N}_t, \quad M_0^a = 1. \tag{3.5}$$

We are in the position to establish the well-known result, which states that under $\mathbb{P}^*$ the process $N_t,\ t \in [0,T]$, follows a Poisson process with the constant intensity $\lambda^* = (1+\kappa)\lambda$.

Proposition 3.4 *Assume that under $\mathbb{P}$ a process N is a Poisson process with intensity λ with respect to the filtration $\mathbb{G}$. Suppose that the probability measure $\mathbb{P}^*$ is defined on $(\Omega, \mathcal{G}_T)$ through* (3.2) *and* (3.3) *for some $\kappa > -1$.*
(i) *The process $N_t,\ t \in [0,T]$, follows a Poisson process under $\mathbb{P}^*$ with respect to $\mathbb{G}$ with the constant intensity $\lambda^* = (1+\kappa)\lambda$.*
(ii) *The compensated process $N_t^*,\ t \in [0,T]$, defined as*

$$N_t^* = N_t - \lambda^* t = N_t - (1+\kappa)\lambda t = \hat{N}_t - \kappa\lambda t,$$

follows a $\mathbb{P}^*$-martingale with respect to $\mathbb{G}$.

Proof. From remark (iii) after Proposition 3.2, we know that it suffices to find λ^* such that, for any fixed $b \in \mathbb{R}$, the process $\tilde{M}^b$, given as

$$\tilde{M}_t^b := e^{bN_t - \lambda^* t(e^b - 1)}, \quad \forall\, t \in [0,T], \tag{3.6}$$

follows a $\mathbb{G}$-martingale under $\mathbb{P}^*$. By standard arguments, the process $\tilde{M}^b$ is a $\mathbb{P}^*$-martingale if and only if the product $\tilde{M}^b \eta$ is a martingale under the original probability measure $\mathbb{P}$. But in view of (3.4), we have

$$\tilde{M}_t^b \eta_t = \exp\Big(N_t\big(b + \ln(1+\kappa)\big) - t\big(\kappa\lambda + \lambda^*(e^b - 1)\big)\Big).$$

Let us write $a = b + \ln(1+\kappa)$. Since b is an arbitrary real number, so is a. Then, by virtue of part (iii) in Proposition 3.2, we necessarily have

$$\kappa\lambda + \lambda^*(e^b - 1) = \lambda(e^a - 1).$$

After simplifications, we conclude that, for any fixed real number b, the process $\tilde{M}^b$ defined by (3.6) is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ if and only if $\lambda^* = (1+\kappa)\lambda$. In other words, the intensity λ^* of N under $\mathbb{P}^*$ satisfies $\lambda^* = (1+\kappa)\lambda$. Also the second statement is clear. □

Remarks. Assume that $\mathbb{G} = \mathbb{F}^N$, i.e., the filtration $\mathbb{G}$ is generated by some Poisson process N. Then any strictly positive $\mathbb{G}$-martingale η under $\mathbb{P}$ is known to satisfy SDE (3.3) for some $\mathbb{G}$-predictable process κ.

Assume that W is a Brownian motion and N follows a Poisson process under $\mathbb{P}$ with respect to $\mathbb{G}$. Let η satisfy

$$d\eta_t = \eta_{t-}\big(\beta_t\, dW_t + \kappa\, d\hat{N}_t\big), \quad \eta_0 = 1, \tag{3.7}$$

for some $\mathbb{G}$-predictable stochastic process β and some constant $\kappa > -1$. A simple application of the Itô's product rule shows that if processes η^1 and η^2 satisfy:

$$d\eta^1_t = \eta^1_{t-}\beta_t\, dW_t, \quad d\eta^2_t = \eta^2_{t-}\kappa\, d\hat{N}_t,$$

then the product $\eta_t := \eta^1_t\eta^2_t$ satisfies (3.7). Taking the uniqueness of solutions to the linear SDE (3.7) for granted, we conclude that the unique solution to this SDE is given by the expression:

$$\eta_t = \exp\Big(\int_0^t \beta_u\, dW_u - \frac{1}{2}\int_0^t \beta_u^2\, du\Big) \exp\big(N_t \ln(1+\kappa) - \kappa\lambda t\big). \tag{3.8}$$

The proof of the next result is left to the reader as exercise.

Proposition 3.5 *Let the probability $\mathbb{P}^*$ be given by* (3.2) *and* (3.8) *for some constant $\kappa > -1$ and a $\mathbb{G}$-predictable process β, such that $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$.*
(i) *The process $W^*_t = W_t - \int_0^t \beta_u\, du$, $t \in [0,T]$, follows a Brownian motion under $\mathbb{P}^*$, with respect to the filtration $\mathbb{G}$.*
(ii) *The process N_t, $t \in [0,T]$, follows a Poisson process with the constant intensity $\lambda^* = (1+\kappa)\lambda$ under $\mathbb{P}^*$, with respect to the filtration $\mathbb{G}$.*
(iii) *Processes W^* and N are mutually independent under $\mathbb{P}^*$.*

3.2 Poisson Process with Deterministic Intensity

Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be any non-negative, locally integrable function such that $\int_0^\infty \lambda(u)\, du = \infty$. By definition, the process N (with $N_0 = 0$) is the Poisson process with *intensity function* λ if for every $0 \le s < t$ the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$, and has the Poisson law with parameter $\Lambda(t) - \Lambda(s)$, where the *hazard function* Λ equals $\Lambda(t) = \int_0^t \lambda(u)\, du$.

More generally, let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-continuous, increasing function with $\Lambda(0) = 0$ and $\Lambda(\infty) = \infty$. The Poisson process with the hazard function Λ satisfies, for every $0 \le s < t$ and every $k = 0, 1, \ldots$

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s\} = \mathbb{P}\{N_t - N_s = k\} = \frac{(\Lambda(t) - \Lambda(s))^k}{k!}\, e^{-(\Lambda(t)-\Lambda(s))}.$$

Example 3.1 The most convenient and widely used method of constructing a Poisson process with a hazard function Λ runs as follows: we take a Poisson process $\tilde{N}$ with the constant intensity $\lambda = 1$, with respect to some filtration $\tilde{\mathbb{G}}$, and we define the time-changed process $N_t := \tilde{N}_{\Lambda(t)}$. The process N is easily seen to follow a Poisson process with the hazard function Λ, with respect to the time-changed filtration $\mathbb{G}$, where $\mathcal{G}_t = \tilde{\mathcal{G}}_{\Lambda(t)}$ for every $t \in \mathbb{R}_+$.

Since for arbitrary $0 \le s < t$

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_t - N_s) = \Lambda(t) - \Lambda(s),$$

it is clear that the compensated Poisson process $\hat{N}_t = N_t - \Lambda(t)$ follows a $\mathbb{G}$-martingale under $\mathbb{P}$. A suitable generalization of Proposition 3.3 shows that a Poisson process with the hazard function Λ and a Brownian motion with respect to $\mathbb{G}$ follow mutually independent processes under $\mathbb{P}$. The proof of the next lemma relies on a direct application of the Itô formula, and so it is omitted.

Lemma 3.2 *Let Z be an arbitrary bounded, $\mathbb{G}$-predictable process. Then the process M^Z, given by the formula*

$$M_t^Z = \exp\Big(\int_{]0,t]} Z_u\, dN_u - \int_0^t (e^{Z_u} - 1)\, d\Lambda(u)\Big),$$

follows a $\mathbb{G}$-martingale under $\mathbb{P}$. Moreover, M^Z is the unique solution to the SDE

$$dM_t^Z = M_{t-}^Z(e^{Z_t} - 1)\, d\hat{N}_t, \quad M_0^Z = 1.$$

In case of a Poisson process with intensity function λ, it can be easily deduced from Lemma 3.2 that, for any (Borel measurable) function $\kappa : \mathbb{R}_+ \to (-1, \infty)$, the process

$$\zeta_t = \exp\Big(\int_{]0,t]} \ln(1 + \kappa(u))\, dN_u - \int_0^t \kappa(u)\lambda(u)\, du\Big)$$

is the unique solution to the SDE

$$d\zeta_t = \zeta_{t-}\kappa(t)\, d\hat{N}_t, \quad \eta_0 = 1.$$

Using similar arguments as in the case of constant κ, one can show that the unique solution to the SDE

$$d\eta_t = \eta_{t-}\big(\beta_t\, dW_t + \kappa(t)\, d\hat{N}_t\big), \quad \eta_0 = 1,$$

is given by the following expression:

$$\eta_t = \zeta_t \exp\Big(\int_0^t \beta_u\, dW_u - \frac{1}{2}\int_0^t \beta_u^2\, du\Big). \tag{3.9}$$

The next result generalizes Proposition 3.5. Again, the proof is left to the reader.

Proposition 3.6 *Let $\mathbb{P}^*$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, such that the density process η in* (3.2) *is given by* (3.9). *Then, under $\mathbb{P}^*$ and with respect to $\mathbb{G}$:*
(i) *the process $W_t^* = W_t - \int_0^t \beta_u\, du$, $t \in [0, T]$, follows a Brownian motion,*
(ii) *the process N_t, $t \in [0, T]$, is a Poisson process with the intensity function $\lambda^*(t) = 1 + \kappa(t)\lambda(t)$,*
(iii) *processes W^* and N are mutually independent under $\mathbb{P}^*$.*

3.3 Conditional Poisson Process

We start by assuming that we are given a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$ and a certain sub-filtration $\mathbb{F}$ of $\mathbb{G}$. Let Φ be an $\mathbb{F}$-adapted, right-continuous, increasing process, with $\Phi_0 = 0$ and $\Phi_\infty = \infty$. We refer to Φ as the *hazard process.* In some cases, we have $\Phi_t = \int_0^t \phi_u\, du$ for some $\mathbb{F}$-progressively measurable process ϕ with locally integrable sample paths. Then the process ϕ is called the *intensity process.* We are in a position to state the definition of the $\mathbb{F}$-conditional Poisson process associated with Φ. Slightly different, but essentially equivalent, definition of a conditional Poisson process (also known as the doubly stochastic Poisson process) can be found in Brémaud (1981) and Last and Brandt (1995).

Definition 3.2 A process N defined on a probability space $(\Omega, \mathbb{G}, \mathbb{P})$ is called the $\mathbb{F}$-*conditional Poisson process* with respect to $\mathbb{G}$, associated with the hazard process Φ, if for any $0 \le s < t$ and every $k = 0, 1, \ldots$

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty\} = \frac{(\Phi_t - \Phi_s)^k}{k!}\, e^{-(\Phi_t - \Phi_s)}, \tag{3.10}$$

where $\mathcal{F}_\infty = \sigma(\mathcal{F}_u : u \in \mathbb{R}_+)$.

At the intuitive level, if a particular sample path $\Phi_\cdot(\omega)$ of the hazard process is known, the process N has exactly the same properties as the Poisson process with respect to $\mathbb{G}$ with the (deterministic) hazard function $\Phi_\cdot(\omega)$. In particular, it follows from (3.10) that

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty\} = \mathbb{P}\{N_t - N_s = k \,|\, \mathcal{F}_\infty\},$$

i.e., conditionally on the σ-field $\mathcal{F}_\infty$ the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$.

Similarly, for any $0 \le s < t \le u$ and every $k = 0, 1, \dots,$ we have

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_u\} = \frac{(\Phi_t - \Phi_s)^k}{k!}\, e^{-(\Phi_t - \Phi_s)}. \tag{3.11}$$

In other words, conditionally on the σ-field $\mathcal{F}_u$ the process N_t, $t \in [0, u]$, behaves like a Poisson process with the hazard function Φ. Finally, for any $n \in \mathbb{N}$, any non-negative integers $k_1, \dots, k_n$, and arbitrary non-negative real numbers $s_1 < t_1 \le s_2 < t_2 \le \ldots \le s_n < t_n$ we have

$$\mathbb{P}\Big(\bigcap_{i=1}^n \{N_{t_i} - N_{s_i} = k_i\}\Big) = \mathbb{E}_{\mathbb{P}}\Big(\prod_{i=1}^n \frac{\big(\Phi_{t_i} - \Phi_{s_i}\big)^{k_i}}{k_i!}\, e^{-(\Phi_{t_i} - \Phi_{s_i})}\Big).$$

Let us notice that in all conditional expectations above, the reference filtration $\mathbb{F}$ can be replaced by the filtration $\mathbb{F}^\Phi$ generated by the hazard process. In fact, an $\mathbb{F}$-conditional Poisson process with respect to $\mathbb{G}$ follows also a conditional Poisson process with respect to the filtrations: $\mathbb{F}^N \vee \mathbb{F}$ and $\mathbb{F}^N \vee \mathbb{F}^\Phi$ (with the same hazard process).

We shall henceforth postulate that $\mathbb{E}_{\mathbb{P}}(\Phi_t) < \infty$ for every $t \in \mathbb{R}_+$.

Lemma 3.3 *The compensated process $\hat{N}_t = N_t - \Phi_t$ follows a martingale with respect to $\mathbb{G}$.*

Proof. It is enough to notice that, for arbitrary $0 \le s < t$,

$$\mathbb{E}_{\mathbb{P}}(N_t - \Phi_t \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(N_t - \Phi_t \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty) \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_s - \Phi_s \,|\, \mathcal{G}_s) = N_s - \Phi_s,$$

where in the second equality we have used the property of a Poisson process with deterministic hazard function. □

Given the two filtrations $\mathbb{F}$ and $\mathbb{G}$ and the hazard process Φ, it is not obvious whether we may find a process N, which would satisfy Definition 3.2. To provide a simple construction of a conditional Poisson process, we assume that the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$, endowed with a reference filtration $\mathbb{F}$, is sufficiently large to accommodate for the following stochastic processes: a Poisson process $\tilde{N}$ with the constant intensity $\lambda = 1$ and an $\mathbb{F}$-adapted hazard process Φ. In addition, we postulate that the Poisson process $\tilde{N}$ is independent of the filtration $\mathbb{F}$

Remark. Given a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, it is always possible to enlarge it in such a way that there exists a Poisson process $\tilde{N}$ with $\lambda = 1$, independent of the filtration $\mathbb{F}$, and defined on the enlarged space.

Under the present assumptions, for every $0 \le s < t$, any $u \in \mathbb{R}_+$, and any non-negative integer k, we have

$$\mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k \,|\, \mathcal{F}_\infty\} = \mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k \,|\, \mathcal{F}_u\} = \mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k\}$$

and

$$\mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k \,|\, \mathcal{F}_s^{\tilde{N}} \vee \mathcal{F}_s\} = \mathbb{P}\{\tilde{N}_t - \tilde{N}_s = k\} = \frac{(t-s)^k}{k!}\, e^{-(t-s)}.$$

The next result describes an explicit construction of a conditional Poisson process. This construction is based on a random time change associated with the increasing process Φ.

Proposition 3.7 *Let $\tilde{N}$ be a Poisson process with the constant intensity $\lambda = 1$, independent of a reference filtration $\mathbb{F}$, and let Φ be an $\mathbb{F}$-adapted, right-continuous, increasing process. Then the process $N_t = \tilde{N}_{\Phi_t}$, $t \in \mathbb{R}_+$, follows the $\mathbb{F}$-conditional Poisson process with the hazard process Φ with respect to the filtration $\mathbb{G} = \mathbb{F}^N \vee \mathbb{F}$.*

Proof. Since $\mathcal{G}_s \vee \mathcal{F}_\infty = \mathcal{F}_s^N \vee \mathcal{F}_\infty$, it suffices to check that

$$\mathbb{P}\{N_t - N_s = k \,|\, \mathcal{F}_s^N \vee \mathcal{F}_\infty\} = \frac{(\Phi_t - \Phi_s)^k}{k!}\, e^{-(\Phi_t - \Phi_s)}$$

or, equivalently,

$$\mathbb{P}\{\tilde{N}_{\Phi_t} - \tilde{N}_{\Phi_s} = k \,|\, \mathcal{F}_{\Phi_s}^{\tilde{N}} \vee \mathcal{F}_\infty\} = \frac{(\Phi_t - \Phi_s)^k}{k!}\, e^{-(\Phi_t - \Phi_s)}.$$

The last equality follows from the assumed independence of $\tilde{N}$ and $\mathbb{F}$. □

Remark. Within the setting of Proposition 3.7, any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale, so that Condition (M.1) is satisfied.

The total number of jumps of the conditional Poisson process is obviously unbounded with probability 1. In some financial models, only the properties of the first jump are relevant, though. There exist many ways of constructing the conditional Poisson process, but Condition (F.1) is always satisfied by the first jump of such a process, since it follows directly from Definition 3.2. In effect, if we denote $\tau = \tau_1$, then for any $t \in \mathbb{R}_+$ and $u \geq t$ we have:

$$\mathbb{P}\{\tau \leq t \,|\, \mathcal{F}_u\} = \mathbb{P}\{N_t \geq 1 \,|\, \mathcal{F}_u\} = \mathbb{P}\{N_t - N_0 \geq 1 \,|\, \mathcal{G}_0 \vee \mathcal{F}_u\} = \mathbb{P}\{\tau \leq u \,|\, \mathcal{F}_\infty\},$$

where the last equality follows from (3.11). It is also clear, once more by (3.11), that $\mathbb{P}\{\tau \leq t \,|\, \mathcal{F}_u\} = e^{-\Phi_u}$ for every $0 \leq t \leq u$.

Example 3.2 *Cox process.* In some applications, it is natural to consider a special case of an $\mathbb{F}$-conditional Poisson process, with the filtration $\mathbb{F}$ generated by a certain stochastic process, representing the *state variables.* To be more specific, on considers a conditional Poisson process with the intensity process ϕ given as $\phi_t = g(t, Y_t)$, where Y is an $\mathbb{R}^d$-valued stochastic process independent of the Poisson process $\tilde{N}$, and $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is a (continuous) function. The reference filtration $\mathbb{F}$ is typically chosen to be the natural filtration of the process Y; that is, we take $\mathbb{F} = \mathbb{F}^Y$. In such a case, the resulting $\mathbb{F}$-conditional Poisson process is referred to as the *Cox process* associated with the state variables process Y, and the intensity function g.

Our last goal is to examine the behavior of an $\mathbb{F}$-conditional Poisson process N under an equivalent change of a probability measure. Let us assume, for the sake of simplicity, that the hazard process Φ is continuous, and the reference filtration $\mathbb{F}$ is generated by a process W, which follows a Brownian motion with respect to $\mathbb{G}$. For a fixed $T > 0$, we define the probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{G}_T)$ by setting:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \eta_T, \quad \mathbb{P}\text{-a.s.}, \tag{3.12}$$

where the Radon-Nikodým density process η_t, $t \in [0, T]$, solves the SDE

$$d\eta_t = \eta_{t-}\big(\beta_t \, dW_t + \kappa_t \, d\hat{N}_t\big), \quad \eta_0 = 1, \tag{3.13}$$

for some $\mathbb{G}$-predictable processes β and κ such that $\kappa > -1$ and $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$. An application of Itô's product rule shows that the unique solution to (3.13) is equal to the product $\nu_t \zeta_t$, where

$d\nu_t = \nu_t \beta_t \, dW_t$ and $d\zeta_t = \zeta_{t-} \kappa_t \, d\hat{N}_t$, with $\nu_0 = \zeta_0 = 1$. The solutions to the last two equations are

$$\nu_t = \exp\Big(\int_0^t \beta_u \, dW_u - \frac{1}{2} \int_0^t \beta_u^2 \, du \Big)$$

and

$$\zeta_t = \exp(U_t) \prod_{0<u\le t} (1 + \Delta U_u) \exp(-\Delta U_u),$$

respectively, where we denote $U_t = \int_{]0,t]} \kappa_u \, d\hat{N}_u$. It is useful to observe that ζ admits the following representations:

$$\zeta_t = \exp\Big(- \int_0^t \kappa_u \, d\Phi_u \Big) \prod_{0<u\le t} (1 + \kappa_u \Delta N_u),$$

and

$$\zeta_t = \exp\Big(\int_{]0,t]} \ln(1+\kappa_u) \, dN_u - \int_0^t \kappa_u \, d\Phi_u \Big).$$

Proposition 3.8 *Let the Radon-Nikodým density of $\mathbb{P}^*$ with respect to $\mathbb{P}$ be given by* (3.12)–(3.13). *Then the process $W_t^* = W_t - \int_0^t \beta_u \, du$, $t \in [0,T]$, follows a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{P}^*$, and the process*

$$N_t^* = \hat{N}_t - \int_0^t \kappa_u \, d\Phi_u = N_t - \int_0^t (1+\kappa_u) \, d\Phi_u, \quad \forall \, t \in [0,T], \tag{3.14}$$

follows a $\mathbb{G}$-martingale under $\mathbb{P}^*$. If, in addition, the process κ is $\mathbb{F}$-adapted, then the process N follows under $\mathbb{P}^*$ an $\mathbb{F}$-conditional Poisson process with respect to $\mathbb{G}$, and the hazard process of N under $\mathbb{P}^*$ equals

$$\Phi_t^* = \int_0^t (1+\kappa_u) \, d\Phi_u.$$

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WARSAW UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 4

4 Defaultable Claims

We fix a finite horizon date $T^* > 0$, and we suppose that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with some filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}$, is sufficiently rich to support the following objects:

- the *short-term interest rate* process r,
- the *firm's value process* V, which models the total value of the firm's assets,
- the *barrier process* v, which will serve to specify the default time,
- the *promised contingent claim* X representing the firm's liabilities to be redeemed at time $T \le T^*$,
- the process A, which models the *promised dividends,* i.e., the firm's liabilities stream that is redeemed continuously or discretely over time to the holder of a defaultable claim,
- the *recovery claim* $\tilde{X}$, which represents the recovery payoff received at time T, if default occurs prior to or at the claim's maturity date T,
- the *recovery process* Z, which specifies the recovery payoff at time of default, if it occurs prior to or at the maturity date T.

The probability measure $\mathbb{P}$ is assumed to represent the *real-world* (or *statistical*) probability, as opposed to the *spot martingale measure* (or the *risk-neutral probability*). The latter probability is denoted by $\mathbb{P}^*$ in what follows.

4.1 Technical Assumptions

We postulate that the processes V, Z, A, and v are progressively measurable with respect to the filtration $\mathbb{F}$, and that the random variables X and $\tilde{X}$ are $\mathcal{F}_T$-measurable. In addition, A is assumed to be a process of finite variation, with $A_0 = 0$. We assume without mentioning that all random objects introduced above satisfy suitable integrability conditions that are needed for evaluating the functionals defined in the sequel.

4.2 Default Time

Let us denote by τ the random time of default. At this stage, it is essential to stress that the various approaches to valuing and hedging of defaultable securities differ between themselves with regard to the ways in which the default event – and thus also the default time τ – are modeled. In the structural approach, the default time τ will be typically defined in terms of the value process V and the barrier process v. Specifically, we shall set

$$\tau := \inf \{ t > 0 : t \in \mathcal{T},\, V_t < v_t \} \tag{4.1}$$

with the usual convention that the infimum over the empty set equals $+\infty$. In (4.1), the set $\mathcal{T}$ is assumed to be a Borel measurable subset of the time interval $[0, T]$ (or $[0, \infty)$ in the case of perpetual claims). From the mathematical standpoint, we shall frequently be justified in substituting the strict inequality '$<$' with the '$\leq$' in (4.1), and in analogous definitions, without altering the probabilistic content of the definition. Furthermore, τ will be an $\mathbb{F}$-stopping time, and since the underlying filtration $\mathbb{F}$ in most structural models is generated by a standard Brownian motion, τ will be an $\mathbb{F}$-predictable stopping time (as any stopping time with respect to a Brownian filtration).

The latter property means that within the framework of the structural approach there exists a sequence of increasing stopping times announcing the default time; in this sense, the default time can be forecasted with some degree of certainty. By contrast, in the intensity-based approach, the default time will not be a predictable stopping time with respect to the 'enlarged' filtration, denoted by $\mathbb{G}$ in Part III of the text. In typical examples, the filtration $\mathbb{G}$ will encompass some Brownian filtration $\mathbb{F}$, but $\mathbb{G}$ will be strictly larger than $\mathbb{F}$. At the intuitive level, in the intensity-based approach the occurrence of the default event comes as a total surprise. For any date t, the present value of the default intensity yields the conditional probability of the occurrence of default over an infinitesimally small time interval $[t, t+dt]$.

4.3 Recovery Rules

If default occurs after time T, the promised claim X is paid in full at time T. Otherwise, depending on the adopted model, either the amount Z_τ is paid at time τ, or the amount $\tilde{X}$ is paid at the maturity date T. In a general setting, we consider simultaneously both kinds of recovery payoff, and thus a defaultable claim is formally defined as a quintuple $DCT = (X, A, \tilde{X}, Z, \tau)$. In most practical situations, however, we shall deal with only one type of recovery payoff – that is, we shall set either $\tilde{X} = 0$ or $Z \equiv 0$. Thus, a typical defaultable claim can be seen either the quadruplet $DCT^1 = (X, A, \tilde{X}, \tau)$ or as $DCT^2 = (X, A, Z, \tau)$, depending on the recovery scheme. The former is called a defaultable claim with *recovery at maturity* (DCT of the *first type*), and the latter a defaultable claim with *recovery at default* (DCT of the *second type*). The absence of the superscript i suggests that a particular expression is valid for a generic defaultable claim. Notice that the date T, the information structure $\mathbb{F}$ and the real-world probability $\mathbb{P}$ are also intrinsic components of the definition of a defaultable claim.

4.4 Risk-Neutral Valuation Formula

Suppose now that our underlying financial market model is arbitrage-free, in the sense that there exists a *spot martingale measure* $\mathbb{P}^*$ (also referred to as a *risk-neutral probability*), meaning that price process of any tradeable security, which pays no coupons or dividends, follows an $\mathbb{F}$-martingale under $\mathbb{P}^*$, when discounted by the *savings account* B, given as

$$B_t := \exp\Big(\int_0^t r_u \, du\Big).$$

We introduce the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$, and we denote by D the process that models all the cash flows received by the owner of a defaultable claim. Let us set $X^d(T) = X \mathbb{1}_{\{\tau > T\}} + \tilde{X} \mathbb{1}_{\{\tau \leq T\}}$.

Definition 4.1 The *dividend process* D of a defaultable contingent claim $DCT = (X, A, \tilde{X}, Z, \tau)$, which settles at time T, equals

$$D_t = X^d(T) \mathbb{1}_{\{t \geq T\}} + \int_{]0,t]} (1 - H_u) \, dA_u + \int_{]0,t]} Z_u \, dH_u.$$

It is clear that D is a process of finite variation over $[0,T]$. Since

$$\int_{]0,t]} (1 - H_u)\, dA_u = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}}\, dA_u = A_{\tau-} \mathbb{1}_{\{\tau \le t\}} + A_t \mathbb{1}_{\{\tau > t\}},$$

it is apparent that in case the default occurs at some date t, the promised dividend $A_t - A_{t-}$, that is due to be paid at this date, is not actually passed over to the holder of a defaultable claim. Furthermore, we have

$$\int_{]0,t]} Z_u\, dH_u = Z_{\tau \wedge t} \mathbb{1}_{\{\tau \le t\}} = Z_\tau \mathbb{1}_{\{\tau \le t\}},$$

where $\tau \wedge t = \min(\tau, t)$. At the formal level, the promised payoff X could be considered as a part of the promised dividends process A. However, such a convention would be inconvenient, since in practice the recovery rules concerning the promised dividends A and the promised claim X are generally different. For instance, in the case of a defaultable coupon bond, it is frequently postulated that in case of default the future coupons are lost (formally, they are subject to the zero recovery scheme), but a strictly positive fraction of the bond's face value is usually received by the bondholder. We adopt the following definition of the ex-dividend price $X^d(t,T)$ of a defaultable claim. At any time t, the random variable $X^d(t,T)$ is meant to represent the current value of all future cash flows associated with a given defaultable claim DCT. In particular, we always have $X^d(T,T) = 0$. A formal justification for expression (4.2) is postponed to Sect. 4.6.

Definition 4.2 The (ex-dividend) *price process* of the defaultable claim $DCT = (X, A, \tilde{X}, Z, \tau)$, which settles at time T, is given as

$$X^d(t,T) = B_t\, \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{F}_t \Big), \quad \forall\, t \in [0,T]. \tag{4.2}$$

One easily recognizes (4.2) as a variant of the *risk-neutral valuation formula* that is known to give the arbitrage price of attainable contingent claims. Attainability of a defaultable claim DCT is not obvious, though. Structural models typically assume that assets of the firm represent a tradeable security (in practice, the total market value of firm's shares is usually taken as the proxy for V). Consequently, the issue of existence of replicating strategies for defaultable claims can be analyzed in a similar way as in standard default-free financial models. In particular, it is essential to assume that the reference filtration $\mathbb{F}$ is generated by the price processes of tradeable assets. Otherwise, for instance, when the default time τ is the first passage time of V to a lower threshold, which does not represent the price of a tradeable asset (so that τ is not a stopping time with respect to the filtration generated by some tradeable assets), the issue of attainability of defaultable contingent claims becomes more delicate. To summarize, the validity of the valuation formula (4.2) is not obvious a priori, so that it needs to be examined on a case by case basis.

For the ease of future reference, we shall now examine in some detail the two special cases of expression (4.2). It follows immediately from (4.2) that the price process $X^{d,i}(\cdot,T)$ of a defaultable claim DCT^i equals, for $i = 1,2$:

$$X^{d,i}(t,T) := B_t\, \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1}\, dD_u^i \,\Big|\, \mathcal{F}_t \Big), \tag{4.3}$$

where

$$D^1_t = \big(X\mathbb{1}_{\{\tau>T\}} + \tilde{X}\mathbb{1}_{\{\tau\le T\}}\big)\mathbb{1}_{\{t\ge T\}} + \int_{]0,t]} (1-H_u)\,dA_u,$$

and

$$D^2_t = X\mathbb{1}_{\{\tau>T\}}\mathbb{1}_{\{t\ge T\}} + \int_{]0,t]} (1-H_u)\,dA_u + \int_{]0,t]} Z_u\,dH_u.$$

Consider first a defaultable claim with recovery at maturity – that is, the claim DCT^1. In the absence of the promised dividends (i.e., when $A \equiv 0$), the valuation formula (4.3) becomes, for $0 \le t < T$,

$$X^{d,1}(t,T) := B_t\,\mathbb{E}_{\mathbb{P}^*}\big(B_T^{-1}X^{d,1}(T)\,\big|\,\mathcal{F}_t\big), \tag{4.4}$$

where the terminal payoff $X^{d,1}(T)$, which equals

$$X^{d,1}(T) = X\mathbb{1}_{\{\tau>T\}} + \tilde{X}\mathbb{1}_{\{\tau\le T\}}, \tag{4.5}$$

represents the cash flow at time T of a given defaultable claim with recovery at maturity. It is thus clear that, in the absence of promised dividends, the discounted price process $X^{d,1}(t,T)/B_t$, $t < T$, follows an $\mathbb{F}$-martingale under $\mathbb{P}^*$, provided, of course, that a usual integrability condition is imposed on $X^{d,1}(T)$.

4.5 Self-Financing Trading Strategies

We are now going to provide a formal justification of Definition 4.2, based on the no-arbitrage arguments. We write S^i, $i = 1,\dots,k$ to denote the price processes of k primary securities in an arbitrage-free financial model. We make the standard assumption that the processes S^i, $i = 1,\dots,k-1$ follow semimartingales. In addition, we set $S^k_t = B_t$ so that S^k represents the value process of the savings account. For the sake of convenience, we assume that S^i, $i = 1,\dots,k-1$ are non-dividend-paying assets, and we introduce the discounted price processes $\tilde{S}^i$ by setting $\tilde{S}^i_t = S^i_t/B_t$.

Let us now also assume that we have an additional security that pays dividends during its lifespan – assumed to be the time interval $[0,T]$ – according to a process of finite variation D, with $D_0 = 0$. Let S^0 denote the yet unspecified price process of this security. In particular, we refrain from postulating that S^0 follows a semimartingale. Of course, we do not necessarily need to interpret S^0 as the value process of a defaultable claim, though we have here this particular interpretation in mind.

Let an $\mathbb{F}$-predictable process $\phi = (\phi^0,\dots,\phi^k)$ stand for a trading strategy. At this stage, it will be enough to examine a simple trading strategy involving a defaultable claim. In fact, since we do not assume a priori that S^0 follows a semimartingale, we are not yet in a position to consider general trading strategies involving the defaultable claim anyway.

Suppose that we purchase at time 0 one unit of the 0^{th} asset at the initial price S^0_0, we hold it until time T, and we invest all the proceeds from dividends in a savings account. More specifically, we consider a buy-and-hold strategy $\psi = (1,0,\dots,0,\psi^k)$. The associated *wealth process* $U(\psi)$ equals:

$$U_t(\psi) = S^0_t + \psi^k_t B_t, \quad \forall\, t \in [0,T], \tag{4.6}$$

with some initial value $U_0(\psi) = S^0_0 + \psi^k_0$. We assume that the strategy ψ introduced above is *self-financing*; i.e., we postulate that for every $t \in [0,T]$

$$U_t(\psi) - U_0(\psi) = S^0_t - S^0_0 + D_t + \int_{]0,t]} \psi^k_u\,dB_u. \tag{4.7}$$

Lemma 4.1 *The discounted wealth* $\tilde{U}_t(\psi) = B_t^{-1}U_t(\psi)$ *of a self-financing trading strategy* ψ *satisfies, for every* $t \in [0,T]$,

$$\tilde{U}_t(\psi) = \tilde{U}_0(\psi) + \tilde{S}_t^0 - \tilde{S}_0^0 + \int_{]0,t]} B_u^{-1} dD_u. \tag{4.8}$$

Proof. We define an auxiliary process $\hat{U}(\psi)$ by setting $\hat{U}_t(\psi) := U_t(\psi) - S_t^0 = \psi_t^k B_t$. In view of (4.7), we have

$$\hat{U}_t(\psi) = \hat{U}_0(\psi) + D_t + \int_{]0,t]} \psi_u^k \, dB_u,$$

and so the process $\hat{U}(\psi)$ follows a semimartingale.

An application of Itô's product rule yields

$$\begin{aligned} d\big(B_t^{-1}\hat{U}_t(\psi)\big) &= B_t^{-1} d\hat{U}_t(\psi) + \hat{U}_t(\psi)\, dB_t^{-1} \\ &= B_t^{-1} dD_t + \psi_t^k B_t^{-1} dB_t + \psi_t^k B_t \, dB_t^{-1} \\ &= B_t^{-1} dD_t, \end{aligned}$$

where we have used the obvious equality $B_t^{-1}dB_t + B_t\, dB_t^{-1} = 0$. Integrating the last equality, we obtain

$$B_t^{-1}\big(U_t(\psi) - S_t^0\big) = B_0^{-1}\big(U_0(\psi) - S_0^0\big) + \int_{]0,t]} B_u^{-1} dD_u,$$

and this immediately yields (4.8). □

In view of Lemma 4.1, for every $t \in [0,T]$ we also have:

$$\tilde{U}_T(\psi) - \tilde{U}_t(\psi) = \tilde{S}_T^0 - \tilde{S}_t^0 + \int_{]t,T]} B_u^{-1} \, dD_u. \tag{4.9}$$

4.6 Martingale Measures

We are ready to derive the risk-neutral valuation formula for the ex-dividend price S_t^0. To this end, we assume that our model is arbitrage-free, meaning here that it admits a (not necessarily unique) spot martingale measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$. In particular, this implies that the discounted price $\tilde{S}^i$ of any non-dividend paying primary security, as well as the discounted wealth process $\tilde{U}(\phi)$ of any *admissible* self-financing trading strategy $\phi = (0, \phi^1, \dots, \phi^k)$, follow martingales under $\mathbb{P}^*$. In addition, we postulate that the trading strategy ψ introduced in Sect. 4.5 is also *admissible,* so that the discounted wealth process $\tilde{U}(\psi)$ follows a $\mathbb{P}^*$-martingale with respect to the filtration $\mathbb{F}$.

We make an assumption that the market value at time t of the 0^{th} security comes exclusively from the future dividends stream; this amounts to postulate that $S_T^0 = \tilde{S}_T^0 = 0$. In view of this convention, we shall refer to S^0 as the *ex-dividend price* of the 0^{th} asset, e.g., a defaultable claim.

Proposition 4.1 *The ex-dividend price process* S^0 *satisfies, for* $t \in [0,T]$,

$$S_t^0 = B_t \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{F}_t \Big). \tag{4.10}$$

Proof. In view of the martingale property of the discounted wealth process $\tilde{U}(\psi)$, for any $t \in [0,T]$ we have

$$\mathbb{E}_{\mathbb{P}^*}\big(\tilde{U}_T(\psi) - \tilde{U}_t(\psi) \,\big|\, \mathcal{F}_t\big) = 0.$$

Taking into account (4.9), we thus obtain

$$\tilde{S}^0_t = \mathbb{E}_{\mathbb{P}^*}\Big(\tilde{S}^0_T + \int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{F}_t\Big).$$

Since by assumption $S^0_T = \tilde{S}^0_T = 0$, the last formula yields (4.10). □

Let us now examine a general trading strategy $\phi = (\phi^0, \ldots, \phi^k)$. The associated *wealth process* $U(\phi)$ equals $U_t(\phi) = \sum_{i=0}^k \phi^i_t S^i_t$. A strategy ϕ is said to be *self-financing* if $U_t(\phi) = U_0(\phi) + G_t(\phi)$ for every $t \in [0,T]$, where the *gains process* $G(\phi)$ is defined as follows:

$$G_t(\phi) := \int_{]0,t]} \phi^0_u\, dD_u + \sum_{i=0}^k \int_{]0,t]} \phi^i_u\, dS^i_u.$$

Corollary 4.1 *For any self-financing trading strategy ϕ, the discounted wealth process $\tilde{U}(\phi) := B_t^{-1}U_t(\phi)$ follows a local martingale under $\mathbb{P}^*$.*

Proof. Since B is a continuous process of finite variation, Itô's product rule gives

$$d\tilde{S}^i_t = S^i_t\, dB_t^{-1} + B_t^{-1} dS^i_t$$

for $i = 0, \ldots, k$, and so

$$\begin{aligned}
d\tilde{U}_t(\phi) &= U_t(\phi)\, dB_t^{-1} + B_t^{-1} dU_t(\phi) \\
&= U_t(\phi)\, dB_t^{-1} + B_t^{-1}\Big(\sum_{i=0}^k \phi^i_t\, dS^i_t + \phi^0_t\, dD_t\Big) \\
&= \sum_{i=0}^k \phi^i_t\big(S^i_t\, dB_t^{-1} + B_t^{-1} dS^i_t\big) + \phi^0_t B_t^{-1} dD_t \\
&= \sum_{i=1}^{k-1} \phi^i_t\, d\tilde{S}^i_t + \phi^0_t\big(d\tilde{S}^0_t + B_t^{-1} dD_t\big) = \sum_{i=1}^{k-1} \phi^i_t\, d\tilde{S}^i_t + \phi^0_t\, d\hat{S}^0_t,
\end{aligned}$$

where the process $\hat{S}^0$ is given by the formula

$$\hat{S}^0_t := \tilde{S}^0_t + \int_{]0,t]} B_u^{-1}\, dD_u.$$

To conclude, it suffices to observe that in view of (4.10) the process $\hat{S}^0$ satisfies

$$\hat{S}^0_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]0,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{F}_t\Big),$$

and thus it follows a martingale under $\mathbb{P}^*$. □

It is worth noticing that $\hat{S}^0_t$ represents the discounted cum-dividend price at time t of the 0^{th} asset.

WARSAW UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 5

5 Merton's (1974) Model of Corporate Debt

In his pathbreaking paper, Merton (1974) considers a firm with a single liability carrying a promised (deterministic) terminal payoff L. Several standard conditions are imposed on the continuous-time Black-Scholes-type *frictionless market*. Let us recall the most important assumptions:

- trading takes place continuously in time,
- all traded assets are infinitely divisible,
- an unrestricted borrowing and lending of funds is possible at the same interest rate,
- no restrictions on the short-selling of traded securities are present,
- the transaction costs and taxes (or tax benefits) are disregarded,
- the bankruptcy and/or reorganization costs in case of default are negligible.

5.1 Merton's Model with Deterministic Interest Rates

One of the simplifying assumptions in the original Merton's model is that the short-term interest rate is constant and equals r. Therefore, the price at time t of the unit default-free zero-coupon bond with maturity T is easily seen to be $B(t,T) = e^{-r(T-t)}$. The latter formula can be extended to the case of a deterministic continuously compounded interest rate $r : \mathbb{R}_+ \to \mathbb{R}$. In this case, the price of a T-maturity zero-coupon bond equals:

$$B(t,T) = \exp\Big(- \int_t^T r(u)\, du \Big), \quad \forall\, t \in [0,T].$$

In the sequel, we denote by $E(V_t)$ ($D(V_t)$, resp.) the value of the firm's equity (debt, resp.) at time t; hence, the total value of firm's assets satisfies $V_t = E(V_t) + D(V_t)$. We postulate that the firm's value process V follows a geometric Brownian motion under the spot martingale measure $\mathbb{P}^*$, specifically,

$$dV_t = V_t\big((r-\kappa)\, dt + \sigma_V\, dW_t^*\big), \tag{5.1}$$

where σ_V is the constant volatility coefficient of the value process V and the constant κ represents the payout ratio, provided that it is non-negative. Otherwise, κ reflects an inflow of capital to the firm. The process W^* is the one-dimensional standard Brownian motion under $\mathbb{P}^*$, with respect to some reference filtration $\mathbb{F}$ (it is common to take $\mathbb{F} = \mathbb{F}^{W^*}$; this is not essential, though). Notice that dynamics (5.1) is justified only under the assumption that the total value of the firm's assets represents a traded security.

We postulate that the default event may only occur at the debt's maturity date T. Specifically, if at the maturity T the total value V_T of the firm's assets is less than the notional value L of the firm's debt, the firm defaults and the bondholders receive the amount V_T.

Otherwise, the firm does not default, and its liability is repaid in full. We are thus dealing here with a rather elementary example of a defaultable claim with recovery at maturity.

In terms of our generic model, we have:

$$X = L,\ A \equiv 0,\ \tilde{X} = V_T,\ \tau = T\mathbb{1}_{\{V_T<L\}} + \infty\mathbb{1}_{\{V_T\geq L\}},$$

where, as usual, $\infty \times 0 = 0$. Put another way,

$$X^{d,1}(T) = L\mathbb{1}_{\{\tau>T\}} + V_T\mathbb{1}_{\{\tau\leq T\}} = L\mathbb{1}_{\{V_T\geq L\}} + V_T\mathbb{1}_{\{V_T<L\}}$$

or, equivalently,

$$X^{d,1}(T) = \min(V_T, L)\mathbb{1}_{\{V_T\geq L\}} + \min(V_T, L)\mathbb{1}_{\{V_T<L\}} = \min(V_T, L).$$

The fixed amount L may be interpreted as the face value (or par value) of a corporate zero-coupon bond maturing at time T. Since

$$X^{d,1}(T) = \min(V_T, L) = L - (L - V_T)^+,$$

where $x^+ = \max(x, 0)$ for every $x \in \mathbb{R}$, the price process $X^{d,1}(t,T)$ of a defaultable zero-coupon bond is manifestly equal to the difference of the value of a default-free zero-coupon bond with the face value L and the value of a European put option written on the firm's assets, with the strike price L and the exercise date T. This put option, with the terminal payoff $(L - V_T)^+$, is commonly referred in the present context as the *put-to-default*. Formally, the value of the firm's debt at time t thus equals

$$D(V_t) = D(t,T) = LB(t,T) - P_t, \tag{5.2}$$

where P_t is the price of the put-to-default, and where, for the sake of notational convenience, we write $D(t,T)$ to denote the price of a defaultable bond:

$$D(t,T) := X^{d,1}(t,T) = B_t\,\mathbb{E}_{\mathbb{P}^*}(B_T^{-1}X^{d,1}(T)\,|\,\mathcal{F}_t).$$

It is apparent from (5.2) that the value at time t of the firm's equity satisfies

$$E(V_t) = V_t - D(V_t) = V_t - LB(t,T) + P_t = C_t, \tag{5.3}$$

where C_t stands in turn for the price at time t of a call option written on the firm's assets, with the strike price L and the exercise date T. To justify the last equality in (5.3), we may observe that at time T we have

$$E(V_T) = V_T - D(V_T) = V_T - \min(V_T, L) = (V_T - L)^+,$$

and thus the firm's equity can be seen as a call option on the firm's assets. Alternatively, we may directly use the so-called *put-call parity* relationship for European-style options:

$$C_t - P_t = V_t - LB(t,T).$$

Combining (5.2) with the classic Black-Scholes formula for the arbitrage price of a European put option, Merton (1974) derived a closed-form expression for the arbitrage price of a corporate bond. In what follows, N denotes the standard Gaussian cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du, \quad \forall\, x \in \mathbb{R}.$$

Proposition 5.1 *For every $0 \le t < T$ we have*

$$D(t,T) = V_t e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big) + LB(t,T)N\big(d_2(V_t,T-t)\big), \tag{5.4}$$

where

$$d_{1,2}(V_t, T-t) = \frac{\ln(V_t/L) + \big(r-\kappa \pm \frac{1}{2}\sigma_V^2\big)(T-t)}{\sigma_V\sqrt{T-t}}\,. \tag{5.5}$$

Proof. Suppose first that we take the classic Black-Scholes options valuation formula for granted. Recall that the Black-Scholes price of a European put option with the strike price L, written on a dividend-paying stock equals:

$$P_t = LB(t,T)N\big(-d_2(V_t,T-t)\big) - V_t e^{-\kappa(T-t)} N\big(-d_1(V_t,T-t)\big),$$

so that

$$D(t,T) = V_t e^{-\kappa(T-t)} N\big(-d_1(V_t,T-t)\big) + LB(t,T)\big(1 - N\big(-d_2(V_t,T-t)\big)\big).$$

Since obviously $N(-x) = 1-N(x)$, the last expression is easily seen to be equivalent to Merton's formula (5.4).

For the reader's convenience, we provide below the direct derivation of expression (5.4), based on the risk-neutral valuation formula:

$$S_t^0 = B_t\,\mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1}\,dD_u \,\Big|\, \mathcal{F}_t\Big).$$

For the sake of notational convenience, we shall write σ rather than σ_V, and we denote $\tilde{r} = r-\kappa$. When applied to a defaultable bond, the last formula yields, for every $0 \le t < T$,

$$D(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{P}^*}\big(L\mathbb{1}_{\{V_T \ge L\}} + V_T\mathbb{1}_{\{V_T<L\}} \,\big|\, \mathcal{F}_t\big),$$

so that

$$D(t,T) = LB(t,T)\,\mathbb{P}^*\{V_T \ge L \,|\, \mathcal{F}_t\} + B(t,T)\,\mathbb{E}_{\mathbb{P}^*}(V_T\mathbb{1}_{\{V_T<L\}} \,|\, \mathcal{F}_t). \tag{5.6}$$

Put another way, $D(t,T) = LB(t,T)J_1 + B(t,T)J_2$ with

$$J_1 = \mathbb{P}^*\{V_T \ge L \,|\, \mathcal{F}_t\}, \quad J_2 = \mathbb{E}_{\mathbb{P}^*}(V_T\mathbb{1}_{\{V_T<L\}} \,|\, \mathcal{F}_t).$$

Solving SDE (5.1), for every $t \in [0,T]$ we obtain

$$V_T = V_t \exp\big(\sigma(W_T^* - W_t^*) + (\tilde{r} - \tfrac{1}{2}\sigma^2)(T-t)\big). \tag{5.7}$$

For J_1, we have (recall that $L > 0$)

$$\begin{aligned}
J_1 &= \mathbb{P}^*\Big\{ V_t \exp\big(\sigma(W_T^* - W_t^*) + (\tilde{r} - \tfrac{1}{2}\sigma^2)\,(T-t)\big) \ge L \,\Big|\, \mathcal{F}_t\Big\} \\
&= \mathbb{P}^*\Big\{ -\sigma(W_T^* - W_t^*) \le \ln(V_t/L) + (\tilde{r} - \tfrac{1}{2}\sigma^2)\,(T-t) \,\Big|\, \mathcal{F}_t\Big\} \\
&= \mathbb{P}^*\Big\{ \xi \le \frac{\ln(x/L) + (\tilde{r} - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \Big\}_{x=V_t} \\
&= N\big(d_2(V_t, T-t)\big),
\end{aligned}$$

since the random variable $\xi := -(W^*_T - W^*_t)/\sqrt{T-t}$ is independent of the σ-field $\mathcal{F}_t$, and has the standard Gaussian law $N(0,1)$ under $\mathbb{P}^*$.

To evaluate J_2, it is convenient to introduce an auxiliary probability measure $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ by setting

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}^*} = \exp\big(\sigma W^*_T - \tfrac{1}{2}\sigma^2 T\big) =: \eta_T, \quad \mathbb{P}^*\text{-a.s.}$$

It is well known that for every $t \in [0,T]$ we have

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}^*}\Big|_{\mathcal{F}_t} = \exp\big(\sigma W^*_t - \tfrac{1}{2}\sigma^2 t\big) = \eta_t, \quad \mathbb{P}^*\text{-a.s.}$$

Let us denote $A = \{V_T < L\}$. It is clear that

$$J_2 = \mathbb{E}_{\mathbb{P}^*}(V_T \mathbb{1}_A \,|\, \mathcal{F}_t) = V_0\, e^{\tilde{r}T}\, \mathbb{E}_{\mathbb{P}^*}(\eta_T \mathbb{1}_A \,|\, \mathcal{F}_t).$$

Consequently, using the abstract Bayes rule, we obtain

$$J_2 = V_0\, e^{\tilde{r}T}\, \eta_t\, \bar{\mathbb{P}}\{A \,|\, \mathcal{F}_t\} = B^{-1}(t,T) V_t e^{-\kappa(T-t)}\, \bar{\mathbb{P}}\{A \,|\, \mathcal{F}_t\},$$

where the last equality is a consequence of the following chain of equalities:

$$V_0\, e^{\tilde{r}T}\, \eta_t = V_0\, \exp\big(\sigma W^*_t - \tfrac{1}{2}\sigma^2 t + (r-\kappa)T\big) = V_t e^{(r-\kappa)(T-t)}.$$

By virtue of Girsanov's theorem, the process $\bar{W}_t = W^*_t - \sigma t$ follows a standard Brownian motion on the space $(\Omega, \mathbb{F}, \bar{\mathbb{P}})$. The dynamics of V under $\bar{\mathbb{P}}$ are

$$dV_t = V_t\big((\tilde{r} + \sigma^2)\, dt + \sigma\, d\bar{W}_t\big),$$

and thus for every $t \in [0,T]$ we have

$$V_T = V_t \exp\big(\sigma(\bar{W}_T - \bar{W}_t) + (\tilde{r} + \tfrac{1}{2}\sigma^2)(T-t)\big).$$

Consequently,

$$\begin{aligned}
\bar{\mathbb{P}}\{A \,|\, \mathcal{F}_t\} &= \bar{\mathbb{P}}\Big\{V_t \exp\big(\sigma(\bar{W}_T - \bar{W}_t) + (\tilde{r} + \tfrac{1}{2}\sigma^2)(T-t)\big) < L \,\Big|\, \mathcal{F}_t\Big\} \\
&= \bar{\mathbb{P}}\Big\{\sigma(\bar{W}_T - \bar{W}_t) < -\ln(V_t/L) - (\tilde{r} + \tfrac{1}{2}\sigma^2)\,(T-t) \,\Big|\, \mathcal{F}_t\Big\} \\
&= \bar{\mathbb{P}}\Big\{\, \bar{\xi} < \frac{-\ln(x/L) - (\tilde{r} + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \Big\}_{x=V_t} \\
&= N\big(-d_1(V_t, T-t)\big),
\end{aligned}$$

since $\bar{\xi} := (\bar{W}_T - \bar{W}_t)/\sqrt{T-t}$ is independent of the σ-field $\mathcal{F}_t$, and has the standard Gaussian law $N(0,1)$ under $\bar{\mathbb{P}}$. We conclude that

$$J_2 = B^{-1}(t,T) V_t e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big).$$

This completes the derivation of formula (5.4). □

From the proof of Proposition 5.1, we deduce also that

$$D(t,T) = LB(t,T)\, \mathbb{P}^*\{V_T \geq L \,|\, \mathcal{F}_t\} + V_t e^{-\kappa(T-t)}\, \bar{\mathbb{P}}\{V_T < L \,|\, \mathcal{F}_t\}.$$

When $\kappa = 0$, it is not difficult to verify that $\bar{\mathbb{P}}$ is a martingale measure corresponding to the choice of V as a discount factor. In other words, $\bar{\mathbb{P}}$ is equivalent to $\mathbb{P}^*$ and the process B_t/V_t follows a martingale under $\bar{\mathbb{P}}$. Notice that the conditional probabilities of default are:

$$p_t^* = \mathbb{P}^*\{V_T < L \,|\, \mathcal{F}_t\} = N\big(-d_2(V_t, T-t)\big),$$

and

$$\bar{p}_t = \bar{\mathbb{P}}\{V_T < L \,|\, \mathcal{F}_t\} = N\big(-d_1(V_t, T-t)\big).$$

It is customary to refer to p_t^* as the conditional risk-neutral probability of default. When $\kappa = 0$, $\bar{p}_t$ can also be seen as the 'risk-neutral probability of default' (associated with a different choice of the discount factor, however). Merton's valuation formula can be re-expressed as follows:

$$D(t,T) = L_t(1 - p_t^*) + L_t p_t^* \delta_t^* = L_t(1 - p_t^* w_t^*),$$

where $L_t = LB(t,T)$ is the present value of the promised claim (as well as the present value of the exposure at default), and δ_t^* is the conditional risk-neutral expected recovery rate upon default. Specifically,

$$\delta_t^* := \frac{\mathbb{E}_{\mathbb{P}^*}\{V_T \mathbb{1}_{\{V_T < L\}} \,|\, \mathcal{F}_t\}}{L\,\mathbb{P}^*\{V_T < L \,|\, \mathcal{F}_t\}} = \frac{V_t e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big)}{LB(t,T) N\big(-d_2(V_t, T-t)\big)}\,.$$

Recall also that $w_t^* = 1 - \delta_t^*$ is called the conditional risk-neutral expected writedown rate upon default. Let $l_t := L_t/V_t$ stand for the firm's *leverage ratio*. In terms of the process l_t, formula (5.4) becomes

$$\frac{D(t,T)}{L_t} = l_t^{-1} e^{-\kappa(T-t)} N\big(-h_1(l_t, T-t)\big) + N\big(h_2(l_t, T-t)\big), \tag{5.8}$$

where

$$h_{1,2}(l_t, T-t) = \frac{-\ln l_t - \kappa(T-t) \pm \frac{1}{2}\sigma_V^2(T-t)}{\sigma_V \sqrt{T-t}}\,. \tag{5.9}$$

Notice that the quantity l_t gives the 'nominal' value of the firm's leverage ratio. Indeed, L_t represents the default-free value of the firm's debt, as opposed to the actual market value $D(t,T)$ of the firm's debt.

5.2 Hedging of a Corporate Bond

Since Merton's formula can be seen as a variant of the Black-Scholes valuation result, the form of the replicating (self-financing) trading strategy for a defaultable bond can be easily deduced from the well-known expressions for the Black-Scholes hedging strategy for a European put option. For the sake of completeness, we state the following corollary to Proposition 5.1, in which we write $D(t,T) = u(V_t, t)$.

Corollary 5.1 *The unique replicating strategy for a defaultable bond involves holding at any time* $0 \le t < T$ *the* $\phi_t^1 V_t$ *units of cash invested in the firm's value and* $\phi_t^2 B(t,T)$ *units of cash invested in default-free bonds, where*

$$\phi_t^1 = u_V(V_t, t) = e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big)$$

and

$$\phi_t^2 = \frac{D(t,T) - \phi_t^1 V_t}{B(t,T)} = LN\big(d_2(V_t, T-t)\big).$$

5.3 Credit Spreads

An important characteristic of a defaultable bond is the difference between its yield and the yield of an equivalent default-free bond, i.e., the *credit spread.* Recall that the credit spread $S(t,T)$ is defined through the formula $S(t,T) = Y^d(t,T) - Y(t,T)$, where $Y^d(t,T)$ and $Y(t,T)$ are given by:

$$Y(t,T) = -\frac{\ln B(t,T)}{T-t}, \quad Y^d(t,T) = -\frac{\ln D(t,T)}{T-t},$$

In Merton's model the yield on a default-free bond is equal to the short-term interest rate; i.e., $Y(t,T) = r$. Using (5.8) with $L = 1$, we arrive at the following representation for the credit spread in Merton's model

$$S(t,T) = -\frac{\ln\Big(l_t^{-1} e^{-\kappa(T-t)} N\big(-h_1(l_t, T-t)\big) + N\big(h_2(l_t, T-t)\big)\Big)}{T-t}.$$

Let us now analyze the behavior of the credit spread when time converges to the debt's maturity. For this purpose, observe that: $\lim_{t\to T} l_t = L/V_T$,

$$\lim_{t\to T} N\big(-h_1(l_t, T-t)\big) = \begin{cases} 1, & \text{on } \{V_T < L\}, \\ 0, & \text{on } \{V_T > L\}, \end{cases}$$

and

$$\lim_{t\to T} N\big(h_2(l_t, T-t)\big) = \begin{cases} 0, & \text{on } \{V_T < L\}, \\ 1, & \text{on } \{V_T > L\}. \end{cases}$$

The reader can readily verify that

$$\lim_{t\to T} S(t,T) = \begin{cases} +\infty, & \text{on } \{V_T < L\}, \\ 0, & \text{on } \{V_T > L\}. \end{cases} \tag{5.10}$$

An essential feature of Merton's model is that the default time τ appears to be a predictable stopping time with respect to the filtration $\mathbb{F}^V$ generated by the value process V, as it is announced, for instance, by the following sequence of $\mathbb{F}^V$-stopping times:

$$\tau_n = \{\, t \geq T - \tfrac{1}{n} : V_t < L\} \tag{5.11}$$

with the usual convention that $\inf \emptyset = \infty$.

References

[1] Merton, R. (1974) On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance* 29, 449–470.

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6 Zhou's (1996) Model

Zhou (1996) extends Merton's approach by modelling the firm's value process V as a geometric jump-diffusion process.[2] The main purpose of Zhou's study was to address the issue of predictability of the default time τ, inherent in Merton's model (the time of default remains predictable within the so-called simplified version of Zhou's model presented in this section, though). To state Zhou's equation for the dynamics of the value process V, we need to introduce a Poisson process N with the intensity λ under the probability measure $\mathbb{P}^*$ and a sequence of independent identically distributed random variables $(U_i)_{i\geq 1}$ with the finite expected value $\nu = \mathbb{E}_{\mathbb{P}^*}(U_i)$. We assume that the σ-fields generated by the processes W^*, N and the sequence $(U_i)_{i\geq 1}$ are mutually independent under $\mathbb{P}^*$. The equation for the dynamics of V under the risk-neutral measure $\mathbb{P}^*$ now takes the following form:

$$dV_t = V_{t-}\big((r-\lambda\nu)\,dt + \sigma_V\,dW^*_t + d\pi_t\big), \tag{6.1}$$

where π is a jump process whose jump times are specified by the jump times of the Poisson process N, and the size of the i^{th} jump is U_i. In other words, the process π is a marked Poisson process:

$$\pi_t = \sum_{i=1}^{N_t} U_i, \quad \forall\, t\in[0,T].$$

We endow our underlying probability space with the filtration $\mathbb{F}$ generated by processes W^* and π. It is not difficult to check that the compensated process $\tilde{\pi}_t = \pi_t - \lambda\nu t$ is a $\mathbb{P}^*$-martingale with respect to this filtration. Consequently, the process $V^*_t = e^{-rt}V_t$, which is easily seen to satisfy

$$dV^*_t = V^*_{t-}\big(\sigma_V\,dW^*_t + d\tilde{\pi}_t\big), \tag{6.2}$$

also follows a martingale under $\mathbb{P}^*$ with respect to $\mathbb{F}$. Equation (6.2) can be solved explicitly, yielding

$$V^*_t = V^*_0 \exp\big(\tilde{\pi}_t + \sigma_V W^*_t - \tfrac{1}{2}\sigma_V^2 t\big)\prod_{u\leq t}(1+\Delta\tilde{\pi}_u)\exp(-\Delta\tilde{\pi}_u),$$

where $\Delta\tilde{\pi}_u = \tilde{\pi}_u - \tilde{\pi}_{u-}$ or, equivalently,

$$V_t = V_0\exp\big(\sigma_V W^*_t + (r - \tfrac{1}{2}\sigma_V^2 - \lambda\nu)t\big)\prod_{i=1}^{N_t}(1+U_i). \tag{6.3}$$

From now on, we assume, in addition, that $U_i + 1$ has the log-normal distribution under $\mathbb{P}^*$ – that is, $\ln(U_i+1) \sim N(\mu,\sigma)$. This implies that

$$\nu := \mathbb{E}_{\mathbb{P}^*}(U_i) = \exp\big(\mu + \tfrac{1}{2}\sigma^2\big) - 1.$$

[2] Mason and Bhattacharya (1981) use a pure jump process for the firm's value.

The case considered in Sect. 2 of Zhou (1996) corresponds to a defaultable claim with recovery at maturity

$$X = L,\ A \equiv 0,\ \tilde{X} = L(1 - \bar{w}(V_T/L)),\ \tau = T\mathbb{1}_{\{V_T<L\}} + \infty\mathbb{1}_{\{V_T \geq L\}},$$

where $\bar{w} : \mathbb{R}_+ \to \mathbb{R}$, referred to as the *writedown function,* determines the recovery value in case of default. If we choose $\bar{w}(x) = 1 - x$, $\tilde{X}$ reduces to the recovery structure of the original Merton model. In general, we have

$$X^{d,1}(T) = L\mathbb{1}_{\{\tau>T\}} + L(1 - \bar{w}(V_T/L))\mathbb{1}_{\{\tau \leq T\}}$$

or, equivalently,

$$X^{d,1}(T) = L\big(\mathbb{1}_{\{V_T \geq L\}} + \bar{\delta}(V_T/L)\mathbb{1}_{\{V_T<L\}}\big),$$

where $\bar{\delta}(V_T/L) = 1 - \bar{w}(V_T/L)$ is the recovery rate of the defaulted bond. The following auxiliary result establishes the conditional probability law of the default event $\{\tau = T\}$ with respect to the σ-field $\mathcal{F}_t$.

Lemma 6.1 *The risk-neutral conditional probability of default satisfies*

$$\mathbb{P}^*\{\tau = T \,|\, \mathcal{F}_t\} = \sum_{i=0}^{\infty} e^{-\lambda(T-t)} \, \frac{\big(\lambda(T-t)\big)^i}{i!} \, N\big(-d_{2,i}(V_t, T-t)\big),$$

where, for every $i \in \mathbb{N}$ and $t \in \mathbb{R}_+$,

$$d_{2,i}\big(V, t\big) = \frac{\ln(V/L) + \mu_i(t)}{\sigma_i(t)}$$

with

$$\mu_i(t) = (r - \tfrac{1}{2}\sigma_V^2 - \lambda\nu)t + i\mu, \quad \sigma_i^2(t) = \sigma_V^2 t + i\sigma^2.$$

Proof. Obviously $\mathbb{P}^*\{\tau = T \,|\, \mathcal{F}_t\} = \mathbb{P}^*\{V_T < L \,|\, \mathcal{F}_t\}$. In view of the assumed independence of the Brownian motion W^* and the jump component π, it is enough to consider the conditional probability with respect to the number of jumps in the interval $[t, T]$ and to use the formula for the total probability. In view of (6.3), on the set $\{N_T - N_t = i\}$ the random variable V_T can be represented as follows

$$V_T = V_t \exp\Big(\sigma_V(W_T^* - W_t^*) + (r - \tfrac{1}{2}\sigma_V^2 - \lambda\nu)(T-t) + \sum_{j=1}^{i} \zeta_j\Big),$$

where ζ_j, $j = 0, \ldots, i$, are independent identically distributed random variables with the Gaussian law $N(\mu, \sigma)$. In addition, ζ_js are independent of W^*. Put another way, $V_T = V_t\, e^{\zeta}$, where ζ is a Gaussian random variable, independent of $\mathcal{F}_t$, with the expected value:

$$\mathbb{E}_{\mathbb{P}^*}(\zeta) = (r - \tfrac{1}{2}\sigma_V^2 - \lambda\nu)(T-t) + i\mu,$$

and the variance:

$$\mathrm{Var}_{\mathbb{P}^*}(\zeta) = \sigma_V^2(T-t) + i\sigma^2.$$

The above representation for the random variable V_T leads directly to the asserted formula. The details are left to the reader. □

6.1 Defaultable bond

We define the price $D(t,T)$ of a defaultable bond by setting

$$D(t,T) = B_t\, \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} X^{d,1}(T) \,|\, \mathcal{F}_t). \tag{6.4}$$

Due to the presence of the jump component in the dynamics of V, it is clear that an analytical approach to the valuation of defaultable claims in Zhou's framework requires solving an integro-differential PDE involving the infinitesimal generator of V, and this does not seem to be an easy matter. On the other hand, the valuation of a defaultable bond through the probabilistic approach presents no difficulties. Of course, it still remains a problem of validity of formula (6.4), because it is not supported in Zhou's set-up by the existence of a replicating strategy for a defaultable bond. Thus, in contrast to Merton's valuation formula, expression (6.4) should be seen as the formal definition of the price process of a defaultable bond.

Proposition 6.1 *Assume that $\bar{w}(x) = 1 - x$. Then for any $t \in [0,T]$ we have*

$$D(t,T) = LB(t,T)\Big\{1 - \sum_{i=0}^{\infty} e^{-\lambda(T-t)}\, \frac{\big(\lambda(T-t)\big)^i}{i!}\, N\big(-d_{2,i}(V_t, T-t)\big)$$
$$+ \frac{V_t}{L} \sum_{i=0}^{\infty} e^{\mu_i(T-t) + \sigma_i^2(T-t)/2 - \lambda(T-t)}\, \frac{\big(\lambda(T-t)\big)^i}{i!}\, N\big(-d_{1,i}(V_t, T-t)\big)\Big\},$$

where, for every $i \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we denote

$$\mu_i(t) = (r - \tfrac{1}{2}\sigma_V^2 - \lambda\nu)t + i\mu, \quad \sigma_i^2(t) = \sigma_V^2 t + i\sigma^2,$$

and

$$d_{2,i}\big(V_t, t\big) = \frac{\ln(V_t/L) + \mu_i(t)}{\sigma_i(t)}, \quad d_{1,i}(V_t, t) = d_{2,i}(V_t, t) + \sigma_i(t).$$

Proof. It suffices to apply the valuation formula established in Merton (1973). It extends the Black-Scholes formula to the case of a European put option written on a stock, whose price follows a jump-diffusion process given by (6.1). For a more direct proof, notice that $X^{d,1}(T)$ equals:

$$X^{d,1}(T) = L - L\mathbb{1}_{\{V_T < L\}} + V_T \mathbb{1}_{\{V_T < L\}},$$

so that

$$D(t,T) = LB(t,T) - L\mathbb{P}^*\{V_T < L \,|\, \mathcal{F}_t\} + B(t,T)\, \mathbb{E}_{\mathbb{P}^*}(V_T \mathbb{1}_{\{V_T < L\}} \,|\, \mathcal{F}_t).$$

The second term in the last formula can be found using Lemma 6.1. For the last term, it suffices to first condition with respect to the number of jumps of N in the interval $[t,T]$. On the set $\{N_T - N_t = i\}$, we obtain

$$\mathbb{E}_{\mathbb{P}^*}(V_T \mathbb{1}_{\{V_T < L\}} \,|\, \mathcal{F}_t) = V_t\, \mathbb{E}_{\mathbb{P}^*}(e^{\zeta} \mathbb{1}_{\{xe^{\zeta} < L\}})_{\,|x = V_t},$$

where ζ is an auxiliary Gaussian random variable that was introduced in the proof of Lemma 6.1. The valuation formula now follows directly from the elementary Lemma 6.2. □

Lemma 6.2 *Let ζ be a Gaussian random variable under $\mathbb{P}$ with the expected value m and the variance σ^2. Then for any strictly positive x we have*

$$\mathbb{E}_{\mathbb{P}}(e^{\zeta} \mathbb{1}_{\{e^{\zeta} < x\}}) = e^{m + \sigma^2/2} N\left(\frac{\ln x - m - \sigma^2}{\sigma}\right). \tag{6.5}$$

Proof. Equality (6.5) can be established by elementary integration. Alternatively, we can make use of Girsanov's theorem. It is clear that

$$\begin{aligned} I &:= \mathbb{E}_{\mathbb{P}}(e^{\zeta} \mathbb{1}_{\{e^{\zeta}<x\}}) = \mathbb{E}_{\mathbb{P}}(e^{m+\sigma W_1} \mathbb{1}_{\{e^{m+\sigma W_1}<x\}}) \\ &= e^{m+\sigma^2/2}\, \mathbb{E}_{\mathbb{P}}(e^{\sigma W_1-\sigma^2/2} \mathbb{1}_{\{e^{m+\sigma W_1}<x\}}), \end{aligned}$$

where W follows a standard Brownian motion on some filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Let $\tilde{\mathbb{P}}$ be a probability measure, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_1)$, with the following Radon-Nikodým density:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\sigma W_1 - \tfrac{1}{2}\sigma^2\right), \quad \mathbb{P}\text{–a.s.}$$

From Girsanov's theorem, the process $\tilde{W}_t = W_t - \sigma t$ follows a standard Brownian motion under $\tilde{\mathbb{P}}$, and thus

$$\begin{aligned} I &= e^{m+\sigma^2/2}\, \tilde{\mathbb{P}}\{e^{m+\sigma W_1} < x\} = e^{m+\sigma^2/2}\, \tilde{\mathbb{P}}\{e^{m+\sigma \tilde{W}_1+\sigma^2} < x\} \\ &= e^{m+\sigma^2/2}\, \tilde{\mathbb{P}}\{\sigma \tilde{W}_1 < \ln x - m - \sigma^2\}. \end{aligned}$$

This immediately yields (6.5). □

Observe that in the case of no jumps – that is, for $\lambda = 0$ – the formula established in Proposition 6.1 reduces to Merton's result. It is noteworthy that the closed-form expressions for the value of a defaultable bond can also be derived for other natural choices of the writedown function, such as: $\bar{w}(x) = w_0 - w_1 x$, $\bar{w}(x) = \min(1, w_0 - w_1 x)$, etc.

Remarks. We have discussed only a special Merton-like case of Zhou's approach. The general model examined by Zhou (1996) belongs to the class of first-passage-time models that are studied at some length in the next chapter. He postulates that the default time τ is the first passage time of the firm's value to a constant barrier. More specifically,

$$\tau = \inf\{\, t \in [0,T) : V_t \leq \bar{v}\},$$

where $\bar{v} > 0$ is a positive constant. Furthermore, if default occurs prior to the bond's maturity T, the owner receives the payoff $\tilde{X} = L(1 - \bar{w}(V_\tau/L))$ at time T; equivalently, he gets the amount $Z_\tau = B(\tau,T)\tilde{X}$ at default time. An analytical result for the price of a defaultable bond is not available in this set-up; Zhou (1996) provides a tractable way of valuing such a bond, though.

References

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[2] Merton, R.C. (1973) Theory of rational option pricing. *Bell Journal of Economics and Management Sciences* 4, 141–183.

[3] Zhou, C. (1996) A jump-diffusion approach to modeling credit risk and valuing defaultable securities. Working paper, Federal Reserve Board, Washington.

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7 Properties of First Passage Times

We have already briefly discussed the risk-neutral valuation formulae for corporate bonds. It should be stressed that several results in the existing literature rely on the probabilistic approach. For instance, bond valuation formulae in Longstaff and Schwartz (1995) and Saá-Requejo and Santa-Clara (1999) correspond to the following generic expression

$$D(t,T) = B(t,T)\,\mathbb{P}_T\{\tau > T \,|\, \mathcal{F}_t\} + \delta B(t,T)\,\mathbb{P}_T\{\tau \le T \,|\, \mathcal{F}_t\}$$

in which the default time is defined as the first passage time of the value process to a (constant or variable) barrier, and $\mathbb{P}_T$ is the forward martingale probability measure. Direct computations based on the above formula require, of course, the knowledge of conditional distribution of the default time τ with respect to the σ-field $\mathcal{F}_t$. In this lecture, we provide a few classic results related to this issue.

Let us first consider two one-dimensional Itô processes X^1 and X^2 with respective dynamics under the probability measure $\mathbb{P}^*$ given by

$$dX^i_t = X^i_t\big(\mu_i(t)\,dt + \sigma_i(t)\,dW^i_t\big), \quad X^i_0 = x^i > 0, \tag{7.1}$$

for $i = 1, 2$, where W^i, $i = 1, 2$ are mutually independent d-dimensional standard Brownian motions with respect to the underlying filtration $\mathbb{F}$, and $\mu_i : \mathbb{R}_+ \to \mathbb{R}$, $\sigma_i : \mathbb{R}_+ \to \mathbb{R}^d$ are such that the SDEs (7.1) possess unique, strong, global solutions. Let us also assume that $x^1 > x^2$. Frequently, the default time τ is modeled as $\tau = \inf\{\, t \ge 0 : X^1_t \le X^2_t\}$. It is convenient to introduce the log-ratio process $Y_t := \ln(X^1_t/X^2_t)$, so that $\tau = \inf\{\, t \ge 0 : Y_t \le 0\}$. The dynamics of Y are described in the next lemma. Since the proof of Lemma 7.1 relies on a straightforward application of Itô's formula, it is omitted.

Lemma 7.1 *The process Y satisfies*

$$dY_t = \nu(t)\,dt + \sigma_1(t)\,dW^1_t - \sigma_2(t)\,dW^2_t \tag{7.2}$$

with

$$\nu(t) = \mu_1(t) - \mu_2(t) + \tfrac{1}{2}|\sigma_2(t)|^2 - \tfrac{1}{2}|\sigma_1(t)|^2, \tag{7.3}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^d$.

Suppose now that the coefficients μ_i and σ_i, $i = 1, 2$ are constant vectors in $\mathbb{R}^d$. In this case, the process Y follows a Brownian motion with the standard deviation σ and the drift ν, specifically: $dY_t = \nu\,dt + \sigma\,dW^*_t$, $Y_0 = y_0$, where

$$\nu = \mu_1 - \mu_2 + \tfrac{1}{2}|\sigma_2|^2 - \tfrac{1}{2}|\sigma_1|^2, \quad \sigma^2 = |\sigma_1|^2 + |\sigma_2|^2,$$

and W^* is a standard (one-dimensional) Brownian motion under $\mathbb{P}^*$ with respect to $\mathbb{F}$. Put another way:

$$Y_t = y_0 + \sigma W^*_t + \nu t, \quad \forall\, t \in \mathbb{R}_+, \tag{7.4}$$

for some constants $\nu \in \mathbb{R}$ and $\sigma > 0$. Let us notice that Y inherits from W^* a strong Markov property with respect to $\mathbb{F}$.

7.1 Probability Law of the First Passage Time

Let τ stand for the *first passage time to zero* by the process Y, that is, $\tau := \inf\{ t \geq 0 : Y_t = 0\}$. It is well known that in an arbitrarily small time interval $[0,t]$ the sample path of the Brownian motion started at 0 passes through origin infinitely many times (see, for instance, Page 42 in Krylov (1995)). Using Girsanov's theorem and the strong Markov property of the Brownian motion, it is thus easy to deduce that first passage time by Y to zero coincides with the first crossing time by Y of the level 0, that is, with probability 1:

$$\tau = \inf\{ t \geq 0 : Y_t < 0\} = \inf\{ t \geq 0 : Y_t \leq 0\}.$$

The following result is standard.

Lemma 7.2 *Let $\sigma > 0$ and $\nu \in \mathbb{R}$. Then for every $x > 0$ we have*

$$\mathbb{P}^*\big\{ \sup_{0\leq u\leq s} (\sigma W^*_u + \nu u) \leq x\big\} = N\left(\frac{x-\nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x} N\left(\frac{-x-\nu s}{\sigma\sqrt{s}}\right) \tag{7.5}$$

and for every $x < 0$

$$\mathbb{P}^*\big\{ \inf_{0\leq u\leq s} (\sigma W^*_u + \nu u) \geq x\big\} = N\left(\frac{-x+\nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x} N\left(\frac{x+\nu s}{\sigma\sqrt{s}}\right). \tag{7.6}$$

Proof. To derive the first equality, we shall employ Girsanov's theorem and the reflection principle for a Brownian motion. Assume first that $\sigma = 1$. Let $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F}_s)$ given by

$$\frac{d\mathbb{P}}{d\mathbb{P}^*} = e^{-\nu W^*_s - \frac{\nu^2}{2}s}, \quad \mathbb{P}^*\text{-a.s.},$$

so that the process $W_t := X_t = W^*_t + \nu t$, $t \in [0,s]$, follows a standard Brownian motion under $\mathbb{P}$, and

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{\nu W_s - \frac{\nu^2}{2}s}, \quad \mathbb{P}\text{-a.s.}$$

Moreover

$$\mathbb{P}^*\{ \sup_{0\leq u\leq s} (W^*_u + \nu u) > x,\, W^*_s + \nu s \leq x\} = \mathbb{E}_{\mathbb{P}}\Big(e^{\nu W_s - \frac{\nu^2}{2}s} 1\!\!1_{\{\sup_{0\leq u\leq s} W_u > x,\, W_s \leq x\}}\Big).$$

Let us set $\tau_x = \inf\{ t \geq 0 : W_t = x\}$, and let us introduce an auxiliary process $\tilde{W}_t$, $t \in [0,s]$, by setting:

$$\tilde{W}_t = W_t 1\!\!1_{\{\tau_x \geq t\}} + (2x - W_t) 1\!\!1_{\{\tau_x < t\}}.$$

By virtue of the reflection principle, the process $\tilde{W}$ is a standard Brownian motion under $\mathbb{P}$. Moreover, we have

$$\{ \sup_{0\leq u\leq s} \tilde{W}_u > x,\, \tilde{W}_s \leq x\} = \{W_s \geq x\} \subset \{\tau_x \leq s\}.$$

We obtain

$$\mathbb{P}^*\big\{ \sup_{0\leq u\leq s} (W^*_u + \nu u) \leq x\big\} \;=\; \mathbb{P}^*\{W^*_s + \nu s \leq x\} - \mathbb{P}^*\big\{ \sup_{0\leq u\leq s} (W^*_u + \nu u) > x,\, W^*_s + \nu s \leq x\big\}$$

$$\begin{aligned}
&= \mathbb{P}^*\{W_s^* + \nu s \le x\} - \mathbb{E}_{\mathbb{P}}\Big(e^{\nu W_s - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{\sup_{0\le u\le s} W_u > x,\, W_s \le x\}}\Big)\\
&= \mathbb{P}^*\{W_s^* + \nu s \le x\} - \mathbb{E}_{\mathbb{P}}\Big(e^{\nu \tilde W_s - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{\sup_{0\le u\le s} \tilde W_u > x,\, \tilde W_s \le x\}}\Big)\\
&= \mathbb{P}^*\{W_s^* + \nu s \le x\} - \mathbb{E}_{\mathbb{P}}\Big(e^{\nu (2x - W_s) - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W_s \ge x\}}\Big)\\
&= \mathbb{P}^*\{W_s^* + \nu s \le x\} - e^{2\nu x}\,\mathbb{E}_{\mathbb{P}}\Big(e^{\nu W_s - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W_s \le -x\}}\Big)\\
&= \mathbb{P}^*\{W_s^* + \nu s \le x\} - e^{2\nu x}\,\mathbb{P}^*\{W_s^* + \nu s \le -x\}\\
&= N\left(\frac{x-\nu s}{\sqrt{s}}\right) - e^{2\nu x} N\left(\frac{-x-\nu s}{\sqrt{s}}\right).
\end{aligned}$$

This ends the proof of the first equality for $\sigma = 1$. For any $\sigma > 0$ we have

$$\mathbb{P}^*\big\{ \sup_{0\le u\le s} (\sigma W_u^* + \nu u) \le x \big\} = \mathbb{P}^*\big\{ \sup_{0\le u\le s} (W_u^* + \nu\sigma^{-1} u) \le x\sigma^{-1} \big\},$$

and this implies (7.5). Since $-W^*$ follows a standard Brownian motion under $\mathbb{P}^*$, for any $x < 0$ we have

$$\mathbb{P}^*\big\{ \inf_{0\le u\le s} (\sigma W_u^* + \nu u) \ge x \big\} = \mathbb{P}^*\big\{ \sup_{0\le u\le s} (\sigma W_u^* - \nu u) \le -x \big\},$$

and thus the second equality is a simple consequence of the first. □

Proposition 7.1 *Let Y be given by* (7.4)*, where $\nu \in \mathbb{R}$, $\sigma > 0$, and W^* is a standard Brownian motion under $\mathbb{P}^*$. Then the random variable τ has an inverse Gaussian probability distribution under $\mathbb{P}^*$. More specifically, for any $0 < s < \infty$,*

$$\mathbb{P}^*\{\tau \le s\} = \mathbb{P}^*\{\tau < s\} = N(h_1(s)) + e^{-2\nu\sigma^{-2}y_0} N(h_2(s)), \tag{7.7}$$

where N is the standard Gaussian cumulative distribution function, and

$$h_1(s) = \frac{-y_0 - \nu s}{\sigma\sqrt{s}}, \quad h_2(s) = \frac{-y_0 + \nu s}{\sigma\sqrt{s}}.$$

Proof. Notice first that

$$\mathbb{P}^*\{\tau \ge s\} = \mathbb{P}^*\{ \inf_{0\le u\le s} Y_u \ge 0\} = \mathbb{P}^*\{ \inf_{0\le u\le s} X_u \ge -y_0\}, \tag{7.8}$$

where $X_u = \sigma W_u^* + \nu u$. We know from Lemma 7.2 that for every $x < 0$ we have

$$\mathbb{P}^*\{ \inf_{0\le u\le s} X_u \ge x\} = N\left(\frac{-x+\nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x} N\left(\frac{x+\nu s}{\sigma\sqrt{s}}\right).$$

When combined with (7.8), this yields (7.7). □

The following corollary is a consequence of Proposition 7.1 and the strong Markov property of the process Y with respect to the filtration $\mathbb{F}$.

Corollary 7.1 *Under the assumptions of Proposition* 7.1 *for any $t < s$ we have, on the set $\{\tau > t\}$,*

$$\mathbb{P}^*\{\tau \le s \,|\, \mathcal{F}_t\} = N\left(\frac{-Y_t - \nu(s-t)}{\sigma\sqrt{s-t}}\right) + e^{-2\nu\sigma^{-2}Y_t} N\left(\frac{-Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right).$$

We are in a position to apply the foregoing results to specific examples of default times. In the first example, we examine the case of a constant lower threshold.

Example 7.1 Suppose that the short-term interest rate process is constant, i.e., $r_t = r,\ t \geq 0$. In addition, let the value process V follow:

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma_V\,dW_t^*\big) \tag{7.9}$$

with constant coefficients κ and $\sigma_V > 0$. Let us also assume that the barrier process v is constant and equal to $\bar{v}$, where the constant $\bar{v}$ satisfies $\bar{v} < V_0$. We set

$$\tau = \inf\{\, t \geq 0 : V_t \leq \bar{v}\} = \inf\{\, t \geq 0 : V_t < \bar{v}\}.$$

Now, letting $X_t^1 = V_t$ and $X_t^2 = \bar{v}$, so that $Y_t = \ln(V_t/\bar{v})$, and identifying the terms in (7.1), we obtain

$$\mu_1 \equiv r - \kappa, \quad \sigma_1 \equiv \sigma_V, \quad x^1 = V_0$$

and

$$\mu_2 \equiv 0, \quad \sigma_2 \equiv 0, \quad x^2 = \bar{v}.$$

Consequently, $\nu = r - \kappa - \frac{1}{2}\sigma_V^2$ and $\sigma = \sigma_V$ in (7.4). Applying Corollary 7.1, we obtain for every $s > t$, on the set $\{\tau > t\}$,

$$\mathbb{P}^*\{\tau \leq s \,|\, \mathcal{F}_t\} = N\left(\frac{\ln\frac{\bar{v}}{V_t} - \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) + \left(\frac{\bar{v}}{V_t}\right)^{2a} N\left(\frac{\ln\frac{\bar{v}}{V_t} + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right),$$

where

$$a = \frac{\nu}{\sigma_V^2} = \frac{r-\kappa-\frac{1}{2}\sigma_V^2}{\sigma_V^2}. \tag{7.10}$$

The last result was used in Leland and Toft (1996).

Example 7.2 Assume that the value process V and the short-term interest rate r are as in Example 7.1. For a fixed γ, let the barrier function be defined as $\bar{v}(t) = Ke^{-\gamma(T-t)}$ for $t \in \mathbb{R}_+$, so that $\bar{v}(t)$ satisfies

$$d\bar{v}(t) = \gamma\bar{v}(t)\,dt, \quad \bar{v}(0) = Ke^{-\gamma T}.$$

Letting $X_t^1 = V_t$, $X_t^2 = \bar{v}(t)$ and identifying the terms in (7.1), we obtain

$$\mu_1 \equiv r - \kappa, \quad \sigma_1 \equiv \sigma_V, \quad x^1 = \bar{v}(0)$$

and

$$\mu_2 \equiv \gamma, \quad \sigma_2 \equiv 0, \quad x^2 = Ke^{-\gamma T},$$

so that the drift and diffusion coefficients in (7.4) are $\tilde{\nu} \equiv r - \kappa - \gamma - \frac{1}{2}\sigma_V^2$ and $\sigma \equiv \sigma_V$. We define the stopping time τ as $\tau = \inf\{\, t \geq 0 :\ V_t \leq \bar{v}(t)\}$. From Corollary 7.1, we obtain for every $t < s$, on the set $\{\tau > t\}$,

$$\mathbb{P}^*\{\tau \leq s \,|\, \mathcal{F}_t\} = N\left(\frac{\ln\frac{\bar{v}(t)}{V_t} - \tilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right) + \left(\frac{\bar{v}(t)}{V_t}\right)^{2\tilde{a}} N\left(\frac{\ln\frac{\bar{v}(t)}{V_t} + \tilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right),$$

where

$$\tilde{a} = \frac{\tilde{\nu}}{\sigma_V^2} = \frac{r-\kappa-\gamma-\frac{1}{2}\sigma_V^2}{\sigma_V^2}. \tag{7.11}$$

The last formula was used in Black and Cox (1976).

7.2 Joint Probability Law of Y and τ

We shall now establish the joint law of Y and τ. To be more specific, we shall find, for every $y \geq 0$,

$$I := \mathbb{P}^*\{Y_s \geq y,\, \tau \geq s \,|\, \mathcal{F}_t\} = \mathbb{P}^*\{Y_s \geq y,\, \tau > s \,|\, \mathcal{F}_t\},$$

where $\tau = \inf\{\, t \geq 0 : Y_t \leq 0\} = \inf\{\, t \geq 0 : Y_t < 0\}$. Given a one-dimensional standard Brownian motion W^* under $\mathbb{P}^*$, and let us denote by $M_s^{W^*}$ and $m_s^{W^*}$ the running maximum and minimum, respectively. More explicitly, $M_s^{W^*} = \sup_{0 \leq u \leq s} W_u^*$ and $m_s^{W^*} = \inf_{0 \leq u \leq s} W_u^*$. It is well known that for every $s > 0$ we have

$$\mathbb{P}^*\{M_s^{W^*} > 0\} = 1, \quad \mathbb{P}^*\{m_s^{W^*} < 0\} = 1. \tag{7.12}$$

The following well-known result, commonly referred to as the *reflection principle,* is a straightforward consequence of the strong Markov property of the Brownian motion.

Lemma 7.3 *The following equality:*

$$\mathbb{P}^*\{W_s^* \leq x,\, M_s^{W^*} \geq y\} = \mathbb{P}^*\{W_s^* \geq 2y - x\} = \mathbb{P}^*\{W_s^* \leq x - 2y\}$$

is valid for every $s > 0$, $y \geq 0$ and $x \leq y$.

We need to examine the case of a slightly more general process – namely, a Brownian motion with non-zero drift. Consider the process X that equals $X_t = \nu t + \sigma W^*{}_t$. We write $M_s^X = \sup_{0 \leq u \leq s} X_u$ and $m_s^X = \inf_{0 \leq u \leq s} X_u$. By virtue of Girsanov's theorem, the process X is a Brownian motion (up to an appropriate rescaling) under an equivalent probability measure and thus (cf. (7.12))

$$\mathbb{P}^*\{M_s^X > 0\} = 1, \quad \mathbb{P}^*\{m_s^X < 0\} = 1,$$

for every $s > 0$.

Lemma 7.4 *For every $s > 0$, the joint distribution of X_s and M_s^X is given by the formula*

$$\mathbb{P}^*\{X_s \leq x,\, M_s^X \geq y\} = e^{2\nu y \sigma^{-2}}\, \mathbb{P}^*\{X_s \geq 2y - x + 2\nu s\}, \tag{7.13}$$

for every $x, y \in \mathbb{R}$ such that $y \geq 0$ and $x \leq y$.

Proof. Since

$$I := \mathbb{P}^*\{X_s \leq x,\, M_s^X \geq y\} = \mathbb{P}^*\{X_s^\sigma \leq x\sigma^{-1},\, M_s^{X^\sigma} \geq y\sigma^{-1}\},$$

where $X_t^\sigma = W^*{}_t + \nu t \sigma^{-1}$, it is clear that we may assume, without loss of generality, that $\sigma = 1$. It is convenient to employ the technique of an equivalent change of probability measure. It follows from Girsanov's theorem that X is a standard Brownian motion under the probability measure $\mathbb{P}$, which is given on $(\Omega, \mathcal{F}_s)$ by setting (recall that $\sigma = 1$)

$$\frac{d\mathbb{P}}{d\mathbb{P}^*} = e^{-\nu W^*{}_s - \frac{\nu^2}{2} s}, \quad \mathbb{P}^*\text{-a.s.}$$

Notice also that

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{\nu W_s - \frac{\nu^2}{2} s}, \quad \mathbb{P}\text{-a.s.},$$

where $W_t := X_t = W_t^* + \nu t,\ t \in [0,s]$, follows a standard Brownian motion under $\mathbb{P}$, and

$$I = \mathbb{E}_{\mathbb{P}}\Big(e^{\nu W_s - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{X_s \le x,\, M_s^X \ge y\}}\Big) = \mathbb{E}_{\mathbb{P}}\Big(e^{\nu W_s - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{W_s \le x,\, M_s^W \ge y\}}\Big).$$

Since W is a standard Brownian motion under $\mathbb{P}$, an application of the reflection principle (7.3) gives

$$\begin{aligned} I &= \mathbb{E}_{\mathbb{P}}\Big(e^{\nu(2y-W_s) - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{2y-W_s \le x,\, M_s^W \ge y\}}\Big) \\ &= \mathbb{E}_{\mathbb{P}}\Big(e^{\nu(2y-W_s) - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{W_s \ge 2y-x\}}\Big) \\ &= e^{2\nu y}\,\mathbb{E}_{\mathbb{P}}\Big(e^{-\nu W_s - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{W_s \ge 2y-x\}}\Big), \end{aligned}$$

since clearly $2y - x \ge y$. Let us define still another equivalent probability measure $\tilde{\mathbb{P}}$ by setting

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\nu W_s - \frac{\nu^2}{2}s}, \quad \mathbb{P}\text{-a.s.}$$

Is is clear that

$$I = e^{2\nu y}\,\mathbb{E}_{\mathbb{P}}\Big(e^{-\nu W_s - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{W_s \ge 2y-x\}}\Big) = e^{2\nu y}\,\tilde{\mathbb{P}}\{W_s \ge 2y - x\}.$$

Furthermore, the process $\tilde{W}_t = W_t + \nu t,\ t \in [0,s]$, follows a standard Brownian motion under $\tilde{\mathbb{P}}$ and we have:

$$I = e^{2\nu y}\,\tilde{\mathbb{P}}\{\tilde{W}_s + \nu s \ge 2y - x + 2\nu s\}.$$

The last equality easily yields (7.13). □

It is worthwhile to observe that (a similar remark applies to all formulae below)

$$\mathbb{P}^*\{X_s \le x,\, M_s^X \ge y\} = \mathbb{P}\{X_s < x,\, M_s^X > y\}.$$

The following result is a straightforward consequence of Lemma 7.4.

Proposition 7.2 *For every $x, y \in \mathbb{R}$ which satisfy $y \ge 0$ and $x \le y$, we have*

$$\mathbb{P}^*\{X_s \le x,\, M_s^X \ge y\} = e^{2\nu y\sigma^{-2}}\, N\left(\frac{x - 2y - \nu s}{\sigma\sqrt{s}}\right). \tag{7.14}$$

Hence,

$$\mathbb{P}^*\{X_s \le x,\, M_s^X \le y\} = N\left(\frac{x - \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y\sigma^{-2}}\, N\left(\frac{x - 2y - \nu s}{\sigma\sqrt{s}}\right) \tag{7.15}$$

for every $x, y \in \mathbb{R}$ such that $x \le y$ and $y \ge 0$.

Proof. For the first equality, note that

$$\mathbb{P}^*\{X_s \ge 2y - x + 2\nu s\} = \mathbb{P}^*\{-\sigma W_s^* \le x - 2y - \nu s\} = N\left(\frac{x - 2y - \nu s}{\sigma\sqrt{s}}\right),$$

since $-\sigma W_t^*$ has Gaussian law with zero mean and variance $\sigma^2 t$. For (7.15), it is enough to observe that

$$\mathbb{P}^*\{X_s \le x,\, M_s^X \le y\} + \mathbb{P}\{X_s \le x,\, M_s^X \ge y\} = \mathbb{P}\{X_s \le x\}$$

and to apply (7.14). This completes the proof. □

It is clear that

$$\mathbb{P}^*\{M_s^X \geq y\} = \mathbb{P}^*\{X_s \geq y\} + \mathbb{P}^*\{X_s \leq y,\, M_s^X \geq y\}$$

for every $y \geq 0$, and thus

$$\mathbb{P}^*\{M_s^X \geq y\} = \mathbb{P}^*\{X_s \geq y\} + e^{2\nu y \sigma^{-2}}\, \mathbb{P}^*\{X_s \geq y + 2\nu s\}. \tag{7.16}$$

Consequently,

$$\mathbb{P}^*\{M_s^X \leq y\} = 1 - \mathbb{P}^*\{M_s^X \geq y\} = \mathbb{P}^*\{X_s \leq y\} - e^{2\nu y \sigma^{-2}}\, \mathbb{P}^*\{X_s \geq y + 2\nu s\}.$$

This leads to the following corollary.

Corollary 7.2 *The following formula holds for every $y \geq 0$*

$$\mathbb{P}^*\{M_s^X \leq y\} = N\left(\frac{y - \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}}\, N\left(\frac{-y - \nu s}{\sigma\sqrt{s}}\right). \tag{7.17}$$

Let us now focus on the law of the minimal value of X. Observe that for any $y \leq 0,$ we have

$$\mathbb{P}^*\{\sup_{0\leq u\leq s}(\sigma W_u^* - \nu u) \geq -y\} = \mathbb{P}^*\{\inf_{0\leq u\leq s}(-\sigma W_u^* + \nu u) \leq y\} = \mathbb{P}^*\{\inf_{0\leq u\leq s} X_u \leq y\},$$

where the last equality follows from the symmetry of the Brownian motion. Consequently, for every $y \leq 0$ we have $\mathbb{P}^*\{m_s^X \leq y\} = \mathbb{P}\{M_s^{\tilde{X}} \geq -y\},$ where the process $\tilde{X}$ equals $\tilde{X}_t = \sigma W_t^* - \nu t$. The following result is thus not difficult to prove.

Proposition 7.3 *For every $s > 0,$ the joint distribution of (X_s, m_s^X) satisfies*

$$\mathbb{P}^*\{\, X_s \geq x,\, m_s^X \geq y\} = N\left(\frac{-x + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}}\, N\left(\frac{2y - x + \nu s}{\sigma\sqrt{s}}\right)$$

for every $x, y \in \mathbb{R}$ such that $y \leq 0$ and $y \leq x$.

Corollary 7.3 *The following formula is valid for every $y \leq 0$*

$$\mathbb{P}^*\{m_s^X \geq y\} = N\left(\frac{-y + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}}\, N\left(\frac{y + \nu s}{\sigma\sqrt{s}}\right).$$

Recall that we denote $Y_t = y_0 + X_t$, where $X_t = \nu t + \sigma W_t^*$. We write

$$m_s^X = \inf_{0\leq u\leq s} X_u, \quad m_s^Y = \inf_{0\leq u\leq s} Y_u.$$

Corollary 7.4 *For any $s > 0$ and $y \geq 0$ we have*

$$\mathbb{P}^*\{Y_s \geq y,\, \tau \geq s\} = N\left(\frac{-y + y_0 + \nu s}{\sigma\sqrt{s}}\right) - e^{-2\nu \sigma^{-2} y_0}\, N\left(\frac{-y - y_0 + \nu s}{\sigma\sqrt{s}}\right).$$

Proof. Since

$$\mathbb{P}^*\{Y_s \geq y,\, \tau \geq s\} = \mathbb{P}^*\{\, Y_s \geq y,\, m_s^Y \geq 0\} = \mathbb{P}^*\{\, X_s \geq y - y_0,\, m_s^X \geq -y_0\},$$

the formula is obvious. □

More generally, the Markov property of Y justifies the following result.

Lemma 7.5 *Under the assumptions of Proposition* 7.1, *for any* $t < s$ *and* $y \geq 0$ *we have, on the set* $\{\tau > t\}$,

$$\mathbb{P}^*\{Y_s \geq y,\, \tau \geq s \,|\, \mathcal{F}_t\} = N\left(\frac{-y + Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right) - e^{-2\nu\sigma^{-2}Y_t} N\left(\frac{-y - Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right).$$

Example 7.3 Assume, as before, that the dynamics of V are

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma_V\, dW_t^*\big) \tag{7.18}$$

and $\tau = \inf\{\, t \geq 0 : V_t \leq \bar{v}\} = \inf\{\, t \geq 0 : V_t < \bar{v}\}$, where the constant $\bar{v}$ satisfies $\bar{v} < V_0$. By applying Lemma 7.5 to $Y_t = \ln(V_t/\bar{v})$ and $y = \ln(x/\bar{v})$, we obtain the following result, which is valid for $x \geq \bar{v}$, on the set $\{\tau > t\}$,

$$\begin{aligned}\mathbb{P}^*\{V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t\} &= N\left(\frac{\ln(V_t/x) + \nu(s-t)}{\sigma\sqrt{s-t}}\right)\\ &- \left(\frac{\bar{v}}{V_t}\right)^{2a} N\left(\frac{\ln \bar{v}^2 - \ln(xV_t) + \nu(s-t)}{\sigma\sqrt{s-t}}\right),\end{aligned}$$

where $\nu = r - \kappa - \frac{1}{2}\sigma_V^2$ and $a = \nu\sigma_V^{-2}$.

Example 7.4 Assume that V satisfies (7.18) and that the barrier function equals $\bar{v}(t) = Ke^{-\gamma(T-t)}$ for some positive constant K. Using again Lemma 7.5, but this time with $Y_t = \ln(V_t/\bar{v}(t))$ and $y = \ln(x/\bar{v}(s))$, we find that for every $t < s \leq T$ and $x \geq \bar{v}(s)$ we have, on the set $\{\tau > t\}$,

$$\begin{aligned}\mathbb{P}^*\{V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t\} &= N\left(\frac{\ln(V_t/\bar{v}(t)) - \ln(x/\bar{v}(s)) + \tilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right)\\ &- \left(\frac{\bar{v}(t)}{V_t}\right)^{2\tilde{a}} N\left(\frac{-\ln(V_t/\bar{v}(t)) - \ln(x/\bar{v}(s)) + \tilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right),\end{aligned}$$

where $\tilde{\nu} = r - \kappa - \gamma - \frac{1}{2}\sigma_V^2$ and $\tilde{a} = \tilde{\nu}\sigma_V^{-2}$. Upon simplification, this yields

$$\mathbb{P}^*\{V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t\} = N\left(\frac{\ln(V_t/x) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) - \left(\frac{\bar{v}(t)}{V_t}\right)^{2\tilde{a}} N\left(\frac{\ln \bar{v}^2(t) - \ln(xV_t) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right),$$

where $\nu = r - \kappa - \frac{1}{2}\sigma_V^2$. In particular, by setting $t = 0$ and $s = T$, we obtain for $x \geq \bar{v}(T)$

$$\mathbb{P}^*\{V_T \geq x,\, \tau \geq T\} = N\left(\frac{\ln(V_0/x) + \nu T}{\sigma_V\sqrt{T}}\right) - \left(\frac{\bar{v}(0)}{V_0}\right)^{2\tilde{a}} N\left(\frac{\ln \bar{v}^2(0) - \ln(xV_0) + \nu T}{\sigma_V\sqrt{T}}\right).$$

Remarks. Notice that if we take $x = \bar{v}(s) = Ke^{-\gamma(T-s)}$, then clearly

$$1 - \mathbb{P}^*\{V_s \geq \bar{v}(s),\, \tau \geq s \,|\, \mathcal{F}_t\} = \mathbb{P}^*\{\tau < s \,|\, \mathcal{F}_t\} = \mathbb{P}^*\{\tau \leq s \,|\, \mathcal{F}_t\}.$$

On the other hand, we have

$$1 - N\left(\frac{\ln(V_t/\bar{v}(s)) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) = N\left(\frac{\ln(\bar{v}(t)/V_t) - \tilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right)$$

and

$$N\left(\frac{\ln \bar{v}^2(t) - \ln(\bar{v}(s)V_t) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) = N\left(\frac{\ln(\bar{v}(t)/V_t) + \tilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right).$$

Notice that by setting $x = \bar{v}(s)$ we rediscover the formula previously established in Example 7.2.

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WARSAW UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 8

8 Black and Cox (1976) Model

The original Merton model does not allow for a premature default, in the sense that the default may only occur at the maturity of the claim. Several authors put forward structural-type models in which this restrictive and unrealistic feature is relaxed. In most of these models, the time of default is given as the first passage time of the value process V to a deterministic or random barrier. The default may thus occur at any time before or on the bond's maturity date T. The challenge here is to appropriately specify the lower threshold v, the recovery process Z, and to compute the corresponding functional that appears on the right-hand side of the risk-neutral valuation formula:

$$X^d(t,T) := B_t \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{F}_t \Big).$$

As one might easily guess, this is a non-trivial problem, in general. In addition, the practical problem of the lack of direct observations of the value process V largely limits the applicability of the first-passage-time models. The aim of this lecture is to present the first-passage-time structural model put forward by Black and Cox (1976). As a rule, the default time is denoted by τ; the symbols $\bar{\tau}$, $\hat{\tau}$ and $\tilde{\tau}$ being reserved to some auxiliary random times.

8.1 Corporate Zero-Coupon Bond

Black and Cox (1976) extend Merton's (1974) research in several directions. In particular, they make account for specific features of debt contracts as: safety covenants, debt subordination, and restrictions on the sale of assets. They assume that the firm's stockholders (or bondholders) receive a continuous dividend payment, proportional to the current value of the firm. Specifically, they postulate that

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma_V \, dW_t^*\big), \tag{8.1}$$

where the constant $\kappa \geq 0$ represents the payout ratio, and $\sigma_V > 0$ is the constant volatility coefficient. The short-term interest rate is assumed to be non-random, specifically, $r_t = r$, where r is a constant. This means that the interest rate risk is disregarded in the original Black and Cox (1976) model.

8.1.1 Safety covenants

Let us first focus on the safety covenants in the firm's indenture provisions. Generally speaking, safety covenants provide the firm's bondholders with the right to force the firm to bankruptcy or reorganization if the firm is doing poorly according to a set standard. The standard for a poor performance is set in Black and Cox (1976) in terms of a time-dependent deterministic barrier $\bar{v}(t) = Ke^{-\gamma(T-t)}$, $t \in [0,T)$, for some constant $K > 0$. They postulate that as soon as the value of firm's assets crosses this lower threshold, the bondholders take over the firm.

Otherwise, default takes place at debt's maturity or not depending on whether $V_T < L$ or not. Let us set:

$$v_t = \begin{cases} \bar{v}(t), & \text{for } t < T, \\ L, & \text{for } t = T. \end{cases} \tag{8.2}$$

The default event occurs at the first time $t \in [0, T]$ at which the firm's value V_t falls below the level v_t, or the default event does not occur at all. The default time τ thus equals (as usual, $\inf \emptyset = +\infty$):

$$\tau = \inf \{ t \in [0, T] : V_t < v_t \}.$$

The recovery process Z and the recovery payoff $\tilde{X}$ are proportional to the value process, specifically, $Z \equiv \beta_2 V$ and $\tilde{X} = \beta_1 V_T$ for some constants $\beta_1, \beta_2 \in [0, 1]$. The classic case examined by Black and Cox (1976) corresponds to $\beta_1 = \beta_2 = 1$. To summarize, we consider the following model:

$$X = L, \ A \equiv 0, \ Z \equiv \beta_2 V, \ \tilde{X} = \beta_1 V_T, \ \tau = \bar{\tau} \wedge \hat{\tau},$$

where the *early default time* $\bar{\tau}$ equals

$$\bar{\tau} = \inf \{ t \in [0, T) : V_t < \bar{v}(t) \},$$

and $\hat{\tau}$ stands for Merton's default time: $\hat{\tau} = T \mathbb{1}_{\{V_T < L\}} + \infty \mathbb{1}_{\{V_T \geq L\}}$.

Remarks. Assume that $V_0 > \bar{v}(0)$. It is important to notice that since the process V satisfies (8.1) and $\bar{v}$ is a smooth function, $\bar{\tau}$ is also the first passage time of the value process V to the deterministic barrier $\bar{v}$, specifically,

$$\bar{\tau} = \inf \{ t \in [0, T) : V_t \leq \bar{v}(t) \} = \inf \{ t \in [0, T) : V_t = \bar{v}(t) \}.$$

The choice of a strict or large inequality in the definition of the early default time $\bar{\tau}$ is thus a matter of convention. The same observation applies to other examples of first-passage-time structural models considered in the sequel.

In addition, we postulate that $\bar{v}(t) \leq LB(t, T)$ or, more explicitly,

$$Ke^{-\gamma(T-t)} \leq Le^{-r(T-t)}, \quad \forall \, t \in [0, T], \tag{8.3}$$

so that, in particular, $K \leq L$. Condition (8.3) ensures that the payoff to the bondholder at the default time τ never exceeds the face value of debt, discounted at a risk-free rate. Since the interest rate r is assumed to be constant, the pricing function $u = u(V, t)$ of a defaultable bond solves the following PDE:

$$u_t(V, t) + (r - \kappa) V u_V(V, t) + \tfrac{1}{2} \sigma_V^2 V^2 u_{VV}(V, t) - ru(V, t) = 0$$

with the boundary condition $u(Ke^{-\gamma(T-t)}, t) = \beta_2 K e^{-\gamma(T-t)}$ and the terminal condition $u(V, T) = \min(\beta_1 V, L)$. To find an explicit solution to this problem, we prefer to rely on a probabilistic approach, though. To this end, we notice that for any $t < T$ the price $D(t, T) = u(V_t, t)$ of a defaultable bond admits the following probabilistic representation, on the set $\{\tau > t\} = \{\bar{\tau} > t\}$

$$\begin{aligned} D(t, T) = {} & \mathbb{E}_{\mathbb{P}^*} \Big(Le^{-r(T-t)} \mathbb{1}_{\{\bar{\tau} \geq T, \, V_T \geq L\}} \, \Big| \, \mathcal{F}_t \Big) \\ & + \mathbb{E}_{\mathbb{P}^*} \Big(\beta_1 V_T e^{-r(T-t)} \mathbb{1}_{\{\bar{\tau} \geq T, \, V_T < L\}} \, \Big| \, \mathcal{F}_t \Big) \\ & + \mathbb{E}_{\mathbb{P}^*} \Big(K \beta_2 e^{-\gamma(T-\bar{\tau})} e^{-r(\bar{\tau}-t)} \mathbb{1}_{\{t < \bar{\tau} < T\}} \, \Big| \, \mathcal{F}_t \Big). \end{aligned}$$

After default – that is, on the set $\{\tau \le t\} = \{\bar\tau \le t\}$, we clearly have

$$D(t,T) = \beta_2 \bar v(\tau) B^{-1}(\tau,T) B(t,T) = K\beta_2 e^{-\gamma(T-\tau)} e^{r(t-\tau)}.$$

The first two conditional expectations in the valuation formula for defaultable bond can be computed by using the formula for the conditional probability $\mathbb{P}^*\{V_s \ge x,\, \tau \ge s \,|\, \mathcal{F}_t\}$, established in Example 1.4 of Lecture 4. To evaluate the third conditional expectation, we shall employ the conditional probability law of the first passage time of the process V to the barrier $\bar v(t)$ – this law was already found in Example 1.2 of Lecture 4. We are thus in a position to establish the following valuation result, due to Black and Cox (1976). Recall that we denote:

$$\nu = r - \kappa - \tfrac{1}{2}\sigma_V^2, \quad \tilde\nu = \nu - \gamma = r - \kappa - \gamma - \tfrac{1}{2}\sigma_V^2,$$

and $\tilde a = \tilde\nu \sigma_V^{-2}$. For the sake of brevity, in the statement and the proof of Proposition 8.1 we shall write σ instead of σ_V.

Proposition 8.1 *Assume that $\tilde\nu^2 + 2\sigma^2(r-\gamma) > 0$. Then the price process $D(t,T) = u(V_t,t)$ of a defaultable bond equals, on the set $\{\tau > t\}$,*

$$\begin{aligned} D(t,T) &= LB(t,T)\big(N\big(h_1(V_t,T-t)\big) - R_t^{2\tilde a} N\big(h_2(V_t,T-t)\big)\big) \\ &+ \beta_1 V_t e^{-\kappa(T-t)}\big(N\big(h_3(V_t,T-t)\big) - N\big(h_4(V_t,T-t)\big)\big) \\ &+ \beta_1 V_t e^{-\kappa(T-t)} R_t^{2\tilde a+2}\big(N\big(h_5(V_t,T-t)\big) - N\big(h_6(V_t,T-t)\big)\big) \\ &+ \beta_2 V_t\big(R_t^{\theta+\zeta} N\big(h_7(V_t,T-t)\big) + R_t^{\theta-\zeta} N\big(h_8(V_t,T-t)\big)\big), \end{aligned}$$

where $R_t = \bar v(t)/V_t$,

$$\theta = \tilde a + 1, \quad \zeta = \sigma^{-2}\sqrt{\tilde\nu^2 + 2\sigma^2(r-\gamma)}$$

and

$$\begin{aligned} h_1(V_t,T-t) &= \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\ h_2(V_t,T-t) &= \frac{\ln \bar v^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\ h_3(V_t,T-t) &= \frac{\ln(L/V_t) - (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ h_4(V_t,T-t) &= \frac{\ln(K/V_t) - (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ h_5(V_t,T-t) &= \frac{\ln \bar v^2(t) - \ln(LV_t) + (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ h_6(V_t,T-t) &= \frac{\ln \bar v^2(t) - \ln(KV_t) + (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ h_7(V_t,T-t) &= \frac{\ln(\bar v(t)/V_t) + \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\ h_8(V_t,T-t) &= \frac{\ln(\bar v(t)/V_t) - \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

Before proceeding to the proof of Proposition 8.1, we state an elementary lemma.

Lemma 8.1 *For any $a \in \mathbb{R}$ and $b > 0$ we have, for every $y > 0$,*

$$\int_0^y x\, dN\left(\frac{\ln x + a}{b}\right) = e^{\frac{1}{2}b^2 - a}\, N\left(\frac{\ln y + a - b^2}{b}\right) \tag{8.4}$$

and

$$\int_0^y x\, dN\left(\frac{-\ln x + a}{b}\right) = e^{\frac{1}{2}b^2 + a}\, N\left(\frac{-\ln y + a + b^2}{b}\right). \tag{8.5}$$

Let a, b, $c \in \mathbb{R}$ satisfy $b < 0$ and $c^2 > 2a$. Then for every $y > 0$

$$\int_0^y e^{ax}\, dN\left(\frac{b - cx}{\sqrt{x}}\right) = \frac{d + c}{2d}\, g(y) + \frac{d - c}{2d}\, h(y), \tag{8.6}$$

where $d = \sqrt{c^2 - 2a}$ and

$$g(y) = e^{b(c-d)}\, N\left(\frac{b - dy}{\sqrt{y}}\right), \quad h(y) = e^{b(c+d)}\, N\left(\frac{b + dy}{\sqrt{y}}\right).$$

Proof. The proof of (8.4)–(8.5) is standard. For (8.6), observe that

$$f(y) := \int_0^y e^{ax}\, dN\left(\frac{b - cx}{\sqrt{x}}\right) = \int_0^y e^{ax}\, n\left(\frac{b - cx}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} - \frac{c}{2\sqrt{x}}\right) dx,$$

where n is the probability density function of the standard Gaussian law. On the other hand,

$$\begin{aligned} g'(x) &= e^{b(c - \sqrt{c^2 - 2a})}\, n\left(\frac{b - \sqrt{c^2 - 2a}x}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} - \frac{\sqrt{c^2 - 2a}}{2\sqrt{x}}\right) \\ &= e^{ax}\, n\left(\frac{b - cx}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} - \frac{d}{2\sqrt{x}}\right) \end{aligned}$$

and

$$\begin{aligned} h'(x) &= e^{b(c + \sqrt{c^2 - 2a})}\, n\left(\frac{b + \sqrt{c^2 - 2a}x}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} + \frac{\sqrt{c^2 - 2a}}{2\sqrt{x}}\right) \\ &= e^{ax}\, n\left(\frac{b - cx}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} + \frac{d}{2\sqrt{x}}\right). \end{aligned}$$

Consequently,

$$g'(x) + h'(x) = -e^{ax}\frac{b}{x^{3/2}}\, n\left(\frac{b - cx}{\sqrt{x}}\right)$$

and

$$g'(x) - h'(x) = -e^{ax}\frac{d}{x^{1/2}}\, n\left(\frac{b - cx}{\sqrt{x}}\right).$$

Thus, f can be represented as follows:

$$f(y) = \frac{1}{2}\int_0^y \left(g'(x) + h'(x) + \frac{c}{d}\left(g'(x) - h'(x)\right)\right) dx.$$

Since $\lim_{y\to 0+} g(y) = \lim_{y\to 0+} h(y) = 0$, we conclude that for every $y > 0$ we have

$$f(y) = \frac{1}{2}(g(y) + h(y)) + \frac{c}{2d}\,(g(y) - h(y)).$$

This end the proof of the lemma. □

Proof of Proposition 8.1. Since the proof relies on calculations that are rather standard, though lengthy, we shall merely sketch the proof. We need to find the following conditional expectations:

$$\begin{aligned}
D_1(t,T) &= LB(t,T)\,\mathbb{P}^*\{V_T \ge L,\, \bar\tau \ge T \,|\, \mathcal{F}_t\},\\
D_2(t,T) &= \beta_1 B(t,T)\,\mathbb{E}_{\mathbb{P}^*}\big(V_T \mathbb{1}_{\{V_T<L,\,\bar\tau\ge T\}} \,\big|\, \mathcal{F}_t\big),\\
D_3(t,T) &= K\beta_2 B_t e^{-\gamma T}\,\mathbb{E}_{\mathbb{P}^*}\big(e^{(\gamma-r)\bar\tau}\mathbb{1}_{\{t<\bar\tau<T\}} \,\big|\, \mathcal{F}_t\big).
\end{aligned}$$

For the sake of notational convenience, we set $t = 0$. Let us first evaluate $D_1(0,T)$ – that is, the part of the bond's value corresponding to no-default event. From Example 1.4 of Lecture 4, we know that if $L \ge \bar v(T) = K$ then

$$\mathbb{P}^*\{V_T \ge L,\, \bar\tau \ge T\} = N\left(\frac{\ln\frac{V_0}{L} + \nu T}{\sigma\sqrt{T}}\right) - R_0^{2\tilde a} N\left(\frac{\ln\frac{\bar v^2(0)}{LV_0} + \nu T}{\sigma\sqrt{T}}\right)$$

with $R_0 = \bar v(0)/V_0$. It is thus clear that

$$D_1(0,T) = LB(0,T)\big(N\big(h_1(V_0,T)\big) - R_0^{2\tilde a}N\big(h_2(V_0,T)\big)\big).$$

Let us now examine $D_2(0,T)$ – that is, the part of the bond's value associated with default at time T. It is clear that

$$\frac{D_2(0,T)}{\beta_1 B(0,T)} = \mathbb{E}_{\mathbb{P}^*}\big(V_T\mathbb{1}_{\{V_T<L,\,\bar\tau\ge T\}}\big) = \int_K^L x\,d\mathbb{P}^*\{V_T < x,\, \bar\tau \ge T\}.$$

Using again Example 1.4 of Lecture 4 and the fact that $\mathbb{P}^*\{\bar\tau \ge T\}$ does not depend on x, we get, for every $x \ge K$,

$$d\mathbb{P}^*\{V_T < x,\, \bar\tau \ge T\} = dN\left(\frac{\ln\frac{x}{V_0} - \nu T}{\sigma\sqrt{T}}\right) + R_0^{2\tilde a}\, dN\left(\frac{\ln\frac{\bar v^2(0)}{xV_0} + \nu T}{\sigma\sqrt{T}}\right).$$

Let us denote

$$K_1(0) = \int_K^L x\,dN\left(\frac{\ln x - \ln V_0 - \nu T}{\sigma\sqrt{T}}\right)$$

and

$$K_2(0) = \int_K^L x\,dN\left(\frac{2\ln\bar v(0) - \ln x - \ln V_0 + \nu T}{\sigma\sqrt{T}}\right).$$

Using (8.4)–(8.5), we obtain

$$K_1(0) = V_0 e^{(r-\kappa)T}\left(N\left(\frac{\ln\frac{L}{V_0} - \hat\nu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{K}{V_0} - \hat\nu T}{\sigma\sqrt{T}}\right)\right),$$

where $\hat{\nu} = \nu + \sigma^2 = r - \kappa + \frac{1}{2}\sigma^2$, and

$$K_2(0) = V_0 R_0^2 e^{(r-\kappa)T} \left(N\left(\frac{\ln \frac{\bar{v}^2(0)}{LV_0} + \hat{\nu}T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln \frac{\bar{v}^2(0)}{KV_0} + \hat{\nu}T}{\sigma\sqrt{T}} \right) \right).$$

Since

$$D_2(0,T) = \beta_1 B(0,T)\big(K_1(0) + R_0^{\tilde{a}} K_2(0)\big),$$

we conclude that

$$\begin{aligned} D_2(0,T) &= \beta_1 V_0 e^{-\kappa T}\big(N\big(h_3(V_0,T)\big) - N\big(h_4(V_0,T)\big)\big) \\ &\quad + \beta_1 V_0 e^{-\kappa T} R_0^{2\tilde{a}+2}\big(N\big(h_5(V_0,T)\big) - N\big(h_6(V_0,T)\big)\big). \end{aligned}$$

It remains to find $D_3(0,T)$ – that is, the part of bond's value associated with the possibility of forced bankruptcy before the bond's maturity date T. To this end, it is enough to calculate the following expected value

$$\bar{v}(0)\,\mathbb{E}_{\mathbb{P}^*}\left(e^{(\gamma-r)\bar{\tau}}\mathbb{1}_{\{\bar{\tau}<T\}}\right) = \bar{v}(0)\int_0^T e^{(\gamma-r)s}\,d\mathbb{P}^*\{\bar{\tau} \le s\},$$

where (see Example 1.2 of Lecture 4)

$$\mathbb{P}^*\{\bar{\tau} \le s\} = N\left(\frac{\ln(\bar{v}(0)/V_0) - \tilde{\nu}s}{\sigma\sqrt{s}}\right) + \left(\frac{\bar{v}(0)}{V_0}\right)^{2\tilde{a}} N\left(\frac{\ln(\bar{v}(0)/V_0) + \tilde{\nu}s}{\sigma\sqrt{s}}\right).$$

Notice that $\bar{v}(0) < V_0$, and thus $\ln(\bar{v}(0)/V_0) < 0$. Using (8.6), we obtain

$$\begin{aligned} &\bar{v}(0)\int_0^T e^{(\gamma-r)s}\,dN\left(\frac{\ln(\bar{v}(0)/V_0) - \tilde{\nu}s}{\sigma\sqrt{s}}\right) \\ &\quad = \frac{V_0(\tilde{a}+\zeta)}{2\zeta} R_0^{\theta-\zeta} N\big(h_8(V_0,T)\big) - \frac{V_0(\tilde{a}-\zeta)}{2\zeta} R_0^{\theta+\zeta} N\big(h_7(V_0,T)\big) \end{aligned}$$

and

$$\begin{aligned} &\frac{\bar{v}(0)^{2\tilde{a}+1}}{V_0^{2\tilde{a}}}\int_0^T e^{(\gamma-r)s}\,dN\left(\frac{\ln(\bar{v}(0)/V_0) + \tilde{\nu}s}{\sigma\sqrt{s}}\right) \\ &\quad = \frac{V_0(\tilde{a}+\zeta)}{2\zeta} R_0^{\theta+\zeta} N\big(h_7(V_0,T)\big) - \frac{V_0(\tilde{a}-\zeta)}{2\zeta} R_0^{\theta-\zeta} N\big(h_8(V_0,T)\big). \end{aligned}$$

Consequently,

$$D_3(0,T) = \beta_2 V_0\big(R_0^{\theta+\zeta} N\big(h_7(V_0,T)\big) + R_0^{\theta-\zeta} N\big(h_8(V_0,T)\big)\big). \tag{8.7}$$

This completes the proof of the proposition. □

The financial interpretation of the coefficients β_1 and β_2 is that they reflect the bankruptcy (or reorganization) costs incurred at the time of default. It is clear that as soon as $\beta_1 < 1$ and/or $\beta_2 < 1$ the value of a defaultable bond is less than in case of zero bankruptcy costs, i.e., when $\beta_1 = \beta_2 = 1$. In some circumstances, the values $\beta_1 < 1$ and/or $\beta_2 < 1$ can be interpreted as reflecting the violation of the strict priority rule.

It should be noted that, similarly as in the case of the Merton model, the Black and Cox model produces credit spreads close to zero for small maturities, a feature that is inconsistent with empirical studies. The reason again is that the default time is predictable with respect to the natural filtration of the value process V.

8.1.2 Strict priority rule

For the sake of simplicity, we shall assume that $\beta_1 = \beta_2 = 1$, i.e., no bankruptcy/reorganization costs are present. Suppose that the firm's debt can be classified into *senior* bonds and (subordinated) *junior* bonds, with the same maturity date T. At debt's maturity, payments can be made to the holders of junior bonds only if the promised payment to the holders of senior bonds has been made. Such a convention is commonly referred to as the *strict* (or *absolute*) *priority rule*. Assume that the total face value L of the firm's liabilities equals $L = L_s + L_j$, where L_s (L_j, resp.) is the face value of senior bonds (of junior bonds, resp.) Let $u(V_t, t; L, \bar{v})$ stand for the price $D(t,T)$ – given by Proposition 8.1 – of a defaultable bond in the Black and Cox model, where, for the sake of convenience, we have introduced in the notation the face value L and the barrier function $\bar{v}$.

It is clear that the value $D_s(t,T)$ at time $t < T$ of the senior debt equals, on the set $\{\tau > t\}$,

$$D_s(t,T) = u(V_t, t; L_s, \bar{v})$$

and it amounts to $\min\left(\bar{v}(\tau), L_s B(\tau,T)\right)$ at time of default, provided that default has occurred prior to the maturity date. The total value of firm's debt equals, on the set $\{\tau > t\}$,

$$D(t,T) = u(V_t, t; L, \bar{v})$$

and it equals $\bar{v}(\tau)$ at time of default. Thus, the value of the junior debt is

$$D_j(t,T) = D(t,T) - D_s(t,T) = u(V_t, t; L, \bar{v}) - u(V_t, t; L_s, \bar{v})$$

on the set $\{\tau > t\}$, and it equals $\min\left(\bar{v}(\tau) - L_s B(\tau,T), L_j B(\tau,T)\right)$ at time of default, provided that the default has occurred prior to the maturity date. For instance, if $\bar{v}(t) = KB(t,T)$ for some constant $K \le L$ then we have, on the set $\{\tau > t\}$,

$$D_j(t,T) = \begin{cases} L_j B(t,T), & \text{if } K = L, \\ D(t,T) - L_s B(t,T), & \text{if } L_s \le K < L, \\ D(t,T) - D_s(t,T), & \text{if } K < L_s. \end{cases}$$

As one might easily guess, the above analysis can be extended to cover the case of several classes of subordinated debt.

8.1.3 Special cases

Let us now analyze some special cases of the Black-Cox valuation formula. We shall assume that $\beta_1 = \beta_2 = 1$, and the barrier function $\bar{v}$ is chosen in such a way that $K = L$. Then necessarily $\gamma \ge r$ (otherwise, condition (8.3) would be violated). Obviously, if $K = L$, then $K_1(t) = K_2(t) = 0$, and thus $D(t,T) = D_1(t,T) + D_3(t,T)$, where:

$$D_1(t,T) = LB(t,T)\big(N\big(h_1(V_t, T-t)\big) - R_t^{2\tilde{a}} N\big(h_2(V_t, T-t)\big)\big) \tag{8.8}$$

and

$$D_3(t,T) = V_t\big(R_t^{\theta+\zeta} N\big(h_7(V_t, T-t)\big) + R_t^{\theta-\zeta} N\big(h_8(V_t, T-t)\big)\big). \tag{8.9}$$

Case $\gamma = r$. If we also assume that $\gamma = r$, then $\zeta = -\sigma^{-2}\tilde{\nu}$, and thus

$$V_t R_t^{\theta+\zeta} = LB(t,T), \quad V_t R_t^{\theta-\zeta} = V_t R_t^{2\tilde{a}+1} = LB(t,T) R_t^{2\tilde{a}}.$$

Moreover, it is also easy to see that in this case

$$h_1(V_t, T-t) = \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}} = -h_7(V_t, T-t),$$

while

$$h_2(V_t, T-t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}} = h_8(V_t, T-t).$$

We conclude that if $\bar{v}(t) = Le^{-r(T-t)} = LB(t,T)$, then $D(t,T) = LB(t,T)$. This result is quite intuitive; a defaultable bond with a safety covenant represented by the barrier function, which equals the discounted value of the bond's face value, is obviously equivalent to a default-free bond with the same face value and maturity. Notice also that when $\gamma = r$ but $K < L$, then we have: $D_3(t,T) = KB(t,T)\mathbb{P}^*\{\tau < T \,|\, \mathcal{F}_t\}$.

Case $\gamma > r$. If $K = L$ but $\gamma > r$ then one would expect that $D(t,T)$ would be smaller than $LB(t,T)$. We shall show that when γ tends to infinity (all other parameters being fixed), then the Black and Cox price converges to Merton's price, that is,

$$\lim_{\gamma\to\infty} D(t,T) = V_t e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big) + LB(t,T)\big(d_2(V_t, T-t)\big).$$

First, it is clear that $h_1(V_t, T-t) = d_2(V_t, T-t)$. Furthermore, straightforward calculations show that

$$\lim_{\gamma\to\infty} R_t^{2\tilde{a}} N\big(h_2(V_t, T-t)\big) = \lim_{\gamma\to\infty} R_t^{\theta-\zeta} N\big(h_8(V_t, T-t)\big) = 0$$

and thus the second term on the right-hand side of (8.8), as well as the second term on the right-hand side of (8.9), vanish. Finally,

$$\lim_{\gamma\to\infty} R_t^{\theta+\zeta} N\big(h_8(V_t, T-t)\big) = e^{-\kappa(T-t)}\, N\big(-d_1(V_t, T-t)\big),$$

since $\lim_{\gamma\to\infty} R_t^{\theta+\zeta} = e^{-\kappa(T-t)}$ and $\lim_{\gamma\to\infty} h_7(V_t, T-t) = -d_1(V_t, T-t)$.

References

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WARSAW UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 9

9 Black and Cox Model with Random Interest Rates

We shall examine a natural generalization of the Black and Cox (1976) approach, which takes into account both the credit and interest rate risk. Formally, our goal is to extend the bond valuation formula of Proposition 8.1 to the case of stochastic term structure of interest rates, as specified by the Heath et al. (1992) approach. We make the following standing assumptions:
(i) the default triggering barrier $\bar{v}$ equals $\bar{v}(t) = KB(t,T)f(t)$ for some constant K, and some function $f : [0,T) \to \mathbb{R}_+$,
(ii) the volatility of the forward value of the firm follows a deterministic function.
To guarantee the existence of a closed-form solution for the value of a defaultable bond, the function f in (ii) needs to be chosen in a judicious way (see expression (9.3) below). On the other hand, to satisfy the second requirement above, we find it convenient to place ourselves in the Gaussian Heath-Jarrow-Morton setup. More specifically, we assume that the bond price volatility is a deterministic function.

We assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, is rich enough to support the short-term interest rate process r and the value process V. Let us fix a finite time horizon $T > 0$. The dynamics under the spot martingale measure $\mathbb{P}^*$ of the firm's value and of the price of a default-free zero-coupon bond $B(t,T)$ are

$$dV_t = V_t\big((r_t - \kappa(t))\,dt + \sigma(t)\,dW_t^*\big), \tag{9.1}$$

and

$$dB(t,T) = B(t,T)\big(r_t\,dt + b(t,T)\,dW_t^*\big), \tag{9.2}$$

respectively, where W^* is a d-dimensional standard Brownian motion. Furthermore, $\kappa : [0,T] \to \mathbb{R}$, $\sigma : [0,T] \to \mathbb{R}^d$ and $b(\cdot,T) : [0,T] \to \mathbb{R}^d$ are assumed to be bounded functions. In view of (9.1)-(9.2), the *forward value* $F_V(t,T) := V_t/B(t,T)$ of the firm satisfies under the forward martingale measure $\mathbb{P}_T$

$$dF_V(t,T) = -\kappa(t)F_V(t,T)\,dt + F_V(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T,$$

where the process W^T, given by the formula

$$W_t^T = W_t^* - \int_0^t b(u,T)\,du, \quad \forall\, t \in [0,T],$$

is known to follow a d-dimensional standard Brownian motion under $\mathbb{P}_T$. Let us introduce an auxiliary process $F_V^\kappa(t,T)$ by setting, for $t \in [0,T]$,

$$F_V^\kappa(t,T) = F_V(t,T)e^{-\int_t^T \kappa(u)\,du}.$$

It is clear that $F_V^\kappa(t,T)$ follows a lognormally distributed martingale under $\mathbb{P}_T$, specifically,

$$dF_V^\kappa(t,T) = F_V^\kappa(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T.$$

Furthermore, it is apparent that $F_V^\kappa(T,T) = F_V(T,T) = V_T$. We consider the following modification of the Black and Cox approach:

$$X = L,\ Z_t = \beta_2 V_t,\ \tilde{X} = \beta_1 V_T,\ \tau = \inf\{\, t \in [0,T] : V_t < v_t\},$$

where $\beta_2,\ \beta_1 \in [0,1]$ are constants, and the barrier v is given by the formula

$$v_t := \begin{cases} KB(t,T)e^{\int_t^T \kappa(u)\,du}, & \text{for } t < T, \\ L, & \text{for } t = T, \end{cases} \tag{9.3}$$

where the constant K satisfies $0 < K \le L$. Let us denote, for any $t \le T$,

$$\kappa(t,T) = \int_t^T \kappa(u)\,du, \quad \sigma^2(t,T) = \int_t^T |\sigma(u) - b(u,T)|^2 du,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. We write briefly $F_t = F_V(t,T)$, and we denote

$$\eta_+(t,T) = \kappa(t,T) + \tfrac{1}{2}\sigma^2(t,T), \quad \eta_-(t,T) = \kappa(t,T) - \tfrac{1}{2}\sigma^2(t,T).$$

Proposition 9.1 *Let the barrier process v be given by (9.3). For any $t < T$, the forward price $F_D(t,T) = D(t,T)/B(t,T)$ of a defaultable bond with the face value L and the maturity date T equals, on the set $\{\tau > t\} = \{\bar{\tau} > t\}$,*

$$\begin{aligned} F_D(t,T) = {} & L\big(N\big(\hat{h}_1(F_t,t,T)\big) - (F_t/K)e^{-\kappa(t,T)}N\big(\hat{h}_2(F_t,t,T)\big)\big) \\ & + \beta_1 F_t e^{-\kappa(t,T)}\big(N\big(\hat{h}_3(F_t,t,T)\big) - N\big(\hat{h}_4(F_t,t,T)\big)\big) \\ & + \beta_1 K\big(N\big(\hat{h}_5(F_t,t,T)\big) - N\big(\hat{h}_6(F_t,t,T)\big)\big) \\ & + \beta_2 K J_1(F_t,t,T) + \beta_2 F_t e^{-\kappa(t,T)} J_2(F_t,t,T), \end{aligned}$$

where

$$\begin{aligned} \hat{h}_1(F_t,t,T) &= \frac{\ln(F_t/L) - \eta_+(t,T)}{\sigma(t,T)}, \\ \hat{h}_2(F_t,T,t) &= \frac{2\ln K - \ln(LF_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \hat{h}_3(F_t,t,T) &= \frac{\ln(L/F_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \hat{h}_4(F_t,t,T) &= \frac{\ln(K/F_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \hat{h}_5(F_t,t,T) &= \frac{2\ln K - \ln(LF_t) + \eta_+(t,T)}{\sigma(t,T)}, \\ \hat{h}_6(F_t,t,T) &= \frac{\ln(K/F_t) + \eta_+(t,T)}{\sigma(t,T)}, \end{aligned}$$

and for any fixed $0 \le t < T$ and $F_t > 0$

$$J_{1,2}(F_t,t,T) = \int_t^T e^{\kappa(u,T)}\,dN\left(\frac{\ln(K/F_t) + \kappa(t,T) \pm \frac{1}{2}\sigma^2(t,u)}{\sigma(t,u)}\right).$$

Remarks. Let us assume that $\beta_2 = \beta_1 = 1$. It can be checked that if $b \equiv 0$ and the coefficients κ and σ are assumed to be constant, the term $J_{1,2}(F_t, t, T)$ can be evaluated explicitly, and the valuation formula of Proposition 9.1 reduces to the special case of the formula obtained in Proposition 8.1 with $\gamma = r - \kappa$. It is worthwhile to stress that the choice of a barrier in the following form:

$$v_t = \begin{cases} \bar{v}(t) = Ke^{-\gamma(T-t)}, & \text{for } t < T, \\ L, & \text{for } t = T, \end{cases}$$

instead of (9.3) does not lead to a closed-form solution, in general.

Before we proceed to the proof of Proposition 9.1, let us recall an auxiliary result. Assume that $\tilde{Y}_t,\ t \in [0, U]$, follows under $\tilde{\mathbb{P}}$ a generalized Brownian motion with drift with respect to the filtration $\tilde{\mathbb{F}}$. Specifically,

$$\tilde{Y}_t = Y_0 + \sigma \tilde{W}_t + \nu t, \quad \tilde{Y}_0 = \tilde{y}_0 > 0, \tag{9.4}$$

where $\tilde{W}_t,\ t \in [0, U]$, follows under $\tilde{\mathbb{P}}$ a standard one-dimensional Brownian motion with respect to $\tilde{\mathbb{F}}$, and the coefficients $\sigma > 0$ and $\nu \in \mathbb{R}$ are constants.

Lemma 9.1 *Let $\tilde{\tau}$ be the first passage time to zero by the process $\tilde{Y}$ given by formula* (9.4)*, specifically,*

$$\tilde{\tau} = \inf\{\, t < U : \tilde{Y}_t = 0\}.$$

Then for any $0 < s \le U$

$$\tilde{\mathbb{P}}\{\tilde{\tau} < s\} = N\left(\frac{-\tilde{y}_0 - \nu s}{\sigma\sqrt{s}}\right) + e^{-2\nu\sigma^{-2}\tilde{y}_0} N\left(\frac{-\tilde{y}_0 + \nu s}{\sigma\sqrt{s}}\right),$$

where N is the standard normal cumulative distribution function. Furthermore, for any $0 \le u < s \le U$ and any $y \ge 0$, we have, on the set $\{\tilde{\tau} > u\}$,

$$\begin{aligned}\tilde{\mathbb{P}}\{\tilde{Y}_s \ge y,\ \tilde{\tau} \ge s \,|\, \tilde{\mathcal{F}}_u\} &= N\left(\frac{-y + \tilde{Y}_u + \nu(s-u)}{\sigma\sqrt{s-u}}\right) \\ &\quad - e^{-2\nu\sigma^{-2}\tilde{Y}_u} N\left(\frac{-y - \tilde{Y}_u + \nu(s-u)}{\sigma\sqrt{s-u}}\right).\end{aligned}$$

Proof of Proposition 9.1. Under the present assumptions, a defaultable bond is formally equivalent to the contingent claim X which settles at the bond's maturity date T, and is given by the expression:

$$X := \beta_1 F_V^\kappa(T,T) 1\!\!1_{\{\bar{\tau} \ge T,\, V_T < L\}} + L 1\!\!1_{\{\bar{\tau} \ge T,\, V_T \ge L\}} + \beta_2 v_{\bar{\tau}} B^{-1}(\bar{\tau}, T) 1\!\!1_{\{t < \bar{\tau} < T\}}.$$

Consequently, the forward price of a defaultable bond admits the following representation

$$\begin{aligned}F_D(t,T) &= \mathbb{E}_{\mathbb{P}_T}\Big(\beta_1 F_V^\kappa(T,T) 1\!\!1_{\{\bar{\tau} \ge T,\, V_T < L\}} + L 1\!\!1_{\{\bar{\tau} \ge T,\, V_T \ge L\}} \,\Big|\, \mathcal{F}_t\Big) \\ &\quad + \beta_2\, \mathbb{E}_{\mathbb{P}_T}\Big(v_{\bar{\tau}} B^{-1}(\bar{\tau}, T) 1\!\!1_{\{t < \bar{\tau} < T\}} \,\Big|\, \mathcal{F}_t\Big).\end{aligned}$$

The representation above is an immediate consequence of the definition of the forward martingale measure $\mathbb{P}_T$. We conclude that we have, on the set $\{\bar{\tau} > t\}$,

$$\begin{aligned}F_D(t,T) &= L\, \mathbb{P}_T\big\{F_V^\kappa(T,T) \ge L,\ \bar{\tau} \ge T \,\big|\, \mathcal{F}_t\big\} \\ &\quad + \beta_1\, \mathbb{E}_{\mathbb{P}_T}\Big(F_V^\kappa(T,T) 1\!\!1_{\{F_V^\kappa(T,T) < L,\, \bar{\tau} \ge T\}} \,\Big|\, \mathcal{F}_t\Big) \\ &\quad + \beta_2 K\, \mathbb{E}_{\mathbb{P}_T}\Big(e^{\kappa(\bar{\tau}, T)}\, 1\!\!1_{\{t < \bar{\tau} < T\}} \,\Big|\, \mathcal{F}_t\Big) =: I_1(t) + I_2(t) + I_3(t),\end{aligned}$$

where $\bar\tau$ equals (as usual, $\inf\emptyset = +\infty$)

$$\bar\tau = \inf\{\, t < T : F^{\kappa}_V(t,T) \le K\} = \inf\{\, t < T : \, Y_t \le 0\},$$

where in turn $Y_t := \ln\big(F^{\kappa}_V(t,T)/K\big)$ for $t \in [0,T]$. It is clear that

$$Y_t = Y_0 + \int_0^t (\sigma(u) - b(u,T))\, dW^T_u - \frac{1}{2}\int_0^t |\sigma(u) - b(u,T)|^2 du.$$

We consider the following deterministic time change $A : [0,T] \to \mathbb{R}_+$ associated with Y:

$$A_t = \int_0^t |\sigma(u) - b(u,T)|^2 du.$$

Let $A^{-1} : [0, A_T] \to [0,T]$ stand for the inverse time change. Then the time-changed process $\tilde Y_t := Y_{A^{-1}_t}$, $t \in [0, A_T]$, follows under $\mathbb{P}_T$ a one-dimensional Brownian motion with the drift coefficient $-1/2$, with respect to the time-changed filtration $\tilde{\mathbb{F}}$, where we set $\tilde{\mathcal{F}}_t = \mathcal{F}_{A^{-1}_t}$ for $t \in [0, A_T]$ (cf. Revuz and Yor (1991)). More explicitly, $\tilde Y$ satisfies

$$\tilde Y_t = Y_0 + \tilde W_t - \tfrac{1}{2}t, \quad \forall\, t \in [0, A_T],$$

for a certain $(\mathbb{P}_T, \tilde{\mathbb{F}})$-standard Brownian motion $\tilde W$.

We shall first examine $I_1(t)$. Let us denote $\tilde L = \ln(L/K)$, and let us set $\tilde\tau := \inf\{\, t < A_T : \tilde Y_t \le 0\}$. Notice that for any fixed $t < T$, we have, on the set $\{\bar\tau > t\} = \{\tilde\tau > A_t\}$,

$$\mathbb{P}_T\big\{F^{\kappa}_V(T,T) \ge L,\, \bar\tau \ge T \,\big|\, \mathcal{F}_t\big\} = \mathbb{P}_T\big\{\, \tilde Y_{A_T} \ge \tilde L,\, \tilde\tau \ge A_T \,\big|\, \tilde{\mathcal{F}}_{A_t}\big\}.$$

Making use of Lemma 9.1, with $\tilde{\mathbb{P}} = \mathbb{P}_T$, $\sigma = 1$, $\nu = -1/2$, $u = A_t$ and $s = A_T$, we obtain

$$\begin{aligned}
&\mathbb{P}_T\big\{\, \tilde Y_{A_T} \ge \tilde L,\, \tilde\tau \ge A_T \,\big|\, \tilde{\mathcal{F}}_{A_t}\big\}\\
&\quad = N\left(\frac{\ln(K/L) + \tilde Y_{A_t} - \frac{1}{2}(A_T - A_t)}{\sqrt{A_T - A_t}}\right)\\
&\qquad - e^{\tilde Y_{A_t}}\, N\left(\frac{\ln(K/L) - \tilde Y_{A_t} - \frac{1}{2}(A_T - A_t)}{\sqrt{A_T - A_t}}\right).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
I_1(t) &= L\mathbb{P}_T\big\{\, \tilde Y_{A_T} \ge \tilde L,\, \tilde\tau \ge A_T \,\big|\, \tilde{\mathcal{F}}_{A_t}\big\}\\
&= LN\left(\frac{\ln(F_t/L) - \kappa(t,T) - \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)}\right)\\
&\quad - e^{-\kappa(t,T)}\frac{LF_t}{K}\, N\left(\frac{2\ln K - \ln(F_tL) + \kappa(t,T) - \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)}\right).
\end{aligned}$$

This shows that

$$I_1(t) = L\Big(N\big(\hat h_1(F_t,t,T)\big) - (F_t/K)e^{-\kappa(t,T)}N\big(\hat h_2(F_t,t,T)\big)\Big),$$

as expected.

To simplify the notation, we shall evaluate $I_2(t)$ and $I_3(t)$ for $t = 0$ only. The case of $t > 0$ follows by similar arguments as those used in the derivation of the formula for $I_1(t)$, and thus it presents no difficulties.

Let us focus on $I_2(0)$. In view of the definition of the processes $\tilde{Y}$ and A, we have

$$\mathbb{E}_{\mathbb{P}_T}\Big(F_V^\kappa(T,T)\mathbb{1}_{\{F_V^\kappa(T,T)<L,\,\tilde{\tau}\geq T\}}\Big) = K\mathbb{E}_{\mathbb{P}_T}\Big(e^{\tilde{Y}_{A_T}}\mathbb{1}_{\{\tilde{Y}_{A_T}<\tilde{L},\,\tilde{\tau}\geq A_T\}}\Big),$$

and thus we may re-express $I_2(0)$ as follows:

$$I_2(0) = \beta_1 K \int_0^{\tilde{L}} e^x \, d\mathbb{P}_T\{\tilde{Y}_{A_T} < x,\, \tilde{\tau} \geq A_T\}.$$

Using again Lemma 9.1, we obtain

$$\begin{aligned}
&d\mathbb{P}_T\big\{\,\tilde{Y}_{A_T} < x,\, \tilde{\tau} \geq A_T\big\}\\
&\quad = dN\left(\frac{x - \tilde{Y}_0 + \frac{1}{2}A_T}{\sqrt{A_T}}\right) + e^{\tilde{Y}_0}\, dN\left(\frac{-x - \tilde{Y}_0 - \frac{1}{2}A_T}{\sqrt{A_T}}\right)\\
&\quad = dN\left(\frac{x - \ln(F_0/K) + \kappa(0,T) + \frac{1}{2}\sigma^2(0,T)}{\sigma(0,T)}\right)\\
&\qquad + e^{-\kappa(0,T)}\frac{F_0}{K}\, dN\left(\frac{-x - \ln(F_0/K) + \kappa(0,T) - \frac{1}{2}\sigma^2(0,T)}{\sigma(0,T)}\right).
\end{aligned}$$

Therefore, $I_2(0) = I_{21}(0) + I_{22}(0)$, where, by standard calculations

$$\begin{aligned}
I_{21}(0) &= \beta_1 K \int_0^{\tilde{L}} e^x \, dN\left(\frac{x - \ln(F_0/K) + \kappa(0,T) + \frac{1}{2}\sigma^2(0,T)}{\sigma(0,T)}\right)\\
&= \beta_1 F_0 e^{-\kappa(0,T)} N\left(\frac{\ln(L/F_0) + \kappa(0,T) - \frac{1}{2}\sigma^2(0,T)}{\sigma(0,T)}\right)\\
&\quad - \beta_1 F_0 e^{-\kappa(0,T)} N\left(\frac{\ln(K/F_0) + \kappa(0,T) - \frac{1}{2}\sigma^2(0,T)}{\sigma(0,T)}\right)\\
&= \beta_1 F_0 e^{-\kappa(0,T)}\big(N\big(\hat{h}_3(F_0,0,T)\big) - N\big(\hat{h}_4(F_0,0,T)\big)\big)
\end{aligned}$$

and

$$\begin{aligned}
I_{22}(0) &= \beta_1 e^{-\kappa(0,T)} F_0 \int_0^{\tilde{L}} e^x \, dN\left(\frac{-x - \ln(F_0/K) + \kappa(0,T) - \frac{1}{2}\sigma^2(0,T)}{\sigma(0,T)}\right)\\
&= \beta_1 K N\left(\frac{2\ln K - \ln(LF_0) + \kappa(0,T) + \frac{1}{2}\sigma^2(0,T)}{\sigma(0,T)}\right)\\
&\quad - \beta_1 K N\left(\frac{\ln(K/F_0) + \kappa(0,T) + \frac{1}{2}\sigma^2(0,T)}{\sigma(0,T)}\right)\\
&= \beta_1 K\big(N\big(\hat{h}_5(F_0,0,T)\big) - N\big(\hat{h}_6(F_0,0,T)\big)\big).
\end{aligned}$$

To establish the last two formulae, note that for any $c \neq 0$, and $a, b, d \in \mathbb{R}$, we have (we set here $\tilde{d} = d - c^{-1}$)

$$\int_a^b e^x \, dN(cx + d) = e^{\frac{1}{2}(\tilde{d}^2 - d^2)}\big(N(cb + \tilde{d}) - N(ca + \tilde{d})\big).$$

Observe that $I_{21}(0) > 0$ and $I_{22}(0) < 0$; we always have $I_2(0) > 0$, though. It remains to evaluate $I_3(0)$, where

$$I_3(0) = \beta_2 K\, \mathbb{E}_{\mathbb{P}_T}\big(e^{\kappa(\bar\tau,T)}\, \mathbb{1}_{\{\bar\tau<T\}}\big) = \beta_2 K \int_0^T e^{\kappa(t,T)}\, d\mathbb{P}_T\{\bar\tau < t\}.$$

In view of Lemma 9.1, we have

$$\mathbb{P}_T\{\tilde\tau < s\} = N\left(\frac{-\tilde Y_0 + \frac{1}{2}s}{\sqrt{s}}\right) + e^{\tilde Y_0} N\left(\frac{-\tilde Y_0 - \frac{1}{2}s}{\sqrt{s}}\right),$$

where $\tilde Y_0 = Y_0$, and, as before, $\tilde\tau = \inf\{\, t < A_T : \tilde Y_t \le 0\}$. Since clearly $\mathbb{P}_T\{\bar\tau < t\} = \mathbb{P}_T\{\tilde\tau < A_t\}$, we obtain

$$\begin{aligned}\mathbb{P}_T\{\bar\tau < t\} &= N\left(\frac{-Y_0 + \frac{1}{2}A_t}{\sqrt{A_t}}\right) + e^{Y_0} N\left(\frac{-Y_0 - \frac{1}{2}A_t}{\sqrt{A_t}}\right)\\ &= N\left(\frac{\ln\frac{K}{F_0} + \kappa(0,T) + \frac{1}{2}A_t}{\sqrt{A_t}}\right) + e^{-\kappa(0,T)}\frac{F_0}{K}\, N\left(\frac{\ln\frac{K}{F_0} + \kappa(0,T) - \frac{1}{2}A_t}{\sqrt{A_t}}\right).\end{aligned}$$

We conclude that $I_3(0) = I_{31}(0) + I_{32}(0)$, where

$$\begin{aligned}I_{31}(0) &= \beta_2 K \int_0^T e^{\kappa(t,T)}\, dN\left(\frac{\ln(K/F_0) + \kappa(0,T) + \frac{1}{2}\sigma^2(0,t)}{\sigma(0,t)}\right)\\ &= \beta_2 K J_1(F_0, 0, T)\end{aligned}$$

and

$$\begin{aligned}I_{32}(0) &= \beta_2 F_0 e^{-\kappa(0,T)} \int_0^T e^{\kappa(t,T)}\, dN\left(\frac{\ln(K/F_0) + \kappa(0,T) - \frac{1}{2}\sigma^2(0,t)}{\sigma(0,t)}\right)\\ &= \beta_2 F_0 e^{-\kappa(0,T)} J_2(F_0, 0, T).\end{aligned}$$

This completes the proof of Proposition 9.1. □

To the best of our knowledge, explicit formulae for $J_1(F_t,t,T)$ and $J_2(F_t,t,T)$ are not available in the general time-dependent setup (even when, e.g., the dividend ratio κ is constant). Incidentally, quite simple expressions for these two terms can be obtained provided that we set $\kappa = 0$; that is, in the absence of dividends. The following result is an immediate corollary to Proposition 9.1.

Corollary 9.1 *Under the assumptions of Proposition* 9.1, *if* $\kappa \equiv 0$ *then*

$$\begin{aligned}F_D(t,T) = {}& L\big(N\big(-d_1(F_t,t,T)\big) - (F_t/K)N\big(d_6(F_t,t,T)\big)\big)\\ &+ \beta_1 F_t\big(N\big(d_2(F_t,t,T)\big) - N\big(d_4(F_t,t,T)\big)\big)\\ &+ \beta_1 K\big(N\big(d_5(F_t,t,T)\big) - N\big(d_3(F_t,t,T)\big)\big)\\ &+ \beta_2 K N\big(d_3(F_t,t,T)\big) + \beta_2 F_t N\big(d_4(F_t,t,T)\big),\end{aligned}$$

where

$$d_1(F_t,t,T) = \frac{\ln(L/F_t) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_2(F_t,t,T) + \sigma(t,T),$$
$$d_3(F_t,t,T) = \frac{\ln(K/F_t) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_4(F_t,t,T) + \sigma(t,T),$$
$$d_5(F_t,t,T) = \frac{\ln(K^2/F_tL) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_6(F_t,t,T) + \sigma(t,T).$$

Proof. Since the inequality $F_t > K$ is satisfied on the set $\{\bar{\tau} > t\}$, we have

$$J_1(F_t,t,T) = \int_t^T dN\left(\frac{\ln(K/F_t) + \frac{1}{2}\sigma^2(t,u)}{\sigma(t,u)}\right) = N\left(\frac{\ln(K/F_t) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)}\right)$$

and

$$J_2(F,t,T) = \int_t^T dN\left(\frac{\ln(K/F_t) - \frac{1}{2}\sigma^2(t,u)}{\sigma(t,u)}\right) = N\left(\frac{\ln(K/F_t) - \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)}\right).$$

The formula now follows from simple calculations. □

Let us observe that the formula of Corollary 9.1 covers as a special case the valuation result established by Briys and de Varenne (1997). In some other recent studies of first passage time models, in which the triggering barrier is assumed to be either a constant or an unspecified stochastic process, typically no closed-form solution for the value of a corporate debt is available, and thus a numerical approach is required. The interested reader is referred to, among others, Kim et al. (1993), Longstaff and Schwartz (1995), Nielsen et al. (1993), or Saá-Requejo and Santa-Clara (1999).

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WARSAW UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 10

10 Intensity-Based Valuation of Defaultable Claims

In this lecture, we present basic results that can be obtained through the intensity-based approach to the valuation of defaultable claims. We assume that we are given the underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ (of course, $\mathcal{F}_t \subseteq \mathcal{G}$ for any t). The probability measure $\mathbb{Q}^*$ is interpreted as a *spot martingale measure* for our model of securities market; the real-world probability measure will be denoted by $\mathbb{Q}$. All processes introduced below are defined on the probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$.

We formally identify a *defaultable claim* with a quintuple $DCT = (X, A, \tilde{X}, Z, \tau)$. The *default time* τ is an arbitrary non-negative random variable, which is defined on the underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$; in particular, $\mathbb{Q}^*\{\tau < +\infty\} = 1$. For the sake of convenience, we shall usually assume that $\mathbb{Q}^*\{\tau = 0\} = 0$ and $\mathbb{Q}^*\{\tau > t\} > 0$ for every $t \in \mathbb{R}_+$. For a given default time τ, we introduce the associated jump process H by setting $H_t = \mathbb{1}_{\{\tau \leq t\}}$ for $t \in \mathbb{R}_+$. We shall refer to H as the *default process.* It is obvious that H is a right-continuous process. Let $\mathbb{H}$ be the filtration generated by the process H – i.e., $\mathcal{H}_t = \sigma(H_u : u \leq t) = \sigma(\{\tau \leq u\} : u \leq t)$.

An essential role is played by the enlarged filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. By definition, for every t we set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$. It should be emphasized that the *default time* τ is not necessarily a stopping time with respect to the filtration $\mathbb{F}$. On the other hand, τ is, of course, a stopping time with respect to the filtration $\mathbb{G}$. In most intensity-based models, the underlying filtration $\mathbb{G}$ encompasses a certain Brownian filtration $\mathbb{F}$; $\mathbb{G}$ is usually strictly larger than $\mathbb{F}$, though. In this case, the default time is usually modeled in such a way that it is not a predictable stopping time with respect to the filtration $\mathbb{G}$. Recall that if τ is a stopping time with the Brownian filtration $\mathbb{F}$, then it is necessarily a predictable stopping time.

The *short-term interest rate* process r follows an $\mathbb{F}$-progressively measurable process, such that the savings account B, given by the usual expression:

$$B_t = \exp\Big(\int_0^t r_u \, du\Big), \quad \forall \, t \in \mathbb{R}_+,$$

is well defined.

We introduce the following random variables and processes that specify the cash flows associated with a defaultable claim:

- the *promised contingent claim* X, representing the payoff received by the owner of the claim at time T, if there was no default prior to or at time T,
- the process A representing the *promised dividends* – that is, the stream of (continuous or discrete) cash flows received by the owner of the claim prior to default,
- the *recovery process* Z, representing the recovery payoff at the time of default, if default occurs prior to or at the maturity date T,
- the *recovery claim* $\tilde{X}$, representing the recovery payoff at time T, if default occurs prior to or at the maturity date T.

We shall postulate throughout that the processes Z and A are predictable with respect to the reference filtration $\mathbb{F}$, and that the random variables X and $\tilde{X}$ are $\mathcal{F}_T$-measurable. By assumption, the promised dividends process A follows a process of finite variation, with $A_0 = 0$. As usual, the sample paths of all processes are assumed to be right-continuous functions, with finite left-hand limits, with probability one. We shall assume without mentioning that all the above random objects satisfy suitable integrability conditions that are needed for evaluating the functionals introduced later on.

10.1 Risk-Neutral Valuation Formula

We place ourselves within the framework of an arbitrage-free financial market model. Specifically, we postulate that the underlying probability measure $\mathbb{Q}^*$ is the *spot martingale measure* (or the *risk-neutral probability*), meaning that the price process of any tradeable security, which pays no coupons or dividends, necessarily follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$, when discounted by the savings account B. Let us first recall the definitions of the dividend process and the price process of a defaultable claim (see Definition 4.1).

Definition 10.1 *The dividend process D of a defaultable claim $DCT = (X, A, \tilde{X}, Z, \tau)$ equals*

$$D_t = X^d(T)\mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u)\, dA_u + \int_{]0,t]} Z_u\, dH_u, \tag{10.1}$$

where $X^d(T) = X\mathbb{1}_{\{\tau > T\}} + \tilde{X}\mathbb{1}_{\{\tau \le T\}}$.

The next definition mimics Definition 4.2 of the *price process* (or the *value process*) of a defaultable claim. Expression (10.2) is henceforth referred to as the *risk-neutral valuation formula.*

Definition 10.2 *The (ex-dividend) price process $X^d(\cdot, T)$ of a defaultable claim $DCT = (X, A, \tilde{X}, Z, \tau)$, which settles at time T, is given as*

$$X^d(t,T) = B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big), \quad \forall\, t \in [0,T]. \tag{10.2}$$

Before presenting the no-arbitrage arguments supporting Definition 10.2, let us consider a few special cases of the risk-neutral valuation formula (10.2). For the sake of brevity, we shall write $S_t^0 = X^d(t,T)$. Combining (10.1) with (10.2), we obtain

$$S_t^0 = B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1}(1 - H_u)\, dA_u + \int_{]t,T]} B_u^{-1} Z_u\, dH_u + B_T^{-1} X^d(T) \,\Big|\, \mathcal{G}_t \Big).$$

where, as before, we denote

$$X^d(T) = \tilde{X}\mathbb{1}_{\{\tau \le T\}} + X\mathbb{1}_{\{\tau > T\}} = \tilde{X} H_T + X(1 - H_T).$$

If the claim does not pay any dividends prior to default – that is, if $A \equiv 0$, and if $\tilde{X} = 0$, the risk-neutral valuation formula simplifies to:

$$S_t^0 = B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(B_\tau^{-1} Z_\tau \mathbb{1}_{\{t < \tau \le T\}} + B_T^{-1} X \mathbb{1}_{\{\tau > T\}} \,\Big|\, \mathcal{G}_t \Big). \tag{10.3}$$

It is apparent that in this case $S^0_t = 0$ on the set $\{\tau \le t\}$, and so

$$S^0_t = 1\!\!1_{\{\tau > t\}} B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(B_\tau^{-1} Z_\tau 1\!\!1_{\{t<\tau\le T\}} + B_T^{-1} X 1\!\!1_{\{\tau > T\}} \,\Big|\, \mathcal{G}_t \Big). \tag{10.4}$$

It should be stressed that we do not postulate here that a defaultable claim is attainable. In fact, within the framework of the intensity-based approach, a defaultable claim typically cannot be duplicated by trading in default-free securities, so that the standard arguments based on the existence of a replicating strategy do not apply in this setting. On the other hand, the valuation formula (10.2) can be supported by suitable no-arbitrage arguments. Let us briefly summarize these arguments.

To this end, we assume that $S^1, \ldots, S^n$ are price processes of n non-dividend paying primary assets in our market model, with $S^n = B$. We do not need to be more specific about the nature of primary assets here. It suffices to assume that the savings account B is well-defined. Let the 0^{th} asset correspond to the defaultable claim so that $S^0_t = X^d(t,T)$. We write $\phi = (\phi^0, \ldots, \phi^k)$ to denote an $\mathbb{G}$-predictable process representing a trading strategy. The *wealth process* $U(\phi)$ of a strategy ϕ is given by the formula

$$U_t(\phi) = \sum_{i=0}^{k} \phi^i_t S^i_t, \quad \forall\, t \in [0,T].$$

A strategy ϕ is called *self-financing,* provided that $U_t(\phi) = U_0(\phi) + G_t(\phi)$ for every $t \in [0,T]$, where the *gains process* $G(\phi)$ is defined as follows

$$G_t(\phi) := \int_{]0,t]} \phi^0_u \, dD_u + \sum_{i=0}^{k} \int_{]0,t]} \phi^i_u \, dS^i_u.$$

The following result is merely a reformulation of Corollary 4.1.

Proposition 10.1 *For any self-financing trading strategy* $\phi = (\phi^0, \ldots, \phi^k)$, *the discounted wealth process* $\tilde{U}_t(\phi) = B_t^{-1} U_t(\phi)$, $t \in [0,T]$, *follows a local martingale under* $\mathbb{Q}^*$ *with respect to* $\mathbb{G}$.

It is customary to restrict the class of trading strategies, by postulating that the discounted wealth of an *admissible* strategy follows a martingale under $\mathbb{Q}^*$ (to this end, it suffices, for instance, to consider only strategies with non-negative wealth processes). Proposition 10.1 shows that if the original securities market model is arbitrage-free, and the ex-dividend price process of an additional security (i.e., of a defaultable claim) is given by Definition 10.2, then the arbitrage-free feature of the securities market model is preserved.

10.2 Valuation via the Hazard Process

Before stating the definition of the $\mathbb{F}$-hazard process, let us quote the following useful formula:

$$\mathbb{Q}^*\{t < \tau \le T \,|\, \mathcal{G}_t\} = 1\!\!1_{\{\tau > t\}} \frac{\mathbb{Q}^*\{t < \tau \le T \,|\, \mathcal{F}_t\}}{\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\}}. \tag{10.5}$$

We denote $F_t = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}$, and we shall postulate throughout that the inequality $F_t < 1$ is valid for every $t \in \mathbb{R}_+$. The *survival process* G of the random time τ with respect to the reference filtration $\mathbb{F}$ equals

$$G_t := 1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\}, \quad \forall\, t \in \mathbb{R}_+.$$

Since $\{\tau \le t\} \subseteq \{\tau \le s\}$, for any $0 \le t \le s$ we have:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*}(F_s \,|\, \mathcal{F}_t) &= \mathbb{E}_{\mathbb{Q}^*}(\mathbb{Q}^*\{\tau \le s \,|\, \mathcal{F}_s\} \,|\, \mathcal{F}_t) \\ &= \mathbb{Q}^*\{\tau \le s \,|\, \mathcal{F}_t\} \ge \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\} = F_t,\end{aligned}$$

and so the process F (the survival process G, resp.) follows a bounded, non-negative $\mathbb{F}$-submartingale ($\mathbb{F}$-supermartingale, resp.) under $\mathbb{Q}^*$. The hazard process of the default time, given the flow of information represented by the filtration $\mathbb{F}$, is formally introduced through the following definition.

Definition 10.3 *The $\mathbb{F}$-hazard process of τ under $\mathbb{Q}^*$, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$ or, equivalently,*

$$\Gamma_t := -\ln G_t = -\ln(1 - F_t), \quad \forall\, t \in \mathbb{R}_+.$$

Since $G_0 = 1$, it is clear that $\Gamma_0 = 0$. In view of the equality $\mathbb{Q}^*\{\tau < +\infty\} = 1$, it is also easy to see that $\lim_{t\to\infty} \Gamma_t = \infty$. For the sake of conciseness, we shall refer to Γ as the $\mathbb{F}$-hazard process of τ, rather than the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}^*$. If no risk of ambiguity arises, we shall simply call it the hazard process of τ. Combining formula (10.5) with the definition of the hazard process, we obtain

$$\begin{aligned}\mathbb{Q}^*\{t < \tau \le T \,|\, \mathcal{G}_t\} &= \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}^*}\big(e^{-\Gamma_t} - e^{-\Gamma_T} \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}^*}\big(1 - e^{\Gamma_t - \Gamma_T} \,\big|\, \mathcal{F}_t\big).\end{aligned}$$

It is evident that the hazard process Γ is continuous if and only if the submartingale F, and thus also the supermartingale G, follow continuous processes. Assume, in addition, that the sample paths of F are non-decreasing functions; this amounts to postulating that the martingale part of F vanishes. We adopt the widely used convention of calling such a process an *increasing continuous process.* In this case, the hazard process Γ of τ also follows an increasing continuous process. The following result is standard.

Proposition 10.2 *Let Y be a $\mathcal{G}$-measurable random variable and let $t \le s$. Then*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{F}_t)}{\mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\}}. \tag{10.6}$$

Furthermore,

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > s\}}\, e^{\Gamma_t} Y \,|\, \mathcal{F}_t) \tag{10.7}$$

and

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t < \tau \le s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t < \tau \le s\}}\, e^{\Gamma_t} Y \,|\, \mathcal{F}_t). \tag{10.8}$$

If Y is $\mathcal{F}_s$-measurable, then

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > s\}}\, Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}_{\mathbb{P}}\big(e^{\Gamma_t - \Gamma_s} Y \,|\, \mathcal{F}_t\big). \tag{10.9}$$

10.2.1 Stochastic intensity

In most reduced-form models of credit risk, the hazard process Γ of a default time is postulated to have absolutely continuous sample paths (with respect to the Lebesgue measure on $\mathbb{R}_+$). Specifically, it is assumed that the hazard process Γ of τ admits the following integral representation

$$\Gamma_t = \int_0^t \gamma_u \, du, \quad \forall\, t \in \mathbb{R}_+,$$

for some non-negative, $\mathbb{F}$-progressively measurable stochastic process γ, with integrable sample paths. In addition, we assume that $\int_0^\infty \gamma_u \, du = \infty$, $\mathbb{Q}^*$-a.s. The process γ is called the $\mathbb{F}$-*hazard rate* or the $\mathbb{F}$-*intensity* of τ. It is also customary to refer to γ as the *stochastic intensity* of τ, especially when the choice of the reference filtration $\mathbb{F}$ is clear from the context.

In terms of the stochastic intensity of a default time, the conditional probability of the default event $\{t < \tau \leq T\}$, given the information $\mathcal{G}_t$ available at time t, equals

$$\mathbb{Q}^*\{t < \tau \leq T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}^*}\Big(1 - e^{-\int_t^T \gamma_u du} \,\Big|\, \mathcal{F}_t\Big), \tag{10.10}$$

and the conditional probability of the non-default event $\{\tau > T\}$ equals

$$\mathbb{Q}^*\{\tau > T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}^*}\Big(e^{-\int_t^T \gamma_u du} \,\Big|\, \mathcal{F}_t\Big). \tag{10.11}$$

10.2.2 Intensity function

In some instances, the intensity of a default time is non-random; in such cases, it is referred to as the *intensity function* of τ. The concept of intensity function appears, for instance, when the trivial filtration is chosen as the reference filtration $\mathbb{F}$, so that $\mathbb{G} = \mathbb{H}$. To emphasize the deterministic character of the hazard function, we shall write $\gamma(t)$, rather than γ_t, and so formulae (10.10)–(10.11) become

$$\mathbb{Q}^*\{t < \tau < T \,|\, \mathcal{H}_t\} = \mathbb{1}_{\{\tau > t\}} \Big(1 - e^{-\int_t^T \gamma(u) du}\Big), \tag{10.12}$$

and

$$\mathbb{Q}^*\{\tau > T \,|\, \mathcal{H}_t\} = \mathbb{1}_{\{\tau > t\}} e^{-\int_t^T \gamma(u) du}, \tag{10.13}$$

respectively. Recall that $\mathcal{H}_t = \sigma(H_u : u \leq t) = \sigma(\{\tau \leq u\} : u \leq t)$, and thus $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ is the natural filtration of the random time τ. The assumption that the filtration $\mathbb{H}$ models the flow of information available to a trader amounts to saying that he has no access to the market data other than the occurrence of the default time τ.

In some more general circumstances – for instance, when the default time τ is independent of a (non-trivial) filtration $\mathbb{F}$ – it has a deterministic intensity with respect to $\mathbb{F}$, and equalities (10.12)–(10.13) remain valid with the σ-field $\mathcal{H}_t$ replaced by a strictly larger σ-field $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$.

10.3 Canonical Construction of a Default Time

We shall now briefly describe the most commonly used construction of a default time associated with a given hazard process Γ. It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with a given $\mathbb{F}$-hazard process Γ. We assume that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ defined on a filtered probability space $(\tilde{\Omega}, \mathbb{F}, \mathbb{P}^*)$. As usual, we assume that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In many instances, Γ is given by the equality

$$\Gamma_t = \int_0^t \gamma_u \, du, \quad \forall\, t \in \mathbb{R}_+,$$

for some non-negative, $\mathbb{F}$-progressively measurable intensity process γ.

To construct a random time τ such that Γ is the $\mathbb{F}$-hazard process of τ, we need to enlarge the underlying probability space $\tilde{\Omega}$. This also means that Γ is not the $\mathbb{F}$-hazard process of τ under $\mathbb{P}^*$, but rather the $\mathbb{F}$-hazard process of τ under a suitable extension $\mathbb{Q}^*$ of the probability measure $\mathbb{P}^*$. Let ξ be a random variable defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{Q}})$, uniformly distributed on the interval $[0,1]$ under $\hat{\mathbb{Q}}$. We consider the product space $\Omega = \tilde{\Omega} \times \hat{\Omega}$, endowed with the product σ-field $\mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathcal{F}}$ and the product probability measure $\mathbb{Q}^* = \mathbb{P}^* \otimes \hat{\mathbb{Q}}$. The latter equality means that for arbitrary events $A \in \mathcal{F}_\infty$ and $B \in \hat{\mathcal{F}}$ we have $\mathbb{Q}^*\{A \times B\} = \mathbb{P}^*\{A\}\hat{\mathbb{Q}}\{B\}$.

An alternative way of achieving basically the same goal relies on postulating that the underlying probability space $(\tilde{\Omega}, \mathbb{F}, \mathbb{P}^*)$ is sufficiently rich to support a random variable ξ, uniformly distributed on the interval $[0,1]$, and independent of the filtration $\mathbb{F}$ under $\mathbb{P}^*$. In this version of the canonical construction, Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbb{P}^*$.

We define the random time $\tau : \Omega \to \mathbb{R}_+$ by setting

$$\tau = \inf \{\, t \in \mathbb{R}_+ : e^{-\Gamma_t} \le \xi \,\} = \inf \{\, t \in \mathbb{R}_+ : \Gamma_t \ge \eta \,\}, \tag{10.14}$$

where the random variable $\eta = -\ln \xi$ has a unit exponential law under $\mathbb{Q}^*$. It is not difficult to find the process $F_t = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}$. Indeed, since clearly $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and the random variable Γ_t is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} = \mathbb{Q}^*\{\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty\} = \hat{\mathbb{Q}}\{\xi < e^x\}_{x=\Gamma_t} = e^{-\Gamma_t}. \tag{10.15}$$

Consequently, we have

$$1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} = \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t\big) = e^{-\Gamma_t}, \tag{10.16}$$

and so F is an $\mathbb{F}$-adapted, right-continuous, increasing process. It is also clear that the process Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}^*$. As an immediate consequence of (10.15) and (10.16), we obtain the following interesting property of the canonical construction of the default time:

$$\mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_\infty\} = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}, \quad \forall\, t \in \mathbb{R}_+. \tag{10.17}$$

Let us now analyze some important consequences of (10.17). First, we obtain

$$\mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_\infty\} = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_u\} = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\} = e^{-\Gamma_t} \tag{10.18}$$

for arbitrary two dates $0 \le t \le u$. Notice that only the last equality in (10.18) is necessarily satisfied by the $\mathbb{F}$-hazard process Γ of τ; the first two equalities are additional features of the canonical construction of τ, meaning that they are not necessarily valid in a general set-up. Equality (10.18) entails the conditional independence under $\mathbb{Q}^*$ of the σ-fields $\mathcal{H}_t$ and $\mathcal{F}_t$, given the σ-field $\mathcal{F}_\infty$. Such a property of the two filtrations $\mathbb{H}$ and $\mathbb{F}$ is termed Condition (F.1). It can be shown that Condition (F.1) is equivalent to Condition (M.1), which can be stated as follows: an arbitrary $\mathbb{F}$-martingale also follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$. The latter condition was previously studied by, among others, Brémaud and Yor (1978), Kusuoka (1999) and Elliott et al. (2000). We have the following result. Its proof is left to the reader.

Lemma 10.1 *Assume that the process Γ is continuous. Then the $(\mathbb{F},\mathbb{G})$-martingale hazard process Λ of the random time τ, given by* (10.14), *coincides with the $\mathbb{F}$-hazard process Γ of τ.*

Remarks. In most credit risk models, the reference filtration $\mathbb{F}$ is generated by the process W that follows a Brownian motion under $\mathbb{P}^*$. In view of the martingale invariance property, the canonical construction ensures that the Brownian motion process W remains a continuous martingale (and thus a Brownian motion) under the extended probability measure $\mathbb{Q}^*$ and with respect to the enlarged filtration $\mathbb{G}$. Let us stress again that $\mathbb{Q}^*\{A \times \hat{\Omega}\} = \mathbb{P}^*\{A\}$ for any event $A \in \mathcal{F}_\infty$; that is, the restriction of the probability measure $\mathbb{Q}^*$ to the σ-field $\mathcal{F}_\infty$ coincides with $\mathbb{P}^*$.

Example 10.1 *Deterministic hazard process.* Let us assume that the underlying filtration $\mathbb{F}$ is non-trivial, but the $\mathbb{F}$-hazard process Γ is postulated to follow a deterministic function; that is, the $\mathbb{F}$-hazard process equals Γ for some function $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$. Assume that the default time τ is defined as before – i.e.,

$$\tau = \inf \{ t \in \mathbb{R}_+ : e^{-\Gamma(t)} \leq \xi \}.$$

We claim that the default process H is independent of the filtration $\mathbb{F}$ or, equivalently, that the filtration $\mathbb{H}$ generated by the default process H is independent of the filtration $\mathbb{F}$ under $\mathbb{Q}^*$. It suffices to check that we have, for any fixed $t \in \mathbb{R}_+$ and arbitrary $0 \leq u \leq t$,

$$\mathbb{Q}^*\{\tau \leq u \,|\, \mathcal{F}_t\} = \mathbb{Q}^*\{\tau \leq u\}. \tag{10.19}$$

Equality (10.19) easily follows from (10.18). In effect, we have

$$\mathbb{Q}^*\{\tau \leq u \,|\, \mathcal{F}_t\} = \mathbb{Q}^*\{\tau \leq u \,|\, \mathcal{F}_u\} = 1 - e^{-\Gamma(u)} = \mathbb{Q}^*\{\tau \leq u\},$$

where the last equality is a consequence of the assumption that the hazard process is deterministic.

If the default process H is independent of the filtration $\mathbb{F}$ then any $\mathbb{F}$-adapted process Y is independent of H under $\mathbb{Q}^*$. In particular, since the short-term rate r follows an $\mathbb{F}$-adapted process, processes H and r are mutually independent under $\mathbb{Q}^*$ when the $\mathbb{F}$-hazard process of τ is deterministic, and the default time τ is constructed through the canonical approach.

Example 10.2 *State variables.* In some financial models, it is assumed that the reference filtration $\mathbb{F}$ is generated by some stochastic process, Y say. More specifically, the $\mathbb{F}$-intensity of the default time is given by the equality

$$\Gamma_t = \int_0^t g(u, Y_u)\, du, \quad \forall\, t \in \mathbb{R}_+,$$

for some function $g : \mathbb{R}_+ \times \mathcal{Y} \to \mathbb{R}_+$ satisfying mild technical assumptions, where $\mathcal{Y}$ denotes the state space for the process Y (typically, $\mathcal{Y} = \mathbb{R}^d$).

10.4 Integral Representation of the Value Process

Our next goal is to establish a convenient representation for the pre-default value of a defaultable claim in terms of the hazard process Γ of the default time. For the sake of conciseness, we denote

$$I_t(A) = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1}(1 - H_u)\, dA_u \,\Big|\, \mathcal{G}_t\Big)$$

and

$$J_t(Z) = B_t \, \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau\le T\}} B_\tau^{-1} Z_\tau \,\big|\, \mathcal{G}_t\big).$$

We shall also write

$$\tilde{K}_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} \tilde{X} \mathbb{1}_{\{\tau\le T\}} \,\big|\, \mathcal{G}_t\big), \quad K_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \,\big|\, \mathcal{G}_t\big).$$

It is thus clear that $S_t^0 = I_t(A) + J_t(Z) + \tilde{K}_t + K_t$. Let us stress that we do not need to assume here that the default time τ was constructed through the canonical method.

Proposition 10.3 *The pre-default value process S_t^0 of a defaultable claim $(X, A, 0, Z, \tau)$ admits the following representation for $t \in [0,T]$*

$$S_t^0 = \mathbb{1}_{\{\tau>t\}} G_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1}(G_u\, dA_u - Z_u\, dG_u) + G_T B_T^{-1} X \,\Big|\, \mathcal{F}_t\Big).$$

If the survival process G, and thus also the hazard process Γ, are continuous, then

$$S_t^0 = \mathbb{1}_{\{\tau>t\}} B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} e^{\Gamma_t - \Gamma_u}\,(dA_u + Z_u\, d\Gamma_u) + B_T^{-1} X e^{\Gamma_t - \Gamma_T} \,\Big|\, \mathcal{F}_t\Big).$$

Proofs of the next two auxiliary results are omitted (for details, see Appendix 2)

Lemma 10.2 *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a bounded, continuous function. Then for any $t < s \le \infty$*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} h(\tau) \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \, \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} h(u)\, dF_u \,\Big|\, \mathcal{F}_t\Big). \tag{10.20}$$

Let Z be a bounded, $\mathbb{F}$-predictable process. Then for any $t < s \le \infty$

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} Z_\tau \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \, \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} Z_u\, dF_u \,\Big|\, \mathcal{F}_t\Big). \tag{10.21}$$

Lemma 10.3 *Assume that A is a bounded, $\mathbb{F}$-predictable process of finite variation. Then for every $t \le s$*

$$\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} (1 - H_u)\, dA_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} (1 - F_u)\, dA_u \,\Big|\, \mathcal{F}_t\Big)$$

or, equivalently,

$$\mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} (1 - H_u)\, dA_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}}\Big(\int_{]t,s]} e^{\Gamma_t - \Gamma_u}\, dA_u \,\Big|\, \mathcal{F}_t\Big).$$

Proof of Proposition 10.3. Since $\tilde{X} = 0$, it is obvious that $\tilde{K}_t = 0$ for $t \in [0,T]$, and so the value process satisfies: $S_t^0 = I_t(A) + J_t(Z) + K_t$. By applying Lemma 7.3 to the process of finite variation $\int_{]0,t]} B_u^{-1}\, dA_u$, we obtain

$$I_t(A) = 1\!\!1_{\{\tau>t\}} G_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} G_u\, dA_u \,\Big|\, \mathcal{F}_t \Big)$$

or, equivalently,

$$I_t(A) = 1\!\!1_{\{\tau>t\}} B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} e^{\Gamma_t - \Gamma_u}\, dA_u \,\Big|\, \mathcal{F}_t \Big).$$

Furthermore, formula (7.21) of Lemma 7.2 yields

$$J_t(Z) = -1\!\!1_{\{\tau>t\}} G_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} Z_u\, dG_u \,\Big|\, \mathcal{F}_t \Big).$$

If, in addition, the survival process G is a continuous (and thus a decreasing) process, the hazard process Γ is an increasing continuous process, and

$$J_t(Z) = 1\!\!1_{\{\tau>t\}} B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T B_u^{-1} e^{\Gamma_t - \Gamma_u} Z_u\, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big).$$

Finally, it follows from (7.6) that

$$K_t = 1\!\!1_{\{\tau>t\}} G_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}^*}(1\!\!1_{\{\tau>T\}} B_T^{-1} X \,|\, \mathcal{F}_t). \tag{10.22}$$

Since the random variables X and B_T are $\mathcal{F}_T$-measurable, we also have (see (7.9))

$$K_t = 1\!\!1_{\{\tau>t\}} G_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}^*}(G_T B_T^{-1} X \,|\, \mathcal{F}_t) = 1\!\!1_{\{\tau>t\}} B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} X e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t\big).$$

Both formulae of the proposition are obtained upon summation. □

Corollary 10.1 *Assume that the $\mathbb{F}$-hazard process Γ follows a continuous process of finite variation. Then the pre-default value of a defaultable claim $(X, A, 0, Z, \tau)$ coincides with the pre-default value of a defaultable claim $(X, \hat{A}, 0, 0, \tau)$, where $\hat{A}_t = A_t + \int_0^t Z_u\, d\Gamma_u$.*

Remarks. We have omitted in Proposition 10.3 the recovery payoff $\tilde{X}$, since the expression based on the hazard process of the default time does not easily cover the case of a general $\mathcal{F}_T$-measurable random variable. However, in the special case when $\tilde{X} = \delta$ for some constant δ, it suffices to substitute $\tilde{X}$ with an equivalent payoff $\delta B(\tau, T)$ at time of default.

Let us return to the case of the default time that admits the stochastic intensity γ. The second formula of Proposition 10.3 now takes the following form

$$\begin{aligned} S_t^0 &= 1\!\!1_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} e^{-\int_t^u (r_v+\gamma_v)dv} (dA_u + \gamma_u Z_u\, du) \,\Big|\, \mathcal{F}_t \Big) \\ &\quad + 1\!\!1_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(e^{-\int_t^T (r_v+\gamma_v)dv} X \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

To get a more concise representation for the last expression, we introduce the *default-risk-adjusted interest rate* $\tilde{r} = r + \gamma$ and the associated *default-risk-adjusted savings account* $\tilde{B}$, given by the formula

$$\tilde{B}_t = \exp\Big(\int_0^t \tilde{r}_u\,du\Big), \quad \forall\, t \in \mathbb{R}_+. \tag{10.23}$$

Although $\tilde{B}_t$ does not represent the price of a tradeable security, it has similar features as the savings account B; in particular, $\tilde{B}$ also follows an $\mathbb{F}$-adapted, continuous process of finite variation. In terms of the process $\tilde{B}$, we have

$$S_t^0 = \mathbb{1}_{\{\tau>t\}} \tilde{B}_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} \tilde{B}_u^{-1}\,dA_u + \int_t^T \tilde{B}_u^{-1} Z_u \gamma_u\,du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t\Big). \tag{10.24}$$

It is noteworthy that the default time τ does not appear explicitly in the conditional expectation on the right-hand side of (10.24).

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Faculty of Mathematics and Information Science
Credit Risk Modelling: Lecture 11

11 Various Recovery Schemes

In this lecture, we shall examine few application the intensity-based approach to the valuation of basic examples of defaultable contingent claim. For further examples, we refer to, e.g., Jarrow and Turnbull (1995), Duffie et al. (1997), Duffie (1998), Lando (1998), and Schönbucher (1998).

11.1 Case of a Deterministic Intensity

For the sake of simplicity, we shall assume in this section that the default time τ admits the intensity function γ with respect to $\mathbb{F}$, and the continuously compounded interest rate r is deterministic. In view of the latter assumption, at time t the price of a unit default-free zero-coupon bond of maturity T equals

$$B(t,T) = e^{-\int_t^T r(v)\,dv}, \quad \forall\, t \in [0,T].$$

Our goal is to derive some integral representations for the pre-default values of simple defaultable claims. We take $A \equiv 0$, $\tilde{X} = 0$ and $Z_\tau = h(\tau)$ for some continuous function $h : \mathbb{R}_+ \to \mathbb{R}$. If, in addition, the promised payoff X is non-random, the pre-default value of the claim equals

$$S_t^0 = \mathbb{1}_{\{\tau > t\}} B_t \Big(\int_t^T e^{-\int_t^u \gamma(v)dv} B_u^{-1} \gamma(u) h(u)\,du + B_T^{-1} X e^{-\int_t^T \gamma(v)dv} \Big)$$

or, equivalently,

$$S_t^0 = \mathbb{1}_{\{\tau > t\}} \Big(\int_t^T e^{-\int_t^u \tilde{r}(v)dv} \gamma(u) h(u)\,du + X e^{-\int_t^T \tilde{r}(v)dv} \Big), \tag{11.1}$$

where $\tilde{r}(v) = r(v) + \gamma(v)$.

Remarks. Let us again stress that S_t^0 represents only the pre-default value of a defaultable claim. At any date t, the discounted payoff of the defaultable claim introduced above is given by the following expression

$$Y_t = \mathbb{1}_{\{\tau \le T\}}\, h(\tau) e^{-\int_t^\tau r(v)dv} + \mathbb{1}_{\{\tau > T\}} X e^{-\int_t^T r(v)dv}.$$

Thus, the 'full' value at time t of a defaultable claim equals

$$\mathbb{E}_{\mathbb{Q}^*}(Y_t \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} \Big(\int_t^T e^{-\int_t^u \tilde{r}(v)dv} h(u) \gamma(u)\,du + X e^{-\int_t^T \tilde{r}(v)dv} \Big) + \mathbb{1}_{\{\tau \le t\}} h(\tau) e^{\int_\tau^t r(v)dv}.$$

The additional third term in the last formula represents the current value of the recovery cash flow $h(\tau)$ received by the owner of the claim at the time of default, and reinvested in the savings account.

Let us consider few examples of corporate zero-coupon bonds with maturity date T that are subject to various recovery schemes. In the next section, we shall study these schemes in the context of general contingent claims. In all cases examined below, the pre-default value of a corporate bond appears to be proportional to the bond's face value, L. In what follows, when referring to the pre-default values of corporate bonds, we shall usually set $L = 1$ and we shall suppress L from the notation.

11.1.1 Zero recovery

Let us first consider a corporate zero-coupon bond with *zero recovery* at default. This corresponds to the choice of $h = 0$ and $X = L = 1$ in (11.1). Denoting by $D^0(t,T)$ the pre-default value at time t of such a bond, for every $t \in [0,T]$ we obtain:

$$D^0(t,T) = \mathbb{1}_{\{\tau>t\}} e^{-\int_t^T (r(v)+\gamma(v))dv} = \mathbb{1}_{\{\tau>t\}} B(t,T) e^{-\int_t^T \gamma(v)dv}.$$

Under the zero recovery scheme, the corporate bond becomes, of course, valueless as soon as the default occurs.

11.1.2 Fractional recovery of par value

Let us assume that the recovery function h satisfies $h = \delta L = \delta$ for some constant recovery coefficient $0 \le \delta \le 1$. The corresponding recovery scheme is aptly termed the *fractional recovery of par value.* The pre-default value at time t of a corporate bond that is subject to this recovery scheme, denoted by $\tilde{D}^\delta(t,T)$, equals

$$\tilde{D}^\delta(t,T) = \mathbb{1}_{\{\tau>t\}} \Big(\int_t^T e^{-\int_t^u \tilde{r}(v)\,dv} \delta\gamma(u)\,du + e^{-\int_t^T \tilde{r}(v)\,dv} \Big).$$

Notice that $\tilde{D}^\delta(t,T)$ represents the value before default of a corporate bond that pays at time of default a constant payoff proportional to the bond's face value, in case the bond defaults before or at the bond's maturity date T. However, it is clear that constant coefficient δ can be replaced by some function $\delta(t)$ of time (the same remark applies to the next recovery scheme).

11.1.3 Fractional recovery of Treasury value

Let us finally assume that the recovery function equals

$$h(\tau) = \delta L e^{-\int_\tau^T r(v)\,dv} = \delta e^{-\int_\tau^T r(v)\,dv}. \tag{11.2}$$

The above specification of the recovery function describes a corporate zero-coupon bond with the so-called *fractional recovery of Treasury value.* Indeed, since the payoff $h(\tau)$ can be invested in the savings account, we may formally postulate that the bond pays the constant payoff δ at maturity T if default occurs before maturity (otherwise, it pays the nominal value $L = 1$). We may thus equally well postulate that the recovery payoff at default equals

$$h(\tau) = \delta B(\tau,T). \tag{11.3}$$

Under the present assumptions, the two alternative specifications, (11.2) and (11.3), yield an identical pre-default value of a corporate bond with the fractional recovery of Treasury value.

It is thus interesting to notice that the latter specification is much more convenient if random character of interest rates is taken into account. Indeed, in all models the current value bond price $B(\tau, T)$ is known at time τ, and thus one can always define the recovery process Z by setting $Z_t = \delta B(t,T)$ for every $t \in [0,T]$. On the other hand, the right-hand side of (11.2) is not observed at time τ under the uncertainty of interest rates.

Let us denote by $D^\delta(t,T)$ the pre-default value of a unit corporate bond with the fractional recovery of Treasury value. By plugging (11.2) into the general formula (11.1), we obtain

$$D^\delta(t,T) = \mathbb{1}_{\{\tau>t\}} \Big(\int_t^T e^{-\int_t^T r(v)dv} e^{-\int_t^u \gamma(v)dv} \delta\gamma(u)\,du + e^{-\int_t^T \tilde{r}(v)dv} \Big),$$

that is,

$$D^\delta(t,T) = \mathbb{1}_{\{\tau>t\}} B(t,T) \Big\{ \delta \Big(1 - e^{-\int_t^T \gamma(v)\,dv} \Big) + e^{-\int_t^T \gamma(v)\,dv} \Big\}.$$

We end this example by noticing that the pre-default value $D^\delta(t,T)$ of a corporate bond with the fractional recovery of Treasury value can also be expressed as follows (see (10.10))

$$D^\delta(t,T) = B(t,T) \big(\delta\, \mathbb{Q}^*\{t < \tau \le T \,|\, \mathcal{G}_t\} + \mathbb{Q}^*\{\tau > T \,|\, \mathcal{G}_t\} \big).$$

It is worth stressing that the last representation, though apparently universal, in fact hinges on the non-random character of interest rates postulated in this section. In a more general setting, we need to impose some further assumptions, as well as to substitute the spot martingale measure $\mathbb{Q}^*$ with the associated forward martingale measure $\mathbb{Q}_T$.

11.2 Implied Probabilities of Default

Simple valuation formulae based on the intensity function are frequently used by practitioners in order to calibrate the model. The basic idea is to derive the default probabilities implicit in market quotes of traded defaultable securities (corporate bonds, default swaps, etc.), and to subsequently use these probabilities to value defaultable securities that are not quoted in the market. It is apparent that such an approach to model's calibration parallels the widely popular method of using implied volatilities of publicly traded (or at least liquid) options to value these exotic options for which the market quotes are either not available, or not reliable. Typically, it is postulated that:

- the interest rate process and the default process are mutually independent under the spot martingale measure $\mathbb{Q}^*$,
- the default can only be observed at some date from a given finite collection of dates $0 < T_1 < \cdots < T_n = T^*$, for some horizon date T^*,
- we are given the default-free term structure of interest rates, formally identified here with the prices $B(0, T_i)$, $i = 1, \dots, n$ of zero-coupon Treasury bonds.

The first assumption means that we are interested in finding the intensity function, as opposed to the intensity process with respect to some non-trivial filtration $\mathbb{F}$. The second means that we are not interested in the exact behavior of the intensity function between the 'observed default dates,' so that we may adopt a convention that the intensity function is constant between each two dates T_i and T_{i+1}.

In view of the preceding discussion, we may, and do, assume that the intensity function $\lambda : [0, T^*] \to \mathbb{R}_+$ satisfies $\lambda(t) = \sum_{i=1}^n \alpha_i \mathbb{1}_{[T_{i-1}, T_i[}(t)$ for some positive constants α_i, $i = 0, \ldots, n-1$. This in turn amounts to postulate that the probabilities of survival satisfy, for $j = 1, \ldots, n$,

$$G(T_j) = \mathbb{Q}^*\{\tau \geq T_j\} = 1 - \exp\Big(- \sum_{i=1}^{j} \alpha_i (T_i - T_{i-1})\Big)$$

or, equivalently, that for every $j = 1, \ldots, n$

$$q_j^* := \mathbb{Q}^*\{T_{j-1} < \tau \leq T_j\} = G(T_j) - G(T_{j-1}) = \exp\big(- \alpha_j (T_j - T_{j-1})\big).$$

Notice that in general the inequality $\sum_{j=1}^n q_j^* = \mathbb{Q}^*\{\tau \leq T_n\} \leq 1$ is valid (we set $T_0 = 0$). In other words, we do not need to assume that the default will definitely happen before or at the horizon date T^*.

Example 11.1 Let us assume, for instance, that our goal is to calibrate the model to market quotes of a family of default swaps. For simplicity, we assume that $T_1, \ldots, T_n$ are payment dates, and T_n is the maturity of the contract. Under the independence assumption, the present value at time 0 of the default payment leg is

$$I_1 = \sum_{i=1}^{n} B(0, T_i) X_{T_i} q_i^* = \sum_{i=1}^{n} B(0, T_i) X_{T_i} \big(G(T_i) - G(T_{i-1})\big),$$

where the (non-random) payoffs X_{T_i} are typically expressed either as some fixed amounts, or as a percentage of the present value of the future coupons and the face value of the underlying bond discounted at the risk-free rate.

The premium payment leg is defined as a stream of fixed cash flows κ that are paid until the maturity of the contract or until default, whichever comes first. The present value of these cash flows at time 0 equals

$$\begin{aligned} I_2 &= B_0 \, \mathbb{E}_{\mathbb{Q}^*} \Big(\sum_{i=1}^{n} \mathbb{1}_{\{T_{i-1} < \tau \leq T_i\}} \sum_{j=1}^{i} B_{T_j}^{-1} \kappa \,\Big|\, \mathcal{G}_0 \Big) \\ &= \kappa \sum_{i=1}^{n} q_i^* \sum_{j=1}^{i} B(0, T_j) = \kappa \sum_{i=1}^{n} B(0, T_i) \sum_{j=1}^{i} q_j^* \\ &= \kappa \sum_{i=1}^{n} B(0, T_i) \mathbb{Q}^*\{\tau \leq T_i\} = \kappa \sum_{i=1}^{n} B(0, T_i)(1 - G(T_i)). \end{aligned}$$

Given a portfolio of default swaps and their market quotes, we may search for the values of α_i, $i = 0, \ldots, n-1$. The calibration procedure relies on solving non-linear equation of the form $I_1 = I_2$. In principle, the quoted default swaps should be repriced correctly within the calibrated model. Using this approach, we may not only value new issues of contracts that are exposed to the default risk of the underlying entity, but we may also mark to market outstanding default swaps and other defaultable contracts.

11.3 Exogenous Recovery Rules

We shall now return to the case of a defaultable claim $DCT = (X, A, 0, Z, \tau)$. In Sect. 11.1, we have briefly presented several alternative recovery rules in the context of intensity-based valuation of a corporate bond. As expected, these schemes can be extended to the case of an arbitrary defaultable claim. We shall now examine these extensions in some detail.

11.3.1 Fractional recovery of par value

We need to assume here that the par value (or the face value) of a defaultable claim is well defined. Denoting by L the constant representing the claim's par value and by δ the claim's recovery rate, we set $Z_t = \delta L$. Therefore, the pre-default value, denoted by $\tilde{D}^\delta_t$, equals

$$\tilde{D}^\delta_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1}(1 - H_u)\, dA_u + \int_{]t,T]} B_u^{-1}\delta L \, dH_u + B_T^{-1} X \mathbb{1}_{\{\tau > T\}} \,\Big|\, \mathcal{G}_t \Big).$$

Consequently, by virtue of Proposition 10.3,

$$\tilde{D}^\delta_t = \mathbb{1}_{\{\tau > t\}} G_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1}(G_u \, dA_u - \delta L \, dG_u) + G_T B_T^{-1} X \,\Big|\, \mathcal{F}_t \Big),$$

where G is the survival process of τ with respect to the reference filtration $\mathbb{F}$. In the case of a continuous survival process G, the last formula yields

$$\tilde{D}^\delta_t = \mathbb{1}_{\{\tau > t\}} B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} e^{\Gamma_t - \Gamma_u} \, (dA_u + \delta L \, d\Gamma_u) + B_T^{-1} X e^{\Gamma_t - \Gamma_T} \,\Big|\, \mathcal{F}_t \Big),$$

where $\Gamma_t = -\ln G_t$ is the $\mathbb{F}$-hazard process of the default time.

Example 11.2 Let us first assume that $A \equiv 0$. We shall write $U(t, T)$ to denote the price of a *digital default put* – that is, a default-risk sensitive security, which pays one unit of cash at time τ if default occurs prior to or at T, and pays zero otherwise. Formally, a digital default put corresponds to a defaultable claim of the form $(0, 0, 0, 1, \tau)$. We have $U(t, T) = \tilde{D}^1(t, T) - D^0(t, T)$ or, more explicitly,

$$U(t, T) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_\tau^{-1} \mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{G}_t).$$

Let $\mathbb{Q}_T$ be the T-forward martingale measure, associated with $\mathbb{Q}^*$ through the formula

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}^*} = \frac{1}{B(0, T) B_T}, \quad \mathbb{Q}^*\text{-a.s.}$$

Using the abstract Bayes rule, we obtain the following representation for the price of a defaultable claim in terms of the forward martingale measure

$$\tilde{D}^\delta_t = \mathbb{1}_{\{\tau > t\}} \delta L S(t, T) + \mathbb{1}_{\{\tau > t\}} B(t, T) \, \mathbb{E}_{\mathbb{Q}_T} \big(X e^{\Gamma_t - \Gamma_T} \,\big|\, \mathcal{F}_t \big).$$

Notice that the hazard process of τ is not affected by the change of probability measure from $\mathbb{Q}^*$ to $\mathbb{Q}_T$. If the promised dividends process A does not vanish, we need to add an extra term on the right-hand side of the last equality.

As already observed in Corollary 10.1, if the $\mathbb{F}$-hazard process Γ follows a continuous process of finite variation, we may set $Z \equiv 0$ and substitute the promised dividends process A with the process $\hat{A}_t = A_t + \delta L \Gamma_t$. In other words, from the point of view of arbitrage-free valuation the two defaultable claims $(X, A, \tilde{X}, \delta L, \tau)$ and $(X, A + \delta L\Gamma, \tilde{X}, 0, \tau)$ are essentially equivalent if Γ is a continuous process of finite variation.

Finally, if the default time τ admits the $\mathbb{F}$-intensity process γ, then we have (cf. (10.24))

$$\tilde{D}_t^\delta = 1\!\!1_{\{\tau>t\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} \tilde{B}_u^{-1} \, dA_u + \delta L \int_t^T \tilde{B}_u^{-1} \gamma_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big),$$

where the default-risk-adjusted savings account $\tilde{B}$ is given by (10.23). If, in addition, the sample paths of the process A are absolutely continuous functions: $A_t = \int_0^t a_u \, du$, then

$$\begin{aligned} \tilde{D}_t^\delta &= 1\!\!1_{\{\tau>t\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T \tilde{B}_u^{-1} (a_u + \delta L \gamma_u) \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big) \\ &= 1\!\!1_{\{\tau>t\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T \tilde{B}_u^{-1} (a_u \gamma_u^{-1} + \delta L) \gamma_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big), \end{aligned}$$

where the last equality holds, provided that $\gamma > 0$. We may here choose, without loss of generality, $\mathbb{F}$-predictable versions of processes a and γ. In view of the considerations above, we are in a position to state the following corollary, which furnishes still another equivalent representation of a defaultable claim with fractional recovery of par value.

Corollary 11.1 *Assume that $A_t = \int_0^t a_u \, du$ and $\Gamma_t = \int_0^t \gamma_u \, du$ with $\gamma > 0$. Then a defaultable claim $(X, A, \tilde{X}, \delta L, \tau)$ is equivalent to a defaultable claim $(X, 0, \tilde{X}, \hat{Z}, \tau)$, where $\hat{Z}_t = \delta L + a_t \gamma_t^{-1}$.*

11.3.2 Fractional recovery of no-default value

In case of a general contingent claim, the counterpart of the fractional recovery of Treasury value scheme is referred to as the *fractional recovery of no-default value.* In this scheme, it is assumed that the owner of a defaultable claim receives at time of default a fixed fraction of a market value of an equivalent non-defaultable security. By definition, the *no-default value* (also known as the *Treasury value*) of a defaultable claim $(X, A, \tilde{X}, Z, \tau)$ is equal to the expected discounted value of the promised dividends A and the promised contingent claim X, specifically:

$$U_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{[t,T]} B_u^{-1} \, dA_u + B_T^{-1} X \,\Big|\, \mathcal{G}_t \Big). \tag{11.4}$$

Notice that U includes also the dividends paid at time t. When valuing a defaultable claim $(X, A, \tilde{X}, Z, \tau)$ with fractional recovery of no-default value, we set $\tilde{X} = 0$ and $Z_t = \delta U_t$, where U is given by the last formula. Put more explicitly, the pre-default value equals

$$D_t^\delta = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} (1 - H_u) \, dA_u + \int_{]t,T]} B_u^{-1} \delta U_u \, dH_u + B_T^{-1} X 1\!\!1_{\{\tau>T\}} \,\Big|\, \mathcal{G}_t \Big).$$

Proposition 11.1 *For any $t < T$ we have $D_t^\delta = (1-\delta)\tilde{D}_t^0 + 1\!\!1_{\{\tau>t\}} \delta \tilde{U}_t$, where the process $\tilde{D}_t^0$, which equals*

$$\tilde{D}_t^0 = 1\!\!1_{\{\tau>t\}} G_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} G_u \, dA_u + G_T B_T^{-1} X \,\Big|\, \mathcal{F}_t \Big),$$

represents the pre-default value of a defaultable claim $(X, A, 0, 0, \tau)$ *with zero recovery and* $\tilde{U}_t$ *is given by*

$$\tilde{U}_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} \, dA_u + B_T^{-1} X \,\Big|\, \mathcal{G}_t \Big). \tag{11.5}$$

Proof. We shall sketch the proof. Since manifestly

$$D_t^{\delta} = \tilde{D}_t^0 + \delta B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} U_u \, dH_u \,\Big|\, \mathcal{G}_t \Big),$$

it suffices to show that the following equality is valid:

$$1\!\!1_{\{\tau > t\}} \tilde{U}_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} (1 - H_u) \, dA_u + B_T^{-1} X \, 1\!\!1_{\{\tau > T\}} \,\Big|\, \mathcal{G}_t \Big) + J,$$

where we have set

$$J := B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} U_u \, dH_u \,\Big|\, \mathcal{G}_t \Big).$$

But

$$\begin{aligned} J &= B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{[u,T]} B_v^{-1} \, dA_v + B_T^{-1} X \,\Big|\, \mathcal{G}_u \Big) dH_u \,\Big|\, \mathcal{G}_t \Big) \\ &= B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} \Big(\int_{[u,T]} B_v^{-1} \, dA_v + B_T^{-1} X \Big) dH_u \,\Big|\, \mathcal{G}_t \Big) \\ &= B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(1\!\!1_{\{\tau > t\}} \int_{]t,T]} B_u^{-1} H_u \, dA_u + B_T^{-1} X \, 1\!\!1_{\{t < \tau \le T\}} \,\Big|\, \mathcal{G}_t \Big), \end{aligned}$$

where we have used, in particular, Fubini's theorem. □

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