CREDIT RISK: MODELLING, VALUATION AND HEDGING

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VALUE-OF-THE-FIRM APPROACH

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VALUE-OF-THE-FIRM APPROACH

Advantages:

- An approach based on the volatility of the total value of a firm. The credit risk is thus measured in a standard way.
- The random time of default is defined in an intuitive way; it reflects the notion of the firm's insolvency.
- Valuation of defaultable claims relies on similar techniques as the valuation of exotic options in the Black-Scholes setup.
- The concept of the distance to default, which measures the obligor's leverage relative to the volatility of its assets values, may serve to reflect credit ratings.
- Dependent defaults are easy to handle through correlation of processes corresponding to different names.

Disadvantages:

- Assumes the total value of the firm's assets can be easily observed.
- Postulates that the total value of the firm's assets is a tradable security.
- Generates low credit spreads for corporate bonds close to maturity.
- Requires a judicious specification of the default barrier in order to get a good fit with the observed spread curves.

VALUE-OF-THE-FIRM APPROACH

1 Basic Assumptions

We fix a finite horizon date $T^* > 0$, and we suppose that the underlying probability space $(\Omega, \mathcal{F}, \mathsf{P})$, endowed with some filtration $\mathsf{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$, is sufficiently rich to support the following objects:

- the *short-term interest rate* process r,
- the *firm's value process* V, which models the total value of the firm's assets,
- the *barrier process* v, which will serve to specify the default time,
- the *promised contingent claim* X representing the firm's liabilities to be redeemed at time $T \leq T^*$,
- the process C, which models the *promised dividends,* i.e., the firm's liabilities stream that is redeemed continuously or discretely over time to the holder of a defaultable claim,
- the *recovery claim* $\tilde{X}$, which represents the recovery payoff received at time T, if default occurs prior to or at the claim's maturity date T,
- the *recovery process* Z, which specifies the recovery payoff at time of default, if it occurs prior to or at the maturity date T.

1.1 Defaultable Claims

Technical Assumptions:

We postulate that the processes V, Z, C, and v are progressively measurable with respect to the filtration F, and that the random variables X and $\tilde{X}$ are $\mathcal{F}_T$-measurable. In addition, C is assumed to be a process of finite variation, with $C_0 = 0$. We assume without mentioning that all random objects introduced above satisfy suitable integrability conditions.

1.1.1 Probabilities P and P*

The probability measure P is assumed to represent the *real-world* (or *statistical*) probability, as opposed to the *spot martingale measure* (or the *risk-neutral probability*). The latter probability is denoted by P* in what follows.

1.1.2 Default Time

Let us denote by τ the random time of default. It is essential to emphasise that the various approaches to valuing and hedging of defaultable securities differ between themselves with regard to the ways in which the default event – and thus also the default time τ – are modeled. In the structural approach, the default time τ will be typically defined in terms of the value process V and the barrier process v.

We set

$$\tau := \inf \{ t > 0 : t \in \mathcal{T}, V_t \le v_t \}$$

with the usual convention that the infimum over the empty set equals $+\infty$. The set $\mathcal{T}$ is assumed to be a Borel measurable subset of the time interval $[0, T]$ (or $[0, \infty)$ in the case of perpetual claims).

In most structural models, the default time τ is given by the formula:

$$\tau := \inf \{ t > 0 : t \in [0, T], V_t \le \bar{v}(t) \},$$

where $\bar{v} : [0, T] \to \mathsf{R}_+$ is some deterministic function, known as the barrier.

Predictability of τ

Typically, τ will be an F-stopping time, and since the underlying filtration F in most structural models is generated by a standard Brownian motion, τ will be an F-predictable stopping time (as any stopping time with respect to a Brownian filtration).

The latter property means that within the framework of the structural approach there exists a sequence of increasing stopping times announcing the default time; in this sense, the default time can be forecasted with some degree of certainty.

In the intensity-based approach, the default time will not be a predictable stopping time with respect to the 'enlarged' filtration, denoted by G in Part 3. In typical examples, the filtration G will encompass some Brownian filtration F, but G will be strictly larger than F. At the intuitive level, in the intensity-based approach the occurrence of the default event comes as a total surprise. For any date t, the value γ_t of the default intensity yields the conditional probability of the occurrence of default over an infinitesimally small time interval $[t, t+dt]$.

1.1.3 Recovery Rules

If default occurs after time T, the promised claim X is paid in full at time T. Otherwise, depending on the adopted model, either (1) the amount $\tilde{X}$ is paid at the maturity date T, or (2) the amount Z_τ is paid at time τ.

In a general setting, we consider simultaneously both kinds of recovery payoff, and thus a defaultable claim is formally defined as a quintuple

$$DCT = (X, C, \tilde{X}, Z, \tau).$$

In most practical situations, however, we shall deal with only one type of recovery payoff – that is, we shall set either $\tilde{X} = 0$ or $Z \equiv 0$.

1.2 Risk-Neutral Valuation Formula

Suppose that our financial market model is arbitrage-free, in the sense that there exists a *spot martingale measure* (*risk-neutral probability*) P^*, meaning that price process of any tradeable security, which pays no coupons or dividends, follows an $\mathbb{F}$-martingale under P^*, when discounted by the *savings account* B, given as

$$B_t := \exp\big(\int_0^t r_u \, du \big).$$

We introduce the jump process $H_t = \mathbb{1}_{\{\tau \le t\}}$, and we denote by D the process that models all cash flows received by the owner of a defaultable claim. Let us denote

$$X^d(T) = X \, \mathbb{1}_{\{\tau > T\}} + \tilde{X} \, \mathbb{1}_{\{\tau \le T\}}.$$

Definition 1 The *dividend process* D of a defaultable contingent claim $DCT = (X, C, \tilde{X}, Z, \tau)$, which settles at time T, equals

$$D_t = X^d(T) \, \mathbb{1}_{\{t \ge T\}} + \int_{]0,t]} (1 - H_u) \, dC_u + \int_{]0,t]} Z_u \, dH_u.$$

Notice that D is a process of finite variation, and

$$\int_{]0,t]} (1 - H_u) \, dC_u = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}} \, dC_u = C_{\tau-} \, \mathbb{1}_{\{\tau \le t\}} + C_t \, \mathbb{1}_{\{\tau > t\}}.$$

It is apparent that if default occurs at some date t, the promised dividend $C_t - C_{t-}$, which is due to be paid at this date, is not received by the holder of a defaultable claim. Furthermore, if we set $\tau \wedge t = \min(\tau, t)$ then

$$\int_{]0,t]} Z_u \, dH_u = Z_{\tau \wedge t} \, \mathbb{1}_{\{\tau \le t\}} = Z_\tau \, \mathbb{1}_{\{\tau \le t\}}.$$

The promised payoff X could be incorporated into the promised dividends process C. However, this would be inconvenient, since in practice the recovery rules concerning the promised dividends C and the promised claim X are different, in general. For instance, in the case of a defaultable coupon bond, it is frequently postulated that in case of default the future coupons are lost, but a strictly positive fraction of the face value is usually received by the bondholder.

We are in a position to define the ex-dividend price $X^d(t,T)$ of a defaultable claim. At any time t, the random variable $X^d(t,T)$ is meant to represent the current value of all future cash flows associated with a given defaultable claim DCT. In particular, we always have $X^d(T,T) = 0$.

Definition 2 The (ex-dividend) *price process* of the defaultable claim $DCT = (X, C, \tilde{X}, Z, \tau)$ is given as

$$X^d(t,T) = B_t \, \mathsf{E}_{\mathsf{P}^*} \left(\int_{]t,T]} B_u^{-1} \, dD_u \,\middle|\, \mathcal{F}_t \right).$$

1.3 Corporate Zero-Coupon Bond

Assume that $C \equiv 0$ and $X = L$ for some positive constant $L > 0$. Then the value process represents the arbitrage price of a defaultable (corporate) zero-coupon bond with the face value L.

The price $D(t,T)$ of such a bond equals

$$D(t,T) = B_t \, \mathsf{E}_{\mathsf{P}^*}(B_T^{-1}(L \, \mathbb{1}_{\{\tau > T\}} + \tilde{X} \, \mathbb{1}_{\{\tau \le T\}}) \,|\, \mathcal{F}_t).$$

It is convenient to rewrite the last formula as follows:

$$D(t,T) = L B_t \, \mathsf{E}_{\mathsf{P}^*}(B_T^{-1}(\mathbb{1}_{\{\tau > T\}} + \delta(T) \, \mathbb{1}_{\{\tau \le T\}}) \,|\, \mathcal{F}_t),$$

where the random variable $\delta(T) = \tilde{X}/L$ represents the so-called *recovery rate upon default.* It is natural to assume that $0 \le \tilde{X} \le L$ so that $\delta(T)$ satisfies $0 \le \delta(T) \le 1$.

Alternatively, we may re-express the bond price as follows:

$$D(t,T) = L(B(t,T) - B_t \, \mathsf{E}_{\mathsf{P}^*}(B_T^{-1} w(T) \, \mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{F}_t)),$$

where

$$B(t,T) := B_t \, \mathsf{E}_{\mathsf{P}^*}(B_T^{-1} \,|\, \mathcal{F}_t)$$

denotes the price of a unit default-free zero-coupon bond, and $w(T) := 1 - \delta(T)$ is the *writedown rate upon default.*

We conclude the value of a corporate bond depends on the joint probability distribution under P^* of the three-dimensional random variable $(B_T, \delta(T), \tau)$ or, equivalently, of the three-dimensional random variable $(B_T, w(T), \tau)$.

Example. Merton's (1974) model postulates that the recovery payoff upon default equals $\widehat{X} = V_T$, where the random variable V_T represents the value of the firm at time T. Consequently, the random recovery rate upon default equals $\delta(T) = V_T/L$, and the writedown rate upon default equals $w(T) = 1 - V_T/L$.

Expected writedowns

Assume that the savings account B is non-random, that is, the short-term rate r is deterministic. Then the price of a default-free zero-coupon bond is $B(t,T) = B_t B_T^{-1}$, and the price of a zero-coupon corporate bond equals

$$D(t,T) = L_t(1 - w^*(t,T)),$$

where $L_t = LB(t,T)$ is the present value of future liabilities, and $w^*(t,T)$ is the *conditional expected writedown rate* under P^*, given by the following equality:

$$w^*(t,T) = \mathsf{E}_{\mathsf{P}^*}(w(T)\,\mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{F}_t).$$

Notice that we may set $w(T) = 0$ on the event $\{\tau > T\}$.

The *conditional expected writedown rate upon default* equals, under P^*,

$$w_t^* := \frac{\mathsf{E}_{\mathsf{P}^*}(w(T)\,\mathbb{1}_{\{\tau\le T\}}\,|\,\mathcal{F}_t)}{\mathsf{P}^*\{\tau \le T\,|\,\mathcal{F}_t\}} = \frac{w^*(t,T)}{p_t^*}\,,$$

where

$$p_t^* := \mathsf{P}^*\{\tau \le T\,|\,\mathcal{F}_t\}$$

is the *conditional risk-neutral probability of default.* Finally, let $\delta_t^* := 1 - w_t^*$ be the *conditional expected recovery rate upon default* under P^*. In terms of p_t^*, δ_t^* and p_t^*, we obtain

$$D(t,T) = L_t(1-p_t^*) + L_t p_t^* \delta_t^* = L_t(1 - p_t^* w_t^*).$$

If the random variables $w(T)$ and τ are conditionally independent with respect to the σ-field $\mathcal{F}_t$ under P^*, then we have $w_t^* = \mathsf{E}_{\mathsf{P}^*}(w(T)\,|\,\mathcal{F}_t)$.

Example. In most intensity-based models, the recovery rate is assumed to be non-random. Let the recovery rate $\delta(T)$ be constant, specifically, $\delta(T) = \delta$ for some real number δ. In this case, the writedown rate $w(T) = w := 1-\delta$ is non-random as well. Then $w^*(t,T) = wp_t^*$ and $w_t^* = w$ for every $0 \le t \le T$. Furthermore,

$$D(t,T) = L_t(1-p_t^*) + \delta L_t p_t^* = L_t(1 - wp_t^*).$$

2 Classic Models

In most classic models, it is assumed that the value of the firm process V is governed by the SDE:

$$dV_t = V_t\left((r-\kappa)\,dt + \sigma_V\,dW_t^*\right),$$

where κ is the constant payout ratio, and W^* follows the Wiener process under the martingale measure P^*.

2.1 Merton's Model

Basic assumption: A firm has a single liability with promised terminal payoff L, interpreted as the zero-coupon bond with maturity T. The ability of the firm to redeem its debt is determined by the total value V_T of firm's assets at time T. Default may occur at time T only, and the default event corresponds to the event $\{V_T < L\}$ so that the stopping time τ equals

$$\tau = T\,\mathbb{1}_{\{V_T<L\}} + \infty\,\mathbb{1}_{\{V_T\geq L\}}.$$

Moreover $C = 0$, $Z = 0$, and

$$\tilde{X} = V_T\,\mathbb{1}_{\{V_T<L\}} + L\,\mathbb{1}_{\{V_T\geq L\}}$$

so that the payoff from the *defaultable* (*corporate*) bond at maturity equals

$$D_T = \min\left(V_T, L\right) = L - \max\left(L - V_T, 0\right).$$

Notice that in Merton's setup, the valuation of the corporate bond is equivalent to the valuation of a European put (or call) option written on the firm's value with strike equal to the bond's face value.

Let $D(t, T)$ be the price at time $t < T$ of a defaultable (corporate) bond, and let $B(t, T)$ stand for the price of a risk-free (Treasury) bond.

It is clear that the value $D(V_t)$ of the firm's debt equals

$$D(V_t) = D(t, T) = B(t, T) - P_t,$$

where P_t is the price of a put option with strike L and expiration date T. It is apparent from that the value at time t of the firm's equity satisfies

$$E(V_t) := V_t - D(V_t) = V_t - LB(t, T) + P_t = C_t,$$

where C_t stands in turn for the price at time t of a call option written on the firm's assets, with the strike price L and the exercise date T. To justify the last equality above, we may also observe that at time T we have

$$E(V_T) = V_T - D(V_T) = V_T - \min(V_T, L) = (V_T - L)^+.$$

We conclude that the firm's equity can be seen as a call option on the firm's assets.

2.1.1 Merton's Formula

Merton (1974) derived a closed-form expression for the arbitrage price of a corporate bond. Let N denote the standard Gaussian cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \quad \forall\, x \in \mathrm{R}.$$

Proposition 1 *For every $0 \le t < T$ the value $D(t,T)$ of a corporate bond equals*

$$V_t e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big) + LB(t,T) N\big(d_2(V_t, T-t)\big)$$

where

$$d_{1,2}(V_t, T-t) = \frac{\ln(V_t//L) + (r - \kappa \pm \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V \sqrt{T-t}}.$$

The unique replicating strategy for a defaultable bond involves holding at any time $0 \le t < T$ the $\phi_t^1 V_t$ units of cash invested in the firm's value and $\phi_t^2 B(t,T)$ units of cash invested in default-free bonds, where

$$\phi_t^1 = e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big)$$

and

$$\phi_t^2 = \frac{D(t,T) - \phi_t^1 V_t}{B(t,T)} = LN\big(d_2(V_t, T-t)\big).$$

2.1.2 Credit Spreads

For the sake of notational simplicity, we set $\kappa = 0$. Then Merton's formula becomes:

$$D(t,T) = LB(t,T)(N(d - \sigma_V\sqrt{T-t}) + \Gamma_t N(-d)),$$

where $\Gamma_t = V_t/LB(t,T)$ and

$$d = d(V_t, T-t) = \frac{\ln(V_t/L) + (r + \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}}.$$

Notice that $LB(t,T)$ represents the current value of the face value of the firm's debt, so that Γ_t can be seen as a proxy of the asset-to-debt ratio $V_t/D(t,T)$. Let us denote

$$B(t,T) = e^{-r(T-t)}, \quad D(t,T) = e^{-r_t^d(T-t)}.$$

The *credit spread* of the defaultable bond equals

$$r_t^d - r = -(N(d - \sigma_V\sqrt{T-t}) + \Gamma_t N(-d))/(T-t) > 0.$$

This agrees with the well-known fact that risky bonds have an expected return in excess of the risk-free interest rate.

On the other hand, however, when t tends to T, the credit spread in Merton's model tends either to 0 or to infinity, depending on whether $V_t < L$ or $V_t > L$.

2.2 Black and Cox Model

The original Merton model does not allow for a premature default, in the sense that the default may only occur at the maturity of the claim. Several authors put forward structural-type models in which this restrictive and unrealistic feature is relaxed. In most of these models, the time of default is given as the *first passage time* of the value process V to a deterministic or random barrier. The default may thus occur at any time before or on the bond's maturity date T.

The challenge is to appropriately specify the lower threshold v, the recovery process Z, and to compute the corresponding functional that appears on the right-hand side of the risk-neutral valuation formula:

$$X^d(t,T) := B_t \, \mathsf{E}_{\mathsf{P}^*}\big(\int_{]t,T]} B_u^{-1} \, dD_u \,|\, \mathcal{F}_t \big).$$

As one might easily guess, this is a non-trivial problem, in general. In addition, the practical problem of the lack of direct observations of the value process V largely limits the applicability of the first-passage-time models.

Notation: As a rule, the default time will be denoted by τ; the symbols $\bar{\tau}$, $\hat{\tau}$ and $\widehat{\tau}$ are reserved to some auxiliary random times.

2.2.1 Corporate Zero-Coupon Bond

Black and Cox (1976) extend Merton's (1974) research in several directions, by taking into account such specific features of debt contracts as: safety covenants, debt subordination, and restrictions on the sale of assets. They assume that the firm's stockholders (or bondholders) receive a continuous dividend payment, proportional to the current value of the firm. Specifically, they postulate that

$$dV_t = V_t((r - \kappa)\,dt + \sigma_V\,dW_t^*),$$

where the constant $\kappa \geq 0$ represents the payout ratio, and $\sigma_V > 0$ is the constant volatility. The short-term interest rate r is assumed to be constant.

Safety covenants

Safety covenants provide the firm's bondholders with the right to force the firm to bankruptcy or reorganization if the firm is doing poorly according to a set standard. The standard for a poor performance is set by Black and Cox in terms of a time-dependent deterministic barrier $\bar{v}(t) = Ke^{-\gamma(T-t)}$, $t \in [0, T)$, for some constant $K > 0$. As soon as the value of firm's assets crosses this lower threshold, the bondholders take over the firm. Otherwise, default takes place at debt's maturity or not depending on whether $V_T < L$ or not.

Default Time

Let us set:

$$v_t = \begin{cases} \bar{v}(t), & \text{for } t < T, \\ L, & \text{for } t = T. \end{cases}$$

The default event occurs at the first time $t \in [0, T]$ at which the firm's value V_t falls below the level v_t, or the default event does not occur at all. The default time equals ($\inf \emptyset = +\infty$)

$$\tau = \inf \{ t \in [0, T] : V_t < v_t \}.$$

The recovery process Z and the recovery payoff $\tilde{X}$ are proportional to the value process: $Z \equiv \beta_2 V$ and $\tilde{X} = \beta_1 V_T$ for some constants $\beta_1, \beta_2 \in [0, 1]$.

The classic case examined by Black and Cox (1976) corresponds to $\beta_1 = \beta_2 = 1$.

To summarize, we consider the following model:

$$X = L,\ C \equiv 0,\ Z \equiv \beta_2 V,\ \tilde{X} = \beta_1 V_T,\ \tau = \bar{\tau} \wedge \widehat{\tau},$$

where the *early default time* $\bar{\tau}$ equals

$$\bar{\tau} = \inf \{ t \in [0, T) : V_t < \bar{v}(t) \},$$

and $\widehat{\tau}$ stands for Merton's default time:

$$\widehat{\tau} = T\, \mathbb{1}_{\{V_T < L\}} + \infty\, \mathbb{1}_{\{V_T \geq L\}}.$$

2.2.2 Bond Valuation

We postulate, in addition, that $\bar{v}(t) \leq LB(t,T)$ or, more explicitly,

$$Ke^{-\gamma(T-t)} \leq Le^{-r(T-t)}, t \in [0,T], \qquad (1)$$

so that, in particular, $K \leq L$. Condition (1) ensures that the payoff to the bondholder at the default time τ never exceeds the face value of debt, discounted at a risk-free rate.

PDE Approach

The pricing function $u = u(V,t)$ of a defaultable bond solves the following PDE:

$$u_t(V,t) + (r-\kappa)Vu_V(V,t) + \tfrac{1}{2}\sigma_V^2 V^2 u_{VV}(V,t) - ru(V,t) = 0$$

with the boundary condition $u(Ke^{-\gamma(T-t)}, t) = \beta_2 Ke^{-\gamma(T-t)}$ and the terminal condition $u(V,T) = \min(\beta_1 V, L)$.

Probabilistic Approach.

Notice that for any $t < T$ the price $D(t,T) = u(V_t,t)$ of a defaultable bond admits the following probabilistic representation, on the set $\{\tau > t\} = \{\bar{\tau} > t\}$

$$\begin{aligned} D(t,T) \;=\; & \mathsf{E}_{\mathsf{P}^*}(Le^{-r(T-t)}\, \mathbb{1}_{\{\bar{\tau} \geq T,\, V_T \geq L\}} \,|\, \mathcal{F}_t) \\ & + \mathsf{E}_{\mathsf{P}^*}(\beta_1 V_T e^{-r(T-t)}\, \mathbb{1}_{\{\bar{\tau} \geq T,\, V_T < L\}} \,|\, \mathcal{F}_t) \\ & + \mathsf{E}_{\mathsf{P}^*}(K\beta_2 e^{-\gamma(T-\bar{\tau})} e^{-r(\bar{\tau}-t)}\, \mathbb{1}_{\{t < \bar{\tau} < T\}} \,|\, \mathcal{F}_t). \end{aligned}$$

After default – that is, on the set $\{\tau \leq t\} = \{\bar{\tau} \leq t\}$, we clearly have

$$D(t,T) = \beta_2 \bar{v}(\tau) B^{-1}(\tau,T) B(t,T) = K\beta_2 e^{-\gamma(T-\tau)} e^{r(t-\tau)}.$$

Computation:

- the first two conditional expectations in the valuation formula can be computed by using the formula for the conditional probability $\mathsf{P}^*\{V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t\}$,
- to evaluate the third conditional expectation, we shall employ the conditional probability law of the first passage time of the process V to the barrier $\bar{v}(t)$.

We are thus in a position to state the valuation result due to Black and Cox (1976). We denote $\widehat{a} = \widehat{\nu}\sigma_V^{-2}$ and

$$\nu = r - \kappa - \tfrac{1}{2}\sigma_V^2, \quad \widehat{\nu} = \nu - \gamma = r - \kappa - \gamma - \tfrac{1}{2}\sigma_V^2.$$

2.2.3 Black and Cox Formula

For the sake of brevity, in the statement of Proposition 2 we shall write σ instead of σ_V.

Proposition 2 *Assume that* $\widehat{\nu}^2 + 2\sigma^2(r-\gamma) > 0$. *Prior to bond's default, that is: on the set* $\{\tau > t\}$, *the price process* $D(t,T) = u(V_t, t)$ *of a defaultable bond equals*

$$
\begin{aligned}
D(t,T) &= LB(t,T)(N(h_1(V_t,T-t)) - R_t^{2\widehat{a}}N(h_2(V_t,T-t))) \\
&+ \beta_1 V_t e^{-\kappa(T-t)}(N(h_3(V_t,T-t)) - N(h_4(V_t,T-t))) \\
&+ \beta_1 V_t e^{-\kappa(T-t)} R_t^{2\widehat{a}+2}(N(h_5(V_t,T-t)) - N(h_6(V_t,T-t))) \\
&+ \beta_2 V_t(R_t^{\theta+\zeta}N(h_7(V_t,T-t)) + R_t^{\theta-\zeta}N(h_8(V_t,T-t))),
\end{aligned}
$$

where $R_t = \bar{v}(t)/V_t, \theta = \widehat{a} + 1, \quad \zeta = \sigma^{-2}\sqrt{\widehat{\nu}^2 + 2\sigma^2(r-\gamma)}$
and

$$
\begin{aligned}
h_1(V_t,T-t) &= \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\
h_2(V_t,T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\
h_3(V_t,T-t) &= \frac{\ln(L/V_t) - (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_4(V_t,T-t) &= \frac{\ln(K/V_t) - (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_5(V_t,T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_6(V_t,T-t) &= \frac{\ln \bar{v}^2(t) - \ln(KV_t) + (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_7(V_t,T-t) &= \frac{\ln(\bar{v}(t)/V_t) + \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\
h_8(V_t,T-t) &= \frac{\ln(\bar{v}(t)/V_t) - \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

Special Cases

Assume that $\beta_1 = \beta_2 = 1$ and the barrier function $\bar{v}$ is such that $K = L$. Then necessarily $\gamma \geq r$. It can be checked that for $K = L$ we have $D(t,T) = D_1(t,T) + D_3(t,T)$ where:

$$D_1(t,T) = LB(t,T)(N(h_1(V_t, T-t)) - R_t^{2\hat{a}} N(h_2(V_t, T-t)))$$

$$D_3(t,T) = V_t(R_t^{\theta+\zeta} N(h_7(V_t, T-t)) + R_t^{\theta-\zeta} N(h_8(V_t, T-t))).$$

Case $\gamma = r$

If we also assume that $\gamma = r$ then $\zeta = -\sigma^{-2}\widehat{\nu}$, and thus

$$V_t R_t^{\theta+\zeta} = LB(t,T), \quad V_t R_t^{\theta-\zeta} = V_t R_t^{2\hat{a}+1} = LB(t,T) R_t^{2\hat{a}}.$$

It is also easy to see that in this case

$$h_1(V_t, T-t) = \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}} = -h_7(V_t, T-t),$$

while

$$h_2(V_t, T-t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}} = h_8(V_t, T-t).$$

We conclude that if

$$\bar{v}(t) = Le^{-r(T-t)} = LB(t,T)$$

then $D(t,T) = LB(t,T)$.

This result is quite intuitive. A corporate bond with a safety covenant represented by the barrier function, which equals the discounted value of the bond's face value, is equivalent to a default-free bond with the same face value and maturity.

Case $\gamma > r$

For $K = L$ and $\gamma > r$, it is natural to expect that $D(t,T)$ would be smaller than $LB(t,T)$. It is also possible to show that when γ tends to infinity (all other parameters being fixed), then the Black and Cox price converges to Merton's price.

Further Developments

The Black and Cox first-passage-time methodology was later developed by, among others:

Brennan and Schwartz (1977, 1980) - convertible bonds,

Kim et al. (1993) - random barrier and random interest rates (CIR model),

Nielsen et al. (1993) - random barrier and random interest rates (Vasicek's model),

Leland (1994) - optimal capital structure, bankruptcy costs, tax benefits,

Longstaff and Schwartz (1995) - constant barrier and random interest rates (Vasicek's model)

2.2.4 Optimal Capital Structure

We consider a firm that has an interest paying bonds outstanding. We assume that it is a consol bond, which pays continuously coupon rate c. Assume that $r > 0$ and the payout rate κ is equal to zero. This condition can be given a financial interpretation as the restriction on the sale of assets, as opposed to issuing of new equity. Equivalently, we may think about a situation in which the stockholders will make payments to the firm to cover the interest payments. However, they have the right to stop making payments at any time and either turn the firm over to the bondholders or pay them a lump payment of c/r per unit of the bond's notional amount.

Recall that we denote by $E(V_t)$ ($D(V_t)$, resp.) the value at time t of the firm equity (debt, resp.), hence the total value of the firm's assets satisfies $V_t = E(V_t) + D(V_t)$.

Black and Cox (1976) argue that there is a critical level of the value of the firm, denoted as v^*, below which no more equity can be sold. The critical value v^* will be chosen by stockholders, whose aim is to minimize the value of the bonds, and thus to maximize the value of the equity. Notice that v^* is nothing else than a constant default barrier in the problem under consideration; the optimal default time τ^* thus equals $\tau^* = \inf \{ t \geq 0 : V_t \leq v^* \}$.

To find the value of v^*, let us first fix the bankruptcy level $\bar{v}$. The ODE for the pricing function $u^\infty = u^\infty(V)$ of a consol bond takes the following form

$$\tfrac{1}{2}V^2\sigma^2 u^\infty_{VV} + rVu^\infty_V + c - ru^\infty = 0,$$

subject to the lower boundary condition $u^\infty(\bar{v}) = \min(\bar{v}, c/r)$ and the upper boundary condition

$$\lim_{V\to\infty} u^\infty_V(V) = 0.$$

For the last condition, observe that when the firm's value grows to infinity, the possibility of default becomes meaningless, so that the value of the defaultable consol bond tends to the value c/r of the default-free consol bond. The general solution has the following form:

$$u^\infty(V) = \frac{c}{r} + K_1 V + K_2 V^{-\alpha},$$

where $\alpha = 2r/\sigma^2$ and K_1, K_2 are some constants, to be determined from boundary conditions. We find that $K_1 = 0$, and

$$K_2 = \begin{cases} \bar{v}^{\alpha+1} - (c/r)\bar{v}^\alpha, & \text{if } \bar{v} < c/r, \\ 0, & \text{if } \bar{v} \geq c/r. \end{cases}$$

Hence, if $\bar{v} < c/r$ then

$$u^\infty(V_t) = \frac{c}{r} + \left(\bar{v}^{\alpha+1} - \frac{c}{r}\bar{v}^\alpha\right)V_t^{-\alpha} = \frac{c}{r}\left(1 - \left(\frac{\bar{v}}{V_t}\right)^\alpha\right) + \bar{v}\left(\frac{\bar{v}}{V_t}\right)^\alpha.$$

It is in the interest of the stockholders to select the bankruptcy level in such a way that the value of the debt, $D(V_t) = u^\infty(V_t)$, is minimized, and thus the value of firm's equity

$$E(V_t) = V_t - D(V_t) = V_t - \frac{c}{r}(1 - \bar{q}_t) - \bar{v}\bar{q}_t$$

is maximized. It is easy to check that the optimal level of the barrier does not depend on the current value of the firm, and it equals

$$v^* = \frac{c}{r}\frac{\alpha}{\alpha + 1} = \frac{c}{r + \sigma^2/2}.$$

Given the optimal strategy of the stockholders, the price process of the firm's debt (i.e., of a consol bond) takes the form, on the set $\{\tau^* > t\}$,

$$D^*(V_t) = \frac{c}{r} - \frac{1}{\alpha V_t^\alpha}\left(\frac{c}{r + \sigma^2/2}\right)^{\alpha+1}$$

or, equivalently

$$D^*(V_t) = \frac{c}{r}(1 - q_t^*) + v^* q_t^*,$$

where

$$q_t^* = \left(\frac{v^*}{V_t}\right)^\alpha = \frac{1}{V_t^\alpha}\left(\frac{c}{r + \sigma^2/2}\right)^\alpha.$$

3 Stochastic Interest Rates

We assume that the underlying probability space $(\Omega, \mathcal{F}, \mathsf{P})$, endowed with the filtration $\mathsf{F} = (\mathcal{F}_t)_{t\geq 0}$, supports the short-term interest rate process r and the value process V.

The dynamics under the spot martingale measure P^* of the firm's value and of the price of a default-free zero-coupon bond $B(t,T)$ are

$$dV_t = V_t((r_t - \kappa(t))\,dt + \sigma(t)\,dW_t^*),$$

and

$$dB(t,T) = B(t,T)(r_t\,dt + b(t,T)\,dW_t^*),$$

respectively, where W^* is a d-dimensional standard Brownian motion. Furthermore, $\kappa : [0,T] \to \mathsf{R}$, $\sigma : [0,T] \to \mathsf{R}^d$ and $b(\cdot,T) : [0,T] \to \mathsf{R}^d$ are assumed to be bounded functions.

The *forward value* $F_V(t,T) := V_t/B(t,T)$ of the firm satisfies under the forward martingale measure P_T

$$dF_V(t,T) = -\kappa(t)F_V(t,T)\,dt + F_V(t,T)(\sigma(t) - b(t,T))\,dW_t^T,$$

where the process W^T, given by the formula

$$W_t^T = W_t^* - \int_0^t b(u,T)\,du, \quad \forall\, t \in [0,T],$$

is known to follow a d-dimensional SBM under P_T.

For any $t \in [0, T]$, we set

$$F_V^\kappa(t,T) = F_V(t,T)e^{-\int_t^T \kappa(u)\,du}.$$

Then

$$dF_V^\kappa(t,T) = F_V^\kappa(t,T)(\sigma(t) - b(t,T))\,dW_t^T.$$

Furthermore, it is apparent that $F_V^\kappa(T,T) = F_V(T,T) = V_T$.

We consider the following modification of the Black and Cox approach:

$$X = L,\ Z_t = \beta_2 V_t,\ \widehat{X} = \beta_1 V_T,\ \tau = \inf\{\, t \in [0,T] : V_t < v_t\},$$

where $\beta_2,\ \beta_1 \in [0,1]$ are constants, and the barrier v is given by the formula

$$v_t := \begin{cases} KB(t,T)e^{\int_t^T \kappa(u)\,du}, & \text{for } t < T, \\ L, & \text{for } t = T, \end{cases}$$

where the constant K satisfies $0 < K \le L$.

Let us denote, for any $t \le T$,

$$\kappa(t,T) = \int_t^T \kappa(u)\,du, \quad \sigma^2(t,T) = \int_t^T |\sigma(u) - b(u,T)|^2 du,$$

where $|\cdot|$ is the Euclidean norm in R^d. We write $F_t = F_V(t,T)$, and we denote

$$\eta_+(t,T) = \kappa(t,T) + \tfrac{1}{2}\sigma^2(t,T), \quad \eta_-(t,T) = \kappa(t,T) - \tfrac{1}{2}\sigma^2(t,T).$$

Proposition 3 *For any* $t < T$, *the forward price of a defaultable bond* $F_D(t,T) = D(t,T)/B(t,T)$ *equals on the set* $\{\tau > t\}$,

$$\begin{aligned} &L(N(\widehat{h}_1(F_t,t,T)) - (F_t/K)e^{-\kappa(t,T)}N(\widehat{h}_2(F_t,t,T))) \\ &\quad + \beta_1 F_t e^{-\kappa(t,T)}(N(\widehat{h}_3(F_t,t,T)) - N(\widehat{h}_4(F_t,t,T))) \\ &\quad + \beta_1 K(N(\widehat{h}_5(F_t,t,T)) - N(\widehat{h}_6(F_t,t,T))) \\ &\quad + \beta_2 K J_1(F_t,t,T) + \beta_2 F_t e^{-\kappa(t,T)} J_2(F_t,t,T), \end{aligned}$$

where

$$\begin{aligned} \widehat{h}_1(F_t,t,T) &= \frac{\ln(F_t/L) - \eta_+(t,T)}{\sigma(t,T)}, \\ \widehat{h}_2(F_t,T,t) &= \frac{2\ln K - \ln(LF_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \widehat{h}_3(F_t,t,T) &= \frac{\ln(L/F_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \widehat{h}_4(F_t,t,T) &= \frac{\ln(K/F_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \widehat{h}_5(F_t,t,T) &= \frac{2\ln K - \ln(LF_t) + \eta_+(t,T)}{\sigma(t,T)}, \\ \widehat{h}_6(F_t,t,T) &= \frac{\ln(K/F_t) + \eta_+(t,T)}{\sigma(t,T)}, \end{aligned}$$

and for any fixed $0 \le t < T$ *and* $F_t > 0$

$$J_{1,2}(F_t,t,T) = \int_t^T e^{\kappa(u,T)}\, dN\left(\frac{\ln(K/F_t) + \kappa(t,T) \pm \frac{1}{2}\sigma^2(t,u)}{\sigma(t,u)}\right).$$

Corollary 1 *Under the assumptions of Proposition* 3, *if* $\kappa \equiv 0$ *then*

$$\begin{aligned}F_D(t,T) &= L(N(-\,d_1(F_t,t,T)) - (F_t/K)N(d_6(F_t,t,T))) \\ &\quad + \beta_1 F_t(N(d_2(F_t,t,T)) - N(d_4(F_t,t,T))) \\ &\quad + \beta_1 K(N(d_5(F_t,t,T)) - N(d_3(F_t,t,T))) \\ &\quad + \beta_2 K N(d_3(F_t,t,T)) + \beta_2 F_t N(d_4(F_t,t,T)),\end{aligned}$$

where

$$\begin{aligned}d_1(F_t,t,T) &= \frac{\ln(L/F_t) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_2(F_t,t,T) + \sigma(t,T), \\ d_3(F_t,t,T) &= \frac{\ln(K/F_t) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_4(F_t,t,T) + \sigma(t,T), \\ d_5(F_t,t,T) &= \frac{\ln(K^2/F_t L) + \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)} = d_6(F_t,t,T) + \sigma(t,T).\end{aligned}$$

The formula of Corollary 1 covers as a special case the valuation result established by Briys and de Varenne (1997).

Remarks. In some other recent studies of first passage time models, in which the triggering barrier is assumed to be either a constant or an unspecified stochastic process, typically no closed-form solution for the value of a corporate debt is available, and thus a numerical approach is required (see Kim et al. (1993), Longstaff and Schwartz (1995), Nielsen et al. (1993), or Saá-Requejo and Santa-Clara (1999)).

4 Hybrid Models

Madan and Unal (1998) consider the discounted equity value (including reinvested dividends) process $E_t^* = E_t/B_t$ as the unique Markovian state variable in their intensity-based model. The dynamics of E^* under the spot martingale measure P^* are:

$$dE_t^* = \sigma E_t^* \, dW_t^*, \quad E_0^* > 0,$$

for some constant volatility coefficient σ.

Madan and Unal (1998) postulate that the intensity of default satisfies: $\lambda_t = \lambda(E_t^*)$ for some function $\lambda : \mathsf{R}_+ \to \mathsf{R}_+$. The default time τ is specified through the so-called canonical construction, so that it is defined on an enlarged probability space $(\Omega, \mathsf{G}, \mathsf{Q}^*)$, where Q^* is an extension of P^*.

Madan and Unal (1998) propose to take the function

$$\lambda(x) = c \left(\ln(x/\bar{v})\right)^{-2},$$

where c and $\bar{v}$ are strictly positive constants. It is interesting to notice that the stochastic intensity $\lambda_t = \lambda(E_t^*)$ tends to infinity, when the discounted equity value E_t^* approaches, either from above or from below, the critical level $\bar{v}$.

To avoid making a particular choice of a default-free term structure model, Madan and Unal (1998) focus on the futures price of a corporate bond.

It is well known (Duffie and Stanton (1992) or Sect. 15.2 in Musiela and Rutkowski (1997)) that the futures price $\pi^f(X)$ of a contingent claim X, for the settlement date T, is given by the conditional expectation under the spot martingale measure:

$$\pi_t^f(X) = \mathsf{E}_{\mathsf{Q}^*}(X \,|\, \mathcal{G}_t), \quad t \in [0, T].$$

In our case, the futures price $D^f(t,T)$ of a defaultable bond with zero recovery equals $D^f(t,T) = \mathsf{Q}^*\{\tau > T \,|\, \mathcal{G}_t\}$. More explicitly,

$$D^f(t,T) = \mathbb{1}_{\{\tau > t\}} \, \mathsf{E}_{\mathsf{P}^*}(e^{-\int_t^T \lambda(E_u^*, u)\,du} \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau > t\}} v(E_t^*, t)$$

for some function $v : \mathsf{R}_+ \to \mathsf{R}_+$. By virtue of the Feynman-Kac theorem, the function v satisfies, under mild technical assumptions, the following pricing PDE

$$v_t(x,t) + \tfrac{1}{2}\sigma^2(x,t) v_{xx}(x,t) - \lambda(x,t) v(x,t) = 0$$

subject to the terminal condition $v(x,T) = 1$. For the sake of notational simplicity, we have assumed here that the process W^* is one-dimensional.

Madan and Unal (1998) show that under these assumptions the futures price of a corporate bond equals $G(h(E_t^*, T-t))$, where the function h is explicitly known, and the function G satisfies a certain second-order ODE.