# CREDIT RISK: MODELLING, VALUATION AND HEDGING

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# MODELLING OF DEPENDENT DEFAULTS

# SELECTED REFERENCES

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# MAIN ISSUES

Valuation of basket credit derivatives:

- default swap of type F (Duffie 1988, Kijima & Muromachi 2000) – protection against the first default in a basket of defaultable claims,
- default swap of type D (Kijima & Muromachi 2000) – protection against the first two defaults in a basket of defaultable claims,
- $i^{\text{th}}$-to-default claim (Bielecki & Rutkowski 2000) – protection against the first $i$ defaults in a basket of defaultable claims.

Technical issues:

- conditional independence (Kijima & Muromachi 2000),
- simulation of correlated defaults (Duffie & Singleton 1998),
- infectious defaults (Davis & Lo 1999),
- change of a probability measure (Kusuoka, 1999),
- asymmetric default intensities (Jarrow & Yu 2001),
- copulas (Schönbucher and Schubert 2001),
- dependent ratings (Bielecki & Rutkowski 2002).

# 1 Basket Credit Derivatives

Basket credit derivatives are credit derivatives deriving their cash flows values (and thus their values) from credit risks of several reference entities (or prespecified credit events).

We assume that:

- we are given a collection of default times $\tau_1, \ldots, \tau_n$ defined on a common probability space $(\Omega, \mathcal{G}, \mathsf{Q}^*)$,
- $\mathsf{Q}^*\{\tau_i = 0\} = 0$ and $\mathsf{Q}^*\{\tau_i > t\} > 0$ for every $i$ and $t$,
- $\mathsf{Q}^*\{\tau_i = \tau_j\} = 0$ for arbitrary $i \neq j$.

We associate with the collection $\tau_1, \ldots, \tau_n$ the ordered sequence $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(n)}$, where $\tau_{(i)}$ stands for the random time of the $i^{\text{th}}$ default.

Formally:

$$\tau_{(1)} = \min\{\tau_1, \tau_2, \ldots, \tau_n\}$$

and for $i = 2, \ldots, n$

$$\tau_{(i)} = \min\{\tau_k : k = 1, \ldots, n, \ \tau_k > \tau_{(i-1)}\}.$$

In particular,

$$\tau_{(n)} = \max\{\tau_1, \tau_2, \ldots, \tau_n\}.$$

## 1.1 $i^{\text{th}}$-to-Default Contingent Claims

We set $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$ and we denote by $\mathbb{H}^i$ the filtration generated by the process $H^i$, that is, by the observations of the default time $\tau_i$.

In addition, we are given a reference filtration $\mathbb{F}$ on the space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, which models, e.g., the interest rate risk.

We introduce the enlarged filtration $\mathbb{G}$ by setting

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2 \vee \ldots \vee \mathbb{H}^n.$$

The $\sigma$-field $\mathcal{G}_t$ models the information available at time $t$.

A general $i^{\text{th}}$-to-default contingent claim which matures at time $T$ is specified by the following covenants:

- if $\tau_{(i)} = \tau_k \le T$ for some $k = 1, \ldots, n$ then the claimholder gets at time $\tau_{(i)}$ the *recovery payoff* $Z^k_{\tau_{(i)}}$,
- otherwise, that is, if $\tau_{(i)} > T$, the claimholder receives at time $T$ the *promised payoff* $X$.

Technical assumptions:

- $Z^k$ is an $\mathbb{F}$-predictable process,
- $X$ is an $\mathcal{F}_T$-measurable random variable.

## 1.2 Case of Two Entities

For the sake of notational simplicity, we shall frequently consider the case of two reference credit risks.

Cash flows of the first-to-default contract (FDC)

- if $\tau_{(1)} = \min\{\tau_1, \tau_2\} = \tau_i \leq T$ for $i = 1, 2$, the claim pays at time $\tau_i$ the amount $Z^i_{\tau_i}$,
- if $\min\{\tau_1, \tau_2\} > T$, it pays at time $T$ the amount $X$.

Cash flows of the last-to-default contract (LDC)

- if $\tau_{(2)} = \max\{\tau_1, \tau_2\} = \tau_i \leq T$ for $i = 1, 2$, the claim pays at time $\tau_i$ the amount $Z^i_{\tau_i}$,
- if $\max\{\tau_1, \tau_2\} > T$, it pays at time $T$ the amount $X$.

The savings account $B$ equals

$$B_t = \exp\left(\int_0^t r_u \, du\right),$$

and $\mathbb{Q}^*$ stands for the spot martingale measure for our model of the financial market (including defaultable securities, such as: corporate bonds and credit derivatives).

## Values of FDC and LDC

In general, the value at time $t$ of a defaultable claim $(X, Z, \tau)$ is given by the *risk-neutral valuation formula*

$$S_t = B_t \, \mathsf{E}_{\mathsf{Q}^*} \Big( \int_{]t,T]} B_u^{-1} \, dD_u \, | \, \mathcal{G}_t \Big)$$

where $D$ is the *dividend process*, which describes all the cash flows of the claim.

The value at time $t$ of the FDC equals:

$$\begin{aligned} S_t^{(1)} &= B_t \, \mathsf{E}_{\mathsf{Q}^*}\big( B_{\tau_1}^{-1} Z_{\tau_1}^1 \, \mathbb{1}_{\{\tau_1 < \tau_2, \, t < \tau_1 \le T\}} \, | \, \mathcal{G}_t \big) \\ &\quad + B_t \, \mathsf{E}_{\mathsf{Q}^*}\big( B_{\tau_2}^{-1} Z_{\tau_2}^2 \, \mathbb{1}_{\{\tau_2 < \tau_1, \, t < \tau_2 \le T\}} \, | \, \mathcal{G}_t \big) \\ &\quad + B_t \, \mathsf{E}_{\mathsf{Q}^*}\big( B_T^{-1} X \, \mathbb{1}_{\{T < \tau_{(1)}\}} \, | \, \mathcal{G}_t \big). \end{aligned}$$

The value at time $t$ of the LDC equals:

$$\begin{aligned} S_t^{(2)} &= B_t \, \mathsf{E}_{\mathsf{Q}^*}\big( B_{\tau_1}^{-1} Z_{\tau_1}^1 \, \mathbb{1}_{\{\tau_2 < \tau_1, \, t < \tau_1 \le T\}} \, | \, \mathcal{G}_t \big) \\ &\quad + B_t \, \mathsf{E}_{\mathsf{Q}^*}\big( B_{\tau_2}^{-1} Z_{\tau_2}^2 \, \mathbb{1}_{\{\tau_1 < \tau_2, \, t < \tau_2 \le T\}} \, | \, \mathcal{G}_t \big) \\ &\quad + B_t \, \mathsf{E}_{\mathsf{Q}^*}\big( B_T^{-1} X \, \mathbb{1}_{\{T < \tau_{(2)}\}} \, | \, \mathcal{G}_t \big). \end{aligned}$$

Both expressions above are special cases of the general formula. The goal is to derive more explicit representations under various assumptions about $\tau_1$ and $\tau_2$.

# Hazard Process of a Default Time

Let $\tau$ be a non-negative random variable on $(\Omega, \mathcal{G}, \mathsf{Q}^*)$, referred to as the *default time*, and let $\mathbb{F}$ be a reference filtration. We set

$$F_t = \mathsf{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\},$$

so that

$$G_t := 1 - F_t = \mathsf{Q}^*\{\tau > t \,|\, \mathcal{F}_t\}$$

is the *conditional survival probability.* It is easily seen that $F$ is a bounded, non-negative, $\mathbb{F}$-submartingale. We assume that $F_t < 1$ for every $t \in \mathbb{R}_+$.

**Definition 1** The $\mathbb{F}$-*hazard process* $\Gamma$ of $\tau$ equals: $\Gamma_t = -\ln G_t$. If $\Gamma_t = \int_0^t \gamma_u \, du$ then $\gamma$ is called the $\mathbb{F}$-*intensity* of default (in this case, $F$ is an increasing process).

**Lemma 1** *Let $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$. Then for any $s > t$ and any $\mathcal{F}_s$-measurable, integrable, random variable $Y$ we have*

$$\mathsf{E}_{\mathsf{Q}^*}( \mathbb{1}_{\{\tau > s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \, \mathsf{E}_{\mathsf{Q}^*}(e^{\Gamma_t - \Gamma_s} Y \,|\, \mathcal{F}_t).$$

*In particular,*

$$\mathsf{Q}^*\{\tau > s \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \, \mathsf{E}_{\mathsf{Q}^*}(e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t).$$

## 2 Conditionally Independent Default Times

Relatively simple representations for prices of basket credit derivatives can be obtained under the assumption of conditional independence of default times.

**Definition 2** The random times $\tau_i$, $i = 1, \ldots, n$ are said to be *conditionally independent* with respect to F under Q* if and only if for any $T > 0$ and any $t_1, \ldots, t_n \in [0, T]$ we have:

$$\mathsf{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \prod_{i=1}^{n} \mathsf{Q}^*\{\tau_i > t_i \,|\, \mathcal{F}_T\}.$$

Notice that:

- Intuitive meaning of conditional independence: the reference credits (credit names) are subject to common risk factors that may trigger credit (default) events. In addition, each credit name is subject to idiosyncratic risks that are specific for this name.
- Conditional independence of default times means that once the common risk factors are fixed then the idiosyncratic risk factors are independent of each other.
- Conditional independence is not invariant with respect to an equivalent change of a probability measure.

## 2.1 Canonical Construction

Let $\Gamma^i$, $i = 1, \dots, n$ be a given family of $\mathsf{F}$-adapted, increasing, continuous processes, defined on a probability space $(\hat{\Omega}, \mathsf{F}, \mathsf{P}^*)$. We assume that $\Gamma^i_0 = 0$ and $\Gamma^i_\infty = \infty$. If $\Gamma^i_t = \int_0^t \gamma^i_u \, du$ then $\gamma^i$ is the $\mathsf{F}$-intensity of $\tau_i$. Intuitively

$$\mathsf{Q}^*\{\tau_i \in [t, t+dt] \,|\, \mathcal{F}_t \vee \mathcal{H}^i_t\} \approx \mathbb{1}_{\{\tau_i > t\}} \gamma^i_t \, dt.$$

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathsf{P}})$ be an auxiliary probability space with a sequence $\xi_i$, $i = 1, \dots, n$ of mutually independent random variables uniformly distributed on $[0, 1]$. We set

$$\tau_i = \inf \{ \, t \in \mathsf{R}_+ \; : \; \Gamma^i_t(\hat{\omega}) \geq -\ln \xi_i(\tilde{\omega}) \, \}$$

on $(\Omega, \mathcal{G}, \mathsf{Q}^*) = (\hat{\Omega} \times \tilde{\Omega}, \mathcal{F}_\infty \otimes \tilde{\mathcal{F}}, \mathsf{P}^* \otimes \tilde{\mathsf{P}})$. We endow the space $(\Omega, \mathcal{G}, \mathsf{Q}^*)$ with the filtration $\mathsf{G} = \mathsf{F} \vee \mathsf{H}^1 \vee \cdots \vee \mathsf{H}^n$.

**Proposition 1** Default times $\tau_1, \dots, \tau_n$ are conditionally independent with respect to $\mathsf{F}$ under $\mathsf{Q}^*$. The process $\Gamma^i$ is the $\mathsf{F}$-hazard process of $\tau_i$:

$$\mathsf{Q}^*\{\tau_i > s \,|\, \mathcal{F}_t \vee \mathcal{H}^i_t\} = \mathbb{1}_{\{\tau_i > t\}} \, \mathsf{E}_{\mathsf{Q}^*}(e^{\Gamma^i_t - \Gamma^i_s} \,|\, \mathcal{F}_t).$$

We have $\mathsf{Q}^*\{\tau_i = \tau_j\} = 0$ for every $i \neq j$.

## 3 Copula-Based Approach

The concept of a *copula function* allows to produce various multidimensional probability distributions with the same univariate marginal laws.

**Definition 3** $C : [0,1]^n \to [0,1]$ is a *copula function* if:
$C(1,\ldots,1,v_i,1,\ldots,1) = v_i$ for any $i$ and any $v_i \in [0,1]$,
$C$ is an $n$-dimensional cumulative distribution function.

Examples of copulas:

- product copula: $\Pi(v_1,\ldots,v_n) = \Pi_{i=1}^n v_i$,
- Gumbel copula: for $\theta \in [1,\infty)$ we set

$$C(v_1,\ldots,v_n) = \exp\left(-\left[\sum_{i=1}^n (-\ln v_i)^\theta\right]^{1/\theta}\right).$$

**Proposition 2** *For any cumulative distribution function $F$ on $\mathsf{R}^n$ there exists a copula function $C$ such that*

$$F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n))$$

*where $F_i$ is the marginal cumulative distribution function. If, in addition, $F$ is continuous then $C$ is unique.*

## 3.1 Extension of the Canonical Construction

Assume that the c.d.f. of $(\xi_1, \ldots, \xi_n)$ is an $n$-dimensional copula $C$. Then the univariate marginal laws are uniform on $[0,1]$, but the random variables $\xi_1, \ldots, \xi_n$ are not necessarily mutually independent. We still postulate that they are independent of $\mathsf{F}$, and we set:

$$\tau_i = \inf \{ t \in \mathsf{R}_+ : \Gamma_t^i(\hat{\omega}) \geq -\ln \xi_i(\tilde{\omega}) \}.$$

Then:

- the case of default times conditionally independent with respect to $\mathsf{F}$ corresponds to the choice of the product copula $\Pi$. In this case, for $t_1, \ldots, t_n \leq T$ we have

  $$\mathsf{Q}^* \{ \tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T \} = \Pi(Z_{t_1}^1, \ldots, Z_{t_n}^n)$$

  where we set $Z_t^i = e^{-\Gamma_t^i}$,

- in general, for $t_1, \ldots, t_n \leq T$ we obtain

  $$\mathsf{Q}^* \{ \tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T \} = C(Z_{t_1}^1, \ldots, Z_{t_n}^n)$$

  where $C$ is the copula function that was used in the construction of $\tau_1, \ldots, \tau_n$.

## 3.2 Survival Intensities

Schönbucher and Schubert (2001) show that for arbitrary $s \le t$ on the set $\{\tau_1 > s, \ldots, \tau_n > s\}$ we have

$$\mathsf{Q}^*\{\tau_i > t \,|\, \mathcal{G}_s\} = \mathsf{E}_{\mathsf{Q}^*}\left(\frac{C(Z_s^1, \ldots, Z_t^i, \ldots, Z_{t_n}^n)}{C(Z_s^1, \ldots, Z_s^n)} \,|\, \mathcal{F}_s\right).$$

Consequently, the $i^{\text{th}}$ intensity of survival equals, on the set $\{\tau_1 > t, \ldots, \tau_n > t\}$

$$\lambda_t^i = \gamma_t^i \, Z_t^i \, \frac{\partial}{\partial v_i} \ln C(Z_t^1, \ldots, Z_t^n).$$

Here $\lambda_t^i$ is understood as the limit:

$$\lambda_t^i = \lim_{h \downarrow 0} h^{-1} \, \mathsf{Q}^*\{t < \tau_i \le t + h \,|\, \mathcal{F}_t, \tau_1 > t, \ldots, \tau_n > t\}.$$

Schönbucher and Schubert (2001) examine the intensities of survival after the default times of some entities. It appears that, in general, the $i^{\text{th}}$ intensity of survival jumps at time $t$ if the $j^{\text{th}}$ entity defaults at time $t$ for some $j \neq i$.

Remark: Jumps of intensities cannot be efficiently controlled, except for the choice of $C$.

# 4 Jarrow and Yu Approach

For a given finite family of reference credit names, Jarrow and Yu (2001) propose to make a distinction between:

- the primary firms,
- the secondary firms.

At the intuitive level:

- the class of primary firms encompasses these entities whose probabilities of default are influenced by macroeconomic conditions, but not by the credit risk of counterparties. The pricing of bonds issued by primary firms can be done through the standard intensity-based methodology,
- it thus suffices to focus on securities issued by secondary firms, that is, these firms for which the intensity of default depends on the status of some other firms.

Formally, the construction is based on the assumption of asymmetric information. Unilateral dependence is not possible in the case of complete (i.e., symmetric) information.

## 4.1 Model's Construction

Let $\{1, \ldots, n\}$ represent the set of all firms, and let F be the reference filtration. We postulate that:

- for any firm from the set $\{1, \ldots, k\}$ of primary firms, the 'default intensity' depends only on F,
- the 'default intensity' of each firm belonging to the set $\{k+1, \ldots, n\}$ of secondary firms may depend not only on the filtration F, but also on the status (default or no-default) of the primary firms.

## Construction of default times $\tau_1, \ldots, \tau_n$

## First step: defaults of primary firms

We assume that we are given a family of F-adapted 'intensity processes' $\lambda^1, \ldots, \lambda^k$ and we produce a collection $\tau_1, \ldots, \tau_k$ of F-conditionally independent random times through the canonical method:

$$\tau_i = \inf \{ t \in \mathsf{R}_+ \, : \, \int_0^t \lambda_u^i \, du \geq - \ln \xi_i \}$$

where $\xi_i, \ i = 1, \ldots, k$ are mutually independent identically distributed random variables with uniform law on $[0, 1]$ under the spot martingale measure $\mathsf{Q}^*$.

## Second step: defaults of secondary firms

We assume that:

- the probability space $(\Omega, \mathcal{G}, \mathsf{Q}^*)$ is large enough to support a family $\xi_i,\ i = k+1, \dots, n$ of mutually independent random variables, with uniform law on $[0, 1]$,
- these random variables are independent not only of the filtration $\mathsf{F}$, but also of the already constructed in the first step default times $\tau_1, \dots, \tau_k$ of primary firms.

The default times $\tau_i,\ i = k+1, \dots, n$ are also defined by means of the standard formula:

$$\tau_i = \inf \{ t \in \mathsf{R}_+ \,:\, \int_0^t \lambda_u^i \, du \geq -\ln \xi_i \}.$$

However, the 'intensity processes' $\lambda^i$ for $i = k+1, \dots, n$ are now given by the following expression:

$$\lambda_t^i = \mu_t^i + \sum_{l=1}^{k} \nu_t^{i,l} \, \mathbb{1}_{\{\tau_l \leq t\}}$$

where $\mu^i$ and $\nu^{i,l}$ are $\mathsf{F}$-adapted stochastic processes.

If the default of the $j^{\text{th}}$ primary firm does not affect the default intensity of the $i^{\text{th}}$ secondary firm, we set $\nu^{i,j} \equiv 0$.

# Main Features

Let

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \ldots \vee \mathbb{H}^n$$

stand for the enlarged filtration and let

$$\tilde{\mathbb{F}} = \mathbb{F} \vee \mathbb{H}^{k+1} \vee \ldots \vee \mathbb{H}^n$$

be the filtration generated by the reference filtration $\mathbb{F}$ and the observations of defaults of secondary firms.

Then:

- the default times $\tau_1, \ldots, \tau_k$ of primary firms are conditionally independent with respect to $\mathbb{F}$,
- the default times $\tau_1, \ldots, \tau_k$ of primary firms are no longer conditionally independent when we replace the filtration $\mathbb{F}$ by the filtration $\tilde{\mathbb{F}}$,
- in general, the default intensity of a primary firm with respect to the filtration $\tilde{\mathbb{F}}$ differs from the intensity $\lambda^i$ with respect to the filtration $\mathbb{F}$.

Conclusion: defaults of primary firms are also 'dependent' of defaults of secondary firms.

## 4.2 Case of Two Firms

We consider only two firms, A and B say, and we postulate that A is a primary firm, and B is a secondary firm. Let the constant process $\lambda^1_t \equiv \lambda_1$ represent the F-intensity of default for firm A, so that

$$\tau_1 = \inf \{ t \in \mathsf{R}_+ : \int_0^t \lambda^1_u \, du = \lambda_1 t \geq -\ln \xi_1 \}$$

where $\xi_1$ is a random variable independent of F, with the uniform law on $[0, 1]$.

For the second firm, the 'intensity' of default is assumed to satisfy

$$\lambda^2_t = \lambda_2 \, \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \, \mathbb{1}_{\{\tau_1 \leq t\}}$$

for some positive constants $\lambda_2$ and $\alpha_2$, and thus

$$\tau_2 = \inf \{ t \in \mathsf{R}_+ : \int_0^t \lambda^2_u \, du \geq -\ln \xi_2 \}$$

where $\xi_2$ is a r.v. with the uniform law, independent of F, and such that $\xi_1$ and $\xi_2$ are mutually independent.

The following properties hold:

- $\lambda^1$ is the intensity of $\tau_1$ with respect to F,
- $\lambda^2$ is the intensity of $\tau_2$ with respect to $\mathsf{F} \vee \mathsf{H}^1$.
- $\lambda^1$ is not the intensity of $\tau_1$ with respect to $\mathsf{F} \vee \mathsf{H}^2$.

## 4.3 Bond Valuation

The following result was established in Jarrow and Yu (2001), who assume the fractional recovery of Treasury value scheme with the fixed recovery rates $\delta_1$ and $\delta_2$. Let $\lambda = \lambda_1 + \lambda_2$. For $\lambda - \alpha_2 \neq 0$ we denote

$$c_{\lambda_1,\lambda_2,\alpha_2}(u) = \frac{1}{\lambda - \alpha_2}\left(\lambda_1 e^{-\alpha_2 u} + (\lambda_2 - \alpha_2)e^{-\lambda u}\right).$$

For $\lambda - \alpha_2 = 0$ we set

$$c_{\lambda_1,\lambda_2,\alpha_2}(u) = (1 + \lambda_1 u)e^{-\lambda u}.$$

**Proposition 3** *For the bond issued by the primary firm we have*

$$D_1(t,T) = B(t,T)(\delta_1 + (1-\delta_1)e^{-\lambda_1(T-t)}\,\mathbb{1}_{\{\tau_1 > t\}}).$$

*The value of a zero-coupon bond issued by the secondary firm equals, on the set $\{\tau_1 > t\}$, that is, prior to default of the primary firm:*

$$D_2(t,T) = B(t,T)\left(\delta_2 + (1-\delta_2)c_{\lambda_1,\lambda_2,\alpha_2}(T-t)\,\mathbb{1}_{\{\tau_2 > t\}}\right)$$

*and on the set $\{\tau_1 \leq t\}$, that is, after default of the primary firm:*

$$D_2(t,T) = B(t,T)\left(\delta_2 + (1-\delta_2)e^{-\alpha_2(T-t)}\,\mathbb{1}_{\{\tau_2 > t\}}\right).$$

# Special Case: Zero Recovery

Assume that $\lambda_1 + \lambda_2 - \alpha_2 \neq 0$ and the bond is subject to the zero recovery scheme. For the sake of brevity, we set $r = 0$ so that $B(t, T) = 1$ for $t \leq T$.

Then we have the following result:

**Corollary 1** *If $\delta_2 = 0$ then $D_2(t, T) = 0$ on $\{\tau_2 \leq t\}$. On the set $\{\tau_2 > t\}$ we have*

$$D_2(t,T) = \mathbb{1}_{\{\tau_1 \leq t\}}\, e^{-\alpha_2(T-t)} + \mathbb{1}_{\{\tau_1 > t\}}\, \frac{1}{\lambda - \alpha_2}\Big(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2 - \alpha_2) e^{-\lambda(T-t)}\Big)$$

*where we denote $\lambda = \lambda_1 + \lambda_2$.*

Under the present assumptions:

$$D_2(t,T) = \mathsf{Q}^*\{\tau_2 > T \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2\}$$

and the general formula yields

$$D_2(t,T) = \mathbb{1}_{\{\tau_2 > t\}}\, \frac{\mathsf{Q}^*\{\tau_2 > T \,|\, \mathcal{H}_t^1\}}{\mathsf{Q}^*\{\tau_2 > t \,|\, \mathcal{H}_t^1\}}\,.$$

If we set $\Lambda_t^2 = \int_0^t \lambda_u^2\, du$ then

$$D_2(t,T) = \mathbb{1}_{\{\tau_2 > t\}}\, \mathsf{E}_{\mathsf{Q}^*}(e^{\Lambda_t^2 - \Lambda_T^2} \,|\, \mathcal{H}_t^1).$$

## 5 Extension of Jarrow and Yu Results

We shall now argue that the assumption that some firms are primary while other firms are secondary is not relevant. For the sake of simplicity, we assume that:

- $n = 2$; i.e., we consider two firms only,
- the interest rate $r$ is zero: $B(t,T) = 1$ for every $t \le T$,
- the filtration $\mathbb{F}$ is trivial,
- both bonds are subject to the zero-recovery scheme.

Since the situation is symmetric, it suffices to analyze a bond issued by the first firm.

By definition, the price of this bond equals

$$D_1(t,T) = \mathbb{Q}^*\{\tau_1 > T \mid \mathcal{H}_t^1 \vee \mathcal{H}_t^2\}.$$

We shall also evaluate the following values based on partial observations:

$$\tilde{D}_1(t,T) = \mathbb{Q}^*\{\tau_1 > T \mid \mathcal{H}_t^2\}$$

and

$$\hat{D}_1(t,T) = \mathbb{Q}^*\{\tau_1 > T \mid \mathcal{H}_t^1\}.$$

## 5.1 Kusuoka's Construction

Under the original probability measure Q the random times $\tau_i$, $i = 1, 2$ are mutually independent random variables with exponential laws with parameters $\lambda_1$ and $\lambda_2$, resp.

Girsanov's Theorem

For a fixed $T > 0$, we define $\mathsf{Q}^* \sim \mathsf{Q}$ on $(\Omega, \mathcal{G})$

$$\frac{d\mathsf{Q}^*}{d\mathsf{Q}} = \eta_T, \quad \text{Q-a.s.}$$

where $\eta_t$, $t \in [0, T]$, satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \eta_{u-} \kappa_u^i \, d\tilde{M}_u^i$$

where

$$\tilde{M}_t^i = H_t^i - \int_0^{t \wedge \tau_i} \lambda_i \, du$$

$H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$ and processes $\kappa^1$ and $\kappa^2$ satisfy:

$$\kappa_t^1 = \mathbb{1}_{\{\tau_2 < t\}} \left( \frac{\alpha_1}{\lambda_1} - 1 \right), \quad \kappa_t^2 = \mathbb{1}_{\{\tau_1 < t\}} \left( \frac{\alpha_2}{\lambda_2} - 1 \right).$$

It can be checked that the 'martingale intensities' under $\mathsf{Q}^*$ are:

$$\begin{aligned} \lambda_t^1 &= \lambda_1 \, \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \, \mathbb{1}_{\{\tau_2 \le t\}}, \\ \lambda_t^2 &= \lambda_2 \, \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \, \mathbb{1}_{\{\tau_1 \le t\}}. \end{aligned}$$

Let us focus on $\tau_1$. Let $\Lambda^1_t = \int_0^t \lambda^1_u \, du$. Then:

- $\lambda^1$ is an $\mathsf{H}^2$-predictable process and the process

$$M^1_t = H^1_t - \int_0^{t\wedge\tau_1} \lambda^1_u \, du = H^1_t - \Lambda^1_{t\wedge\tau_1}$$

  follows a $\mathsf{G}$-martingale under $\mathsf{Q}^*$.

- $\lambda^1$ is not the intensity of the default time $\tau_1$ with respect to $\mathsf{H}^2$ under $\mathsf{Q}^*$. In general

$$\mathsf{Q}^*\{\tau_1 > s \,|\, \mathcal{H}^1_t \vee \mathcal{H}^2_t\} \neq \mathbb{1}_{\{\tau_1 > t\}} \, \mathsf{E}_{\mathsf{Q}^*}(e^{\Lambda^1_t - \Lambda^1_s} \,|\, \mathcal{H}^2_t).$$

- $\lambda^1$ is the intensity of the default time $\tau_1$ with respect to $\mathsf{H}^2$ under $\mathsf{Q}^1 \sim \mathsf{Q}$, where

$$\frac{d\mathsf{Q}^1}{d\mathsf{Q}} = \tilde{\eta}_T, \quad \mathsf{Q}\text{-a.s.}$$

  and $\tilde{\eta}_t, \; t \in [0,T],$ satisfies

$$\tilde{\eta}_t = 1 + \int_{]0,t]} \tilde{\eta}_{u-} \kappa^2_u \, d\tilde{M}^2_u.$$

  For $s > t$ we have

$$\mathsf{Q}^1\{\tau_1 > s \,|\, \mathcal{H}^1_t \vee \mathcal{H}^2_t\} = \mathbb{1}_{\{\tau_1 > t\}} \, \mathsf{E}_{\mathsf{Q}^1}(e^{\Lambda^1_t - \Lambda^1_s} \,|\, \mathcal{F}_t)$$

  but also

$$\mathsf{Q}^*\{\tau_1 > s \,|\, \mathcal{H}^1_t \vee \mathcal{H}^2_t\} = \mathsf{Q}^1\{\tau_1 > s \,|\, \mathcal{H}^1_t \vee \mathcal{H}^2_t\}.$$

## 5.2 Interpretation of Intensities

Recall that the processes $\lambda_1$ and $\lambda_2$ have jumps if $\alpha_i \neq \lambda_i$, namely:

$$\lambda_t^1 = \lambda_1 \, \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \, \mathbb{1}_{\{\tau_2 \leq t\}}$$

and

$$\lambda_t^2 = \lambda_2 \, \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \, \mathbb{1}_{\{\tau_1 \leq t\}}.$$

The following result shows that the intensities $\lambda^1$ and $\lambda^2$ are 'local intensities' of default with respect to the information available at time $t$.

**Proposition 4** *For $i = 1, 2$ and every $t \in \mathsf{R}_+$ we have*

$$\lambda_i = \lim_{h \downarrow 0} h^{-1} \, \mathsf{Q}^* \{ t < \tau_i \leq t + h \,|\, \tau_1 > t, \, \tau_2 > t \}.$$

*Moreover:*

$$\alpha_1 = \lim_{h \downarrow 0} h^{-1} \, \mathsf{Q}^* \{ t < \tau_1 \leq t + h \,|\, \tau_1 > t, \, \tau_2 \leq t \}.$$

*and*

$$\alpha_2 = \lim_{h \downarrow 0} h^{-1} \, \mathsf{Q}^* \{ t < \tau_2 \leq t + h \,|\, \tau_2 > t, \, \tau_1 \leq t \}.$$

Conclusion: the model can be reformulated as a two-dimensional Markov chain (cf. Lando (1998)).

## 5.3 Bond Valuation

**Proposition 5** *The price $D_1(t,T)$ on $\{\tau_1 > t\}$ equals*

$$D_1(t,T) = \mathbb{1}_{\{\tau_2 \le t\}} e^{-\alpha_1(T-t)} + \mathbb{1}_{\{\tau_2 > t\}} \frac{1}{\lambda - \alpha_1} \Big( \lambda_2 e^{-\alpha_1(T-t)} + (\lambda_1 - \alpha_1) e^{-\lambda(T-t)} \Big).$$

*Furthermore*

$$\tilde{D}_1(t,T) = \mathbb{1}_{\{\tau_2 \le t\}} \frac{(\lambda - \alpha_2)\lambda_2 e^{-\alpha_1(T-\tau_2)}}{\lambda_1 \alpha_2 e^{(\lambda-\alpha_2)\tau_2} + \lambda(\lambda_2 - \alpha_2)} + \mathbb{1}_{\{\tau_2 > t\}} \frac{\lambda - \alpha_2}{\lambda - \alpha_1} \frac{(\lambda_1 - \alpha_1)e^{-\lambda(T-t)} + \lambda_2 e^{-\alpha_1(T-t)}}{\lambda_1 e^{-(\lambda-\alpha_2)t} + \lambda_2 - \alpha_2}$$

*and*

$$\hat{D}_1(t,T) = \mathbb{1}_{\{\tau_1 > t\}} \frac{\lambda_2 e^{-\alpha_1 T} + (\lambda_1 - \alpha_1)e^{-\lambda T}}{\lambda_2 e^{-\alpha_1 t} + (\lambda_1 - \alpha_1)e^{-\lambda t}}.$$

Observe that:

- formula for $D_1(t,T)$ coincides with the Jarrow and Yu formula for the bond issued by a secondary firm,
- $D_1(t,T)$ and $\hat{D}_1(t,T)$ represent ex-dividend values of the bond, so that they vanish after default,
- the latter remark does not apply to $\tilde{D}_1(t,T)$, however.

# 6 Dependent Intensities of Credit Migrations

The goal is to extend the previous analysis and some results to the case of multiple credit ratings.

Assume that the current financial standing of the $i^{\text{th}}$ firm is reflected through the credit ranking process $C^i$ with values in a finite set of credit grades $\mathcal{K}_i = \{1, \ldots, k_i\}$.

For the sake of simplicity, we assume that:

- the reference filtration $\mathsf{F}$ is trivial,
- there are two firms only.

Let $\mathsf{F}^i = \mathsf{F}^{C^i}$, $i = 1, 2$, denote the filtration generated by $C^i$ and let $\mathsf{G} = \mathsf{F}^1 \vee \mathsf{F}^2$. We examine the two following Markovian properties under the martingale measure $\mathsf{Q}^*$.

- the Markov property of $C = (C^1, C^2)$:

$$\mathsf{Q}^*\{C_s^1 = c_1, C_s^2 = c_2 \,|\, \mathcal{G}_t\} = \mathsf{Q}^*\{C_s^1 = c_1, C_s^2 = c_2 \,|\, C_t^1, C_t^2\}.$$

- the $\mathsf{F}^j$-conditional Markov property of $C^i$ for $i \neq j$:

$$\mathsf{Q}^*\{C_s^1 = c_1 \,|\, \mathcal{G}_t) \;=\; \mathsf{Q}^*\{C_s^1 = c_1 \,|\, \sigma(C_t^1) \vee \mathcal{F}_t^2\},$$

$$\mathsf{Q}^*\{C_s^2 = c_2 \,|\, \mathcal{G}_t\} \;=\; \mathsf{Q}^*\{C_s^2 = c_2 \,|\, \sigma(C_t^2) \vee \mathcal{F}_t^1\}.$$

## 6.1 Extension of Kusuoka's Construction

Assume that $k_1 = k_2 = 3$ (three rating grades). We consider the two independent Markov chains $C^i$, $i = 1, 2$ defined on $(\Omega, \mathcal{G}, \mathsf{Q})$ and taking values in $\mathcal{K} = \{1, 2, 3\}$ with generators:

$$\Lambda^i = \begin{pmatrix} -\lambda^i_{12} - \lambda^i_{13} & \lambda^i_{12} & \lambda^i_{13} \\ \lambda^i_{21} & -\lambda^i_{21} - \lambda^i_{23} & \lambda^i_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

The state $k = 3$ is the only absorbing state for each chain. We assume that $(C^1_0, C^2_0) = (1, 1)$.

Next, we define a probability measure $\mathsf{Q}^*$ equivalent to $\mathsf{Q}$. To this end, we introduce processes $\kappa^{i;kk'}$:

$$\kappa^{i;kk'}_t = \sum_{l=2}^{3} \mathbf{H}^{j;kl}_{t-} \left( \frac{\lambda^{i;l}_{kk'}}{\lambda^i_{kk'}} - 1 \right)$$

for $i = 1, 2,\ j \neq i,\ k = 1, 2,\ k' = 1, 2, 3,\ k \neq k'$, where

$$\mathbf{H}^{j;k,l}_t = H^{j;k}_t H^{j;l}_t$$

with $H^{j;k}_t = \mathbb{1}_{\{C^j_t = k\}}$ for $i, j = 1, 2$, $j \neq i$, and $k = 1, 2, 3$. We also define, for $i = 1, 2$ and $k \neq k'$, the transition counting process $H^{i;kk'}_t = \Sigma_{0<u\leq t} H^{i;k}_{u-} H^{i;k'}_u$.

## Associated Martingales

For $i = 1, 2$ the process $M^{i;kk'}$ given by the expression

$$M_t^{i;kk'} = H_t^{i;kk'} - \int_0^t \lambda_{kk'}^i H_u^{i;k}\, du, \quad k \neq k',$$

is known to follow an $\mathsf{F}^i$-martingale under Q, and thus also a G-martingale under Q where $\mathsf{G} = \mathsf{F}^1 \vee \mathsf{F}^2$.

We define a strictly positive martingale under Q:

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \sum_{k=1}^{2} \sum_{k'=1, k' \neq k}^{3} \eta_{u-} \kappa_u^{i;kk'}\, dM_u^{i;kk'}$$

and the probability measure $\mathsf{Q}^*$ equivalent to Q:

$$\frac{d\mathsf{Q}^*}{d\mathsf{Q}} = \eta_T, \quad \text{Q-a.s.}$$

The following result generalizes Kusuoka's construction.

**Proposition 6** *For each $i \neq j$ the process $C^i$ follows an $\mathsf{F}^j$-conditional Markov chain under $\mathsf{Q}^*$. The $\mathsf{F}^j$-transition intensities of $C^i$ under $\mathsf{Q}^*$ are $($for $k \neq k')$*

$$\lambda_{kk'}^{i;*}(t) = (1 + \kappa_t^{i;kk'})\lambda_{kk'}^i$$

*or more explicitly*

$$\lambda_{kk'}^{i;*}(t) = \lambda_{kk'}^i H_t^{j;1} + \sum_{l=2}^{3} \mathbf{H}_t^{j;kl} \lambda_{kk'}^{i;l}.$$

# Conditional Markov Property

The conditional Markov property follows from:

- the fact that the density $\eta$ only depends on $C = (C^1, C^2)$,
- the abstract Bayes formula,
- the fact that the process $C$ is Markovian under Q.

Properties of the model:

- for $i = 1, 2,\ j \neq i$, the process $\lambda^{i;*}_{kk'}$ is the corresponding F$^j$-martingale intensity. In other words, the process $M^{i;*;kk'}$ defined as

$$M^{i;*;kk'}_t = H^{i;kk'}_t - \int_0^t \lambda^{i;*}_{kk'}(u) H^{i;k}_u \, du, \quad k \neq k',$$

  and the process

$$M^{i;*;3}_t = H^{1;3}_{t \wedge s} - \sum_{m=1}^{2} H^{1;m}_{t \wedge s} \lambda^{1;*}_{m3}(t \wedge s)$$

  are G-martingales under Q$^*$.
- the intensities $\lambda^{i;*}_{kk'}$ have the natural interpretation as the 'local intensities' of credit migrations.

## 6.2 Interpretation of Intensities

The intuitive meaning of intensity parameters:

- for original intensities:

$$\lambda^1_{kk'} = \lim_{h\downarrow 0} h^{-1}\, \mathsf{Q}\{C^1_{t+h} = k' \,|\, C^1_t = k,\, C^2_t = 1\},$$

  but also for $l = 2, 3$

$$\lambda^1_{kk'} = \lim_{h\downarrow 0} h^{-1}\, \mathsf{Q}^*\{C^1_{t+h} = k' \,|\, C^1_t = k,\, C^2_t = l\},$$

- for modified intensities: for $l = 2, 3$

$$\lambda^{1;l}_{kk'} = \lim_{h\downarrow 0} h^{-1}\, \mathsf{Q}^*\{C^1_{t+h} = k' \,|\, C^1_t = k,\, C^2_t = l\}.$$

Model's inputs: original generators $\Lambda^1$ and $\Lambda^2$, and the modified matrices:

$$\Lambda^{i;l} = \begin{pmatrix} -\lambda^{i;l}_{12} - \lambda^{i;l}_{13} & \lambda^{i;l}_{12} & \lambda^{i;l}_{13} \\ \lambda^{i;l}_{21} & -\lambda^{i;l}_{21} - \lambda^{i;l}_{23} & \lambda^{i;l}_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

for $i = 1, 2$ and $l = 2, 3$.

## 6.3 First-to-Default Swap

Let $C = (C^1, \ldots, C^n)$. We assume that the payoff occurs at the first change of the credit rating of the firm 1 or 2. The payoff is digital, specifically, if $\tau = \tau_1 \wedge \tau_2$ then

$$Z_\tau = K_1 \, \mathbb{1}_{\{\tau = \tau_1 \leq T\}} + K_2 \, \mathbb{1}_{\{\tau = \tau_2 \leq T\}}.$$

Basic steps of the valuation procedure:

- introduce an auxiliary probability measure $\mathsf{Q}^{1,2}$ equivalent to $\mathsf{Q}^*$,
- verify that any martingale under $\mathsf{Q}^{1,2}$ with respect to $\mathsf{G}^{1,2} = \mathsf{F} \vee \mathsf{H}^3 \vee \ldots \vee \mathsf{H}^n$ is also a $\mathsf{G} = \mathsf{F} \vee \mathsf{H}^1 \vee \ldots \vee \mathsf{H}^n$ martingale under $\mathsf{Q}^{1,2}$,
- use the standard formula to find the $\mathsf{G}^{1,2}$-conditional laws of $\tau_1$ and $\tau_2$ under $\mathsf{Q}^*$ through conditional expectations with respect to $\mathsf{Q}^{1,2}$,
- use the fact that $\tau_1$ and $\tau_2$ are $\mathsf{G}^{1,2}$-conditionally independent under $\mathsf{Q}^*$ in order to value the swap.

Conclusion: We argue that in some cases a high-dimensional (unconditional) expectation can be efficiently evaluated as a low-dimensional conditional expectation under an equivalent probability measure.