Chapter 7
The Pricing of Second Generation Exotics

by Jürgen Hakala, Ghislain Perissé and Tino Senge

After vanilla options and the first generation exotics some more exotic options are of special interest for some clients. Here we present a formula catalogue for computing the Theoretical Value (TV) of such options in the Black–Scholes model.

7.1 Introduction

The pricing and hedging of the second generation exotic options in the Black–Scholes model works the same way as in the case of first generation exotics. One takes a geometric Brownian motion with a risk-neutral drift and computes the discounted expected value of the respective option payoffs. Computing the expectation results in the TV of the option. Trading at the TV is subject to some model risk. However, for second generation exotics, the bid-ask spreads are usually wider. Here we outline the valuation of some exotics such as forward-start options in Section 7.2, ratchet options in Section 7.3, power options in Section 7.4, instalment options in Section 7.5, stairs options in Section 7.6, compound options on a forward-start strategy in Section 7.7, options on the minimum/maximum in Section 7.8 and their generalisation in Section 7.9.

All options are valued in the model for the exchange rate

$$dS_t = S_t[(r^d - r^f)dt + \sigma dW_t] \tag{7.1}$$

7.2 Forward-start options

A forward-start plain vanilla option is an option where the strike of the option is derived at some date in the future relative to the spot at that date. It can be seen as an option where it is paid for now but will start at some time in the future. Let us consider two dates T_F and T_e, $T_F < T_e$, where

T_F is the forward-start date,

T_e is the expiry date.

The holder of the option receives at time T_F an option with expiration date T_e. The strike of this option will be fixed at T_F to αS_{T_F}, where α is a constant which is specified at inception $t = 0$ of the trade. The payoff of a forward-start option is defined as

$$\left(\phi\left(S_{T_e} - \alpha S_{T_F}\right)\right)^{+} \tag{7.2}$$

7.2.1 *Pricing*

In order to price forward-start options we have to distinguish between the time intervals $[0, T_F]$ and $[T_F, T_e]$.

First we consider the value of the option during interval $[T_F, T_e]$. The price of a call option ($\phi = 1$) is given by

$$v(t) = v_{BS}(S_t, \alpha S_{T_F}, T_e - t, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2}), \; t \in [T_F, T_e] \tag{7.3}$$

where $v_{BS}(\cdot)$ is the value of an European call option with spot S_t, strike αS_{T_F}, and time to maturity equal to $T_e - t$.

We now go on pricing the option within interval $[0, T_F]$. We start considering the value of the option at time T_F. At this time the value is equal to an ordinary European option. In case of a call option ($\phi = 1$) we get

$$v(T_F) = v_{BS}(S_{T_F}, \alpha S_{T_F}, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2}) \tag{7.4}$$

$$= e^{r^f_{1,2}(T_e - T_F)} S_{T_F} \mathcal{N}(d_+) - e^{r^d_{1,2}(T_e - T_F)} \alpha S_{T_F} \mathcal{N}(d_-) \tag{7.5}$$

$$= S_{T_F} \; v_{BS}(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2}) \tag{7.6}$$

where

$$d_\pm = \frac{1}{\sigma_{1,2}\sqrt{T_e - T_F}} \left[\ln \frac{S_{T_F}}{\alpha S_{T_F}} + (r^d_{1,2} - r^f_{1,2})(T_e - T_F)\right] \pm \frac{1}{2}\sigma_{1,2}\sqrt{T_e - T_F} \tag{7.7}$$

Using risk-neutral valuation the value of a forward-start call option at time $t = 0$ is given by

$$v(0) = I\!E^{Q_1}\left[e^{-r^d_1 T_1} S_{T_F} v_{BS}(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2})\right] \tag{7.8}$$

where Q_1 is the probability measure such that $\left(e^{-(r^d_1 - r^f_1)t} S_t\right)_{t \in [0, T_F]}$ is a martingale. Since $v_{BS}(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2})$ is deterministic, we use the martingale property and obtain

$$v_0 = e^{-r^f_1 T_1} S_0 v_{BS}(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2}) \tag{7.9}$$

More generally, for $t \in [0, T_F]$, we have

$$v_t = e^{-r^f_1(T_F - t)} S_t \; v_{BS}(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2}) \tag{7.10}$$

$$= e^{-r^f_1(T_F - t)} \; v_{BS}(S_t, \alpha S_t, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2}) \tag{7.11}$$

The same calculation can be done for a forward-start put option ($\phi = -1$).

REMARK 7.1 *The rates r^d_1 and r^f_1 are the instantaneous interest rates. These rates are used for discounting the option value within interval $[0, T_F)$.*

REMARK 7.2 *The rates $r^d_{1,2}$ and $r^f_{1,2}$ are forward rates for interval $[T_F, T_e]$. These rates will be used in the option pricing formula. In the same way $\sigma_{1,2}$ is the forward volatility for time period $[T_F, T_e]$.*

7.2.2 *Greeks*

As in pricing we have to distinguish between two time intervals $[0, T_F]$ and $[T_F, T_e]$.

During the interval $[T_F, T_e]$ the sensitivity parameters are the same as in the case of a plain vanilla option.

Let us focus on interval $[0, T_F]$. Within this period of time the sensitivity parameters delta, gamma and theta for call options are

$$\Delta_t = e^{-r_1^f (T_F - t)} \; v_{BS}(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2}) \tag{7.12}$$

$$\Gamma_t = 0 \tag{7.13}$$

$$\Theta_t = -e^{-r_1^f (T_F - t)} r_1^f S_t \; v_{BS}(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2}) \tag{7.14}$$

In order to calculate vega ($\mathcal{V}$) and rho we have to consider dependencies in the forward rates and forward volatility.

First we recall the definition of the forward volatility. Let σ_1 be the volatility for the interval $[0, T_F]$ and let σ_2 be the volatility within $[0, T_e]$. The forward volatility $\sigma_{1,2}$ within $[T_F, T_e]$ is given by

$$\sigma_{1,2} = \sqrt{\frac{\sigma_2^2 T_2 - \sigma_1^2 T_1}{T_e - T_F}} \tag{7.15}$$

if the term under the square root is non-negative. Using

$$\frac{\partial \sigma_{1,2}}{\partial \sigma_1} = -\frac{1}{\sqrt{T_e - T_F}} \frac{\sigma_1 T_1}{\sqrt{\sigma_2^2 T_2 - \sigma_1^2 T_1}} \tag{7.16}$$

$$\frac{\partial \sigma_{1,2}}{\partial \sigma_2} = \frac{1}{\sqrt{T_e - T_F}} \frac{\sigma_2 T_2}{\sqrt{\sigma_2^2 T_2 - \sigma_1^2 T_1}} \tag{7.17}$$

we can calculate the vegas

$$\mathcal{V}_{\sigma_1 \, t} = -e^{-r_1^f (T_F - t)} S_t \left(\frac{1}{\sqrt{T_e - T_F}} \frac{\sigma_1 (T_F - t)}{\sqrt{\sigma_2^2 (T_e - t) - \sigma_1^2 (T_F - t)}} \right) \times \mathcal{V}_{\sigma_{1,2} \; BS(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2})} \tag{7.18}$$

$$\mathcal{V}_{\sigma_2 \, t} = e^{-r_1^f (T_F - t)} S_t \left(\frac{1}{\sqrt{T_e - T_F}} \frac{\sigma_2 (T_e - t)}{\sqrt{\sigma_2^2 (T_e - t) - \sigma_1^2 (T_F - t)}} \right) \times \mathcal{V}_{\sigma_{1,2} \; BS(1, \alpha, T_e - T_F, \sigma_{1,2}, r^d_{1,2}, r^f_{1,2})} \tag{7.19}$$

Next we recall the definition of the forward rate. Let $r_1^{d/f}$ be the rates for interval $[0, T_F]$ and let $r_2^{d/f}$ be the rates for interval $[0, T_e]$. The forward rates $r_{1,2}^{d/f}$ are given by

$$r_{1,2}^{d/f} = \frac{r_2^{d/f} T_e - r_1^{d/f} T_F}{T_e - T_F} \tag{7.20}$$

Recalling

$$\frac{\partial r_{1,2}^{d/f}}{\partial r_1^{d/f}} = -\frac{T_F}{T_e - T_F} \tag{7.21}$$

$$\frac{\partial r_{1,2}^{d/f}}{\partial r_2^{d/f}} = \frac{T_e}{T_e - T_F} \tag{7.22}$$

we get as derivatives with respect to the rates $r_1^{d/f}$

$$\text{rho}_{r_1^d\, t} = -e^{-r_1^f(T_F-t)} S_t \left(\frac{T_F - t}{T_e - T_F}\right) \text{rho}_{r_{1,2}^d\ BS(1,\alpha,T_e-T_F,\sigma_{1,2},r_{1,2}^d,r_{1,2}^f)} \tag{7.23}$$

$$\begin{aligned}\text{rho}_{r_1^f\, t} = &-e^{-r_1^f(T_F-t)} S_t \left(\frac{T_F - t}{T_e - T_F}\right) \text{rho}_{r_{1,2}^f\ BS(1,\alpha,T_e-T_F,\sigma_{1,2},r_{1,2}^d,r_{1,2}^f)} \\ &- e^{r_1^f(T_F-t)}(T_F - t) S_t\ P_{BS}(1, \alpha, T_e - T_F, \sigma_{1,2}, r_{1,2}^d, r_{1,2}^f)\end{aligned} \tag{7.24}$$

As derivatives with respect to the rates $r_2^{d/f}$ we can derive

$$\text{rho}_{r_2^d\, t} = e^{-r_1^f(T_F-t)} S_t \left(\frac{T_e - t}{T_e - T_F}\right) \text{rho}_{r_{1,2}^d\ BS(1,\alpha,T_e-T_F,\sigma_{1,2},r_{1,2}^d,r_{1,2}^f)} \tag{7.25}$$

$$\text{rho}_{r_2^f\, t} = e^{-r_1^f(T_F-t)} S_t \left(\frac{T_e - t}{T_e - T_F}\right) \text{rho}_{r_{1,2}^f\ BS(1,\alpha,T_e-T_F,\sigma_{1,2},r_{1,2}^d,r_{1,2}^f)} \tag{7.26}$$

REMARK 7.3 *From Equation (7.12) it can be seen that in the case of a forward-start option the deltas of calls and puts are always positive on time interval* $[0, T_F]$.

7.3 Ratchet options

A ratchet option consists of a series of forward-start options. The strike for the next exercise date is set relative to the spot at the previous exercise date. Ratchet options can be presented as a sum of forward-start options. The exercise date of the first component will be the forward-start date of the second, the exercise date of the second component will be the forward-start date of the third, ... Sometimes this type of option is called moving strike option or cliquet option.

Ratchet options can be priced as the a sum of forward-start options. The price of an *n*-period Ratchet option is given by

$$v_{Ratchet}(\phi, t) = \sum_{i=1}^{n} v_{FPV}(\phi, t, T_{F_i}, T_{e_i}, \alpha_i) \tag{7.27}$$

where

$$T_{F_1} = t,\ T_{F_{i+1}} = T_{e_i},\ 1 \leqslant i \leqslant n - 1 \tag{7.28}$$

and v_{FPV} is the price of a forward-start plain vanilla as presented in Section 7.2.

7.4 Power options

A power option pays off the square of a plain vanilla with strike K_S. Since this would in general make the power option very expensive, it is capped at a level K_C. The power call pays

$$\mathbb{I}_{\{K_S < S_{T_e} < K_C\}} \left(S_{T_e} - K_S\right)^2 + \mathbb{I}_{\{K_C \leqslant S_{T_e}\}} (K_C - K_S)^2 \tag{7.29}$$

the power put has payoff

$$\mathbb{I}_{\{K_C < S_{T_e} < K_S\}} \left(K_S - S_{T_e}\right)^2 + \mathbb{I}_{\{S_{T_e} \leqslant K_C\}} (K_S - K_C)^2 \tag{7.30}$$

with expiration date T_e and delivery date $T_d \geqslant T_e$. We denote by t the valuation date (horizon) and assume that the premium is paid at the premium value date T_p.

We model the exchange rate by

$$dS_t = \mu_{t,T_e} S_t\, dt + \sigma_{t,T_e} S_t\, dW \tag{7.31}$$

or equivalently by

$$S_{T_e} = S_t e^{\left(\mu_{t,T_e} - \frac{1}{2}\sigma^2_{t,T_e}\right)(T_e - t) + \sigma W_{T_e - t}} \tag{7.32}$$

with $\mu_{t,T_e} \triangleq r^d_{t,T_e} - r^f_{t,T_e}$.

Risk-neutral valuation leads to the price of the power call

$$\begin{aligned} v(t) &= e^{r^d_{t,T_p}(T_p - t)} \\ &\quad \times I\!E^t\left[e^{-r^d_{T_p,T_d}(T_d - T_p)}\left(I\!I_{\{K_S < S_{T_e} < K_C\}}\left(S_{T_e} - K_S\right)^2 + I\!I_{\{K_C \leqslant S_{T_e}\}}(K_C - K_S)^2\right)\right] \\ &= e^{r^d_{t,T_p}(T_p - t) - r^d_{T_p,T_d}(T_d - T_p)} \\ &\quad \times \left(I\!E^t\left[I\!I_{\{K_S < S_{T_e} < K_C\}} S^2_{T_e}\right] - 2K_S I\!E^t\left[I\!I_{\{K_S < S_{T_e} < K_C\}} S_{T_e}\right]\right. \\ &\quad \left. + \; K_S^2 I\!E^t\left[I\!I_{\left\{K_S < S_{T_e} < K_C\right\}}\right] + (K_C - K_S)^2 I\!E^t\left[I\!I_{\left\{K_C \leqslant S_{T_e}\right\}}\right]\right) \end{aligned} \tag{7.33}$$

Defining

$$d^0_{K_S} \triangleq \frac{1}{\sigma_{t,T_e}\sqrt{T_e - t}}\left(\ln\frac{S_t}{K_S} + \left(\mu_{t,T_e} - \frac{1}{2}\sigma^2_{t,T_e}\right)(T_e - t)\right) \tag{7.34}$$

$$d^0_{K_C} \triangleq \frac{1}{\sigma_{t,T_e}\sqrt{T_e - t}}\left(\ln\frac{S_t}{K_C} + \left(\mu_{t,T_e} - \frac{1}{2}\sigma^2_{t,T_e}\right)(T_e - t)\right) \tag{7.35}$$

$$d^1_{K_S} \triangleq d^0_{K_S} + \sigma_{t,T_e}\sqrt{T_e - t} \tag{7.36}$$

$$d^1_{K_C} \triangleq d^0_{K_C} + \sigma_{t,T_e}\sqrt{T_e - t} \tag{7.37}$$

$$d^2_{K_S} \triangleq d^0_{K_S} + 2\sigma_{t,T_e}\sqrt{T_e - t} \tag{7.38}$$

$$d^2_{K_C} \triangleq d^0_{K_C} + 2\sigma_{t,T_e}\sqrt{T_e - t} \tag{7.39}$$

we can compute the expected values and obtain for the price of the power option

$$\begin{aligned} v(t) &= e^{r^d_{t,T_p}(T_p - t) - r^d_{T_p,T_d}(T_d - T_p)} \\ &\quad \times \left\{S_t^2 e^{\left(2\mu_{t,T_e} + \sigma^2_{t,T_e}\right)(T_e - t)}\left(\mathcal{N}\left(\phi d^2_{K_S}\right) - \mathcal{N}\left(\phi d^2_{K_C}\right)\right)\right. \\ &\quad - \; 2K_S S_t e^{\mu_{t,T_e}(T_e - t)}\left(\mathcal{N}\left(\phi d^1_{K_S}\right) - \mathcal{N}\left(\phi d^1_{K_C}\right)\right) \\ &\quad + \; K_S^2\left(\mathcal{N}\left(\phi d^0_{K_S}\right) - \mathcal{N}\left(\phi d^0_{K_C}\right)\right) \\ &\quad \left. + \; (K_C - K_S)^2 \mathcal{N}\left(\phi d^0_{K_C}\right)\right\} \end{aligned} \tag{7.40}$$

where the binary variable ϕ takes the values $+1$ in case of a call and -1 in case of a put.

7.5 Instalment options

An instalment option is an iterative compound option with

- N working dates $T_1, T_2, \ldots, T_N$ ($T_0 = t$ is the horizon date),
- N strikes, $K_1, K_2, \ldots, K_N$ ($K_i = K_{T_i}$),
- N put–call indicators, $\phi_1, \phi_2, \ldots, \phi_N$ taking the values $+1$ in case of a call and -1 in case of a put.

On each period $[T_i, T_{i+1}]$, the spot $S_i = S_{T_i}$ is modelled by

$$dS_i = \mu_{i-1,i} S_i\, dt + \sigma_{i-1,i} S_i\, dW \tag{7.41}$$

where the numbers $\mu_{i-1,i} \stackrel{\triangle}{=} {r_{i-1,i}}^d - {r_{i-1,i}}^f$ are the forward drifts and $\sigma_{i-1,i}$ the forward volatilities of the asset, which we take to be

$$\sigma_{i-1,i} = \sqrt{\frac{\sigma_i^2 T_i - \sigma_{i-1}^2 T_{i-1}}{T_i - T_{i-1}}} \tag{7.42}$$

$$r_{i-1,i} = \frac{r_i T_i - r_{i-1} T_{i-1}}{T_i - T_{i-1}} \tag{7.43}$$

The quantities r_i^d, r_i^f and σ_i denote the usual forward rates and volatilities.

For $N = 2$, this option is a regular compound option as discussed in Chapter 9. The valuation of instalment options can be carried out using finite differences. We describe this in detail in Section 7.3 of Chapter 22. Here we show how to determine the value of an instalment option by iterative integration.

We introduce the notation

$$d_1^i(x) \stackrel{\triangle}{=} \frac{1}{\sigma_{i+1}\sqrt{T_{i+1} - T_i}} \left(\ln \frac{S_i}{x} + \left(\mu_{i+1} + \frac{1}{2} {\sigma_{i+1}}^2 \right) (T_{i+1} - T_i) \right) \tag{7.44}$$

$$d_2^i(x) \stackrel{\triangle}{=} d_1^i(x) - \sigma_{i+1}\sqrt{T_{i+1} - T_i} \tag{7.45}$$

and denote by

$$\begin{aligned} \mathcal{V}_N(S_{N-1}, K_N) &= I\!E^{T_{N-1}}\left[(\phi_N (S_N - K_N))^+ \right] \\ &= \phi_N S_{N-1} e^{\mu_{N-1,N}(T_N - T_{N-1})} \mathcal{N}\left(\phi_N d_1^N(K_N) \right) - \phi_N K_N \mathcal{N}\left(\phi_N d_2^N(K_N) \right) \end{aligned}$$

the value of an undiscounted vanilla option at time T_{N-1} with strike K_N and maturity T_N when the spot is at S_{N-1}.

Introducing the two functionals

$$\begin{aligned} \mathcal{IO}_{\phi_i=+1}(\mathcal{F})(S_{i-1}) &\stackrel{\triangle}{=} I\!E\left[(\mathcal{F}(S_i) - K_i)^+ | S_{i-1} \right] \\ &= \int_{x_{+1}^i}^{+\infty} \mathcal{F}(S_i(y_i)) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}}\, dy_i \end{aligned} \tag{7.46}$$

$$\begin{aligned} \mathcal{IO}_{\phi_i=-1}(\mathcal{F})(S_{i-1}) &\stackrel{\triangle}{=} I\!E\left[(K_i - \mathcal{F}(S_i))^+ | S_{i-1} \right] \\ &= \int_{-\infty}^{x_{-1}^i} \mathcal{F}(S_i(y_i)) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}}\, dy_i \end{aligned} \tag{7.47}$$

with

$$S_i(y_i) = S_{i-1}e^{\left(\mu_{i-1,i}-\frac{1}{2}\sigma^2_{i-1,i}\right)(T_i-T_{i-1})+\sigma_{i-1,i}\sqrt{T_i-T_{i-1}}y_i} \tag{7.48}$$

$$x^i_{+1} = \inf\{y_i|\mathcal{F}(S_i(y_i)) = K_i\} \tag{7.49}$$

$$x^i_{-1} = \sup\{y_i|\mathcal{F}(S_i(y_i)) = K_i\} \tag{7.50}$$

we can write down the price of an instalment option in the form

$$v(t) = \mathcal{IO}_{\phi_1}\left(\ldots\mathcal{IO}_{\phi_{N-1}}(\mathcal{V}_N)\ldots\right) \tag{7.51}$$

7.6 Stairs options

A stairs option is an option which is working on N periods $[t, T_1]$ $[T_1, T_2], ..., [T_{N-1}, T_N]$, with one, two or no knock-out barrier(s) on each period and final time payoff

$$\left(\phi\left(S_{T_N} - K\right)\right)^+ \tag{7.52}$$

with a strike K.

On each period $[T_{i-1}, T_i]$ we define a lower barrier L_i and a higher barrier H_i. The asset spot is modelled by Equations (7.41), (7.42) and (7.43). Let S_i denote S_{T_i} with $T_0 = t$. In the sequel we recall the formulae for values of vanilla and barrier options as used in the setup of this section. For a derivation see Chapters 1 and 6.

7.6.1 *Last period functionals*

Let us introduce

$$d^i_1(x) \triangleq \frac{1}{\sigma_{i+1}\sqrt{T_{i+1}-T_i}}\left(\ln\frac{S_i}{x} + \left(\mu_{i+1} + \frac{1}{2}{\sigma_{i+1}}^2\right)(T_{i+1}-T_i)\right) \tag{7.53}$$

$$d^i_2(x) \triangleq d^i_1(x) - \sigma_{i+1}\sqrt{T_{i+1}-T_i} \tag{7.54}$$

$$d^i_3(x) \triangleq \frac{1}{\sigma_{i+1}\sqrt{T_{i+1}-T_i}}\left(\ln\frac{x}{S_i} + \left(\mu_{i+1} + \frac{1}{2}{\sigma_{i+1}}^2\right)(T_{i+1}-T_i)\right) \tag{7.55}$$

$$d^i_4(x) \triangleq d^i_3(x) - \sigma_{i+1}\sqrt{T_{i+1}-T_i} \tag{7.56}$$

No barrier The value function of a non-discounted vanilla option is given by

$$\begin{aligned}\mathcal{V}_N(S_{N-1}, K) &= I\!E^{T_{N-1}}\left[(\phi(S_N - K))^+\right]\\ &= \phi S_{N-1}e^{\mu_{N-1,N}(T_N-T_{N-1})}\mathcal{N}\left(\phi d^N_1(K)\right) - \phi K\mathcal{N}\left(\phi d^N_2(K)\right)\end{aligned} \tag{7.57}$$

One barrier The value function of a non-discounted up-and-out call is given by

$$\begin{aligned}UOC_N(S_{N-1}, K) &= I\!E^{T_{N-1}}\left[(S_N - K)^+ I\!I_{\{\max_{[T_{N-1},T_N]} S_s < H_N\}}\right]\\ &= S_{N-1}e^{\mu_{N-1,N}(T_N-T_{N-1})}\mathcal{N}\left(d^N_1(K)\right) - K\mathcal{N}\left(d^N_2(K)\right)\\ &\quad - S_{N-1}e^{\mu_{N-1,N}(T_N-T_{N-1})}\mathcal{N}\left(d^N_1(H_N)\right) + K\mathcal{N}\left(d^N_2(H_N)\right)\\ &\quad - S_{N-1}e^{\mu_{N-1,N}(T_N-T_{N-1})}\left(\frac{H_N}{S_{N-1}}\right)^{\frac{2\mu_{N-1,N}}{\sigma^2_{N-1,N}}+1}\\ &\quad \left\{\mathcal{N}\left(d^N_3(\frac{H^2_N}{K})\right) - \mathcal{N}\left(d^N_3(K)\right)\right\} + K\left(\frac{H_N}{S_{N-1}}\right)^{\frac{2\mu_{N-1,N}}{\sigma^2_{N-1,N}}-1}\\ &\quad \left\{\mathcal{N}\left(d^N_4(\frac{H^2_N}{K})\right) - \mathcal{N}\left(d^N_4(H_N)\right)\right\}\end{aligned} \tag{7.58}$$

The value function of a non-discounted down-and-out put is given by

$$
\begin{aligned}
DOP_N(S_{N-1}, K) &= I\!E^{T_{N-1}}\left[(K - S_N)^+ I\!I_{\{\inf_{[T_{N-1},T_N]} S_s > L_N\}}\right] \\
&= -S_{N-1}e^{\mu_{N-1,N}(T_N-T_{N-1})}\mathcal{N}\left(-d_1^N(K)\right) + K\mathcal{N}\left(-d_2^N(K)\right) \\
&\quad + S_{N-1}e^{\mu_{N-1,N}(T_N-T_{N-1})}\mathcal{N}\left(-d_1^N(L_N)\right) - K\mathcal{N}\left(-d_2^N(L_N)\right) \\
&\quad - S_{N-1}e^{\mu_{N-1,N}(T_N-T_{N-1})}\left(\frac{L_N}{S_{N-1}}\right)^{\frac{2\mu_{N-1,N}}{\sigma^2_{N-1,N}}+1} \\
&\quad \left\{\mathcal{N}\left(d_3^N\left(\frac{L_N^2}{K}\right)\right) - \mathcal{N}\left(d_3^N(K)\right)\right\} + K\left(\frac{L_N}{S_{N-1}}\right)^{\frac{2\mu_{N-1,N}}{\sigma^2_{N-1,N}}-1} \\
&\quad \left\{\mathcal{N}\left(d_4^N(\frac{L_N^2}{K})\right) - \mathcal{N}\left(d_4^N(L_N)\right)\right\}
\end{aligned}
\tag{7.59}
$$

Two barrier The value function of a non-discounted double-knock-out call is given by

$$
\begin{aligned}
&DKOC(S_{T_N}) \\
&= S_{T_{N-1}}\left[\sum_{n=-\infty}^{+\infty} e^{-(\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}}(2nA_{LH_N})+\mu_{N-1,N}(T_N-T_{N-1})}\right. \\
&\quad \times\left\{\mathcal{N}\left(A_{H_N} - (\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}} + 2nA_{LH_N}\right)\right. \\
&\quad \left.-\mathcal{N}\left(A_K - (\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}} + 2nA_{LH_N}\right)\right\} \\
&\quad - \sum_{n=-\infty}^{+\infty} e^{(\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}}(2A_{H_N}+2nA_{LH_N})+\mu_{N-1,N}(T_N-T_{N-1})} \\
&\quad \times\left\{\mathcal{N}\left(-(A_{H_N} + 2nA_{LH_N} + (\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}})\right)\right. \\
&\quad \left.\left.-\mathcal{N}\left(A_K - ((\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}} + 2A_{H_N} + 2nA_{LH_N})\right)\right\}\right] \\
&\quad - K\left[\sum_{n=-\infty}^{+\infty} e^{-\lambda_N\sqrt{T_N-T_{N-1}}(2nA_{LH_N})} \times\left\{\mathcal{N}\left(A_{H_N} - \lambda_N\sqrt{T_N-T_{N-1}} + 2nA_{LH_N}\right)\right.\right. \\
&\quad \left.-\mathcal{N}\left(A_K - \lambda_N\sqrt{T_N-T_{N-1}} + 2nA_{LH_N}\right)\right\} - \sum_{n=-\infty}^{+\infty} e^{\lambda_N\sqrt{T_N-T_{N-1}}(2A_{H_N}+2nA_{LH_N})} \\
&\quad \times\left\{\mathcal{N}\left(-(A_{H_N} + 2nA_{LH_N} + \lambda_N\sqrt{T_N-T_{N-1}})\right)\right. \\
&\quad \left.\left.-\mathcal{N}\left(A_K - (\lambda_N\sqrt{T_N-T_{N-1}} + 2A_{H_N} + 2nA_{LH_N})\right)\right\}\right]
\end{aligned}
\tag{7.60}
$$

and similarly for the non-discounted double-knock-out put

$$
\begin{aligned}
&DKOP(S_{T_N})\\
&= K\Bigg[\sum_{n=-\infty}^{+\infty} e^{-\lambda_N\sqrt{T_N-T_{N-1}}(2nA_{LH_N})}\\
&\times\Big\{\mathcal{N}\Big(A_K-\lambda_N\sqrt{T_N-T_{N-1}}+2nA_{LH_N}\Big)\\
&-\mathcal{N}\Big(A_{L_N}-\lambda_N\sqrt{T_N-T_{N-1}}+2nA_{LH_N}\Big)\Big\}\\
&-\sum_{n=-\infty}^{+\infty} e^{\lambda_N\sqrt{T_N-T_{N-1}}(2A_{H_N}+2nA_{LH_N})}\\
&\times\Big\{\mathcal{N}\Big(A_K-(2A_{H_N}+2nA_{LH_N}+\lambda_N\sqrt{T_N-T_{N-1}})\Big)\\
&-\mathcal{N}\Big(A_{L_N}-(\lambda_N\sqrt{T_N-T_{N-1}}+2A_{H_N}+2nA_{LH_N})\Big)\Big\}\Bigg]\\
&-S_t\Bigg[\sum_{n=-\infty}^{+\infty} e^{-(\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}}(2nA_{LH_N})+\mu_{N-1,N}(T_N-T_{N-1})}\\
&\times\Big\{\mathcal{N}\Big(A_K-(\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}}+2nA_{LH_N}\Big)\\
&-\mathcal{N}\Big(A_{L_N}-(\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}}+2nA_{LH_N}\Big)\Big\}\\
&-\sum_{n=-\infty}^{+\infty} e^{(\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}}(2A_{H_N}+2nA_{LH_N})+\mu_{N-1,N}(T_N-T_{N-1})}\\
&\times\Big\{\mathcal{N}\Big(A_K-(2A_{H_N}+2nA_{LH_N}+(\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}})\Big)\\
&-\mathcal{N}\Big(A_L-((\lambda_N+\sigma_{N-1,N})\sqrt{T_N-T_{N-1}}+2A_{H_N}+2nA_{LH_N})\Big)\Big\}\Bigg] \qquad (7.61)
\end{aligned}
$$

with

$$A_K \triangleq \frac{\ln\frac{K}{S_{N-1}}}{\sigma_{N-1,N}\sqrt{T_N-T_{N-1}}} \qquad (7.62)$$

$$A_{L_N} \triangleq \frac{\ln\frac{L_N}{S_{N-1}}}{\sigma_{N-1,N}\sqrt{T_N-T_{N-1}}} \qquad (7.63)$$

$$A_{H_N} \triangleq \frac{\ln\frac{H_N}{S_{N-1}}}{\sigma_{N-1,N}\sqrt{T_N-T_{N-1}}} \qquad (7.64)$$

$$A_{LH_N} \triangleq \frac{\ln\frac{H_N}{L_N}}{\sigma_{N-1,N}\sqrt{T_N-T_{N-1}}} \qquad (7.65)$$

7.6.2 *Within period functionals*

No barrier We define the functional

$$
\begin{aligned}
Q_i(\mathcal{F})(S_{i-1}) &= I\!E^{T_{i-1}}[\mathcal{F}(S_i)]\\
&= \int_{-\infty}^{+\infty}\mathcal{F}(S_i(y_i))\frac{1}{\sqrt{2\pi}}e^{-\frac{y_i^2}{2}}\,dy_i \qquad (7.66)
\end{aligned}
$$

One barrier We define for $s \in [T_i, T_{i+1}]$

$$Z_i^{\lambda_i} \triangleq \frac{1}{\sigma_{i-1,i}}\ln\frac{S_i}{S_{i-1}} = \lambda_i(s-T_{i-1}) + W_{s-T_{i-1}} \qquad (7.67)$$

$$A_i(y) \triangleq \frac{1}{\sigma_{i-1,i}} \ln \frac{y}{S_{i-1}} \tag{7.68}$$

$$\lambda_i \triangleq \frac{\mu_{i-1,i}}{\sigma_{i-1,i}} - \frac{1}{2}\sigma_{i-1,i} \tag{7.69}$$

and use the joint density

$$I\!P_{Q_Z}\{\max_{s\in[t,t+T]} Z_s^\lambda \geqslant A \text{ and } Z_s^\lambda \in [y, y+dy]\} = \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2A-y)^2}{2T}} e^{y\lambda - \frac{1}{2}\lambda^2 T} dy \tag{7.70}$$

We now define the up-and-out functional by

$$\begin{aligned}
\mathcal{UO}_i(\mathcal{F})(S_{i-1}) &\triangleq I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{\sup_{[T_{i-1},T_i]} S<H_i\}}\right] \\
&= I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{S_i<H_i\}} I\!I_{\{\sup_{[T_{i-1},T_i]} S<H_i\}}\right] \\
&= I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{S_i<H_i\}}\right] \\
&\quad - I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{S_i<H_i\}} I\!I_{\{\sup_{[T_{i-1},T_i]} S \geqslant H_i\}}\right] \\
&= I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{S_i<H_i\}}\right] \\
&\quad - I\!E^{T_{i-1}}_{Q_Z}\left[\mathcal{F}(S_{i-1} e^{\sigma_{i-1,i} Z_i^{\lambda_i}}) I\!I_{\{Z_i^{\lambda_i}<A_i(H_i)\}} I\!I_{\{\max_{[T_{i-1},T_i]} Z_i^{\lambda_i} \geqslant A_i(H_i)\}}\right] \\
&= \int_{-\infty}^{-d_2^{i-1}(H_i)} \mathcal{F}(S_i(y_i)) c \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i \\
&\quad - \left(\frac{H_i}{S_{i-1}}\right)^{\frac{2\mu_{i-1,i}}{\sigma^2_{i-1,i}}-1} \int_{-\infty}^{-d_4^{i-1}(H_i)} \mathcal{F}(S_i(y_i)) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i
\end{aligned} \tag{7.71}$$

where in the first integral

$$S_i(y_i) = S_{i-1} e^{\left(\mu_{i-1,i} - \frac{1}{2}\sigma^2_{i-1,i}\right)(T_i - T_{i-1}) + \sigma_{i-1,i}\sqrt{T_i - T_{i-1}}\, y_i} \tag{7.72}$$

and in the second integral

$$S_i(y_i) = \frac{H_i^2}{S_{i-1}} e^{\left(\mu_{i-1,i} - \frac{1}{2}\sigma^2_{i-1,i}\right)(T_i - T_{i-1}) + \sigma_{i-1,i}\sqrt{T_i - T_{i-1}}\, y_i} \tag{7.73}$$

Similarly we define the down-and-out functional by

$$\begin{aligned}
DO_i(\mathcal{F})(S_{i-1}) &= I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{\inf_{[T_{i-1},T_i]} S>L_i\}}\right] \\
&= I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{S_i>L_i\}} I\!I_{\{\inf_{[T_{i-1},T_i]} S>L_i\}}\right] \\
&= I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{S_i>L_i\}}\right] - I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{S_i>L_i\}} I\!I_{\{\inf_{[T_{i-1},T_i]} S \leqslant L_i\}}\right] \\
&= I\!E^{T_{i-1}}\left[\mathcal{F}(S_i) I\!I_{\{S_i>L_i\}}\right] \\
&\quad - I\!E^{T_{i-1}}_{Q_Z}\left[\mathcal{F}(S_{i-1} e^{-\sigma_{i-1,i} Z_i^{-\lambda_i}}) I\!I_{\{Z_i^{-\lambda_i}<-A_i(L_i)\}} I\!I_{\{\max_{[T_{i-1},T_i]} Z_i^{-\lambda_i} \geqslant -A_i(L_i)\}}\right] \\
&= \int_{-d_2^{i-1}(L_i)}^{+\infty} \mathcal{F}(S_i(y_i)) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i \\
&\quad - \left(\frac{L_i}{S_{i-1}}\right)^{\frac{2\mu_{i-1,i}}{\sigma^2_{i-1,i}}-1} \int_{-d_4^{i-1}(L_i)}^{+\infty} \mathcal{F}(S_i(y_i)) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i
\end{aligned} \tag{7.74}$$

where in the first integral

$$S_i(y_i) = S_{i-1} e^{\left(\mu_{i-1,i} - \frac{1}{2}\sigma_{i-1,i}^2\right)(T_i - T_{i-1}) + \sigma_{i-1,i}\sqrt{T_i - T_{i-1}}\, y_i} \tag{7.75}$$

and in the second integral

$$S_i(y_i) = \frac{L_i^2}{S_{i-1}} e^{\left(\mu_{i-1,i} - \frac{1}{2}\sigma_{i-1,i}^2\right)(T_i - T_{i-1}) + \sigma_{i-1,i}\sqrt{T_i - T_{i-1}}\, y_i} \tag{7.76}$$

Two barriers The distribution of S_{T_i} conditioned on the event that the path reached neither an upper limit H_i nor a lower one L_i on $[T_{i-1}, T_i]$ is known to be

$$\begin{aligned} & e^{-\frac{1}{2}\lambda_i^2 (T_i - T_{i-1}) + \frac{\lambda_i}{\sigma_{i-1,i}} \ln \frac{S_{T_i}}{S_{T_{i-1}}}} \times \\ & \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \left[\exp\left(-\frac{1}{2\sigma_{i-1,i}^2 (T_i - T_{i-1})} \left(\ln \frac{S_{T_i}}{S_{T_{i-1}}} + 2n \ln \frac{H_i}{L_i} \right)^2 \right) \right. \\ & \left. - \exp\left(-\frac{1}{2\sigma_{i,i-1}^2 (T_i - T_{i-1})} \left(\ln \frac{H_i^2}{S_{T_i} S_{T_{i-1}}} + 2n \ln \frac{H_i}{L_i} \right)^2 \right) \right] \mathbb{I}_{\{L_i < S_{T_i} < H_i\}} \end{aligned} \tag{7.77}$$

We define the double-knock-out functional by

$$\begin{aligned} \mathcal{DKO}_i(\mathcal{F})(S_{i-1}) &\triangleq \mathbb{E}^{T_{i-1}} \left[\mathcal{F}(S_i) \mathbb{I}_{\{L_i < \inf_{[T_{i-1}, T_i]} S < \sup_{[T_{i-1}, T_i]} S < H_i\}} \right] \qquad (7.78) \\ &= \int_{L_i}^{H_i} \mathcal{F}(S_i) e^{-\frac{1}{2}\lambda_i^2 (T_i - T_{i-1}) + \frac{\lambda_i}{\sigma_{i-1,i}} \ln \frac{S_{T_i}}{S_{T_{i-1}}}} \\ &\quad \times \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \left[\exp\left(-\frac{1}{2\sigma_{i-1,i}^2 (T_i - T_{i-1})} \left(\ln \frac{S_{T_i}}{S_{T_{i-1}}} + 2n \ln \frac{H_i}{L_i} \right)^2 \right) \right. \\ &\quad \left. - \exp\left(-\frac{1}{2\sigma_{i,i-1}^2 (T_i - T_{i-1})} \left(\ln \frac{H_i^2}{S_{T_i} S_{T_{i-1}}} + 2n \ln \frac{H_i}{L_i} \right)^2 \right) \right] \mathbb{I}_{\{L_i < S_{T_i} < H_i\}} \\ &= \int_{A_{L_i}}^{A_{H_i}} \mathcal{F}\left(S_{i-1} e^{\sigma_{i-1,i}\sqrt{T_i - T_{i-1}}\, x} \right) e^{-\frac{1}{2}\lambda_i^2 (T_i - T_{i-1}) + \lambda_i \sqrt{T_i - T_{i-1}}\, x} \\ &\quad \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \left[\exp\left(-\frac{1}{2} (x - 2nA_{LH_i})^2 \right) \right. \\ &\quad \left. - \exp\left(-\frac{1}{2} (-x + 2A_{H_i} + 2nA_{LH_i})^2 \right) \right] dx \\ &= \sum_{n=-\infty}^{+\infty} \left[e^{\lambda_i \sqrt{T_i - T_{i-1}} (2nA_{LH_i})} \right. \\ &\quad \int_{A_{L_i} - (\lambda_i \sqrt{T_i - T_{i-1}} + 2nA_{LH_i})}^{A_{H_i} - (\lambda_i \sqrt{T_i - T_{i-1}} + 2nA_{LH_i})} \mathcal{F}\left(S_i \left(\frac{H_i}{L_i} \right)^{2n} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i \\ &\quad - e^{\lambda_i \sqrt{T_i - T_{i-1}} (2A_{H_i} + 2nA_{LH_i})} \\ &\quad \int_{A_{L_i} - (\lambda_i \sqrt{T_i - T_{i-1}} + 2(A_{H_i} + 2nA_{LH_i}))}^{-(A_{H_i} + \lambda_i \sqrt{T_i - T_{i-1}} + 2nA_{LH_i})} \mathcal{F}\left(S_i \left(\frac{H_i}{S_{i-1}} \right)^2 \left(\frac{H_i}{L_i} \right)^{2n} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i \end{aligned}$$

where in the integrals

$$S_i(y_i) = S_{i-1} e^{\left(\mu_{i-1,i} - \frac{1}{2}\sigma_{i-1,i}^2\right)(T_i - T_{i-1}) + \sigma_{i-1,i}\sqrt{T_i - T_{i-1}}\, y_i} \tag{7.79}$$

7.6.3 *Price*

We introduce $\mathcal{ST}_i^{\alpha_i}$, where α_i is taken from the set $\alpha_i = \{L_i, H_i\}$, and consider the following four cases

- If $L_i = 0$, $\mathcal{ST}_i^{\alpha_i} = \mathcal{UO}_i$ (Moreover, if $i = N$, $\mathcal{ST}_N^{\alpha_N} = UOC_N$),
- If $H_i = +\infty$, $\mathcal{ST}_i^{\alpha_i} = DO_i$ (Moreover, if $i = N$, $\mathcal{ST}_N^{\alpha_N} = DOP_N$),
- If $L_i = 0$ and $H_i = +\infty$, $\mathcal{ST}_i^{\alpha_i} = Q_i$ (Moreover, if $i = N$, $\mathcal{ST}_N^{\alpha_N} = \mathcal{V}_N$),
- If $L_i \neq 0$ and $H_i \neq +\infty$, $\mathcal{ST}_i^{\alpha_i} = \mathcal{DKO}_i$ (Moreover, if $i = N$, $\mathcal{ST}_N^{\alpha_N} = DKOP_N$ or $DKOC_N$).

Consequently each stair product is described by $\{\alpha_i | i = 1, ..., N\}$ and its price is

$$v(t) = e^{-r_a(T_N - t)} \mathcal{ST}_1^{\alpha_1}\left(\ldots \mathcal{ST}_N^{\alpha_N}\right) \tag{7.80}$$

7.7 Compound on forward-start strategy

This product is a variant of the compound option on a strategy which will include a strategy of forward-starting options all fixing at the maturity of the root option.

A compound option's intrinsic value is the difference of the strike and the Market Value (MV) of the underlying option strategy

$$(\phi(\mathrm{MV} - K))^+ \tag{7.81}$$

This means that the holder of the options receives the underlying option strategy for the price K at the maturity of the root option on exercise of the root option. The relation is valid for both the option strategy being forward-starting or a regular fixed strategy.

In order to price a compound option on a forward-starting option strategy we model the exchange rate by

$$dS_t = \mu_t S_t\, dt + \sigma_t S_t\, dW \tag{7.82}$$

Next we consider the horizon date T_h, the value date for the horizon T_v, the maturity of the root option T_m, and the delivery date for the strike T_d. The fixing date for the forward-start strategy is the same as the maturity T_m. For each option of the strategy there can be a different maturity T and delivery Ti_d. The product has a price

$$v(T_h) = I\!E^{T_h}\left[e^{-r_{T_v,T_d}(T_d - T_v)}\left(\phi\left(\sum_{i=1}^{N} P(S_{T_m^i}) - K\right)\right)^+\right] \tag{7.83}$$

We assume that the forward-start strategy is fixed at maturity of the root option and hence depends only on the spot at maturity. In particular, the price of a forward-start option at the fixing date is homogeneous of order one in the spot (see Section 7.2) and can be written as

$$v(S_{T_m}) = S_{T_m} v(1) \tag{7.84}$$

In such a case the price of the compound is given by

$$v = v_\phi\left(S_{T_h}, \frac{K}{P_f}, T_m, T_d, r^d_{T_vT_d}, r^f_{T_vT_d}, \sigma_{T_hT_m}\right)v_f \tag{7.85}$$

$$v_f = \sum_{i=0}^{N} v_\phi(1, \tilde{\alpha}, T_m, T_d, T^i_m, T^i_d, r^d_{T_dT^i_d}, r^f_{T_dT^i_d}, \sigma_{T_mT^i_m}) \tag{7.86}$$

where v_ϕ denotes the value function of a plain vanilla, v_f the unity-price of the forward-start strategy at fixing date and $\tilde{\alpha}$ a vector of fixing parameters relative to the spot at maturity using the corresponding forward rates and volatilities. This formula is valid for all strategies with a positive forward-start unity-price bounded away from zero.

Examples of forward-start options which have a homogeneous price of order one in the fixing spot are

- plain vanilla options with the strike fixed with $K = \alpha S$,
- single barrier options with the strike fixed with $K = \alpha S$ and the barrier fixed with $B = \beta S$,
- double barrier options with the strike fixed with $K = \alpha S$ and the barriers fixed with $L = \beta S$ and $H = \gamma S$.

After maturity of the root option the product is either worthless (if expired) or the sum of the fixed forward-start options (if exercised).

7.8 Options on the minimum/maximum

This section concerns the products whose underlying is the maximum or minimum of a basket of spots or exotic products. In the first part, we introduce the forward contracts on the maximum or the minimum of a basket of currencies.

7.8.1 *Best-of and worst-of currencies forwards*

Best-of forward Let us consider n exchange rates with the same base currency and n different foreign currencies with their normalisers N_i. The payoff of this product is

$$\max\left(\frac{S_1}{N_1}, .., \frac{S_n}{N_n}\right) - 1 \tag{7.87}$$

We model the spots by

$$dS_i = \mu_i S_i\,dt + \sigma_i S_i\,dW^\dagger_i \tag{7.88}$$

where the Brownian increments $dW^\dagger_i$ are correlated (ie $< dW^\dagger_i, dW^\dagger_j > = (\Omega\Omega^T)_{ij}\,dt = \rho_{ij}\sigma_i\sigma_j\,dt$). After a Cholesky decomposition, we can rewrite this as

$$dS_i = \mu_i S_i\,dt + \sum_{j=1}^{N} \Omega_{ij}\,dW_j \tag{7.89}$$

where the Brownian increments dW_i are independent. This implies

$$S_i(T) = S_i(t)\exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)(T-t) + \sum_{j=1}^{N}\Omega_{ij}dW_j\right) \tag{7.90}$$

In the following calculations we use the replacement $W_i = (T-t)x_i$. The price is given by

$$\begin{aligned} v(t) &= \mathbb{E}^t\left[e^{-r_{pd}(T_d-T_p)}\left(\max\left(\frac{S_1(T)}{N_1},..,\frac{S_n(T)}{N_n}\right)-1\right)\right] \\ &= e^{-r_{pd}(T_d-T_p)}\left(\mathbb{E}^t\left[S_1(T)\mathbb{I}_{\{S_1(T)>S_2(T),..,S_n(T)\}}\right]+\dots\right. \\ &\quad \left. + \mathbb{E}^t\left[S_n(T)\mathbb{I}_{\{S_n(T)>S_1(T),..,S_{n-1}(T)\}}\right]-1\right) \end{aligned} \tag{7.91}$$

For the first expectation we get

$$\begin{aligned} &\mathbb{E}^t\left[S_1(T)\mathbb{I}_{\{S_1(T)>S_2(T),..,S_n(T)\}}t\right] \\ &= S_1(t)\int_{x_1,..,x_n=-\infty}^{+\infty}\exp\left(\left(\mu_1-\frac{1}{2}\sigma_1^2\right)(T-t)+\sigma_1\sqrt{T-t}x_1\right) \\ &\quad \mathbb{I}_{\{S_1(T)>S_2(T),...,S_1(T)>S_n(T)\}}\frac{1}{\sqrt{2\pi}^n}\exp\left(-\frac{1}{2}\left(x_1^2+..+x_n^2\right)\right)dx_1\dots dx_n \end{aligned} \tag{7.92}$$

We solve this integral by first working on the region where $S_1 > S_2$ and hence $d_2^1(x_1) > x_2$ with

$$d_2^1(x_1) \triangleq \frac{\ln\frac{S_1(t)}{S_2(t)} + \left(\mu_1-\mu_2-\frac{1}{2}(\sigma_1^2-\sigma_2^2)\right)(T-t)+\sqrt{T-t}(\Omega_{11}-\Omega_{21})x_1}{\Omega_{22}\sqrt{T-t}} \tag{7.93}$$

Moreover, $S_1 > S_3$ gives $d_3^1(x_1, x_2) > x_3$ with

$$d_3^1(x_1,x_2) \triangleq \frac{\ln\frac{S_1(t)}{S_3(t)} + \left(\mu_1-\mu_3-\frac{1}{2}(\sigma_1^2-\sigma_3^2)\right)(T-t)+\sqrt{T-t}((\Omega_{11}-\Omega_{31})x_1)-\Omega_{32}x_2}{\Omega_{33}\sqrt{T-t}} \tag{7.94}$$

Again, $S_1 > S_4$ gives $d_4^1(x_1, x_2, x_3) > x_4$ with

$$d_4(x_1,x_2,x_3) \triangleq \frac{\ln\frac{S_1(t)}{S_4(t)} + \left(\mu_1-\mu_4-\frac{1}{2}(\sigma_1^2-\sigma_4^2)\right)(T-t)+\sqrt{T-t}((\Omega_{11}-\Omega_{41})x_1)-\Omega_{42}x_2-\Omega_{43}x_3}{\Omega_{44}\sqrt{T-t}} \tag{7.95}$$

Continuing this way, we get the price of the forward contract

$$\begin{aligned} v(t) &= e^{-r_{pd}(T_d-T_p)}\left(\sum_{i=1}^{n}S_i(t)\int_{x_i=-\infty}^{+\infty}e^{\left(\mu_i-\frac{1}{2}\sigma_i^2\right)(T-t)+\sigma_i\sqrt{T-t}x_i}\int_{x_1=-\infty}^{d_2^i(x_i)}\dots\int_{x_{n-1}=-\infty}^{d_2^i(x_1,\dots,x_{n-2})}\right. \\ &\quad \left.\mathcal{N}(d_4^i(x_1,\dots,x_{n-1}))\frac{1}{\sqrt{2\pi}^n}\exp\left(-\frac{1}{2}\left(x_1^2+\dots+x_{n-1}^2\right)\right)dx_1\dots dx_{n-1}-1\right) \end{aligned} \tag{7.96}$$

Worst-of forward In the case of a worst-of forward, the value is

$$\begin{aligned} v(t) &= e^{-r_{pd}(T_d-T_p)}\mathbb{E}^t\left[1-\min\left(\frac{S_1}{N_1},\dots,\frac{S_n}{N_n}\right)\right] \\ &= e^{-r_{pd}(T_d-T_p)}\left(1-\sum_{i=1}^{n}S_i(t)\int_{x_i=-\infty}^{+\infty}e^{\left(\mu_i-\frac{1}{2}\sigma_i^2\right)(T-t)+\sigma_i\sqrt{T-t}x_i}\int_{x_1=-\infty}^{-d_2^i(x_i)}\dots\int_{x_{n-1}=-\infty}^{-d_2^i(x_1,\dots,x_{n-2})}\right. \\ &\quad \left.\mathcal{N}(-d_4^i(x_1,\dots,x_{n-1}))\frac{1}{\sqrt{2\pi}^n}\exp\left(-\frac{1}{2}\left(x_1^2+\dots+x_{n-1}^2\right)\right)dx_1\dots dx_{n-1}\right) \end{aligned} \tag{7.97}$$

7.8.2 *Best-of call and worst-of put*

Best-of call The value is given by

$$v(t) = e^{-r_{pd}(T_d - T_p)} I\!E^t \left[\left(\max\left(\frac{S_1}{N_1}, .., \frac{S_n}{N_n} \right) - 1 \right)^+ \right] \tag{7.98}$$

Defining for $i = 1, \ldots, n$

$$d_1^i \triangleq \frac{\ln \frac{S_i}{N_i} + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)(T-t)}{\sqrt{T-t}\sigma_i} \tag{7.99}$$

we obtain for the value

$$v(t) = e^{-r_{pd}(T_d - T_p)} \left(\sum_{i=1}^n S_i(t) \int_{x_i = d_1^i}^{+\infty} \left(e^{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)(T-t) + \sigma_i \sqrt{T-t} x_i} - 1 \right) \int_{x_1 = -\infty}^{d_2^i(x_i)} \cdots \int_{x_{n-1} = -\infty}^{d_2^i(x_1, \ldots, x_{n-2})} \right.$$
$$\left. \mathcal{N}(d_4^i(x_1, \ldots, x_{n-1})) \frac{1}{\sqrt{2\pi}^n} \exp\left(-\frac{1}{2}\left(x_1^2 + \ldots + x_{n-1}^2\right) \right) dx_1 \ldots dx_{n-1} \right) \tag{7.100}$$

We list a numerical example in Table 7.1.

Table 7.1 Values of a one-year best-of call with spot US\$/DM $= 1.6573$, $\sigma = 10.7\%$, $r^d = 3.1953\%$, $r^f = 5.0223\%$ and £/DM $= 2.754173$, $\sigma = 8.5\%$, $r^d = 3.1953\%$, $r^f = 5.4923\%$ for different numbers of steps using Gauss–Legendre integration. The normalisers are the spot values, the strike is 1. The Monte-Carlo simulation used 10 000 000 paths.

Number of Steps	Value
49	417.7713
99	416.7085
149	416.7879
199	416.8573
249	416.6880
299	416.8171
349	416.7681
399	416.7847
Monte Carlo	416.8990

Worst-of put Similarly, we obtain for the value of a worst-of put

$$v(t) = e^{-r_{pd}(T_d - T_p)} I\!E^t \left[\left(1 - \min\left(\frac{S_1}{N_1}, .., \frac{S_n}{N_n} \right) \right)^+ \right]$$
$$= e^{-r_{pd}(T_d - T_p)} \sum_{i=1}^n S_i(t) \int_{x_i = -\infty}^{d_1^i} \left(1 - e^{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)(T-t) + \sigma_i \sqrt{T-t} x_i} \right) \int_{x_1 = -\infty}^{-d_2^i(x_i)} \cdots \int_{x_{n-1} = -\infty}^{-d_2^i(x_1, \ldots, x_{n-2})}$$
$$\mathcal{N}(-d_4^i(x_1, \ldots, x_{n-1})) \frac{1}{\sqrt{2\pi}^n} \exp\left(-\frac{1}{2}\left(x_1^2 + \ldots + x_{n-1}^2\right) \right) dx_1 \ldots dx_{n-1} \tag{7.101}$$

We list a numerical example in Table 7.2.

Table 7.2 Values of a one-year worst-of put with spot US$/DM $= 1.6573$, $\sigma = 10.7\%$, $r = 3.1953\%$, $r = 5.0223\%$, spot £/DM $= 2.754173$, $\sigma = 8.5\%$, $r^d = 3.1953\%$, $r^f = 5.4923\%$ and spot Sfr/DM $= 1.211774$, $\sigma = 5\%$, $r^d = 3.1953\%$, $r^f = 1.6588$ for different numbers of steps using Gauss-Legendre integration. The normalisers are the spot values, the strike is 1. The Monte-Carlo simulation used 25 000 000 paths.

Number of Steps	Value
9	605.7585
19	696.4858
29	699.7040
39	699.8150
49	699.5770
59	699.3771
69	699.0469
99	699.0175
Monte Carlo	699.0960

7.9 Generalised options on the minimum/maximum

We discuss the pricing of generalised options on the minimum/maximum. Generalised means in this context an arbitrary set of spots with a possibly different overall payoff currency.

The instruments S_i with a base B_i and underlying currency U_i, which have possibly no common base currency, enter a Foreign Exchange product with payoff in a pay-out currency B, ie, the payoff is

$$Q\left(\max\left(\frac{S_1\tilde{S}_1}{N_1},\cdots,\frac{S_n\tilde{S}_n}{N_n}\right)-K\right)^+ \tag{7.102}$$

where the numbers N_i are normalisers and $\tilde{S}_i$ stands for the conversion into a common denominator currency and Q for conversion to the global base currency B.

Under the corresponding risk-neutral measure with respect to their domestic measure B_i all the spots follow

$$dS_{U_iB_i} = \mu_{U_iB_i}S_{U_iB_i}dt + \sigma_{U_iB_i}S_{U_iB_i}dW_i^\dagger \tag{7.103}$$

There are three distinct cases

- The base currency B_i is the global base currency B. In this case the pricing formulae need not be altered.
- The underlying currency U_i is the global base currency B. In this case the drift is adjusted.
- Neither base nor underlying currency is the same as the global base currency.

In order to derive expressions for the general case we will expand the number of spots for these contracts with the corresponding set of U_iB and B_iB spots. For these spots all of them share the same base currency and hence their dynamics are described in the base currency measure. The differential equations are

$$dS_{U_iB} = \mu_{U_iB} S_{U_iB} dt + \sigma_{U_iB} S_{U_iB} d\tilde{W}_i^{\dagger} \tag{7.104}$$

$$dS_{B_iB} = \mu_{B_iB} S_{B_iB} dt + \sigma_{B_iB} S_{B_iB} d\tilde{W}_i^{\dagger} \tag{7.105}$$

With the cross volatilities $\sigma_{U_iB_j}, \sigma_{U_iU_j}, \sigma_{B_iB_j}$ the correlation matrix of the expanded system can be calculated. From there the Cholesky decomposition leads to equations using orthogonal Brownian motions

$$dS_{U_iB} = \mu_{U_iB} S_{U_iB} dt + \sigma_{U_iB} S_{U_iB} \sum_k \Omega_{ik} d\tilde{W}_k \tag{7.106}$$

$$dS_{B_iB} = \mu_{B_iB} S_{B_iB} dt + \sigma_{B_iB} S_{B_iB} \sum_k \Omega_{N+ik} d\tilde{W}_k \tag{7.107}$$

Hence the dynamics of the spots $S_{U_iB_i} = \frac{S_{U_iB}}{S_{B_iB}}$ have correlation and drift in the domestic measure

$$\mu_{U_iB_i} = \mu_{U_iB} - \mu_{B_iB} - \frac{1}{2}(\sigma^2_{U_iB} - \sigma^2_{U_iB_i} - \sigma^2_{B_iB}) \tag{7.108}$$

$$\begin{aligned} \sigma_{U_iB_i}\sigma_{U_jB_j}\rho_{U_iB_i/U_jB_j} = & \sum_k \sigma_{U_iB}\sigma_{U_jB}\Omega_{ik}\Omega_{jk} + \sum_k \sigma_{B_iB}\sigma_{B_jB}\Omega_{ik}\Omega_{jk} \\ & - \sum_k \sigma_{B_iB}\sigma_{U_jB}\Omega_{ik}\Omega_{jk} - \sum_k \sigma_{U_iB}\sigma_{B_jB}\Omega_{ik}\Omega_{jk} \end{aligned} \tag{7.109}$$

Using this information one can derive a correlation matrix that gives rise again to a Cholesky decomposition and the pricing formula with these changed coefficients and drifts can be used.

7.9.1 *Example*

We consider the case that all the involved currencies have a common underlying currency which is the global base currency. The equation for the drift terms is

$$\mu_{U_iB_i} = -\mu_{B_iU_i} + (\sigma^2_{U_iB_i}) \tag{7.110}$$

The correlations are determined as

$$\rho_{U_iB_i/U_jB_j} = -\rho_{B_iU_i/U_jB_j} \tag{7.111}$$

$$\rho_{U_iB_i/U_jB_j} = \rho_{B_iU_i/B_jU_j} \tag{7.112}$$

and are the same as in the non-quanto case.

For pricing the existing undiscounted formula is used with adapted risk-neutral drift for each currency

$$\begin{aligned} & Qe^{-r_BT}\text{Price}_{GeneralisedBW}(S_i, r_{B_i}, r_{U_i}, \mu_i = r_{B_i} - r_{U_i}, \sigma_{U_iB_i}, \rho_{U_iB_i/U_jB_j}) \\ & = Qe^{-r_BT}\text{Price}_{BW}(S_i, r_{B_i}, r_{U_i}, \mu_i = r_{B_i} - r_{U_i} + \sigma^2_{U_iB_i}, \sigma_{U_iB_i}, \rho_{U_iB_i/U_jB_j}) \end{aligned} \tag{7.113}$$