Pure Jump Lévy Processes for Asset Price Modelling

Hélyette Geman

Professor of Finance

University Paris IX Dauphine and ESSEC

Abstract

The goal of the paper is to show that some types of Lévy processes such as the hyperbolic motion and the CGMY are particularly suitable for asset price modelling and option pricing. We wish to review some fundamental mathematic properties of Lévy distributions, such as the one of infinite divisibility, and how they translate observed features of asset price returns. We explain how these processes are related to Brownian motion, the central process in finance, through stochastic time changes which can in turn be interpreted as a measure of the economic activity. Lastly, we focus on two particular classes of pure jump Lévy processes, the generalized hyperbolic model and the CGMY models, and report on the goodness of fit obtained both on stock prices and option prices.

I. Introduction

Normality of asset returns has played a central role in financial theory for the last few decades, starting with the Markowitz frontier and the Capital Asset Pricing Model and more recently, as a convenient setting for Value at Risk computations. The normality of distributions has been augmented with the assumption of continuity of trajectories when Samuelson introduced in 1965 the geometric Brownian motion, then used in the seminal papers by Black-Scholes (1973) and Merton (1973). As documented in a considerable number of papers written by academics and practitioners, both normality and continuity assumptions (which are not identical but related in a way that we shall make precise later on in the paper) are contradicted by the data in several pieces of evidence.

Return distributions are more leptokurtic than the normal one as noted by Fama as early as 1963 ; this feature is more accentuated when the holding period becomes shorter and becomes particularly clear on high frequency data. Option prices exhibit the famous volatility smile as well as prices higher than predicted by the Black-Scholes formula for short-dated options. At the same time, jumps may clearly be identified in equity data ; in fact, the inability to trade continuously implies de facto jumps in prices. These jumps contribute to (or may be the source of) stochastic volatility while they lead to *finite variation* trajectories in the absence of a diffusion term, as observed in practice.

The goal of the paper is to go beyond the Brownian-motion compound Poisson process introduced by Merton in 1976 (and usually referred to as jump-diffusion) and analyze as a better alternative two particular classes of pure jump Lévy processes on which several authors have focused attention lately. These processes are the generalized hyperbolic motion and the CGMY process which have been introduced in the finance literature by Eberlein (1995), Barndorff-Nielsen (1998), Keller and Prause (1998) on one hand, Geman, Madan and Yor (2001), Can, Geman, Madan and Yor (2002) on the other hand.

We also show how that they are related to Brownian motion, a process which does exhibit both properties of normality and continuity mentioned. This relationship holds through a stochastic time change which is in fact a measure of the economic activity as conjectured in

Clark's (1973) seminal paper, and demonstrated by Geman and Ané (1996) and Ané and Geman (2000).

The remainder of the paper is organized as follows. Section 2 recalls some fundamental properties of Lévy processes and the financial interpretation attached to them. Section 3 establishes that these processes are necessarily time-changed Brownian motion in a no-arbitrage framework and discusses the properties of the Lévy measure in terms of those of the economic clock. Section 4 describes some pure jump Lévy processes which have been recently demonstrated as providing a conclusive goodness of fit of equity markets. The first class is the class of the generalized hyperbolic motion, which includes the normal inverse Gaussian process and the hyperbolic motion ; the second one contains the VG and the CGMY models. Section 5 contains concluding comments.

II. Some fundamentals of Lévy processes

As usual in the finance literature, we represent the uncertainty of the economy by a filtered probability space, (Ω, F, F_t, P) where F_t is the filtration of information available at time t and P is the real probability measure. There will be no change of probability measure in this section and all definitions and properties of the processes under analysis will hold with respect to P. The Lévy processes, which include Poisson process and Brownian motion as special cases, were the first class of stochastic processes to be studied in the spirit of trajectories, crucial for finance. For notational simplicity, we are going to consider R-valued processes starting at zero but most of the results described below hold for R^n-valued processes.

A process X is called a Lévy process if it has (almost surely) right-continuous paths and if its increments are independent and time-homogeneous. The first condition is somewhat technical and ensures that the paths of X cannot explode ; the second one characterizes Lévy processes from a modelling standpoint and expresses that for any time t greater than s, the distribution of the increment (X_t-X_s) depends only on the length of the interval (t-s) and that (X_t-X_s) is independent of (X_u, u≤s).

An important consequence is the *infinite divisibility* of distributions that is derived from the above property. For simplicity, let us consider X_1 (value at the process at time 1) : for any n greater than 1, we can write

$$X_1 = \left(X_1 - X_{\frac{n-1}{n}} \right) + \left(X_{\frac{n-1}{n}} - X_{\frac{n-2}{n}} \right) + \cdots + \left(X_{\frac{2}{n}} - X_{\frac{1}{n}} \right) + X_{\frac{1}{n}}$$

Hence X_1 can be expressed as the sum of n independent identically distributed random variables, with common law the law of $X_{\frac{1}{n}}$, i.e., the law of X_1 is *infinitely divisible* (the same holds for all X_t). This is to be compared with the well-cited motivation for modelling stock returns by the Gaussian distribution, namely that this distribution is a limiting distribution of sums of n independent random variables (up to a scaling factor) which may be viewed as representing the effects of various shocks in the economy.

A second major result, which has direct implications for option pricing, is the expression of the characteristic function of X_t, known as the Lévy-Khintchine formula :

$$E[iuX_1] = e^{-tf(u)}$$

where f has the following Lévy-Khintchine representation

$$f(u) = \frac{s^2}{2}u^2 - iau + \int_{|x|\geq 1}(1 - e^{iux})k(dx) + \int_{|x|<1}(1 - e^{iux} + iux)k(dx) \quad (2)$$

a is called the drift of the Lévy process, θ is the diffusion coefficient and ν (dx) is a measure on $R - \{0\}$ such that

$$\int \inf(1,x^2)\,\nu(dx) < \infty$$

and called the Lévy measure of the process X.
The Lévy measure of the process X may also be defined by

$$k(A) = E\left\{\sum_{0\leq s\leq 1} 1_A(\Delta X_s)\right\}$$

where A is an arbitrary interval bounded away from zero. The Lévy density has the same mathematical requirements as a probability density, except that is does not need to be integrable and must have zero mass at the origin. Integration of the Lévy density over a particular spatial domain provides the arrival rates of jump sizes in this domain.

Note that in the whole paper, we shall say that the Lévy process X (which represents the log of the stock price and will be essentiallly a pure jump process) has *infinite activity* if the integral of the measure k on the real line is infinite. This expression, also used in turbulence theory, characterizes a "high" rate of arrival of jumps of different sizes and will adequately allow us to dispense with the need to consider an additional and unrelated diffusion component. At the other end of the spectrum, the continuity requirement of diffusion models forces the arrival rates of all jumps sizes to zero and thus reduces the local variation of uncertainty in the price dimension to be explained with a single instantaneous volatility parameter.

It is interesting to observe that there is a one-to-one correspondence between Lévy processes and characteristic functions represented as in (1). More precisely, starting with formula (2), we can build three processes $X^{(1)}, X^{(2)}, X^{(3)}$ as follows :
Denoting $X_t^{(1)} = -at + sW_t$ where (W$_t$) is a standard P-Brownian motion, the characteristic function of $X^{(1)}$ is straightforward and equal to

$$f_1(u) = iau + \frac{1}{2}s^2u^2$$

Now, consider the process $X_t^{(2)} = \sum_{j=1}^{N_t} Y_j$

where N is a Poisson process whose intensity λ is defined by

$$l = \int_{|x|>1} k(dx)$$

and Y_1, Y_2,…,Y_n….. are independent random variables, independent of the process N and with common distribution

$$1_{|x|>1}k(dx)$$

We recognize in $X^{(2)}$ a compound Poisson process whose characteristic function is $e^{-f_2(u)}$, where

$$f_2(u) = -\int \left(e^{iux} - 1\right)1_{|x|>1}k(dx)$$

It may be proved in the same manner that the last term in formula (2) is the characteristic function of a Lévy process $X^{(3)}$ obtained as a limit of compound Poisson processes (different from $X^{(2)}$). Hence

$$X = X^{(1)} + X^{(2)} + X^{(3)} \qquad (3)$$

is the sum of three Lévy processes independent of one another, hence is a Lévy process as well. The same property holds the other way, meaning that any Lévy process can be written as above

$$X = X^{(1)} + X^{(2)} + X^{(3)}$$

Let us observe that the decomposition exhibited in (3) illuminates the fact that, in the same way the instantaneous volatility describes the local uncertainty of a diffusion, the Lévy density describes the local uncertainty of a pure jump process..

Each of $X^{(1)}$, $X^{(2)}$, $X^{(3)}$ is a *semimartingale*, so is X. Hence, any Lévy process is a semimartingale. We know that stock prices have to be semimartingales under the real probability measure P and Lévy processes appear as a wide natural class of candidates for stock prices. The above discussion shows that in order to get continuity for the trajectories of the process X, the components $X^{(2)}$ and $X^{(3)}$ need to be zero and the process X to be reduced to

$$X_t^{(1)} = -at + sW_t.$$

Hence, the important property :
The only Lévy process with continuous paths is the Brownian motion (with drift).
From a finance standpoint, this means that if we start with a Lévy process to describe the return (or natural log of stock), we obtain normality together with continuity ; expressed differently, it means that it is necessary to introduce discontinuous Lévy processes whenever deviations from normality are clearly exhibited by the data.

We have so far gathered the important properties :

a) Lévy processes as the representation of stock returns (or price changes) are consistent with the no-arbitrage assumption.
b) They have the merit of providing infinitely divisible distributions, hence to express price changes as resulting from a great number of shocks in the economy.

c) If deviations from normality are clearly exhibited by the data, it becomes necessary to introduce discontinuous Lévy processes, possibly with a diffusion component.

d) In order to obtain a finite quadratic variation process, the diffusion component must be zero and the process be a pure jump Lévy process.

(The quadratic variation of a process (X_t) over a time interval [a, b] is classically defined as the limit of $\sum_{i=1}^{n}\left[X(t_{i+1})-X(ti)\right]^2$ when the width of the partition $t_0=a<t_1\ldots<t_{n+1}=b$ of the interval [a, b] goes to zero).

Obviously, a finite variation process is a better representation of real stock prices.

III. Economic Activity, Time Changes and Asset Price Modelling

In order to provide an answer to the deviations of stock returns from normality already observed at the time, Mendelbrot (1963) proposed the class of stable Pareto distributions, denoted by $SP(a)$, where a, the characteristic exponent, belongs to the interval]0,2].

In the symmetric case, stable Pareto distributions are defined by the log-characteristic function

$$\ln f(t) = i\boldsymbol{d}t - c|t|^{\boldsymbol{a}}$$

where δ denotes the location parameter, c the scale parameter. When the characteristic exponent α is equal to 2, the stable distribution coincides with the normal distribution ; for $\alpha=1$, it gives the Cauchy distribution. For $\alpha<2$, stable distributions are more peaked around the center than the normal ones and have arbitrarily fat tails since the variance is infinite. For $\alpha\leq1$, even the first moment does not exist. So far, these features have led most authors to reject the stable hypothesis for stock price returns. Since stable processes are Lévy processes, this shows that some additional properties must be taken into account to identify suitable ones.

In 1973, Clark offers a strictly different solution to account for the non normality of returns since the title of his paper mentions " a process model with finite variance". Analyzing a database of cotton Future price returns, he proposes linking them to the existence of variations in volume during different trading period and introduces subordinated processes by writing the return process X(t) as a subordinated process

$$X(t) = Z(T(t)) \qquad (4)$$

According to the definition given by Bochner in 1955, a subordinator T(t) is an almost surely increasing process with independent and stationary increments ; no specific condition is required on the process Z which is sometimes called in finance the directing process.

In his seminal paper, Clark conjectured that Z could be chosen as a Brownian motion, that T(t) be log-normally distributed *and* have the economic interpretation of (cumulated) volume traded in the market. Indeed, using historical data on returns (represented by Y) and volume (represented by T), he was able to show that the distribution of Z did satisfy classical normality tests.

Geman-Ané (1996) and Ané-Geman (2000) validate mathematically and extend Clark's conjecture by exhibiting two types of results :

a) A remarkable theorem by Monroe (1978) establishes that any semimartingale can be written as a time-changed Brownian motion. Since the no-arbitrage assumption implies the existence of a probability measure under which discounted stock prices are martingales, these stock prices have to be semimartingales under the real probability measure. Hence the same property holds for the log of the price and

$$\ln S(t) = W(T(t)) \qquad (5)$$

where we now *know* that the directing process W is a Brownian motion.

Comparing formula (5) with formula (4), we observe that the left-hand sides are similar since the semimartingale property may be written indifferently for S or ln S. In the right-hand sides however, T(t) is in formula (5) a general time change, only constrained by the property of being an almost surely increasing process.

b) Given the considerable amount of empirical literature analyzing the relationship between volume, price changes, volatility, information arrival, number of trades (see for instance Karpoff, 1987, Jones,Kaul and Lipson,1994), Ané and Geman (2000) test the number of trades against volume as the possible representation of the time change .They conclude through the empirical analysis of high frequency data of equity indexes and individual stocks that the (cumulative) number of trades is a better stochastic clock for generating virtually perfect normality of returns.

Formula (5) illuminates how asset prices respond to the arrival of information. Some days, very little news, good or bad, is released; trading is typically slow and prices barely fluctuate. In contrast, when new information arrives and traders adjust their expectations accordingly, trading becomes brisk and the price evolution accelerates.

Before turning to the discussion of the different processes generated for the stock price by interesting choices of the time change T, it is worth noticing that the constant volatility in the arithmetic Brownian W is going to give rise to *stochastic volatility* for the stock price process S when it is "compounded" with a stochastic time T. Geman and Yor (1993) who introduced general time changes for solving the valuation of an Asian option in the classical Black-Scholes setting also observe how these time changes appear as a natural tool to handle stochastic volatility . Taking the Hulland White (1987) model where the squared volatility is supposed to be driven by a geometric Brownian motion, they use a stochastic time change to solve for the average "perfect replication time" of put options in portfolio insurance strategies.

Coming back to formula (5), we observe that two types of randomness come into play in the return process : the Brownian motion and the time change. Due to the *self-similarity* property of Brownian motion

$$W\left(c^2 t\right) = cW(t) \text{ for c } > 0$$

the qualitative features of trajectories do not change if we change the time-scale. In contrast, real stock price paths change dramatically if we look at them on different time scales (see for instance Table 1).

Table 1

Comparison of skewness and kurtosis at different time scales

Mean	Variance	Skewness	Kurtosis
Cisco Systems One-Minute Returns			
4.2292 E-06	4.9975 E-07	2.2655 E-01	4.0852 E+01
Cisco Systems 10-Minute Returns			
4.2287 E-05	3.4759 E-06	3.1943 E-01	17.0438
Intel Five-Minute Returns			
-5.2863 E-06	1.0241 E-06	-9.5237 E-02	21.2283
Intel 15-Minute Returns			
-1.5839 E-05	70.969 E-07	-1.7460 E-01	12.2681

Moments for Cisco Systems and Intel stock returns are presented in this table. The time period goes from January 2, 1997 to December 31,1997 ; the data have been collected from Reuters. The number of observations for the period of analysis is n= 101, 707 and 10, 171 for the Cisco systems series on one-minute and 10-minute intervals and n=20, 352 and 6, 784 for the Intel series on 5-minute and 15 minute intervals.

The stochastic clock in formula (5) will precisely account for the representation of returns by different classes of distributions for different holding periods as already recommended by Fama in 1965. From a modelling standpoint, it is interesting to observe that in some way, Mandelbrot and Clark agreed on their correction of normality of returns since a stable process is also a time-changed Brownian. For a stable process with index a, where a is between 0 and 2, the Lévy measure is

$$n(dx) = \frac{1}{x^{a+1}} dx \quad \text{for x} > 0$$

The characteristic function of a Brownian motion evaluated at an independent increasing stable process T of index α is given by

$$E[\exp(iu\,W(T(t)))] = E\left[\exp\left(\frac{-u^2 T(t)}{2}\right)\right]$$

$$= \exp\left[-t\left(\frac{c}{2}\right)|u|^{2a}\right]$$

Hence the process X(t) = W(T(t)) is a stable process of index 2α.

Another property that we can derive from Monroe's theorem relates to a finding which has been proposed in finance by several authors as an answer to the observed lepkokurtosicity of the return distribution as a *mixture of normals* .

Choosing for simplicity a discrete f_T for T distribution, we can write :

$$P(X(t) \in dx) = \sum_u P(X(t) \in dx / T(t) = u) f_T(u)$$

Now, assuming the independence of the processes W and T, we obtain :

$$P(X(t) \in dx) = \sum_u P(W(u) \in dx) f_T(u)$$

Hence, the distribution of Y appears as a mixture of normal distributions, where the mixing factor is the density of the time change, which itself accounts for the market activity measured by the volume or number of trades. It is interesting to note that in one of the most cited papers on the subject, Richardson and Smith (1994) have as a goal stated in the title *the measure of daily flow information* through the test of the mixture of distributions hypothesis.

Lastly, we can observe that the continuity of the process (S(t)) is equivalent to the continuity of the process T(t). If T(t) is continuous, then it may be written in the following form

$$T(t) = \int_0^t a(u)du + \int_0^t b(u)dZ(u)$$

Because T(t) is increasing, b(u) ≡ 0 and the time change is locally deterministic. This is an undesirable property since we believe T(t) is related to locally random market activity like the arrival of orders or information and T cannot be continuous. This is another argument in favor of the jump processes introduced in the following section.

IV. Pure Jump Lévy Processes in Finance

The normal inverse Gaussian process, the hyperbolic motion, the variance gamma and the CGMY model are Lévy processes which share the property of being pure jump and infinite activity. Their empirical performance in fitting stock prices and equity option prices has been recently evidenced by a number of authors. In all four cases, the tractability of the characteristic function allows to recover option prices through the fast Fourier transform (see Carr and Madan, 1998). Hence, it suffices to assume the same structure of the process with possibly different parameters under the real probability measure P and the risk-neutral probability measure Q to test the goodness of fit both on stock and option data.

Generalized hyperbolic distributions were proposed by Barndorff Nielsen (1977) for modelling the grain size of wind blown sand. They contain as subclasses the hyperbolic distribution and the normal inverse Gaussian model, respectively introduced in finance by Eberlein and Keller (1995) and Barndorff-Nielsen (1998)..

1. The Normal Inverse Gaussian Model

Its characteristic function is defined by three parameters :

$$f_{NIG}(u; a, b, td) = \exp\left[-td\left(\sqrt{a^2 - (b + iu)^2} - \sqrt{a^2 - b^2}\right)\right]$$

The linearity of the log of the characteristic function with respect to time shows that it is an infinitely divisible-process with stationary independent increments.

We can also recognize the general property of being a time-changed Brownian motion is also satisfied by this Lévy process

$$X(t) = W(T(t))$$

with the interesting feature that the time change T may be chosen as an inverse Gaussian process independent of the directing Brownian motion W.

For T positive, define T_t^v the first time that a Brownian motion with drift v reaches the positive level t. The density of T_t^v is *inverse Gaus*sian and its Laplace transform has a simple expression of the form

$$E\left[\exp\left(-\boldsymbol{l}\,T_t^{\boldsymbol{n}}\right)\right] = \exp\left[-t\left(\sqrt{2\boldsymbol{l}+\boldsymbol{n}^2}-\boldsymbol{n}\right)\right]$$

If now we consider the Brownian motion with drift θ and volatility σ computed at this Gaussian time, we define a new process X_{NIG} by

$$X_{NIG}(t;\boldsymbol{s},\boldsymbol{n},\boldsymbol{q}) = \boldsymbol{q}\,T_t^{\boldsymbol{n}} + \boldsymbol{s}\,W\left(T_t^{\boldsymbol{n}}\right)$$

This *normal inverse Gaussian* process has a characteristic function which is fairly simple

$$E\left[e^{iuX_{NIG}(t)}\right] = E\left[\exp\left(iu\boldsymbol{q}\,T_t^{\boldsymbol{n}}\,\frac{-\boldsymbol{s}^2u^2}{2}\,T_t^{\boldsymbol{n}}\right)\right]$$

$$= E\left[\exp\left(iu\boldsymbol{q}\,\frac{-\boldsymbol{s}^2u^2}{2}\right)T_t^{\boldsymbol{n}}\right]$$

$$= \exp\left[-t\sqrt{\boldsymbol{n}^2-2iu\boldsymbol{q}+\boldsymbol{s}^2u^2}-\boldsymbol{n}\right]$$

$$= \exp\left(-t\boldsymbol{s}\sqrt{\frac{\boldsymbol{n}^2}{\boldsymbol{s}^2}+\frac{\boldsymbol{q}^2}{\boldsymbol{s}^4}-\left(\frac{\boldsymbol{q}}{\boldsymbol{s}^2}+iu\right)^2}-\frac{\boldsymbol{n}}{\boldsymbol{s}}\right)$$

Denoting for simplication

$$\begin{cases} \boldsymbol{b} = \dfrac{\boldsymbol{q}}{\boldsymbol{s}^2} \\[2ex] \boldsymbol{a}^2 = \dfrac{\boldsymbol{n}^2}{\boldsymbol{s}^2} + \dfrac{\boldsymbol{q}^2}{\boldsymbol{s}^4} \\[2ex] \boldsymbol{d} = \boldsymbol{s} \end{cases}$$

we can write the NIG process as

$$X_{NIG}(T; \mathbf{a}, \mathbf{b}, \mathbf{d}) = \mathbf{b}\mathbf{d}^2 T_t^{\mathbf{d}\sqrt{\mathbf{a}^2 - \mathbf{b}^2}} + \mathbf{d}\, W\left(T_t^{\mathbf{d}\sqrt{\mathbf{a}^2 - \mathbf{b}^2}} \right)$$

Conditioning on a jump of size u in the time change, the move for the process X is Gaussian with mean $\mathbf{b}\mathbf{d}^2 u$ and variance $\mathbf{d}^2 u$. As seen earlier, the arrival rate of the jumps is given by the Lévy density, which in the case of the inverse Gaussian time T, has the following expression

$$k(u) = \frac{\exp\left[\dfrac{-\mathbf{d}^2\left(\mathbf{a}^2 - \mathbf{b}^2\right)}{2} u \right]}{u^{3/2}}$$

It follows that the Lévy density for the NIG process is

$$\int_0^\infty \frac{1}{\mathbf{d}\sqrt{2pu}} \exp\left(\frac{-\left(x - \mathbf{b}\mathbf{d}^2 u\right)^2}{2\mathbf{d}^2 u} \right) \frac{1}{u^{3/2}} \exp\left(\frac{-\mathbf{d}^2\left(\mathbf{a}^2 - \mathbf{b}^2\right)}{2} u \right) du$$

$$= \frac{1}{\mathbf{d}} \int_0^\infty \frac{1}{\sqrt{2p}} t^{-2} \exp\left(\frac{-\mathbf{d}^2\mathbf{a}^2}{2} t - \frac{x^2}{2\mathbf{d}^2 t} \right) dt$$

$$= \frac{e^{\mathbf{b}x}}{\mathbf{d}} \int_0^\infty \frac{1}{\sqrt{2p}} \exp\left(-s - \frac{x^2\mathbf{a}^2}{4s} \right) s^{-2} \frac{\mathbf{d}^2\mathbf{a}^2}{2} ds$$

We introduce the modified Bessel function K_a (see Abramovitz and Stegun, 1972).

$$K_a(x) = \frac{1}{2}\left(\frac{x}{2} \right)^a \int_0^\infty \exp\left(-\left(t + \frac{x^2}{4t} \right) \right) t^{-a-1} dt$$

which allows to extract

$$\int_0^{\infty} \exp\left(-\left(t + \frac{x^2}{4t}\right)\right) t^{-a-1} dt = 2K_a(x)\left(\frac{2}{x}\right)^a$$

Hence, the NIG Lévy density above written involves the Bessel function K_1 with index 1 and reduces to

$$k_{NIG}(X) = \sqrt{\frac{2}{p}} \, da^2 \, e^{bx} \frac{K_1(|x|)}{|x|} \qquad (6)$$

We can observe that the integral of k_{NIG} over the real line is infinite ; hence, the NIG process has *infinite activity.* Venter and de Jongh (2002) confirm the quality of the fitting obtained with the NIG process on a database of returns of the FT-Actuaries All-Share Index for the UK from January 1965 to December 1995. Morevover, they show that when VAR is the risk measure, the NIG based approach is found to be more robust than the EVT (Extreme Value Theory) method for samples of sizes up to 250 and also in larger samples if the NIG distribution fits. According to these authors, the EVT method should only be used in large samples if the NIG distribution does not fit adequately.

2. The Hyperbolic and Generalized Hyperbolic Distributions

The hyperbolic distribution as a density given by

$$f_{(a,b,d,m)}(x) = \frac{\sqrt{a^2 - b^2}}{2 \, a \, d \, K_1\left(d \sqrt{a^2 - b^2}\right)} e^{-a\sqrt{d^2 + (x-m)^2} + b(x-m)}$$

where K_1 denotes the modified Bessel function with index 1.

Barndorff-Nielsen and Halgreen (1977) show that the hyperbolic distribution can be represented as a mixture of normals, where the mixing distribution is a generalized inverse Gaussian with density

$$h(x) = \frac{\sqrt{w/g}}{2K_1\left(\sqrt{wg}\right)} \exp\left\{-\frac{1}{2}\left(gx^{-1} + wx\right)\right\} \quad \text{for } x > 0$$

Let $g = d^2$ and $w = a^2 - b^2$ and introduce a normal distribution with mean $m + bs^2$ and variance s^2 such that s^2 is a random variable with distribution $h(x)$. Then the mixture is a hyperbolic distribution $f_{a,b,d,m}$ defined above.

We observe that :

- the log-density is a hyperbola, hence the name (as opposed to a parabola for the log-density of the normal distribution)

- it provides heavier tails

- it is charaterized by 4 parameters : $m \in R, d > 0$ and $0 \leq |b| < \alpha$

- α and β determine the shape (β being responsible for skewness)

- δ and μ are respectively scale and location parameters

The moment generating function of the hyperbolic motion is given by

$$M(u) = \frac{e^{mu} \sqrt{a^2 - b^2}}{K_1\left(d\sqrt{a^2 - b^2}\right)} \frac{K_1\left(d\sqrt{a^2 - (b+u)^2}\right)}{\sqrt{a^2 - (b+u)^2}}$$

where $|b + u| < a$

Proof : It is just a straightforward computation of the integral

$$M(u) = \int e^{ux} f_{(a,b,d,m)}(x)\,dx$$

Hence all moments of the hyperbolic distribution exist and can be obtained through the derivatives of M evaluated at t=0.

The Lévy-Khintchine representation of the characteristic function in the symmetric centered case ($\beta=\mu=0$) has the following expression

$$f(u) = \exp \int \left(e^{iux} - 1 - iux\right) g(x)\, dx$$

with $g(x) = \dfrac{1}{|x|}\left[\int_0^\infty \dfrac{e^{-\sqrt{2y + a^2|x|}}\, dy}{p^2 y\left(J_1^2\left(d\sqrt{2y}\right) + Y_1^2\left(d\sqrt{2y}\right)\right)} + e^{-a|x|}\right]$

where J_1 and Y_1 are Bessel Functions.

Using the asymptotics of the various Bessel functions, one can deduce that $g(x) \sim \dfrac{1}{x^2}$ for $x \to 0$; hence every path of the process has infinitely many jumps in any finite interval. However, the magnitude of the jumps is such that the moment generating function exists, which is in contrast to the α-stable Lévy process.

The *generalized hyperbolic distribution* involves an extra-parameter λ and has the following density

$$f_{GH}(x; l, a, b, d, m) = a(l, a, b, d)\left(d^2 + (x - m)^2\right)^{\frac{\left(l - \frac{1}{2}\right)}{2}}$$

$$K_{l - \frac{1}{2}}\left(a\sqrt{d^2 + (x - m)^2}\,\exp(b(x - m))\right)$$

where

$$a(?, a, \beta, d) = \frac{\left(a^2 - \beta^2\right)^{?/2}}{\sqrt{2p}\, a^{? - \frac{1}{2}}\, d^?\, K_?\left(d\sqrt{a^2 - \beta^2}\right)}$$

is the normalizing constant. K_ν denotes as before the Bessel function with index ν. The extra-parmeter λ characterizes certain sub-classes and has essentially an impact on the heaviness of the tails. For $\lambda = 1$, we recover the sub-class of hyperbolic distributions, for $l = -\dfrac{1}{2}$ the normal inverse Gaussian. The fact that generalized hyperbolic distributions are infinitely divisible allows to generate a Lévy process (X_t) such that the distribution of X_1 has the density f_{GH} defined ealier.. This process is defined by Eberlein (1999) as the *generalized hyperbolic Lévy motion*.
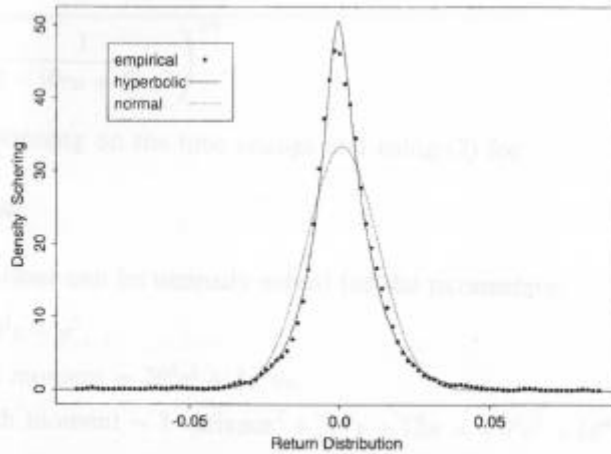
Fig. 1. Repeal Schering stock distribution fitted by the hyperbolic model.

Eberlein, Keller and Prause (1998) show that hyperbolic distributions allow an almost perfect fit to financial data, both in spot and derivative markets ; their empirical analysis investigated in particular major German stocks such as Deutsche Bank, Thyssen, the DAX index as well as options on these various underlyings. Figure 2 shows the distribution of returns for the Schering stock, a pharmaceutical company member of the DAX.

3. The Variance Gamma Model

Madan-Carr-Chang (1998) introduce the process defined by an arithmetic Brownian motion with drift θ and volatility σ, time-changed by an increasing Gamma process with unit mean and variance ν, resulting in the three parameter process

$$X(t;\boldsymbol{s},v,\boldsymbol{q}) = \boldsymbol{q}\, G(t;v) + \boldsymbol{s}\, W\big(G(t;v)\big)$$

where $G(t ;\nu)$ is the Gamma process and $W(t)$ is a standard Brownian motion. The probability density of the gamma process with mean rate t and variance νt is well-known :

$$f(u) = \frac{u^{\frac{t}{\boldsymbol{n}}-1}\, e^{\frac{-u}{\boldsymbol{n}}}}{\boldsymbol{n}^{\frac{t}{\boldsymbol{n}}}\, \Gamma\!\left(\frac{t}{\boldsymbol{n}}\right)}$$

where $\Gamma(x)$ denotes the classical gamma function. Its Laplace transform reduces to

$$E\big[\exp\big(-\boldsymbol{l}\, G_t^{\boldsymbol{n}}\big)\big] = (1 + \boldsymbol{l}\,\boldsymbol{n})^{-t/\boldsymbol{n}} \qquad (7)$$

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It results that the Variance Gamma process has a particularly simple characteristic function

$$f_{vG}(u) = \left(\frac{1}{1 - i\boldsymbol{q}\,vu + \dfrac{\boldsymbol{s}^2 v}{2} u^2} \right)^{t/v}$$

obtained by conditioning on the time change and using (7) for

$$l = \frac{\boldsymbol{s}^2 u^2}{2} - i\boldsymbol{q}\,u$$

The moment equations can be uniquely solved for the parameters

Variance $\quad\quad\quad\quad = \boldsymbol{q}^2 v + \boldsymbol{s}^2$

Central 3$^{\text{rd}}$ moment $\quad = 2\boldsymbol{q}^3 v^2 + 3\boldsymbol{s}^2 \boldsymbol{q}\, v$

Central 4$^{\text{th}}$ moment $\quad = 3\ \text{variance}^2 + 3\ \sigma^4 v + 12\sigma$
$$\quad\quad\quad\quad\quad\quad\quad = +\ \theta^2 v^2 + 6\theta^4 v^3$$

Geman, Madan and Yor (2001) show that the variance gamma process may be expressed as the differences of two independent gamma processes.

X(t)=G$_p$(t)-G$_n$(t)

where G$_p$(t) may be represented as the price change resulting from « positive » shocks and G$_n$(t) the price change resulting from « negative » shocks. It suffices for instance to write

$$\frac{1}{1 - i\boldsymbol{q}vu + \boldsymbol{s}^2 v \dfrac{u^2}{2}} = \left(\frac{1}{1 - iu\boldsymbol{h}_P} \right)\left(\frac{1}{1 + iu\boldsymbol{h}_n} \right)$$

where

$$\boldsymbol{h}_p - \boldsymbol{h}_n = \boldsymbol{qn}$$

$$\boldsymbol{h}_p \boldsymbol{h}_n = \frac{\boldsymbol{s}^2 v}{2}$$

which in turn results into

$$h_p = \left(\sqrt{\frac{q^2 n^2}{4} + \frac{s^2 n}{2}} - \frac{qn}{2} \right)^{-1}$$

$$h_n = \left(\sqrt{\frac{q^2 n^2}{4} + \frac{s^2 n}{2}} + \frac{qn}{2} \right)^{-1}$$

The fact that the VG process X may be written as the difference of the processes $G_p(k)$ and $G_n(k)$ implies that it is a finite variation process. Morevover, the knowledge of the positive and negative moves allows the determination of the Lévy density under the following form

$$k_{VG}(x) = \begin{cases} C \dfrac{\exp(-Mx)}{x} & x > 0 \\[2em] C \dfrac{\exp(-G|x|)}{|x|} & x < 0 \end{cases}$$

where

$$C = \frac{1}{v}; G = \frac{1}{h_n}; M = \frac{1}{h_p}$$

Appearing as a scale factor in the Lévy measure both for positive and negative moves, the constant C characterizes the general activity intensity of the process while the parameters G and M define the speed at which arrival rates decline with the size of the move. In a parallel interpretation, we may come back to the moment equations written earlier and observe that the parameter θ provides skewness to the distribution as it enhances the left tail when negative by both decreasing G and simultaneously increasing M.

The parameter θ affects the skewness of the process ; when θ=0, G= and te distribution is symmetric. Negative values of θ lead to lower values for G and negatively skewed processes (the opposite holding for θ>0).The parameter ν provides kurtosis which, in the absence of skew (θ=0), is equal to 3 (1+ν).
In the spirit of section III, it is interesting to note that ν is non zero whenever the time change G(t,ν) is stochastic, which equivalently corresponds to the situation of excess kurtosis, created by stochastic volatility. Lastly, we can observe that the integral of k on the real line, which is the situation that we described as *infinite activity* for the Lévy measure.

4.The CGMY Process

As seen in the previous paragraphs, the NIG process has infinite activity and infinite variation ; the VG process has finite variation and infinite activity. To represent these various possibilities for different values of the parameter set, Carr,Geman,Madan and Yor (2002) introduce the following Lévy density :

$$k_{CGMY}(x)\begin{cases} \dfrac{C\,e^{-Mx}}{x^{1+Y}} & x > 0 \\[2ex] \dfrac{C\,e^{-G|x|}}{|x|^{1+Y}} & x < 0 \end{cases}$$

The parameter Y captures the « fine » structure of the process in the following way : for low values of Y, the Lévy density integrates to a finite value yielding a process of finite activity. At the same time, the integral of $|x|$ times the Lévy density is also finite and the process has finite variation, as a compound Poisson process. For higher values of Y (typically between 0 and 1), the process has infinite activity and infinite variation, like the VG process. For yet higher values of Y (between 1 and 2), the process has infinite activity and infinite variation like the NIG process.

The CGMY characteristic function is obtained by integration as

$$\log[\boldsymbol{f}_{CGMY}(u)] = C\,t\,\Gamma(-y)\left\{(M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right\}$$

In order to envision the possibility of moves other than jumps, the CGMY process is extended to CGMYe model by addition of a diffusion component

$$X_{CGMY_e}(t) = X_{CGMY}(t) + \boldsymbol{h}\,W(t)$$

where η is a real number.

The CGMYe characteristic function is given by

$$\log[\boldsymbol{f}_{CGMY_e}(u)] = \log[\boldsymbol{f}_{CGMY}(u)] - \boldsymbol{h}^2 u^2 \frac{t}{2}$$

The CGMYe stock price process is defined by

$$S(t) = S(0)\exp\left\{\left(\boldsymbol{m} + \boldsymbol{w} - \frac{\boldsymbol{h}^2}{2}\right)t + X_{CGMY_e}(t)\right\}$$

where

$$\boldsymbol{w} = \frac{-1}{t}\log[\boldsymbol{f}_{CGMY}(I)]$$

ensures that the mean rate of return is μ.

The log characteristic function of the log stock price is

$$\log[\boldsymbol{f}_{\ln s}(u)] = iu\left[(\ln S(0)) + \left(\boldsymbol{m} + \boldsymbol{w} - \frac{\boldsymbol{h}^2}{2}\right)t\right] + \log[\boldsymbol{f}_{CGMY_e}(u)]$$

The higher moments of the CGMYe process are

$$E[X - E(X)]^2 = \boldsymbol{h}^2 + \int_{-\infty}^{+\infty} x^2 k(x)dx$$

$$E[X - E(X)]^3 = \int_{-\infty}^{+\infty} x^3 k(x)dx$$

$$E[X - E(X)]^4 = 3(\mathrm{var}iance) + \int_{-\infty}^{+\infty} x^4 k(x)dx$$

Carr, Geman, Madan and Yor (2002) report on the goodness of fit obtained with the CGMY model on US individual stocks and equity indices (see figure 3). Equity indices such as the SPX behave like processes of infinite activity and finite variation. Quite interestingly, testing the CGMYe against the CGMY representation shows the absence of a diffusion component in equity indices, hence the conjecture that diffusion components possibly existing in individual stock prices are diversified away in the index whose moves are pure jumps. This is a message quite different from the assumption of zero market price of jump risk in jump-diffusion models such as Merton (1976) ; it is in agreement with the perception that the whole market moves together, sometimes because of a sharp price change in a single name, as observed over the recent years. Fig. 2 shows the excellent fit obtained on a database of the SPX index over the period January 1, 1994 to December 31, 1998 using the pure jump CGMY *without any diffusion term.*

The same mathematical structure may be used for the risk-neutral process where the mean is set to equal the interest rate and the three other parameters are determined by matching option prices. More precisely, the stock price process is modelled under the risk-adjusted probability measure *Q* by

$$S(t) = S(0)\exp[(r - q + \boldsymbol{w})t + X_{C'G'M'Y'}(t)]$$

where $? = -(1/t)\ln[f_{C'G'M'Y'}(-i)]$ ensures that the mean rate of return is $(r - q)$, q denoting the dividend yield.
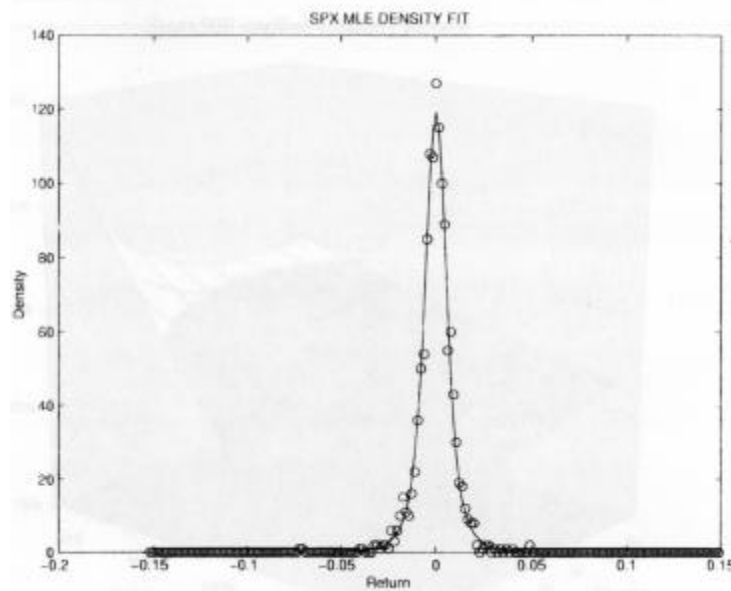

Fig. 2. SPX index distribution fitted by the CGMY model.

The Fourier transform of a standard European call price may be expressed in terms of the log of the strike as

$$\boldsymbol{g}(u) = \int_{-\infty}^{+\infty} e^{iuk} e^{\boldsymbol{a}k} C(k) dk$$

where the call has a strike e^k and the term $e^{\alpha k}$ is meant to ensure the convergence of the integral.

Call prices are then recovered by inversion

$$C(k) = \frac{e^{-\boldsymbol{a}k}}{2\boldsymbol{p}} \int_{-\infty}^{+\infty} e^{-iuk} \boldsymbol{g}(u) du$$

and the paremeters under the risk-neutral measure are derived by calibration to option market prices.

Figures 3, 4 and 5 show the quality of the fitting obtained with a pure jump CGMY (properly risk-neutralized) in the case of the Stox x50E volatility surface. The calibration is superior when adjusting separately short maturities on one hand and long maturities on the other hand. A more complex version of the CGMY model, currently under completion, will allow to handle all strikes and maturities at once.
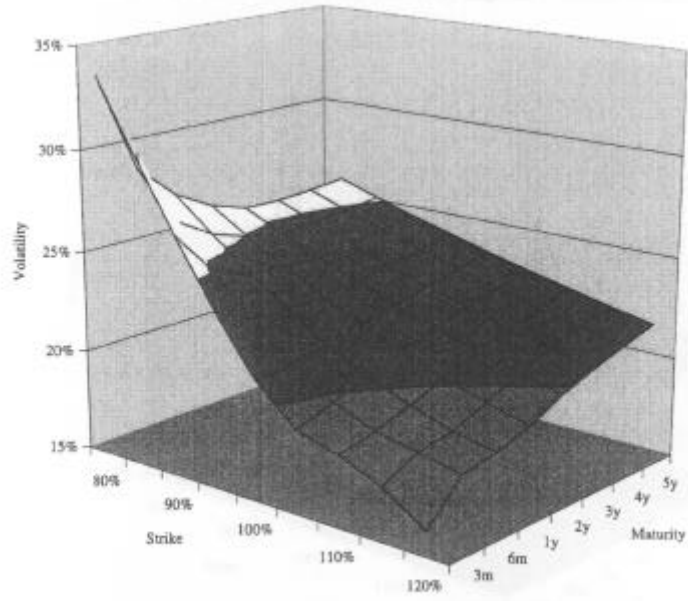
Stoxx50E implied volatility surface

Fig. 3.

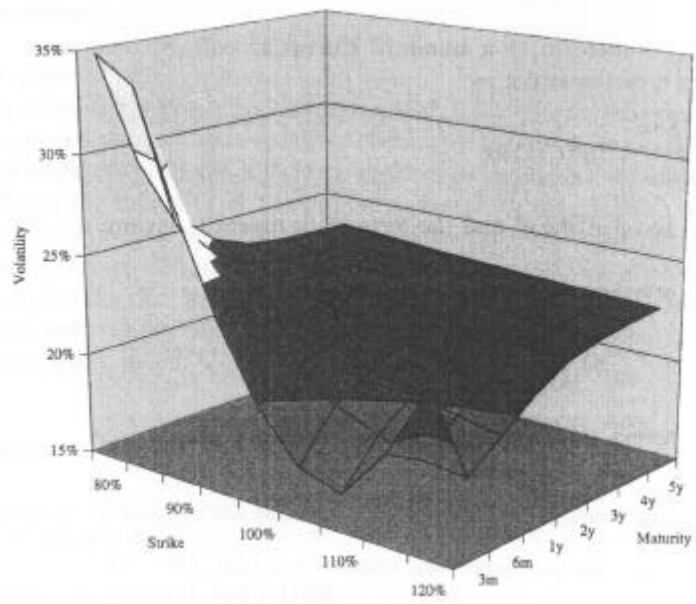CGMY (calibrated to short maturity) implied volatility surface
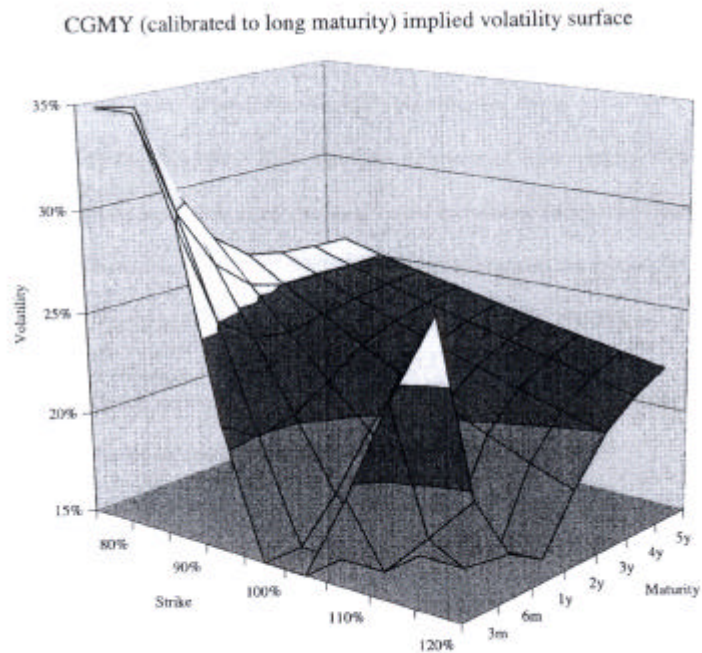

Fig. 4.

CGMY (calibrated to long maturity) implied volatility surface

Fig. 5.

V. Conclusion

We argue in this paper that pure jump Lévy processes with finite variation and infinite activity are better representations of stock price dynamics than the classical diffusions or jump-diffusion models. Their representation as time-changed Brownian motion allows to shift the focus of attention to modelling the time change which itself reflects the intensity of the economic activity through news arrival and trades. We provide examples which have lately been evidenced to perform very well in fitting financial data in US and European equity markets. The existence of explicit expressions for the characteristic functions of the NIG/hyperbolic motion and CGMY processes reduces the complexity of the underlying Lévy measures to a simple representation, quite amenable for statistical estimation.

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