Interacting Defaults and Counterparty Risk: a Markovian Approach

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Abstract

We consider intensity-based dynamic models for dependent defaults. We generalize the standard reduced-form models and assume that the default intensity of a firm is directly affected by the default of other firms in the portfolio. This interaction between defaults, which is termed counterparty risk in the literature, could be due to direct business relations between firms or due to the impact of defaults on the overall credit climate. We construct and study the model using Markov process techniques. We study in detail a model where the interaction between firms is of mean-field type.

Keywords: Credit risk, dependent defaults, Markov processes, mean-field model

1 Introduction

A major cause of concern in the pricing and management of credit risk of a given loan or bond portfolio is the occurrence of disproportionately many defaults of different counterparties in the portfolio, a risk which is directly linked to the dependence between default events. Dependence between defaults stems from at least two sources. First, the profitability and hence the financial health of a firm varies with stochastically fluctuating macroeconomic factors such changes in economic growth. Since different firms are affected by common macroeconomic factors, we have dependence between their defaults. Moreover, dependence between default is caused by direct links between firms such as business relations or a borrower-lender-relationship. For instance, the default probability of a commercial bank is likely to increase if one of its major borrowers or counterparties defaults. Following Jarrow and Yu (2001), in our paper the latter type of interaction between default events is termed counterparty risk.

The dependence between defaults caused by common factors has received a lot of attention in the credit risk literature, as it can and has been modelled in the standard reduced form credit risk models such as Lando (1998) or Duffie and Singleton (1999); for empirical work on the specification of an appropriate factor structure see for instance Duffee (1999) or Driessen (2002). In contrast, researchers became only recently interested in counterparty risk. This interest stems from at least two reasons: first, there is substantial empirical evidence for counterparty risk; for instance Lang and Stulz (1992) have shown that bankruptcy

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filings do impact stock returns (and most likely also default probabilities) of non-defaulted companies. Moreover, as has been pointed out by Hull and White (2001) or Schönbucher and Schubert (2001), the correlation between defaults obtainable in reduced form models are often quite low, so that these models may not be able to mimic the clustering of defaults around economic recessions observed in real data (see for instance Keenan (2000)).[1] Obviously, this calls for an incorporation of other sources of default dependence such as counterparty risk into the model.

At least to our knowledge Jarrow and Yu (2001) are the first to propose an intensity-based model, which allows for counterparty risk. In their framework the impact of defaults on the default intensities of surviving firms is explicitly modelled, which is a very intuitive parametrization of counterparty risk; see also Davis and Lo (2001) for a related approach. Unfortunately, the construction of default processes in Jarrow and Yu (2001) works only for a very special type of interaction between defaults, the so-called primary secondary framework, which excludes many interesting examples of cyclical default dependency. This and other mathematical aspects of the Jarrow-Yu model are discussed in Kusuoka (1999), Bielecki and Rutkowski (2002), and Collin-Dufresne, Goldstein, and Hugonnier (2002). Yu (2002) has carried out an interesting simulation study. He analyzes the default correlations which can be obtained for different parametrizations of the standard reduced form models and of the Jarrow-Yu model. Moreover, he improves upon the original Jarrow-Yu paper and provides a rigorous construction of the model using the general hazard construction from survival analysis. Finally, Gieseke and Weber (2002) use the voter model, which is well-known in the literature on interacting particle systems (see for instance Liggett (1985)) to model interaction between defaults. They come up with a model for the loss distribution of a given portfolio, which is constructed as a mixture of the equilibrium distributions of the voter model. Their model contains an interesting link between credit risk modelling and statistical mechanics. However, the focus on the equilibrium distribution makes their analysis essentially static and hence unsuitable for questions related to the pricing of derivatives.

Counterparty risk is also present in the popular copula model (see for instance Li (2001) or Schönbucher and Schubert (2001)). As shown in the latter paper, in the copula framework the default intensity of the surviving firms typically jumps at the default time of one obligor in the portfolio. However, direction and size of this jump depend on higher order derivatives of the copula, which makes the copula parametrization of counterparty risk quite unintuitive. Moreover, while the calibration of copula models to prices of defaultable bonds is relatively easy, the specification of an appropriate copula is difficult and subject to substantial model risk; see for instance Frey, McNeil, and Nyfeler (2001) for related findings in the context of static models for portfolio credit risk.

In the present paper we propose several extensions to the literature on the Jarrow-Yu type reduced form models for counterparty risk. First, we model the default indicator process of the firms in our portfolio as conditional finite-state Markov chain; the states of this chain are given by the default state of all obligors in the portfolio at a given point in time and the transition rates correspond to the default intensities. This yields a natural and at the same time completely rigorous construction of the Jarrow-Yu model. Moreover, our approach allows us to employ Markov process techniques such as the Kolmogorov equations or special simulation techniques in the analysis of the model.

Our second contribution concerns the modelling of the interaction between the defaults of different firms in the portfolio. This is a major challenge in the Jarrow-Yu framework, in

[1] However, as shown by Yu (2002) this may be related more to an unsatisfactory modelling of state variables driving the reduced form models than to a problem of the reduced form approach per se.

particular if the portfolio is large: the model should capture essential features of counterparty risk, and should at the same time be relatively simple and parsimonious to ensure ease of calibration and simulation. To achieve these goals we split our portfolio in several homogeneous groups and propose a model where the default intensity of a given firm depends essentially on the distribution of defaulted firms in these groups – in the simplest case of a one-group model just the proportion of companies which have defaulted so far. This type of interaction, which is called mean-field interaction in the literature on interacting particle systems,[2] makes immediate sense in the context of portfolio credit risk. For instance, if a financial institution has incurred unusually many losses in its loan portfolio, it is less likely to extend credit lines, if another obligor experiences financial distress. Obviously, this raises the default probability of the remaining obligors. Moreover, unusually many defaults might have a negative impact on the business climate in general, which in turn favors future defaults. From a mathematical viewpoint we are automatically lead to models based on mean-field interaction, if we assume that our portfolio consists of several homogeneous groups within which default times are exchangeable. We will show that homogeneous group models with mean-field interaction are relatively easy to treat, as the size of the state space of the Markov chain can be reduced substantially, making the application of analytical methods such as the Kolmogorov equations feasible even for large portfolios. Using results on the convergence in distribution of Markov processes we study the asymptotic behavior of our mean-field model as the portfolio size becomes large.

Finally we study several practical applications. We provide first results on the pricing of credit risky securities such as corporate bonds or other vulnerable securities. In particular, we show how Markov process techniques can be fruitfully employed to deal with pricing problems in the context of the Jarrow-Yu model. Moreover, in order to quantify the impact of counterparty risk on default correlations and credit loss distribution in the context of our mean-field model we carry out a simulation study. It will turn out that default correlations and in particular quantiles of the loss distributions increase substantially, if we increase the amount of interaction in the portfolio. This shows that our model provides a possible way to overcome the weakness of standard reduced form models and to generate realistic patterns of default correlation.

2 A General Markovian Model

2.1 The Model: Description and General Properties

We consider a portfolio of m firms, indexed by $i \in \{1, \dots, m\}$. The default-state of the portfolio is summarized by a default indicator process $\mathbf{Y} = \big(Y_t(1), \dots, Y_t(m)\big)'_{t \geq 0}$ with values in $\{0,1\}^m$; here $Y_t(i) = 1$ if firm i has defaulted by time t and $Y_t(i) = 0$ else. The corresponding default times are given by $\tau_i = \inf\{t \geq 0 : Y_t(i) = 1\}$. In order to model the dependence of defaults caused by fluctuations in the macroeconomic environment we introduce a d-dimensional state variable process $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_t)_{t \in [0,\infty)}$, representing the evolution of macroeconomic variables such as interest rates, broad share price indices or measures of economic activity. In keeping with most of the literature on reduced-form credit risk models we assume that the dynamics of $\boldsymbol{\Psi}$ are not affected by the evolution of the default indicator process $\mathbf{Y}$. This allows us to construct our model by a two step procedure: first we model the evolution of $\boldsymbol{\Psi}$ and consecutively the conditional distribution of the default indicator

[2] For an inspiring discussion of the relevance of concepts from the literature interacting particle systems for financial modelling we refer to (Föllmer 1994).

process $\mathbf{Y}$ for a given realization of the economic factor process.

The default intensity of a non-defaulted firm i at time t is modelled as a function $\lambda_i(\boldsymbol{\Psi}_t, \mathbf{Y}_t)$ of economic factors and of the default state of the other obligors in the portfolio. Intuitively, given the current default state $\mathbf{Y}_t$ and the trajectory $(\boldsymbol{\Psi}_s)_{s\geq 0}$, the defaults of the firms, which have survived until time t, are independent time-inhomogeneous Poisson events with intensity equal to $\lambda_i(\boldsymbol{\Psi}_s, \mathbf{Y}_t)$ for $s \geq t$; at the first default-time $\tau > t$ the default indicator vector changes and the default intensities of the surviving firms are updated accordingly. This description of the dynamics of $\mathbf{Y}$ is formalized in Assumption 2.1 below, where we postulate that $\mathbf{Y}$ follows a conditional Markov chain with transition rates depending on the realization $(\boldsymbol{\Psi}_s)_{s\geq 0}$ of the factor process.

Notation. $S = \{0,1\}^m$ will be the state space of the default indicator process; elements of S are vectors $\mathbf{y} = (y(1), \ldots, y(m))'$. Note that the cardinality of S is $|S| = 2^m$. For $\mathbf{y} \in S$ we define $\mathbf{y}^i \in S$ by flipping the i-th coordinate, i.e.

$$y^i(i) = 1 - y(i) \text{ and } y^i(j) = y(j), \text{ for } j \in \{1, \ldots, m\} - \{i\}. \tag{1}$$

$\overline{S} = \mathbb{R}^d \times S$ denotes the state space of the pair $(\boldsymbol{\Psi}, \mathbf{Y})$; elements of $\overline{S}$ are denoted by $\boldsymbol{\gamma} = (\boldsymbol{\psi}, \mathbf{y})$. Finally, $\mathbf{D}([0,\infty)), E)$ stands for the Skorohod space of all RCLL functions from $[0,\infty)$ into some Polish space E.

The mathematical model. Define $\Omega_1 := \mathbf{D}([0,\infty), \mathbb{R}^d)$ and $\Omega_2 := \mathbf{D}([0,\infty), S)$, and denote by $\mathcal{F}^i$ the natural σ-field on Ω_i. Our underlying measurable space is given by $(\Omega, \mathcal{F}) := (\Omega_1 \times \Omega_2, \mathcal{F}^1 \times \mathcal{F}^2)$; elements in Ω will be written as $\omega = (\omega_1, \omega_2)$. The coordinate process on Ω_1 will be denoted by $\boldsymbol{\Psi}$, i.e. $\boldsymbol{\Psi}_{\boldsymbol{t}}(\omega_1) = \omega_1(t)$ for $t \geq 0$; it represents the economic factor process; the coordinate process on Ω_2, denoted by $\mathbf{Y}$, models the default indicator process. By $\boldsymbol{\Gamma}_t(\omega)$ we denote the pair $(\boldsymbol{\Psi}_t(\omega_1), \mathbf{Y}_t(\omega_2))$. For $t \in [0,\infty)$ we define $\mathcal{F}^1_t := \sigma(\boldsymbol{\Psi}_s : s \leq t)$, $\mathcal{F}^2_t := \sigma(\mathbf{Y}_s : s \leq t)$ and $\mathcal{F}_t := \mathcal{F}^1_t \times \mathcal{F}^2_t$; moreover, we define the filtration $\{\mathcal{G}_t\}$ by $\mathcal{G}_t := \mathcal{F}^1_\infty \vee \mathcal{F}^2_t$. We assume that investors have access to $\{\mathcal{F}_t\}$, whereas the larger filtration $\{\mathcal{G}_t\}$, which contains information about the default indicator process up to time t and about the whole path $(\boldsymbol{\Psi}_s(\omega_1))_{s\geq 0)}$, serves mainly theoretical purposes.

In this paper we consider a family of probability measures $P_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \overline{S}$ on $(\Omega, \mathcal{F})$ having the structure $P_{\boldsymbol{\gamma}} = \mu_{\boldsymbol{\psi}} \times K_{\mathbf{y}}(\omega_1, d\omega_2)$ for some measure $\mu_{\boldsymbol{\psi}}$ on Ω_1 and some stochastic kernel $K_{\mathbf{y}} : \Omega_1 \times \mathcal{F}^2 \mapsto [0,1]$. The kernel $K_{\mathbf{y}}(\omega_1, d\omega_2)$ models the conditional distribution of the default indicator process $\mathbf{Y}$ for a given realization of ω_1, or equivalently of the trajectory $(\boldsymbol{\Psi}_t(\omega_1))_{t\geq 0}$. Depending on the context $P_{\boldsymbol{\gamma}}$ will represent the historical probability measure (for instance in applications to credit risk management) or an equivalent martingale measure. The appropriate interpretation of P will become clear when we study a particular application.

Assumption 2.1.

(i) Under $\mu_{\boldsymbol{\psi}}$ the process $\boldsymbol{\Psi}$ is a non-exploding process with $\boldsymbol{\Psi}_0 = \boldsymbol{\psi}$ $\mu_{\boldsymbol{\psi}}$ a.s. Moreover, $\boldsymbol{\Psi}$ is autonomous in the following sense. For every bounded, $\mathcal{F}^1_\infty$-measurable rv ξ and every $t \geq 0$ we have $E(\xi \mid \mathcal{F}_t) = E(\xi \mid \mathcal{F}^1_t)$, i.e. observation of the path $(\mathbf{Y}_s(\omega_2))_{0\leq s\leq t}$ does not convey any additional observation on the future evolution of the state variable process.

(ii) Under $K_{\mathbf{y}}(\omega_1, d\omega_2)$ the process $\mathbf{Y}$ is a time-inhomogeneous Markov chain with state space S; moreover $K_{\mathbf{y}}(\omega_1, \mathbf{Y}_0 = \mathbf{y}) = 1$. The infinitesimal generator of the chain $\mathbf{Y}$

is as follows. Recall the definition of $\mathbf{x}^i$ in (1). Let $\lambda_i : \overline{S} \to (0,\infty)$ be continuous functions and define for $\boldsymbol{\psi} \in \mathbb{R}^d$ and any function $f : S \to \mathbb{R}$ the operator

$$G_{[\boldsymbol{\psi}]} f(\mathbf{x}) = \sum_{i=1}^{m} (1 - x(i))\, \lambda_i\left(\boldsymbol{\psi}, \mathbf{x}\right) \big(f(\mathbf{x}^i) - f(\mathbf{x})\big)\,. \tag{2}$$

The infinitesimal generator of $\mathbf{Y}$ under $K_{\mathbf{y}}(\omega_1, d\omega_2)$ at time t is then given by $G_{[\boldsymbol{\Psi}_t(\omega_1)]}$.

When there is no ambiguity we will simply write P, μ, and K and drop the reference to the initial values to ease the notation.

Assumption 2.1 (ii) determines the dynamics of $\mathbf{Y}$ under $K(\omega_1, d\omega_2)$. In particular, the form of $G_{[\boldsymbol{\psi}]}$ implies that default is an absorbing state and that the chain $\mathbf{Y}$ has only transitions from a state $\mathbf{y} \in S$ to neighboring states $\mathbf{y}^i$, which excludes simultaneous defaults. Obviously, these restrictions could easily be removed by considering more general generators. For an explicit construction of a conditional Markov chain or equivalently of a family of kernels $K_{\mathbf{y}}(\omega_1, d\omega_2)$ satisfying Assumption 2.1 (ii) we refer to the literature on Markov chains such as Chapter 11.3 of Bielecki and Rutkowski (2002) or Chapter 2 of Davis (1993). A construction via a change of measure using the Girsanov theorem for point processes is given in Kusuoka (1999) or Bielecki and Rutkowski (2002). Yu (2002) uses the general hazard rate construction from survival analysis as developed for instance in Norros (1986) and Shaked and Shanthikumar (1987).

Markov property and default intensities. We now discuss the (conditional) Markov property for the processes $\mathbf{Y}$ and $\boldsymbol{\Gamma}$. We have for every bounded random variable $F(\boldsymbol{\Psi}, \mathbf{Y}) : \Omega_1 \times \Omega_2 \to \mathbb{R}$

$$E(F(\boldsymbol{\Psi}, \mathbf{Y}) \mid \mathcal{G}_t)(\omega_1, \omega_2) = E^{K(\omega_1,\cdot)}(F(\boldsymbol{\Psi}(\omega_1), \mathbf{Y}) \mid \mathcal{F}_t^2)(\omega_2)\,. \tag{3}$$

Relation (3) is easily shown for $F(\boldsymbol{\Psi}, \mathbf{Y}) = F_1(\boldsymbol{\Psi}) F_2(\mathbf{Y})$ using the definition of $\mathcal{G}_t$ and the fact that $P = \mu \times K$; the extension to general F is done via a monotone class argument. Now define for $t \in [0,\infty)$ and an arbitrary Polish space E the shift operator $\theta_t : \mathbf{D}\left([0,\infty), E\right) \to \mathbf{D}\left([0,\infty), E\right)$ by $\theta_t \omega(s) := \omega(t+s)$. Since $\mathbf{Y}$ is a time-inhomogenous Markov chain under $K(\omega_1, d\omega_2)$, relation (3) yields the following *conditional Markov property* of $\mathbf{Y}$. For all bounded, measurable $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ and all $t \in [0,\infty)$

$$\begin{aligned} E\big(F(\boldsymbol{\Psi}, \mathbf{Y} \circ \theta_t) \mid \mathcal{G}_t\big)(\omega_1, \omega_2) &= E^{K(\omega_1,\cdot)}\left(F(\boldsymbol{\Psi}(\omega_1), \mathbf{Y} \circ \theta_t) \mid \mathcal{F}_t^2\right)(\omega_2) \\ &= E^{K_{\mathbf{Y}_t(\omega_2)}(\theta_t \omega_1,\cdot)}\left(F(\boldsymbol{\Psi}(\omega_1), \mathbf{Y})\right). \end{aligned} \tag{4}$$

Relation (4) immediately yields that $\mathbf{Y}$ forms an time-inhomogeneous Markov chain wrt $\{\mathcal{G}_t\}$ under P. The process $Y_t(i) - \int_0^{t\wedge\tau_i} \lambda_i(\boldsymbol{\Psi}_s, \mathbf{Y}_s)\, ds$ is therefore a $\{\mathcal{G}_t\}$-martingale by the Dynkin formula, and hence an $\{\mathcal{F}_t\}$-martingale, as it is $\{\mathcal{F}_t\}$-adapted. This shows that the process $(\lambda_i(\boldsymbol{\Psi}_t, \mathbf{Y}_t))_{t\geq 0}$ is in fact the *martingale default intensity*[3]of company i.

Suppose now that Assumtion 2.1 holds and that $\boldsymbol{\Psi}$ is moreover a homogeneous Markov process. In that case it is easily seen from (4) that the process $\boldsymbol{\Gamma}$ is Markov wrt $\{\mathcal{F}_t\}$: Define the random variable

$$H : \Omega_1 \times S \to \mathbb{R}\,, \quad H(\omega_1, \mathbf{x}) = E^{K_{\mathbf{x}}(\omega_1,\cdot)}\left(F(\boldsymbol{\Psi}(\omega_1), \mathbf{Y})\right)\,.$$

[3]There are various notions of intensities for random times; see for instance Chapter 6 of Bielecki and Rutkowski (2002)

Using the law of iterated expectations, (4) and the $\mathcal{F}_t^2$-measurability of $\mathbf{Y}_t$ we obtain

$$\begin{aligned} E\left(F(\mathbf{\Psi}\circ\theta_t,\mathbf{Y}\circ\theta_t)\mid\mathcal{F}_t\right)(\omega_1,\omega_2) &= E\left(H(\mathbf{\Psi}\circ\theta_t,\mathbf{Y}_t)\mid\mathcal{F}_t\right)(\omega_1,\omega_2)\\ &= E\left(H(\mathbf{\Psi}\circ\theta_t,\mathbf{Y}_t(\omega_2))\mid\mathcal{F}_t^1\right)(\omega_2)\,. \end{aligned}$$

As as $\mathbf{\Psi}$ is Markov under P this equals $\int_{\Omega_1} H(\omega_1,\mathbf{Y}_t(\omega_2))\mu_{\mathbf{\Psi}_t}(d\omega_1) = E_{\mathbf{\Gamma}_t(\omega)}\big(F(\mathbf{\Psi},\mathbf{Y})\big)$ by definition of H.

Note that the first component of a pair of processes which is jointly Markov is in general neither autonomous nor Markov wrt its own filtration. Under Assumption 2.1 we may evaluate expectations using a two-step procedure, which can be easier than applying Markov process techniques directly to the process $\mathbf{\Gamma}$. This is useful in computing default correlations and risk measures and in the pricing of credit derivatives; see Sections 4 and 5 below.

Conditional transition functions and the Kolmogorov equations. Next we introduce the conditional transition probabilities of the chain $\mathbf{Y}$ under $K(\omega_1,d\omega_2)$. Define for $0\leqslant t\leqslant s<\infty$ and $\mathbf{x},\mathbf{y}\in S$

$$p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1):=E^{K(\omega_1,d\omega_2)}(\mathbf{Y}_s=\mathbf{y}\mid\mathbf{Y}_t=\mathbf{x}). \tag{5}$$

It is well-known that for ω_1 fixed the function $p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1)$ satisfies the Kolmogorov forward and backward equations. These equations will be very useful numerical tools in our analysis of the model. The backward equation is a system of ODE's for the function $(t,\mathbf{x})\to p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1)$, $0\leqslant t\leqslant s$; s and $\mathbf{y}$ are considered as parameters. In its general form the equation is

$$\frac{\partial p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1)}{\partial t}+G_{[\mathbf{\Psi}_t(\omega_1)]}p(t,s,\mathbf{x},\mathbf{y})=0\,,\quad p(s,s,\mathbf{x},\mathbf{y})=1_{\{\mathbf{y}\}}(\mathbf{x})\,. \tag{6}$$

In our model this leads to the following system of ODE's

$$\frac{\partial p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1)}{\partial t}+\sum_{k=1}^{m}(1-x(k))\,\lambda_k(\mathbf{\Psi}_t(\omega_1),\mathbf{x})(p(t,s,\mathbf{x}^k,\mathbf{y}\mid\omega_1)-p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1))=0. \tag{7}$$

The forward-equation is an ODE-System for the function $(s,\mathbf{y})\to p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1)$, $s\geqslant t$. Denote by $G^*_{[\boldsymbol{\psi}]}$ the adjoint operator to $G_{[\boldsymbol{\psi}]}$, operating again on functions from S to $\mathbb{R}$. In its general form the forward equation reads

$$\frac{\partial p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1)}{\partial s}=G^*_{[\mathbf{\Psi}_t(\omega_1)]}p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1),\quad p(t,t,\mathbf{x},\mathbf{y}\mid\omega_1)=1_{\{\mathbf{x}\}}(\mathbf{y}). \tag{8}$$

An explicit form is given in the following lemma.

Lemma 2.2. *Under Assumption 2.1 (ii) the forward equation for the conditional transition rates is*

$$\begin{aligned} \frac{\partial p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1)}{\partial s} =&\sum_{k=1}^{m}y(k)\lambda_k(\mathbf{\Psi}_s(\omega_1),\mathbf{y}^k)p(t,s,\mathbf{x},\mathbf{y}^k\mid\omega_1) \\ &-\sum_{k=1}^{m}\lambda_k(\mathbf{\Psi}_s(\omega_1),\mathbf{y})(1-y(k))p(t,s,\mathbf{x},\mathbf{y}\mid\omega_1). \end{aligned} \tag{9}$$

The proof is given in Appendix A.2. For small m (7) and (9) are easily solved numerically; however, for large m this becomes infeasible as $|S|$ grows exponentially in m.

Remark 2.3 (Simulation). We will see in the sequel that many questions related to the computation of risk measures and the pricing of credit risky securities can be dealt with very efficiently using the Kolmogorov equations. Nonetheless, for working with large inhomogeneous portfolios or for pricing sophisticated credit derivatives one typically has to resort to a simulation approach. Fortunately, the model introduced in Assumption 2.1 is quite easy to simulate from using the standard simulation approach for continuous-time Markov chains; in particular, simulation is no more costly (in terms of computing time) than simulating a standard reduced form model with conditionally independent defaults. We give a detailed description of the simulation algorithm in Appendix A.1.

2.2 Examples for default intensities

Example 2.4 (The primary-secondary framework of Jarrow and Yu). Jarrow and Yu (2001) study a model where default of one firm may influence the default-intensity of other firms. However, in contrast to the present paper they focus on a very special type of interaction, which they call primary-secondary framework. In this framework firms are divided into two classes, primary and secondary firms. The default intensity of primary firms depends only on the factor process $\mathbf{\Psi}$; default intensities of secondary firms depend on $\mathbf{\Psi}$ and on the default state of the primary firms. This simplifying assumption allows Jarrow and Yu to deal with interacting defaults using Cox-process techniques. For concreteness we now present a specific example from Jarrow and Yu (2001). We put $m = 2$ and $d = 1$; Ψ_t is identified with the short rate of interest r_t, which is assumed to follow an extended Vasicek-model. We put

$$\lambda_1(r_t, \mathbf{Y}_t) = \lambda_{1,0} + \lambda_{1,1} r_t \quad \text{and} \quad \lambda_2(r_t, \mathbf{Y}_t) = \lambda_{2,0} + \lambda_{2,1} r_t + \lambda_{2,2} 1_{\{Y_t(1)=1\}};$$

hence company one is a primary firm with default intensity depending only on r_t and company two is a secondary firm.

Example 2.5 (Mean field interaction). The default intensities in Example 2.4 are typical for a model with local interaction, i.e. a model, where for all $i \in \{1, \ldots, m\}$ the default-intensity of firm i depends on the default state of some typically small set $N(i)$ of neighboring firms such as business partners or direct competitors. Alternatively, one can introduce some global or macroeconomic interaction in the sense that individual default intensities depend on the empirical distribution $\rho(\mathbf{Y}_t, \cdot) = \frac{1}{m} \sum_{i=1}^{m} \delta_{Y_t(i)}(\cdot)$ of defaults at time t. Note that $\rho(\mathbf{Y}_t, \{0\})$ respectively $\rho(\mathbf{Y}_t, \{1\})$ give the proportion of surviving respectively defaulted firms in the portfolio at time t. This can be modelled using default intensities of the form

$$\lambda_i\left(\mathbf{\Psi}_t, \mathbf{Y}_t\right) = h_i\big(\mathbf{\Psi}_t, \rho(\mathbf{Y}_t, \{1\})\big) \text{ for functions } h_i : \mathbb{R}^d \times [0, 1] \to \mathbb{R}; \tag{10}$$

As discussed in the introduction this type of global interaction makes immediate sense in the context of portfolio credit risk. Moreover, the points raised in the introduction suggest that the functions h_i in (10) will typically be increasing in their second argument.

3 Models with mean-field interaction

We now analyze variants of the model with mean field interaction introduced in Example 2.5. We begin by introducing a general homogeneous-group model with mean-field interaction.

3.1 A mean-field model with homogeneous groups

Assume that we can divide our portfolio of m firms into k groups (typically $k \ll m$), within which risks are exchangeable. These groups might for instance correspond to firms with different credit rating or to firms from the same industries. Let $\kappa(i) \in \{1, \ldots, k\}$ give the group membership of firm i, $m_\kappa = \sum_{i=1}^m 1_{\{\kappa(i)=\kappa\}}$ the number of firms in group κ, and denote for a given $\mathbf{y} \in S$ by $\rho_\kappa(\mathbf{y}, \cdot) = \frac{1}{m_\kappa} \sum_{i=1}^m 1_{\{\kappa(i)=\kappa\}} \delta_{y(i)}(\cdot)$ the empirical distribution of firms in group κ. Define for $\kappa \in \{1, \ldots, k\}$ the functions $\overline{M}_\kappa(\mathbf{y}) := \rho_\kappa(\mathbf{y}, \{1\})$, put $\overline{\mathbf{M}}(\mathbf{y}) = \big(\overline{M}_1(\mathbf{y}), \ldots, \overline{M}_k(\mathbf{y})\big)'$, and define the process $\overline{\mathbf{M}}_t = \big(\overline{M}_{t,1} \ldots, \overline{M}_{t,k})'$ by $\overline{M}_{t,\kappa} = \overline{M}_\kappa(\mathbf{Y}_t)$; obviously, $\overline{M}_{t,\kappa}$ gives the proportion of firms in group κ which have defaulted by time t. The state space of $\overline{\mathbf{M}}_t$ is given by $S^{\overline{M}} := \big\{\bar{\boldsymbol{l}} = \big(\frac{l_1}{m_1}, \ldots, \frac{l_k}{m_k}\big) : l_\kappa \in \{0, \ldots, m_\kappa\}, 1 \leq \kappa \leq k\big\}$.

Assumption 3.1 (Mean-field model with homogeneous groups). The default intensities of firms in our portfolio belonging to the same group are identical and of the form $\lambda_i\left(\boldsymbol{\psi}, \mathbf{y}\right) = h_{\kappa(i)}\big(\boldsymbol{\psi}, \overline{\mathbf{M}}(\mathbf{y})\big)$ for continuous functions $h_\kappa : \mathbb{R}^d \times S^{\overline{M}} \to \mathbb{R}^+$, $1 \leq \kappa \leq k$.

Assumption 3.1 implies that for all κ the default indicator processes $\{Y_t(i) : 1 \leq i \leq m,\ \kappa(i) = \kappa\}$ of firms belonging to the same group are exchangeable, a fact which we will exploit frequently below. Conversely, consider an arbitrary portfolio of m counterparties with default indicators satisfying Assumtion 2.1, and suppose that the portfolio can be split in $k < m$ homogeneous groups. Homogeneity implies that a) the default intensities are invariant under permutations π of $\{1, \ldots, m\}$, which leave the group structure invariant, i.e. $\lambda_i(\boldsymbol{\psi}, \mathbf{y}) = \lambda_i(\boldsymbol{\psi}, \pi(\mathbf{y}))$ for all i and all permutations π with $\kappa(\pi(j)) = \kappa(j)$ for all $1 \leq j \leq m$, and b) that default intensities of different firms from the same group are identical. Condition a) immediately yields that $\lambda_i(\boldsymbol{\psi}, \mathbf{y}) = h_i(\boldsymbol{\psi}, \mathbf{y})$ for some $h_i : \mathbb{R}^d \times S^{\overline{M}} \to \mathbb{R}^+$ and hence a model of mean-field type; together with condition b) this implies that the default intensities satisfy Assumption 3.1. Hence the mean-field model is the natural counterparty-risk model for portfolios consisting of homogeneous groups.

Example 3.2 (An affine model with counterparty risk). Often we will assume that the default intensities depend only on the overall proportion of defaulted companies given by $\sum_{\kappa=1}^k \frac{m_\kappa}{m} \overline{M}_{t,\kappa}$. A useful example is provided by the following (nearly) affine model with counterparty risk. Given for every group κ nonnegative constants $\lambda_{\kappa,j}$, $j = 0, \ldots, d+1$ and an expected default intensity $\bar{\lambda}_\kappa$ we put

$$h_\kappa(t, \boldsymbol{\psi}, \bar{\boldsymbol{l}}) = \Big[\lambda_{\kappa,0} + \sum_{j=1}^d \lambda_{\kappa,j} \psi_j + \lambda_{\kappa,d+1} \sum_{j=1}^k \frac{m_j}{m} \big(\bar{l}_j - \big(1 - e^{-\bar{\lambda}_j t}\big)\big)\Big]^+ . \tag{11}$$

These default intensities have the following intepretation: $1 - e^{-\bar{\lambda}_j t}$ measures the expected proportion of defaulted firms in group j at time t, and for $\lambda_{\kappa,d+1} > 0$ the default intensity of non-defaulted companies is increased (decreased), if the proportion of defaulted companies is higher (lower) than expected; in particular we have counterparty risk. If $\lambda_{\kappa,d+1} = 0$ for all κ we are in a standard Cox-process framework as studied by Lando (1998) or Duffie and Singleton (1999). Following the latter paper we assume that the factor process follows a square-root diffusion model, i.e.

$$d\Psi_{t,j} = \bar{\kappa}_j(\theta_j - \Psi_{t,j})dt + \sigma_j \sqrt{\Psi_{t,j}}\, dW_{t,j} \tag{12}$$

for a standard Brownian motion $\mathbf{W}_t = (W_{t,1}, \ldots, W_{t,d})'$ and positive constants $\bar{\kappa}_j$, θ_j, σ_j.

Example 3.3 (Intra-industry spillover effects). The following example is proposed by Yu (2002) as a model for similar firms in a concentrated industry. Yu splits the overall portfolio in two groups; group one consists simply of the first firm, group two consists of firms $2, \ldots, m$.[4] Define $\tau^{(1)} := \inf\{\tau_i : 1 \leqslant i \leqslant m\}$ to be the first-to-default time for the firms in our portfolio. The default-intensities are now given by

$$\lambda_t^i = \begin{cases} b + (b' - b)1_{\{\tau^{(1)} \leqslant t\}}, \ i = 1 \\ a + (a' - a)1_{\{\tau^{(1)} \leqslant t\}}, \ i = 2, \ldots, m, \end{cases}$$

for positive constants a, a', b, b'. Note that $\tau^{(1)} \leqslant t$ if and only if $\overline{\mathbf{M}}_t \neq (0,0)$, so that the default intensities can be written as function of $\overline{\mathbf{M}}_t$. Simulation studies reported in Yu (2002) suggest, that the model is able to explain certain features of credit spreads in the market for European telecom bonds.

The next lemma shows that the process $\overline{\mathbf{M}}_t$ is itself conditionally Markov and gives the form of the generator.

Lemma 3.4. *Assume that the default intensities satisfy Assumption 3.1. Then under $K(\omega_1, d\omega_2)$ the process $\overline{\mathbf{M}}_t$ follows a time-inhomogeneous Markov chain with state space $S^{\overline{M}}$. The generator of this chain in t equals $G^{\overline{\mathbf{M}}}_{[\mathbf{\Psi}_t(\omega_1)]}$, where the operator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]}$ is given by*

$$G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]} f\ (\bar{\boldsymbol{l}}) = \sum_{\kappa=1}^{k} m_\kappa (1 - \bar{l}_\kappa) h_\kappa \left(\boldsymbol{\psi}, \bar{\boldsymbol{l}}\right) \left(f\big(\bar{\boldsymbol{l}} + \frac{1}{m_\kappa}\mathbf{e}_\kappa\big) - f\big(\bar{\boldsymbol{l}}\big)\right). \tag{13}$$

Here $\bar{\boldsymbol{l}} = (\bar{l}_1, \ldots, \bar{l}_k) \in S^{\overline{M}}$ and $\boldsymbol{e_\kappa}$ denotes the unit vector κ in $\mathbb{R}^k$.

Proof. Suppose that $\overline{\mathbf{M}}_t = \left(\frac{l_1}{m_1}, \ldots, \frac{l_k}{m_k}\right)'$. Obviously, the component $\overline{M}_{t,\kappa}$ can increase only in steps of size $(m_\kappa)^{-1}$, so that the support of the jump-distribution equals $\{\overline{\mathbf{M}}_t + \frac{1}{m_\kappa} e_\kappa : 1 \leq \kappa \leq k,\ \overline{M}_{t,\kappa} < 1\}$. Now $\overline{\mathbf{M}}_t$ jumps to $\overline{\mathbf{M}}_t + \frac{1}{m_\kappa}\mathbf{e}_\kappa$ if and only if the next defaulting firm belongs to group κ. Hence the transition rate from $\overline{\mathbf{M}}_t$ to $\overline{\mathbf{M}}_t + \frac{1}{m_\kappa}\mathbf{e}_\kappa$ equals

$$\begin{aligned} \sum_{i=1}^{m} 1_{\{\kappa(i)=\kappa\}} \left(1 - Y_t(i)\right) \lambda_i(\mathbf{\Psi}_t, \mathbf{Y}_t) &= h_\kappa(\mathbf{\Psi}_t, \overline{\mathbf{M}}_t) \sum_{i=1}^{m} 1_{\{\kappa(i)=i\}} \left(1 - Y_t(i)\right) \\ &= h_\kappa(\mathbf{\Psi}_t, \overline{\mathbf{M}}_t)\, m_\kappa \left(1 - \overline{M}_{t,\kappa}\right)\big). \end{aligned}$$

The claim follows, as this transition-rate depends on $\mathbf{Y}_t$ only via $\overline{\mathbf{M}}_t$, which shows that $\overline{\mathbf{M}}$ is Markov with respect to $\{\mathcal{G}_t\}$. The form of $G^{\overline{\mathbf{M}}}_{[\mathbf{\Psi}_t(\omega_1)]}$ is obvious from the transition rates. □

Remark 3.5. 1) Note that the size of the state space of $\overline{\mathbf{M}}$ is $\left|S^{\overline{M}}\right| := (m_1 + 1) \cdots (m_k + 1)$. For fixed k $\left|S^{\overline{M}}\right|$ grows at most at rate $(k^{-1}m)^k$ in m, whereas $\left|S\right|$ grows exponentially in m. Hence the conditional distribution of $\overline{\mathbf{M}}_T$ can be inferred using the Kolmogorov equations even for m relatively large.
2) The form of the ODE-system for the backward equation for $\overline{\mathbf{M}}$ follows immediately from the definition of the generator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]}$; the ODE-system for the forward equation can be computed analogously to Lemma 2.2; see Lemma A.1 in the appendix.

[4]This distinction is made by Yu to single out a specific firm for future analysis.

Implications of exchangeability. We can infer individual default probabilities as well as within-group and between-group default correlations from the distribution of the random vector $\overline{\mathbf{M}}_T$ using the fact that within a given group the rv's $Y_T(i)$ are exchangeable under $K(\omega_1, d\omega_2)$ and hence under P. Hence we get for firms with $\kappa(i) = \kappa$ that $P\left(Y_T(i) = 1 \mid \overline{M}_{T,\kappa}\right) = \overline{M}_{T,\kappa}$, and for firms i, j with $i \neq j$, $\kappa(i) = \kappa(j) = \kappa$

$$P\left(Y_T(i) = 1, Y_T(j) = 1 \mid \overline{M}_{T,\kappa} = \frac{M}{m_\kappa}\right) = \frac{\binom{m_\kappa - 2}{M-2}}{\binom{m_\kappa}{M}} = \frac{M(M-1)}{m_\kappa(m_\kappa - 1)}, \tag{14}$$

provided that m_κ and $m_\kappa M \geq 2$; otherwise the lhs of (14) is obviously equal to zero. Finally, we have for obligors i, j belonging to different groups κ_1 and κ_2

$$P\left(Y_T(i) = 1, Y_T(j) = 1 \mid \overline{M}_{T,\kappa_1}, \overline{M}_{T,\kappa_2}\right) = \overline{M}_{T,\kappa_1} \cdot \overline{M}_{T,\kappa_2}$$

Hence we get for oligors i, j in group κ

$$P\left(Y_T(i) = 1\right) = E\left(P(Y_T(i) = 1 \mid \overline{M}_{T,\kappa})\right) = E(\overline{M}_{T,\kappa}), \tag{15}$$

$$P\left(Y_T(i) = 1, Y_T(j) = 1\right) = E\left(\overline{M}_{T,\kappa} \frac{m_\kappa \overline{M}_{T,\kappa} - 1}{m_\kappa - 1}\right), \tag{16}$$

and finally for obligors i, j from different groups κ_1 and κ_2

$$P\left(Y_T(i) = 1, Y_T(j) = 1\right) = E\left(\overline{M}_{T,\kappa_1} \overline{M}_{T,\kappa_2}\right) . \tag{17}$$

Note that for m_κ large the rhs of (16) is approximately equal to $E\left(\left(\overline{M}_{T,\kappa}\right)^2\right)$ for m_κ large. Of course, expressions similar to (16) can also be obtained for higher order default probabilities. More generally, we can even express the probability $P(\mathbf{Y}_T = \overline{\mathbf{y}})$ for some $\overline{\mathbf{y}} \in S$ in terms of the distribution of $\overline{M}_T$. As the distribution of $\mathbf{Y}_T$ is invariant under permutations of $\{1, \ldots, m\}$, which respect the homogeneous group structure, we have with $\bar{\boldsymbol{l}} := \overline{\mathbf{M}}(\overline{\mathbf{y}})$

$$P(\overline{\mathbf{M}}_T = \bar{\boldsymbol{l}}) = \operatorname{card}\{\mathbf{y} \in S : \overline{\mathbf{M}}(\mathbf{y}) = \bar{\boldsymbol{l}}\} P(\mathbf{Y}_T = \overline{\mathbf{y}}) = \binom{m_1}{m_1 \bar{l}_1} \cdots \binom{m_k}{m_k \bar{l}_k} P(\mathbf{Y}_T = \overline{\mathbf{y}}). \tag{18}$$

Of course, since the relations above depend only on the exchangeability of the default indicator processes of firms belonging to the same group, they hold also under the kernel $K(\omega_1, d\omega_2)$.

3.2 Limits for large portfolios

We now consider the limit (in the sense of convergence in distribution) of the model with k homogeneous groups as the size m of the portfolio tends to infinity, assuming that k remains fixed. It will turn out that in the limit the evolution of $\overline{\mathbf{M}}$ becomes deterministic given the evolution of the economic factor process $\mathbf{\Psi}$.

Our setup is as follows. Denote by $\Omega^{(m)} = \mathbf{D}\left([0, \infty), \mathbb{R}^d\right) \times \mathbf{D}([0, \infty), S^{(m)})$ the probability space in model m and define the filtrations $\{\mathcal{F}_t^m\}, \{\mathcal{F}_t^{i,m}\}$, $i = 1, 2$, and $\{\mathcal{G}_t^m\}$ in the obvious way. We assume that for each m the probability measure $P^{(m)} = \mu \times K^{(m)}$ satisfies Assumption 2.1; moreover, μ is assumed to be identical for all m. Denote by $m_\kappa^{(m)}$ the number of obligors in group κ of model m, define the process $\overline{\mathbf{M}}_t^{(m)}$ by $\overline{M}_{t,\kappa}^{(m)} = \overline{M}_\kappa\left(\mathbf{Y}_t^{(m)}\right)$, and assume that the transition rates have the group structure as in Assumption 3.1; in particular the default intensity of company i in model m equals $\lambda_i^{(m)}\big(\boldsymbol{\psi}, \mathbf{y}^{(m)}\big) = h_{\kappa(i)}^{(m)}\big(\boldsymbol{\psi}, \overline{\mathbf{M}}(\mathbf{y}^{(m)})\big)$. According to Lemma 3.4, for given ω_1 the process $\overline{\mathbf{M}}_t^{(m)}$ is Markov under the measure $K^{(m)}(\omega_1, d\omega_2)$. Put $\widetilde{\Omega}_2 := \mathbf{D}([0,\infty),[0,1]^k)$, and denote by $\widetilde{K}^{(m)}(\omega_1, d\widetilde{\omega}_2)$ the distribution of $\overline{\mathbf{M}}_t^{(m)}$ on $\widetilde{\Omega}_2$ under $K^{(m)}(\omega_1, d\omega_2)$.

Next we describe the limiting distribution of $\overline{\mathbf{M}}^{(m)}$. Suppose that for all $\kappa = 1, \dots, k$ the function $h_\kappa^{(m)}$ converges uniformly on compacts to some locally Lipschitz function $h_\kappa^{(\infty)} : \mathbb{R}^d \times [0,1]^k \to \mathbb{R}^+$. Denote by $\overline{\mathbf{M}}_t^{(\infty)}(\omega_1) = \big(\overline{M}_{t,1}^{(\infty)}(\omega_1), \dots, \overline{M}_{t,k}^{(\infty)}(\omega_1)\big)'$ the solution of the following system of ODE's with random coefficients

$$\frac{d}{dt}\,\overline{M}_{t,\kappa}^{(\infty)}(\omega_1) = \big(1 - \overline{M}_{t,\kappa}^{(\infty)}(\omega_1)\big)\, h_\kappa^{(\infty)}\left(\boldsymbol{\Psi}_t(\omega_1), \overline{\mathbf{M}}_t^{(\infty)}(\omega_1)\right), \tag{19}$$

with initial value $\overline{\mathbf{M}}_0^{(\infty)} = \bar{\boldsymbol{l}} \in [0,1]^k$. Note that for fixed $\omega_1 \in \Omega_1$ and $T > 0$ the rhs of (19) is Lipschitz in the second argument, since $[0,1]^k$ is compact and $h_\kappa^{(\infty)}$ is locally Lipschitz; hence a solution of (19) exists. For every ω_1 the trajectory $\big[t \mapsto \overline{\mathbf{M}}_t^{(\infty)}(\omega_1)\big]$ is an element of $\widetilde{\Omega}_2$. Denote by $\delta\big(\overline{\mathbf{M}}^{(\infty)}(\omega_1),\, d\widetilde{\omega}_2\big)$ the Dirac measure on $\widetilde{\Omega_2}$ in the point $\big[t \mapsto \overline{\mathbf{M}}_t^{(\infty)}(\omega_1)\big]$, and define a transition kernel $\widetilde{K}^{(\infty)}$ from Ω_1 to $\widetilde{\Omega_2}$ by $\widetilde{K}^{(\infty)}(\omega_1, d\widetilde{\omega}_2) := \delta\big(\overline{\mathbf{M}}^{(\infty)}(\omega_1),\, d\widetilde{\omega}_2\big)$. Now we have

Proposition 3.6. *Given a sequence of models as above, suppose that* $\lim_{m\to\infty} m_\kappa^{(m)} = \infty$ *for all* $\kappa = 1, \dots, k$ *and that* $\lim_{m\to\infty} \overline{\mathbf{M}}_0^{(m)} = \bar{\boldsymbol{l}}$*. Then for all* ω_1 *the measure* $\widetilde{K}^{(m)}(\omega_1, d\widetilde{\omega}_2)$ *converges weakly to* $\widetilde{K}_{\bar{\boldsymbol{l}}}^{(\infty)}(\omega_1, d\widetilde{\omega}_2)$.

Proof. Denote by $G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(m)}}$ the generator of $\overline{\mathbf{M}}^{(m)}$, and define for $f \in \mathcal{C}^1\left([0,1]^k\right)$ an operator

$$G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(\infty)}} f\left(\bar{\boldsymbol{l}}\right) = \sum_{r=1}^{k} \left(1 - \bar{l}_\kappa\right) h_\kappa^{(\infty)}(\boldsymbol{\psi}, \bar{\boldsymbol{l}}) \frac{\partial}{\partial \bar{l}_\kappa} f(\bar{\boldsymbol{l}}). \tag{20}$$

Note that $G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(\infty)}}$ is the generator of the process $\overline{\mathbf{M}}^{(\infty)}$ defined in (19). It follows from the Lipschitz continuity of $h_\kappa^{(\infty)}$ and the form of $G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(m)}}$ (see (13)), that for all $f \in \mathcal{C}^1\left([0,1]^k\right)$ and every compact set $K \subset \mathbb{R}^d$

$$\lim_{m\to\infty} \sup\left\{ \left| G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(m)}} f\left(\bar{\boldsymbol{l}}\right) \,-\, G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(\infty)}} f(\bar{\boldsymbol{l}}) \right| \,:\, \boldsymbol{\psi} \in K, \bar{\boldsymbol{l}} \in [0,1]^k \right\} = 0\,.$$

This implies that μ almost all ω_1 the transition semigroup of $\overline{\mathbf{M}}^{(m)}$ converges to the semigroup of $\overline{\mathbf{M}}^{(\infty)}$ by Ethier and Kurtz (1986), Theorem 1.6.1, so that the claim follows from Ethier and Kurtz (1986), Theorem 4.2.5. □

Note that the solution of (19) is deterministic given the trajectory $(\boldsymbol{\Psi}_t(\omega_1))_{t\ge 0}$. This shows that for $m \to \infty$ the proportion of defaulted companies is fully determined by the evolution of the economic factors. A similar result has been obtained among others by Frey and McNeil (2003) in the much simpler context of static Bernoulli mixture models for portfolio credit risk.

Next we show that the pair of processes $(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(m)})$ converges in distribution to $(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(\infty)})$.

Corollary 3.7. *Suppose that the hypothesises of Proposition 3.6 hold. Then the sequence $(\mathbf{\Psi}, \overline{\mathbf{M}}^{(m)})$ converges in distribution to $(\mathbf{\Psi}, \overline{\mathbf{M}}^{(\infty)})$, i.e. we have for every bounded and continuous function $F : \mathbf{D}\left([0,\infty), \mathbb{R}^d\right) \times \mathbf{D}\left([0,\infty), [0,1]^k\right) \to \mathbb{R}$*

$$\lim_{m\to\infty} E^{(m)}\left(F\big(\mathbf{\Psi}, \overline{\mathbf{M}}^{(m)}\big)\right) = \int_{\Omega_1} F\big(\mathbf{\Psi}(\omega_1), \overline{\mathbf{M}}^{(\infty)}(\omega_1)\big)\mu(d\omega_1)\,.$$

Proof. Denote by $\widetilde{\mathbf{Y}}$ the coordinate process on $\widetilde{\Omega}_2$. We have

$$E^{(m)}\left(F\big(\mathbf{\Psi}, \overline{\mathbf{M}}^{(m)}\big)\right) = \int_{\Omega_1}\int_{\widetilde{\Omega}_2} F\big(\mathbf{\Psi}(\omega_1), \widetilde{\mathbf{Y}}(\widetilde{\omega}_2)\big)\widetilde{K}^{(m)}(\omega_1, d\widetilde{\omega}_2)\mu(d\omega_1).$$

Now the inner integral on the rhs converges for μ almost all ω_1 to

$$\int_{\widetilde{\Omega}_2} F\left(\mathbf{\Psi}(\omega_1), \widetilde{\mathbf{Y}}(\widetilde{\omega}_2)\right)\widetilde{K}^{(\infty)}(\omega_1, d\widetilde{\omega}_2) = F\big(\mathbf{\Psi}(\omega_1), \overline{\mathbf{M}}^{(\infty)}(\omega_1)\big)$$

by Proposition 3.6. Hence the claim follows from the dominated convergence theorem. $\square$

Corollary 3.7 applies in particular to individual and joint default probabilities as given in (15), (16) or (17) or to prices of credit derivatives as discussed in Section 4.

Example 3.8. We now take up the affine model with counterparty risk introduced in Example 3.2. In order to apply Proposition 3.6, we assume that for all κ the proportion $\frac{m_\kappa^{(m)}}{m}$ of firms in group κ converges to some $\gamma_\kappa \in [0,1]$ as $m \to \infty$. This yields $h_\kappa^{(\infty)}(\boldsymbol{\psi}, \bar{\boldsymbol{l}}) = \left[\lambda_{\kappa,0} + \sum_{j=1}^d \lambda_{\kappa,j}\psi_j + \lambda_{\kappa,d+1}\sum_{r=1}^k \gamma_r\big(\bar{l}_r - (1 - e^{-\bar{\lambda}_r t})\big)\right]^+$, and $\overline{\mathbf{M}}^{(\infty)}$ solves the ODE

$$\frac{d}{dt}\,\overline{M}_{t,\kappa}^{(\infty)} = \big(1 - \overline{M}_{t,\kappa}^{(\infty)}\big)\left[\lambda_{\kappa,0} + \sum_{j=1}^d \lambda_{\kappa,j}\Psi_{t,j} + \lambda_{\kappa,d+1}\sum_{r=1}^k \gamma_r\big(\overline{M}_{t,r}^{(\infty)} - (1 - e^{-\bar{\lambda}_r t})\big)\right]^+. \quad (21)$$

Note that counterparty risk (a positive $\lambda_{\kappa,d+1}$) implies that deviations of $\sum_{r=1}^k \gamma_r \overline{M}_{t,r}^{(\infty)}$ from the expected level $\sum_{r=1}^k \gamma_r(1 - e^{-\bar{\lambda}_r t})$ will have a positive feedback effect on default intensities. Hence the fluctuations in the number of defaults caused by the random evolution of the economic factors are intensified by counterparty risk, so that we should expect heavier tails of the distribution of $\overline{M}_{t,\kappa}^{(\infty)}$. This is further illustrated in simulations in Section 5.

4 Pricing of credit derivatives

In this section we provide first results on the pricing of credit risky securities such as corporate bonds or other vulnerable securities in the context of our counterparty risk model. The main purpose is to show how Markov process techniques can be fruitfully employed to deal with pricing problems; a thorough study of pricing credit derivatives in the context of our model is deferred to further research.

Generalities. Following standard practice in the literature on reduced form credit risk models such as Lando (1998), we take as given a process $(r_t)_{t\in[0,\infty)}$ for the risk-free spot-rate of interest, a money market account B_t with $B_t = \exp(\int_0^t r_u du)$ and an equivalent martingale measure Q, which is used for pricing. We assume that the spot-interest rate is a function of the economic factors, i.e. $r_t = r(\mathbf{\Psi}_t)$ for some function $r : \mathbb{R}^d \to \mathbb{R}$, that Q

satisfies Assumption 2.1, and that $\mathbf{\Psi}$ is a Markov process with generator $\mathcal{L}^{\Psi}$, so that the pair $\mathbf{\Gamma} = (\mathbf{\Psi}, \mathbf{Y})$ is Markov. In this setting default-free zero coupon bond prices are given by

$$p_0(t,T) = E^Q\left(\exp\bigl(-\int_t^T r(\mathbf{\Psi}_s)ds\bigr) \mid \mathcal{F}_t\right),$$

and the price in t of any $\mathcal{F}_T$-measurable claim H is $H_t := E^Q\left(\exp\bigl(-\int_t^T r(\mathbf{\Psi}_s)ds\bigr)\, H \mid \mathcal{F}_t\right)$.

Pricing of vulnerable claims. Consider the pricing of a vulnerable claim of the form $H = f(\mathbf{\Psi}_T)g(\mathbf{Y}_T)$ for suitable functions $f : \mathbb{R}^d \to \mathbb{R}$ and $g : S \to \mathbb{R}$. A prime example is a defaultable zero coupon bond issued by firm i with zero recovery or more generally with recovery of treasury (RT) in the sense of Jarrow and Turnbull (1995) and deterministic recovery rate δ. For this claim we have $f(\boldsymbol{\psi}) = 1$ and $g(\mathbf{y}) = 1_{\{y(i)=0\}} + \delta 1_{\{y(i)=1\}}$.

Using the Markov-property of $\mathbf{\Gamma}_t = (\mathbf{\Psi}_t, \mathbf{Y}_t)$ we get for the price of our vulnerable claim in $t < T$

$$H_t = E^Q_{\mathbf{\Gamma}_t}\left(\exp\Bigl(-\int_0^{T-t} r(\mathbf{\Psi}_s)ds\Bigr)\, f(\mathbf{\Psi}_{T-t})g(\mathbf{Y}_{T-t})\right) =: H(t, \mathbf{\Psi}_t, \mathbf{Y}_t).$$

Now we have two possibilities for computing the function $H(t, \boldsymbol{\psi}, \mathbf{y})$. First we can try to solve directly the backward PDE for the Markov process $\mathbf{\Gamma}$ given by

$$\frac{\partial}{\partial t}H(t,\boldsymbol{\psi},\mathbf{y}) + \mathcal{L}^{\Psi}H(t,\boldsymbol{\psi},\mathbf{y}) + \mathcal{G}_{[\boldsymbol{\psi}]}H(t,\boldsymbol{\psi},\mathbf{y}) = r(\boldsymbol{\psi})H(t,\boldsymbol{\psi},\mathbf{y}), \quad H(T,\boldsymbol{\psi},\mathbf{y}) = f(\boldsymbol{\psi})g(\mathbf{y}).$$

In case that $\mathbf{\Psi}$ follows a diffusion this leads to a linear reaction-diffusion equation; existence results suitable for financial applications are for instance given in Becherer (2003). Alternatively, we may use a two-step approach, which uses only the Kolmogorov equations for the conditional transition probability. Here we get

$$H(t,\boldsymbol{\psi},\mathbf{y}) = E_{\boldsymbol{\psi}}\left(f\left(\mathbf{\Psi}_{T-t}\right)\exp\left(-\int_0^{T-t} r(\mathbf{\Psi}_s)ds\right) E^{K_{\mathbf{y}}(\omega_1,d\omega_2)}\left(g(\mathbf{Y}_{T-t})\right)\right).$$

The inner expectation can be computed using the backward equation for $\mathbf{\Psi}$ or in certain cases for $\overline{\mathbf{M}}$; the integral over Ω_1 is then computed using Monte Carlo simulation. This approach can be advantageous, if the direct numerical solution of the backward equation for $\mathbf{\Gamma}$ is infeasible, for instance because the dimension of the problem is too high.

Simplifications. Further simplifications are possible in the context of the mean-field model of Assumption 3.1. For example, if the payment is contingent on the survival of a particular firm i_0 from group κ_0, i.e. if $g(\mathbf{y}) = 1_{\{y(i_0)=0\}}$, we obtain from (15)

$$E^{K_{\mathbf{y}}(\omega_1,d\omega_2)}\left(1_{\{Y_{T-t}(i_0)=0\}}\right) = E^{K_{\mathbf{y}}(\omega_1,d\omega_2)}\left(\overline{M}_{T-t,\kappa_0}\right),$$

and the expectation on the right hand side can be computed using the backward equation for $\overline{\mathbf{M}}_t$, which leads to a substantial reduction of the size of the state space. In case that we are interested in the simultaneous computation of the prices of several claims with payoff $H = 1_{\{Y_{T_n}(i_0)=0\}}, 1 \leqslant n \leqslant N$ at times $t < T_1 < \cdots < T_N$, such as in the calibration of the model to a given term structure of defaultable zero coupon bonds, it may be advantageous to evaluate the expectations $E^{K_{\mathbf{y}}(\omega_1,d\omega_2)}\left(\overline{M}_{T_n,i_0}\right)$ using the Kolmogorov forward equation for $\overline{\mathbf{M}}_T$ (see Lemma 2.2), as this allows us to complete the different expectations “in one run”.

Remark 4.1. In certain special cases it is possible to obtain analytical expressions for prices of zero coupon bonds in Jarrow-Yu type models for counterparty risk using an ingenious change of measure; see Collin-Dufresne, Goldstein, and Hugonnier (2002). However, the approach works only for small portfolios and a deterministic factor process.

5 Case studies

We now present a number of simulations, which illustrate the impact of counterparty risk on default correlations and quantiles of $\overline{M}$ (the proportion of defaulted companies at some horizon date T) in the mean field model proposed in Section 3. For concreteness we work in the affine model with counterparty risk specified in Example 3.2. In all simulations we consider a homogeneous portfolio with only one group. The economic factor process is modelled as one-dimensional square-root process

$$d\Psi_t = 0.03(0.005 - \Psi_t)dt + 0.016\sqrt{\Psi_t}dW_t\,, \quad \Psi_0 = 0.005\,, \tag{22}$$

and the default intensity equals $h(t, \psi, \bar{l}) = \left[\alpha(0.004 + 5.707\psi) + \lambda_2(\bar{l} - (1 - e^{-\bar{\lambda}t}))\right]^+$; $\bar{\lambda} = 0.03251$ has been chosen so that $1 - e^{-\bar{\lambda}}$ corresponds to the one-year default probability without interaction. For $\lambda_2 = 0$ and $\alpha = 1$ our parameters are the same as in Section C.3 of Yu (2002); they have been taken from the empirical study by Driessen (2002).

We take the horizon to be $T = 1$ year. In our simulations we increase the parameter λ_2 which controls the strength of the interaction from 0 to 3 and adjust α in order to ensure that the one-year default probabilities $P(Y_1(i) = 1)$ remain unchanged as we vary λ_2. We consider portfolios of size $m = 20$, $m = 100$, $m = 500$ and, using the results from Section 3.2, the case $m = \infty$. The distribution of $\overline{M}_1$ is evaluated in two steps: first we simulate $K = 5000$ trajectories of the square root process (22); second we evaluate for each trajectory the conditional distribution of $\overline{M}_1$ by solving numerically the Kolmogorov forward equation using a Runge-Kutta method. The simulation results are presented in Tables 1, 2 and 3 below. Inspection of the tables shows the following observations.

- Quantiles and (except for $m = \infty$) default correlations $\rho_Y = \text{corr}(Y_1(i), Y_1(j), i \neq j$ are increasing in λ_2.
- The increase is more pronounced for smaller portfolios. For instance, for $m = 100$ the 99% quantile of $\overline{M}$ is increased by a factor of almost 4.75 as λ_2 increases from 0 to 3; for $m = \infty$ the factor is only about 1.64.

Both findings make perfect economic sense. In our counterparty risk model a higher (lower) than usual number of defaults in the portfolio leads to an increase (decrease) of the default intensity of the remaining firms in the portfolio and thus to a further increase (decrease) in the ratio of realized versus expected defaults, so that the resulting distribution of $\overline{M}_T$ will have more mass in the tails. Now in our model there are two reasons why the number of defaults should be higher than its theoretical value in the first place: a) we might have a high realization of $\boldsymbol{\Psi}$; b) for a given trajectory of $\boldsymbol{\Psi}$ we might have a realization of the Markov chain with unusually many defaults. As the limit results from Section 3.2 show, for $m \to \infty$ reason b) becomes less and less important, which explains, why the effect of mean-field interaction is more pronounced for small portfolios. However, as observed for instance by Schönbucher and Schubert (2001) and Yu (2002), the inability of the standard reduced form approach to generate sufficient dependency between default is most pronounced for small portfolios, so that our approach generates dependency where it is "most needed".

Note finally that for $m = \infty$ default correlations seem to vary only very little as λ_2 increases whereas quantiles change a lot, so that default probabilities and default correlations alone do not determine high quantiles of the distribution of $\overline{M}_T$. This is interesting, as it contrasts results of Frey and McNeil (2003) in the context of standard static credit risk models.

20 firms									
λ_2	α	$P(Y_1(i) = 1)$	ρ_Y	quantiles					
				80%	90%	95%	97.5%	99%	99.5%
0	1	0.031987	0.000416	0.05	0.1	0.1	0.15	0.15	0.15
0.5	1.0154	0.031979	0.033164	0.05	0.1	0.15	0.15	0.2	0.25
1	1.0365	0.031982	0.088386	0.05	0.1	0.15	0.2	0.3	0.35
1.5	1.0368	0.031979	0.16749	0.05	0.1	0.2	0.3	0.4	0.45
2	1.0063	0.031995	0.26113	0	0.1	0.25	0.35	0.5	0.55
2.5	0.9646	0.031994	0.36174	0	0.05	0.25	0.45	0.6	0.65
3	0.9249	0.031992	0.46202	0	0	0.3	0.5	0.65	0.75

Table 1: The case of 20 firms.

100 firms									
λ_2	α	$P(Y_1(i) = 1)$	ρ_Y	quantiles					
				80%	90%	95%	97.5%	99%	99.5%
0	1	0.031987	0.000416	0.05	0.06	0.06	0.07	0.08	0.09
0.5	1.003	0.031981	0.007453	0.05	0.06	0.08	0.09	0.1	0.11
1	1.007	0.031989	0.020918	0.05	0.07	0.09	0.11	0.13	0.15
1.5	0.9974	0.031995	0.044018	0.06	0.09	0.12	0.14	0.18	0.2
2	0.9612	0.031991	0.078737	0.06	0.1	0.15	0.18	0.23	0.26
2.5	0.9039	0.031996	0.1274	0.05	0.12	0.18	0.23	0.3	0.34
3	0.8353	0.031997	0.19118	0.02	0.12	0.21	0.29	0.38	0.43

Table 2: The case of 100 firms.

500 firms									
λ_2	α	$P(Y_1(i) = 1)$	ρ_Y	quantiles					
				80%	90%	95%	97.5%	99%	99.5%
0	1	0.031987	0.00041579	0.04	0.044	0.046	0.05	0.054	0.056
0.5	1.0004	0.031979	0.0019929	0.042	0.046	0.052	0.056	0.062	0.066
1	1.0014	0.03198	0.0050753	0.044	0.052	0.058	0.066	0.072	0.078
1.5	1.0003	0.031994	0.01093	0.048	0.06	0.07	0.078	0.09	0.098
2	0.9893	0.031992	0.020835	0.052	0.07	0.084	0.098	0.114	0.126
2.5	0.965	0.031993	0.036355	0.056	0.082	0.104	0.122	0.148	0.164
3	0.93	0.031992	0.058283	0.06	0.096	0.128	0.156	0.19	0.214

Table 3: The case of 500 firms.

limit of firms									
λ_2	α	$P(Y_1(i) = 1)$	ρ_Y	quantiles					
				80%	90%	95%	97.5%	99%	99.5%
0	1	0.031987	0.000416	0.03499	0.03666	0.03803	0.03926	0.04076	0.04204
0.5	1	0.031989	0.000413	0.03562	0.03766	0.03926	0.04086	0.04274	0.04403
1	1	0.03199	0.00041	0.03642	0.03898	0.04091	0.04292	0.04516	0.04669
1.5	1	0.031992	0.000405	0.03753	0.04074	0.04312	0.04554	0.04857	0.0505
2	0.9996	0.031993	0.000404	0.03906	0.04319	0.04624	0.04928	0.05304	0.05535
2.5	0.9981	0.031985	0.000421	0.04083	0.0463	0.05022	0.05433	0.0592	0.06256
3	0.9953	0.031982	0.000427	0.04324	0.05031	0.05541	0.06107	0.06694	0.0711

Table 4: The limiting case, where $m = \infty$.

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A Numerical Issues

A.1 Simulation

We now describe an approach how to simulate a trajectory of the process $\mathbf{\Gamma}_t$ with dynamics as in Assumption 2.1 and initial values $\mathbf{\Gamma}_0 = (\boldsymbol{\psi}^{(0)}, \mathbf{y}^{(0)})$ (where typically $\mathbf{y}^{(0)} = (0, \ldots, 0)'$) up to some finite horizon T. The approach follows the standard construction of (conditional) continuous time Markov chains as described for instance in Bielecki and Rutkowski (2002).

First we simulate a trajectory $(\mathbf{\Psi}_t)_{0 \leqslant t \leqslant T}$ of $\mathbf{\Psi}$. Depending on the specific model for $\mathbf{\Psi}$ various approaches can be used; see for instance Duffie and Singleton (1998) or, if $\mathbf{\Psi}$ follows a diffusion process, Kloeden and Platen (1992). In most cases a simple Euler approximation will be sufficient.

Next we have to simulate our first default time $\tau^{(1)} := \inf\{\tau_i : 1 \leqslant i \leqslant m\}$. It is well-known that $\tau^{(1)}$ has hazard-rate process $\lambda_t^{(1)} = \sum_{i=1}^m \big(1 - y^{(0)}(i)\big)\lambda_i(\mathbf{\Psi}_t, \mathbf{y}^{(0)})$. Hence we simply simulate a unit exponential random variable θ_1 independent of $\mathbf{\Psi}$ and put

$$\tau^{(1)} := \inf\Big\{t \geqslant 0 : \int_0^t \lambda_s^{(1)} ds \geqslant \theta_1\Big\}.$$

Now we have to determine the identity $\xi^{(1)} \in \{1, \dots, m\}$ of the firm defaulting in $\tau^{(1)}$. It is shown for instance in Bielecki and Rutkowski (2002) that

$$P(\xi^{(1)} = i \mid \tau^{(1)} = t) = \frac{\big(1 - y^{(0)}(i)\big)\lambda_i(\mathbf{\Psi}_t, \mathbf{y}^{(0)})}{\sum_{j=1}^m \big(1 - y^{(0)}(j)\big)\ \lambda_j(\mathbf{\Psi}_t, \mathbf{y}^{(0)})} =: p_i^{(1)};$$

moreover, given the probabilities $p_i^{(1)}$ the rv $\xi^{(1)}$ is independent of $\mathbf{\Psi}$ and $\tau^{(1)}$. Hence $\xi^{(1)}$ can be simulated as realisation of a random variable ξ, independent of all other variables simulated so far, with $P(\xi = i) = p_i^{(1)}$ for $1 \leqslant i \leqslant m$.

In case that $\tau^{(1)} \geqslant T$ we have accomplished our task and stop. Else we define the vector $\mathbf{y}^{(1)} := (\mathbf{y}^{(0)})^{\xi^{(1)}}$ (recall the notational convention (1)) and for $t \geqslant \tau^{(1)}$ the process $\lambda_t^{(2)} = \sum_{j=1}^m (1 - y^{(1)}(j))\lambda_j(\boldsymbol{\psi}_t, \mathbf{y}^{(1)})$. In analogy to the previous step we put $\tau^{(2)} := \inf\{t \geqslant \tau^{(1)} : \int_{\tau^{(1)}}^t \lambda_s^{(2)} ds \geqslant \theta_2\}$, where θ_2 is again a unit exponential rv, independent of all other variables. $\xi^{(2)}$, the identity of the firm defaulting at time $\tau^{(2)}$, is determined as before, using the identity

$$P(\xi^{(2)} = i \mid \tau^{(2)} = t) = \frac{\big(1 - y^{(1)}(i)\big)\ \lambda_i(\mathbf{\Psi}_t, \mathbf{y}^{(1)})}{\lambda_t^{(2)}}.$$

The algorithm proceeds this way until we have reached some j with $\tau^{(j)} \geqslant T$ or until all companies are default. For typical parameter values we will observe only a few defaults, so that the algorithm to generate a trajectory of $\mathbf{Y}$ proceeds quite fast. Moreover, as pointed out by Duffie and Singleton (1998), the algorithm described above is also an efficient procedure for simulating reduced-form models with conditionally independent defaults.

A.2 Forward equations

Proof of Lemma 2.2. We identify $G_{[\boldsymbol{\psi}]}$ with an $\big|S\big| \times \big|S\big|$ matrix $(\Lambda_{ij}(t \mid \omega_1))_{1 \le i,j \le |S|}$; $G^*_{[\boldsymbol{\psi}]}$ corresponds then to the transpose matrix. For this we choose a bijection $I : \{1, \dots, |S|\} \to S$, $i \mapsto \mathbf{y}_i$. By definition of the generator of $\mathbf{Y}$ we have for $i \neq j$

$$\Lambda_{ij}(t \mid \omega_1) = \begin{cases} (1 - y_i(k))\lambda_k(\mathbf{\Psi}_t(\omega_1), \mathbf{y}_i), & \text{if } \mathbf{y}_j = \mathbf{y}_i^k \text{ for some } k \in \{1, \dots, m\}, \\ 0 \text{ else}. \end{cases} \tag{23}$$

For $i = j$ we put $\Lambda_{ii}(t \mid \omega_1) = -\sum_{j \le |S|, j \neq i} \Lambda_{ij}(t \mid \omega_1)$, so that

$$\Lambda_{ii}(t \mid \omega_1) = -\sum_{k=1}^m (1 - y_i(k))\lambda_k(\mathbf{\Psi}_t(\omega_1), \mathbf{y}_i). \tag{24}$$

Now fix $\mathbf{y} = I(j_0) \in S$. Since $G^*_{[\mathbf{\Psi}_t(\omega_1)]}$ corresponds to multiplication with the transpose matrix $(\Lambda^*_{ij}(t \mid \omega_1))_{1 \leqslant i,j \leqslant |S|}$, the forward equation becomes

$$\frac{\partial p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)}{\partial s} = \sum_{i=1}^{|S|} \Lambda_{ij_0}(s \mid \omega_1) p(t, s, \mathbf{x}, \mathbf{y}_i \mid \omega_1).$$

Using the definition of $\Lambda_{ij}(s \mid \omega_1)$ in (23) and (24) and the relation $(1 - y^k(k)) = y(k)$ we obtain the final version (9) of the forward equation. $\square$

Next we consider forward equations for $\overline{\mathbf{M}}_t$. We have

Lemma A.1. *Assume that the default intensities satisfy Assumption 3.1. Then the adjoint operator $G^{*\overline{\mathbf{M}}}_{[\boldsymbol{\Psi}_t(\omega_1)]}$ to the generator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\Psi}_t(\omega_1)]}$ of $\overline{\mathbf{M}}_t$ is given by*

$$
\begin{aligned}
G^{*\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]} f\,(\bar{\boldsymbol{l}}) = &\sum_{\kappa=1}^{k} 1_{\{\bar{l}_\kappa > 0\}} \big(1 + m_\kappa(1 - \bar{l}_\kappa)\big) h_\kappa\big(\boldsymbol{\psi}, \bar{\boldsymbol{l}} - \frac{1}{m_\kappa}\mathbf{e}_\kappa\big) f\big(\bar{\boldsymbol{l}} - \frac{1}{m_\kappa}\mathbf{e}_\kappa\big) \\
&- \sum_{\kappa=1}^{k} m_\kappa (1 - \bar{l}_\kappa) h_\kappa\left(\boldsymbol{\psi}, \bar{\boldsymbol{l}}\right) f(\bar{\boldsymbol{l}})\,.
\end{aligned}
\tag{25}
$$

Proof. As in the proof of Lemma 2.2 we define a $\left|S^{\overline{M}}\right| \times \left|S^{\overline{M}}\right|$ matrix $\left(\Lambda_{ij}(t|\omega_1)\right)_{i,j=1,\ldots,\left|S^{\overline{M}}\right|}$ and identify the generator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]}$ with the matrix through a bijection $I : \{1, 2, \ldots, \left|S^{\overline{M}}\right|\} \to S^{\overline{M}}$, $I(i) = \bar{\boldsymbol{l}}^{(i)}$. By the definition of the generator of $\overline{\mathbf{M}}$ (see Lemma 3.4) we have for $i \neq j$

$$
\Lambda_{ij}(t \mid \omega_1) = m_\kappa(1 - \bar{l}^{(i)}_\kappa) h_\kappa(\boldsymbol{\Psi}_t, \bar{\boldsymbol{l}}^{(i)}),
\tag{26}
$$

if there is a $\kappa \in \{1, \cdots, k\}$ with $\bar{l}^{(j)}_\kappa = \bar{l}^{(i)}_\kappa + \frac{1}{m_\kappa}$ and $\bar{l}^{(j)}_\gamma = \bar{l}^{(i)}_\gamma$ for $\gamma \neq \kappa$, and $\Lambda_{ij}(t \mid \omega_1) = 0$ else; for $i = j$ we put $\Lambda_{ii}(t \mid \omega_1) = -\sum_{\substack{j=1\\ j\neq i}}^{\left|S^{\overline{M}}\right|} \Lambda_{ij}(t \mid \omega_1) = -\sum_{\kappa=1}^{k} m_\kappa(1 - \bar{l}^{(i)}_\kappa) h_\kappa(\boldsymbol{\Psi}_t, \bar{\boldsymbol{l}}^{(i)})$.

The generator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]}$ corresponds to multiplication with the matrix $\left(\Lambda_{ij}(t|\omega_1)\right)_{i,j=1,\ldots,\left|S^{\overline{M}}\right|}$ and the operator $G^{*\overline{\mathbf{M}}}_{[\boldsymbol{\Psi}_t(\omega_1)]}$ to multiplication with the transpose matrix $\left(\Lambda^*_{ij}(t|\omega_1)\right)_{i,j=1,\ldots,\left|S^{\overline{M}}\right|}$. For a fixed $\bar{\boldsymbol{l}}^{(0)} = I(j_0)$ and a arbitrary $\bar{\boldsymbol{l}}^{(1)} \in S^{\overline{M}}$ we therefore have

$$
\begin{aligned}
G^{*\overline{\mathbf{M}}}_{[\boldsymbol{\Psi}_t(\omega_1)]} f(\bar{\boldsymbol{l}}^{(0)}) &= \sum_{i=1}^{\left|S^{\overline{M}}\right|} \Lambda_{ij_0}(s \mid \omega_1) f(\bar{\boldsymbol{l}}^{(0)}) \\
&= \sum_{\kappa=1}^{k} 1_{\{\bar{l}^{(0)}_\kappa > 0\}} \big(1 + m_\kappa(1 - \bar{l}^{(0)}_\kappa)\big) h_\kappa\big(\boldsymbol{\Psi}_s, \bar{\boldsymbol{l}}^{(0)} - \frac{1}{m_\kappa}\mathbf{e}_\kappa\big) f\big(\bar{\boldsymbol{l}}^{(0)} - \frac{1}{m_\kappa}\mathbf{e}_\kappa \mid \omega_1\big) \\
&\quad - \sum_{\kappa=1}^{k} m_\kappa(1 - \bar{l}^{(0)}_\kappa) h_\kappa\big(\boldsymbol{\Psi}_s, \bar{\boldsymbol{l}}^{(0)}\big) f\big(\bar{\boldsymbol{l}}^{(0)} \mid \omega_1\big)\,.
\end{aligned}
$$

$\square$

The precise form of the forward equation for the transition probabilities of $\overline{\mathbf{M}}_t$ is now immediate.

Usually the probability to have many defaults is very small, especially if we have a large number of firms, i.e. $P(\overline{\mathbf{M}}_T \geq \alpha) \approx 0$ for $\alpha \in [0, 1]$ large enough. Therefore it is useful for the numerical solution of the backward or forward Kolmogorov equation to stop the process $\overline{\mathbf{M}}$ at a level α.

In the following discussion we take $k = 1$; similar simplification can be obtained in the case with more than one groups. The cardinality of the state space for the stopped process is only $\lceil \alpha m \rceil + 1$ instead $m + 1$ of the original process; moreover, as the process $\overline{\mathbf{M}}_t$ is strictly increasing in t we have for $\bar{l} < \alpha$ the equality $P\big(\overline{\mathbf{M}}_t \leq \bar{l}\big) = P\big(\overline{\mathbf{M}}^\alpha_t \leq \bar{l}\big)$, where $\overline{\mathbf{M}}^\alpha_t$ is the stopped process. Moreover, we obtain the stopped process by replacing the default intensity $\tilde{h}(\boldsymbol{\psi}, \bar{l})$ for individual firms with the new default intensity

$$\tilde{h}(\boldsymbol{\psi}, \bar{l}) = 1_{\{\bar{l}<\alpha\}} h(\boldsymbol{\psi}, \bar{l});$$

the equation for the forward and backward operator of the stopped process follows immediately from this representation. Moreover, we have the following error estimate for the expectation

$$E\overline{\mathbf{M}}_t - E\overline{\mathbf{M}}_t^\alpha \leq (1 - \frac{\lceil \alpha m \rceil}{m}) P(\overline{\mathbf{M}}_t^\alpha = \frac{\lceil \alpha m \rceil}{m}), \tag{27}$$

where $\lceil x \rceil := \min\{n \in \mathbb{N}, n \geq x\}$. Relation (27) can be verified as follows. We have

$$\begin{aligned}
E\overline{\mathbf{M}}_t - E\overline{\mathbf{M}}_t^\alpha &= \sum_{i=\lceil \alpha m \rceil}^{m} \frac{i}{m} P(\overline{\mathbf{M}}_t = \frac{i}{m}) - \frac{\lceil \alpha m \rceil}{m} P(\overline{\mathbf{M}}_t \geq \frac{\lceil \alpha m \rceil}{m}) \\
&\leq \sum_{i=\lceil \alpha m \rceil}^{m} P(\overline{\mathbf{M}}_t = \frac{i}{m}) - \frac{\lceil \alpha m \rceil}{m} P(\overline{\mathbf{M}}_t \geq \frac{\lceil \alpha m \rceil}{m}) \\
&= (1 - \frac{\lceil \alpha m \rceil}{m}) P(\overline{\mathbf{M}}_t \geq \frac{\lceil \alpha m \rceil}{m}).
\end{aligned}$$