Chapter 14

How the Greeks would have Hedged Correlation Risk of Foreign Exchange Options[1]

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We show how to compute correlation coefficients in an n-dimensional geometric Brownian motion model for Foreign Exchange (FX) rates, interpret the result geometrically and apply it to eliminate correlation risk when trading multi-asset options.

14.1 Introduction

A multi-asset option is a derivative security whose payoff depends on the future values of several possibly correlated underlying assets. Typical examples are quanto, basket, spread, outside barrier options and options on the minimum/maximum of several assets. The greatest risk in pricing such options lies in the choice of the input parameter correlation, which is not directly observable in market prices unless many rainbow options are traded actively on an exchange. The other market input parameters, interest rates and volatilities, can be obtained from and hedged in the money market and vanilla options market. In an FX market the correlation structure can be computed explicitly in terms of known volatilities – using the interdependence of exchange rates. This implies in particular that the correlation risk of multi exchange rate options can be hedged simply by trading FX volatility.

14.2 Foreign Exchange market model

We assume a constant coefficient geometric Brownian motion in n dimensions

$$dS_t^{(i)} = S_t^{(i)}[\mu_i\, dt + \sigma_i\, dW_t^{(i)}], \quad i = 1, \ldots, n \tag{14.1}$$

where μ_i is the rate of appreciation and σ_i is the volatility of FX rate i, and the correlation matrix $\hat{\rho} = (\rho_{ij})_{i,j=1,\ldots,n}$ is defined by $\mathbf{Cov}(\ln S_t^{(i)}, \ln S_t^{(j)}) = \sigma_i\sigma_j\,\rho_{ij}\,t$. To illustrate the main idea we consider the example of a triangular FX market $S_t^{(1)}$

(£/US\$), $S_t^{(2)}$ (US\$/¥) and $S_t^{(3)}$ (£/¥) satisfying $S_t^{(1)} S_t^{(2)} = S_t^{(3)}$ Computing the variances of the logarithms on both sides yields

$$\rho_{12} = \frac{\sigma_3^2 - \sigma_1^2 - \sigma_2^2}{2\sigma_1\sigma_2} \tag{14.2}$$

This calculation can be repeated and thus visualised in elementary geometry. Labelling the corners of a triangle £, US\$, ¥, the vectors of the edges $\vec{\sigma_1}$ from £ to US\$, $\vec{\sigma_2}$ from US\$ to ¥, $\vec{\sigma_3}$ from £ to ¥, such that $|\vec{\sigma_i}| = \sigma_i$, the law of cosine states that $\cos\phi_{12} = \rho_{12}$, where $\pi - \phi_{12}$ is the angle of the triangle in the US\$-corner. This way edge lengths can be viewed as volatilities and (cosines of) angles as correlations. The correlation structure turns out to be fully determined by the volatilities. Consequently, we *do not* need to estimate correlation coefficients and we can hedge correlation risk merely by trading volatility. Therefore we expect a fairly tight multi exchange rate options market to develop, because the major uncertainty correlation can be completely erased. This may not work in the equity, commodity or fixed income market, although we can imagine admitting one non-currency as long as both domestic and foreign prices and volatilities are available.

14.3 The extension beyond triangular markets

Now we solve the slightly harder question of how to obtain a correlation coefficient of two currency pairs which do not have a common currency. For instance, to compute the correlation between £/¥ and €/US\$, we inflate the market of these two currency pairs to the following market of six currency pairs $S_t^{(1)}$ (£/US\$), $S_t^{(2)}$ (US\$/¥), $S_t^{(3)}$ (£/¥), $S_t^{(4)}$ (€/US\$), $S_t^{(5)}$ (€/£) and $S_t^{(6)}$ (€/¥), denoting by σ_i the volatility of the exchange rate $S_t^{(i)}$. Geometrically we introduce a tetrahedron with triangular sides whose corners are the four currencies £, US\$, ¥, €. The six edge vectors will be labelled $\vec{\sigma_i}$ with lengths σ_i and the direction is always from FX1 to FX2 if the FX rate is named FX1/FX2. We need to determine the $\binom{6}{2} = 15$ correlation coefficients ρ_{ij}, $i = 1, \ldots, 5$, $j = i+1, \ldots, 6$, out of which $12 = 3 \times 4$ are available as

Table 14.1 Matrix of correlation coefficients for a four currency market.

	1	2	3	4	5	6
vol	7.50%	13.45%	14.50%	13.00%	11.65%	16.85%
	£/US\$	US\$/¥	£/¥	€/US\$	€/£	€/¥
£/US\$	1.0000	−0.1333	0.3936	0.4591	−0.1315	0.2478
US\$/¥	−0.1333	1.0000	0.8586	−0.1887	−0.1247	0.6527
£/¥	0.3936	0.8586	1.0000	0.0625	−0.1837	0.7336
€/US\$	0.4591	−0.1887	0.0625	1.0000	0.8203	0.6209
€/£	−0.1315	−0.1247	−0.1837	0.8203	1.0000	0.5333
€/¥	0.2478	0.6527	0.7336	0.6209	0.5333	1.0000

described in the above triangular market. The remaining correlation coefficients ρ_{34}, ρ_{25} and ρ_{16}, can be obtained by using one of the obvious relations

$$\mathbf{Cov}(\ln S_t^{(3)}, \ln S_t^{(4)}) = \mathbf{Cov}(\ln S_t^{(3)}, \ln S_t^{(6)}) - \mathbf{Cov}(\ln S_t^{(3)}, \ln S_t^{(2)}) \tag{14.3}$$

which results in

$$\rho_{34} = \frac{\sigma_1^2 + \sigma_6^2 - \sigma_2^2 - \sigma_5^2}{2\sigma_3\sigma_4} \tag{14.4}$$

The hardest part here is to remember how the signs go, but this will, of course, depend on how one sets up the directions of the volatility vectors or on the style FX rates are quoted in the market. But in principle, we can solve any correlation problem in the FX market by the same method. Also note that given a term structure of volatility one can create a corresponding term structure of correlation.

14.4 Geometric interpretation

If we introduce angles ϕ_{34}, ϕ_{25} and ϕ_{16} via $\cos\phi_{ij} = \rho_{ij}$, we can interpret the three opposite edge correlations as the angles between the three pairs of skew lines generated by the vectors of the tetrahedron. Note that Equation (14.4) is also worth noting as a purely geometric result about tetrahedra, which one might want to name the law of cosine in a tetrahedron, see Figure 14.1.

As a result we find that the correlation structure uniquely corresponds to visible angles of a tetrahedron, where – as often is the case – orthogonality in geometry corresponds to independence in stochastics. We also observe that the FX market of six currency pairs we considered is actually only three-dimensional. Let us point out that there may be a generalisation to n dimensions in mathematics, but for practical use a three-dimensional market suffices.

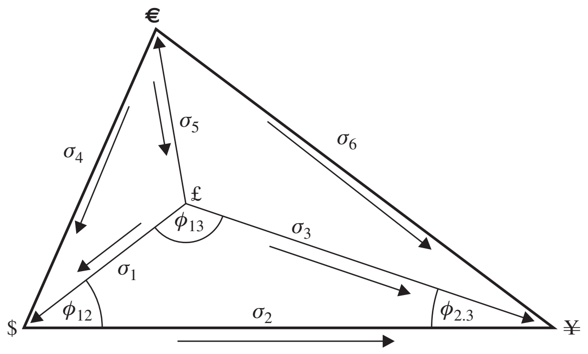

Figure 14.1 Four currency market volatilities represented as a tetrahedron.

14.5 Hedging correlation risk

Computing correlation coefficients based on volatilities as above has a striking implication. Any FX correlation risk can be transformed into a volatility risk.

Table 14.2 Correlation hedge for an outside down-and-out put option.

€/US$ Spot	0.9200	Value (in US$)	0.0111	Simple hedge	Adjusted hedge
£/¥ spot	167.50	correlation risk		−0.0708	0.0000
€/US$ strike	0.9200				
£/¥ barrier	155.00	currency pairs	vols	simple vega	adjusted vega
time (days)	180	£/US$	7.50%	0.0000	−0.2816
correlation	6.25%	US$/¥	13.45%	0.0000	0.5050
US$ rate	5.50%	£/¥	14.50%	−0.0495	−0.0190
€ rate	3.00%	€/US$	13.00%	0.0531	0.0871
¥ rate	0.50%	€/£	11.65%	0.0000	0.4374
£ rate	6.00%	€/¥	16.85%	0.0000	−0.6327

More precisely, suppose we are given an option value function $R(\vec{\sigma}, \hat{\rho})$, then, as we found out, the correlation matrix $\hat{\rho}$ is a redundant parameter which can be written as $\hat{\rho} = \hat{\rho}(\vec{\sigma})$. We can therefore write the value function as $H(\vec{\sigma}) \stackrel{\triangle}{=} R(\vec{\sigma}, \hat{\rho}(\vec{\sigma}))$. The correlation risk can now be hedged by replacing simple vegas $\dfrac{\partial R}{\partial \sigma_i}$ by adjusted vegas of the value function

$$\frac{\partial H}{\partial \sigma_i} = \frac{\partial R}{\partial \sigma_i} + \sum_{j=1}^{5} \sum_{k=j+1}^{6} \frac{\partial R}{\partial \rho_{jk}} \frac{\partial \rho_{jk}}{\partial \sigma_i} \tag{14.5}$$

This also means that the correlation risk of an option on the currency pairs €/US$ and £/¥ can be transferred to volatility risk in all six participating currency pairs, not just in the two €/US$ and $/¥. However, hedging six vegas may be still easier than hedging two vegas and one correlation risk. This is illustrated in the example of an outside barrier option in Table 14.2. The computation is based on Heynen [1].

An alternative approach to hedging correlation risk can be found in Chapter 13, where an explicit relation between correlation risk and cross gamma is presented.

1 This article first appeared in *Wilmott Research Report*, August 2001.

References

[1] **Heynen, R., and H. Kat,** 1994, "Crossing Barriers", *Risk*, 7: 6, pp. 46–51.

[2] **Reiss, O., and U. Wystup,** 2000, "Efficient Computation of Option Price Sensitivities Using Homogeneity and other Tricks", Preprint 584 Weierstrass-Insitite Berlin, URL: http://www.wias-berlin.de/publications/preprints/584.