Pricing Installment Options with an Application to ASX Installment Warrants

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Abstract

Installment options are Bermudan-style options where the holder periodically decides whether to exercise or not and then to keep the option alive or not (by paying the installment). We develop a dynamic programming procedure to price installment options. We derive the range of installments within which the installment option is not redundant with the European contract. Simulations analysis shows the method yields monotonically converging prices, and satisfactory trade-offs between accuracy and computational time. In addition, we examine the flexibility in installment option design that yields various hedging properties. Our approach is applied to installment warrants, which are actively traded on the Australian Stock Exchange. Numerical investigation shows the various capital dilution effects resulting from different installment warrant designs.

1 Introduction

Installment Options (IO) are akin to Bermudan options except that the holder must regularly pay a premium (the "installment") to keep the option alive. The pre-specified dates (thereafter "decision dates") at which the IO may be striked correspond to the installment schedule. Therefore, at each decision date, the holder of the IO must choose between the following

1. to exercise the option, which puts an end to the contract;
2. not to exercise the option and to pay the installment, which keeps the option alive till the next decision date;
3. not to exercise the option and not to pay the installment, which puts an end to the contract.

One of the most actively traded installment options throughout the world are currently the installment warrants on Australian stocks listed on the Australian Stock Exchange (ASX). Installment options are a recent financial innovation that introduces some flexibility in the liquidity management of portfolio strategies. Instead of paying a lump sum for a derivative instrument, the holder of the IO will pay the installments as long as the need for being long in the option is present. In particular, this considerably reduces the cost of entering into a hedging strategy.[1] In addition, the non-payment of an installment suffices to close the position at no transaction cost. This

[1] Risk managers may enter the IO contract at a low initial cost and adjust the installment schedule with respect to their cash forecasts and liquidity constraints. This feature is particularly attractive for corporations which massively hedge interest rate and currency risks with forwards, futures or swaps because standard option contracts imply a cost at entry that may be incompatible with a temporary cash shortage.

reduces the liquidity risk typically associated with other over-the-counter derivatives. Specifically for ASX installment warrants, another advantage is that their holders are entitled to full dividends during the whole life of the product. Also, investors may lodge their shares in return for installments, thereby extracting cash to diversify their portfolios without losing exposure to their shares.

The aim of this paper is threefold. First, we tackle the problem of pricing IOs using Dynamic Programming (DP). Second, we investigate the properties of IOs through theoretical and numerical analysis. Finally, we provide an adaptation of our methodology for ASX installment warrants.

Literature on IOs is scarce. The only research paper we are aware of is that of Davis, Schachermayer and Tompkins (2001). They derive no-arbitrage bounds for the price of the IO and study static versus dynamic hedging strategies within a Black-Scholes framework with stochastic volatility. Their analysis however is restricted to European-style IOs, which allows for an analogy with compound options.

Algorithms based on finite differences have been widely used for pricing options with no known closed-form solution (see e.g. Wilmott, Dewynne and Howison (1993) for a survey). Recently, dynamic programming combined with finite elements has emerged as an alternative for low dimensional option pricing. By contrast to finite difference methods, DP does not require time discretization. Ben Ameur, Breton and L'Écuyer (2002) show this method is particularly well suited for options involving decisions at a limited number of distant dates during the life of the contract. Examples include Bermudan-style options, callables, and convertibles. By construction, IOs allow for both early exercise and installment payment decisions periodically.

The rest of the paper is organized as follows. In section 2, we develop the model in the Black-Scholes setting. In section 3, we solve the Bellman equation and show how to go back through the DP induction. Properties of the value function are derived in section 4. We present simulations analysis in section 5. Adaptation of the methodology to ASX installment warrants is provided in section 6. Section 7 concludes.

2 The model

We consider a Black-Scholes economy. Agents may lend or borrow freely at the constant riskless rate r. The price of the underlying asset $\{S\}$ satisfies the following Stochastic Differential Equation (SDE) under the risk-neutral probability measure

$$dS_t = (r - \delta) S_t dt + \sigma S_t dB_t, \quad \text{for } 0 \leq t \leq T,$$

where δ is the dividend rate, σ the volatility of the return on the underlying asset, and $\{B\}$ a standard Brownian motion. The solution to this SDE is the well-known geometric Brownian motion

$$S_{t''} = S_{t'} \exp\left(\left(r - \delta - \sigma^2/2\right) \Delta t + \sigma \sqrt{\Delta t} Z\right), \quad \text{for } 0 \leq t' \leq t'' \leq T, \quad (1)$$

where $\Delta t = t'' - t'$ and Z is a standard normal random variable independent of the past of $\{S\}$ up to time t'.

Let $t_0 = 0$ be the installment option (IO) inception date and $t_1, t_2, ..., t_n = T$ a collection of decision dates scheduled in the contract. For simplicity, assume that these dates are equally spaced. An installment design is characterized by the vector of premia $\pi = (\pi_1, ..., \pi_{n-1})$ that are to be paid by the holder at dates $t_1, ..., t_{n-1}$ to keep the IO alive.[2] The price of the IO is the upfront payment v_0 required at t_0 to enter the contract.

The *exercise value* of the IO at the decision date t_m, for $m = 1, \ldots, n$, is explicit in the contract and given by

$$v_m^e(s) = \begin{cases} \max(0, s - K), & \text{for an installment call option} \\ \max(0, K - s), & \text{for an installment put option} \end{cases}, \qquad (2)$$

where $s = S_{t_m}$ is the price of the underlying asset at t_m. By the risk-neutral principle, the *holding value* of the option at t_m is

$$v_m^h(s) = E[e^{-r\Delta t} v_{m+1}(S_{t_{m+1}}) \mid S_{t_m} = s], \quad \text{for } m = 0, \ldots, n-1, \qquad (3)$$

where

$$v_m(s) = \begin{cases} v_0^h(s) & \text{for } m = 0 \\ \max\left(v_m^e(s), v_m^h(s) - \pi_m\right) & \text{for } m = 1, \ldots, n-1 \\ v_0^e(s) & \text{for } m = n \end{cases} \qquad (4)$$

The function of $v_m^h(s) - \pi_m$ is called thereafter the *net holding value* at t_m, for $m = 1, \ldots, n-1$.

One way of pricing this IO is via backward iteration: from the known function $v_n = v_n^e$ and using (2)-(4), compute v_{n-1}, then from v_{n-1} compute v_{n-2}, and so on, down to v_0. However, the value function v_m, for $m = 0, \ldots, n-1$, is not known and must be approximated in some way. We propose an approximation method in Section 3 which allows to solve the DP equation (3) in a closed-form for all s and m.

[2] Note that the design $\pi = 0$ corresponds to the case of a Bermudan option.

3 Solving the DP equation

In this section, we compute the expectation in (3). The idea is to partition the positive real axis into a collection of intervals and then to approximate the option value by a piecewise linear interpolation. This yields a closed-form solution to the DP equation (3).

Let $a_0 = 0 < a_1 < \ldots < a_p < a_{p+1} = +\infty$ be a set of points and $R_0, \ldots, R_p$ be a partition of $\mathbb{R}$ into $(p+1)$ intervals

$$R_i = (a_i, a_{i+1}] \quad \text{for } i = 0, \ldots, p.$$

Given an approximation $\widetilde{v}_m$ of the option value v_m at the points a_i and step m, this function is interpolated piecewise linearly, which yields

$$\widehat{v}_m(s) = \sum_{i=0}^{p} (\alpha_i^m + \beta_i^m s)\, I(a_i < s \le a_{i+1}), \tag{5}$$

where I is an indicator function. The local coefficients of this interpolation at step m, that is the α_i^m's and the β_i^m's, are obtained by solving the linear equations

$$\widetilde{v}_m(a_i) = \widehat{v}_m(a_i), \quad \text{for } i = 0, \ldots, p-1.$$

For $i = p$, we take

$$\alpha_p^m = \alpha_{p-1}^m \text{ and } \beta_p^m = \beta_{p-1}^m.$$

Assume now that $\widehat{v}_{m+1}$ is known. Given (1), the expectation in (3) at step m becomes

$$\begin{aligned} & \widetilde{v}_m^h(a_k) \qquad (6) \\ = \;& E\left[e^{-r\Delta t}\widehat{v}_{m+1}\left(S_{t_{m+1}}\right) \mid S_{t_m} = a_k\right] \\ = \;& e^{-r\Delta t}\sum_{i=0}^{p}\alpha_i^{m+1}E\left[I\left(\frac{a_i}{a_k} < e^{\mu\Delta t+\sigma\sqrt{\Delta t}Z} \le \frac{a_{i+1}}{a_k}\right)\right] + \\ & \beta_i^{m+1}a_k E\left[e^{\mu\Delta t+\sigma\sqrt{\Delta t}Z} I\left(\frac{a_i}{a_k} < e^{\mu\Delta t+\sigma\sqrt{\Delta t}Z} \le \frac{a_{i+1}}{a_k}\right)\right], \end{aligned}$$

where $\mu = r - \delta - \sigma^2/2$, and $\widetilde{v}_m^h$ denotes the approximate holding value of the IO.

For $k = 1, \ldots, p$ and $i = 0, \ldots, p$, the first integrals

$$A_{k,i} = E\left[I\left(\frac{a_i}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t}Z} \leq \frac{a_{i+1}}{a_k}\right)\right]$$

can be expressed as

$$\begin{cases} \Phi\left(x_{k,1}\right) & \text{for } i = 0 \\ \Phi\left(x_{k,i+1}\right) - \Phi\left(x_{k,i}\right) & \text{for } 1 \leq i \leq p-1 \\ 1 - \Phi\left(x_{k,p}\right) & \text{for } i = p \end{cases}$$

and the second ones

$$B_{k,i} = E\left[a_k e^{\mu\Delta t + \sigma\sqrt{\Delta t}Z} I\left(\frac{a_i}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t}Z} \leq \frac{a_{i+1}}{a_k}\right)\right]$$

as

$$\begin{cases} a_k \Phi\left(x_{k,1} - \sigma\sqrt{\Delta t}\right) e^{r\Delta t} & \text{for } i = 0 \\ a_k\left[\Phi\left(x_{k,i+1} - \sigma\sqrt{\Delta t}\right) - \Phi\left(x_{k,i} - \sigma\sqrt{\Delta t}\right)\right] e^{r\Delta t} & \text{for } 1 \leq i \leq p-1 \\ a_k\left[1 - \Phi\left(x_{k,p} - \sigma\sqrt{\Delta t}\right)\right] e^{r\Delta t} & \text{for } i = p \end{cases},$$

where $x_{k,i} = \left[\ln\left(a_i/a_k\right) - \mu\Delta t\right] / \left(\sigma\sqrt{\Delta t}\right)$, and Φ stands for the cumulative density function of Z.

We generate the a_k's as the quantiles of S_T, the distribution of the underlying asset price at maturity. The transition parameters, the $A_{k,i}$'s and $B_{k,i}$'s, are then precomputed before doing the first iteration.

The algorithm may be summarized as follows:

1. Compute $\widehat{v}_n\left(s\right)$ for all s using (5);

2. Compute $\widetilde{v}_{n-1}^h\left(a_k\right)$ for all k in a closed-form using (6);

3. Compute $\widetilde{v}_{n-1}(a_k)$ for all k using (4);

4. Compute $\widehat{v}_{n-1}(s)$ for all $s > 0$ using (5);

5. Repeat backward from step $n-1$ to step 0.

Notice that the optimal decisions (exercise and exit strategies) are derived at steps 2 and 3.

4 Theoretical properties

In this section, we derive some theoretical properties related to the design of installment call options. Symmetric results hold for installment put options.

Proposition 1 *The net holding value of the IO call at t_m, $v_m^h(s) - \pi_m$, as a function of $s > 0$, is continuous, differentiable, convex, and monotone with a positive rate less than 1. The value function is null on the exit region $(0, a_m)$, equal to the net holding value on the holding region $[a_m, b_m]$, and equal to the exercise value on the exercise region (b_m, ∞) where a_m and b_m are two thresholds that depend on the IO parameters.*

Proof. The proof proceeds by induction on $m = n-1, \ldots, 0$. At t_{n-1}, the holding value at $s > 0$ is

$$\begin{aligned} v_{n-1}^h(s) &= E\left[e^{-r\Delta t} v_n(S_{t_n}) \mid S_{t_{n-1}} = s\right] \\ &= \int_{-\infty}^{+\infty} e^{-r\Delta t} \left(s e^{\mu\Delta t + \sigma\sqrt{\Delta t} z} - K\right)^+ \phi(z)\, dz, \end{aligned}$$

where ϕ is the density function of the standard normal distribution. Obviously, this function is always strictly positive. By the Lebesgue's dominated convergence theorem (Billingsley, 1995), the holding value appears to be continuous, differentiable for all $s > 0$, and

$$\lim_{s \longrightarrow 0} v_{n-1}^{h}(s) = 0.$$

This function is a convex function of $s > 0$ as a convex combination of convex (piecewise linear) functions of $s > 0$. It is monotone as an integral of an increasing function indexed by $s > 0$. For $s_2 > s_1 > 0$, one has

$$\begin{aligned} & v_{n-1}^{h}(s_2) - v_{n-1}^{h}(s_1) \\ = \; & e^{-r\Delta t} \int_{-\infty}^{+\infty} \left(\left(s_2 e^{\mu \Delta t + \sigma \sqrt{\Delta t} z} - K \right)^{+} - \left(s_1 e^{\mu \Delta t + \sigma \sqrt{\Delta t} z} - K \right)^{+} \right) \phi(z)\, dz \\ \leq \; & (s_2 - s_1)\, e^{-\sigma^2 \Delta t/2} \int_{-\infty}^{+\infty} e^{\sigma \sqrt{\Delta t} z} \phi(z)\, dz \\ = \; & s_2 - s_1. \end{aligned}$$

The increasing rate of the holding value at t_{n-1} is thus less than 1. Consequently, the net holding value reaches 0 at a unique threshold a_{n-1}, and below the exercise value at a unique threshold b_{n-1}, where a_{n-1} and b_{n-1} depend on the IO parameters. Properties of the net holding value and the value functions follow (see figure 1). Now, if one assumes that these properties hold at step $m+1$, the same arguments may be used to proof that they hold at step m (we omit the details). This ends the proof. ■

Figure 1 plots the curve representing the net holding value of the installment call option $v_m^h(s) - \pi_m$ for any decision date m. This curve intersects the x-axis at a_m which separates the exit region from the holding region. Since its slope is less than 1, it necessarily intersects the curve of the call intrinsic value at b_m, which separates the holding region from the exercise region.

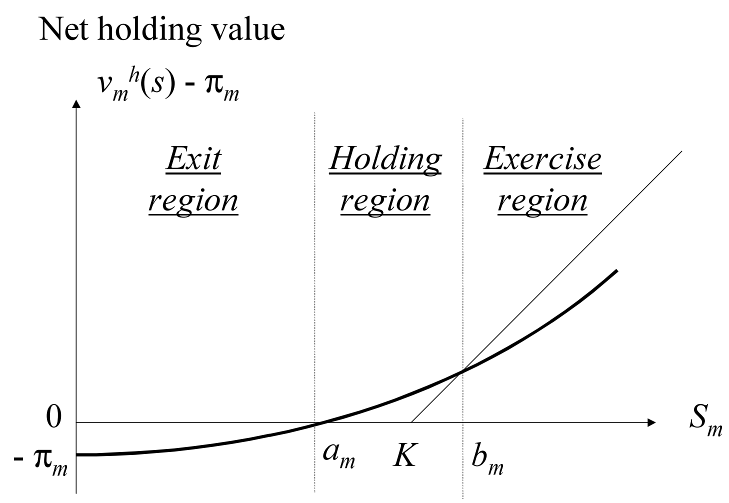

Figure 1

Lemma 2 *Let s and σ be the price and the volatility of the underlying asset, r the interest rate, K, $T = t_n = n\Delta t$, and $t_{n^*} = t_1 = \Delta t$ the strike, the maturity, and the first exercise date of the installment call option with an installment vector $\pi = (\pi_1, \ldots, \pi_{n-1})$ to be paid at $t_1, \ldots, t_{n-1}$. For $k = 1, \ldots, n-1$, assume that $\pi_m \geq c(K, \sigma, K, \Delta t, r)$, for all $m \geq k$. One has*

$$v_k(s) = v_k^e(s), \quad \text{for all } s > 0,$$

where $c(s, \sigma, K, \Delta t, r)$ is the Black-Scholes price of the European call option with parameters s, σ, K, Δt, and r.

Proof. The proof is established by induction on $m = n-1, \ldots, 1$. At time t_{n-1}, one has

$$\begin{aligned} v_{n-1}^{h}(K) &= E\left[e^{-r\Delta t} v_n\left(S_{t_n}\right) \mid S_{t_{n-1}} = K\right] \\ &= E\left[e^{-r\Delta t} v_n^{e}\left(S_{t_n}\right) \mid S_{t_{n-1}} = K\right] \\ &= c\left(K, \sigma, K, \Delta t, r\right). \end{aligned}$$

Recall that the holding value is a monotone function of $s > 0$ with a positive rate less than 1 for the call (see Proposition 1). For $\pi_{n-1} \geq c\left(K, \sigma, K, \Delta t, r\right)$, the net holding value at t_{n-1}, $v_{n-1}^{h}(s) - \pi_{n-1}$, is always lower than the exercise value, $v_{n-1}^{e}(s)$, as shown in Figure 2.

At step $k+1$, assume that $\pi_m \geq c\left(K, \sigma, K, \Delta t, r\right)$ and $v_m(s) = v_m^{e}(s)$, for all $s > 0$ and $m \geq k+1$. One has

$$\begin{aligned} v_{k}^{h}(K) &= E\left[e^{-r\Delta t} v_{k+1}\left(S_{t_{k+1}}\right) \mid S_{t_k} = s\right] \\ &= E\left[e^{-r\Delta t} v_{k+1}^{e}\left(S_{t_{k+1}}\right) \mid S_{t_k} = s\right] \\ &= c\left(K, \sigma, K, \Delta t, r\right). \end{aligned}$$

The same argument used at step $n-1$ may be used again at step k to show that

$$v_k(s) = v_k^{e}(s), \quad \text{for all } s > 0,$$

if $\pi_k \geq c\left(K, \sigma, K, \Delta t, r\right)$. ■

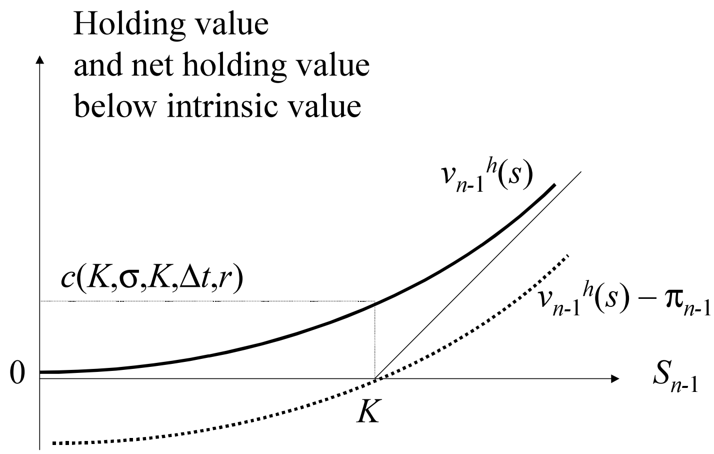

Figure 2

Figure 2 plots the case where the installment is equal to $c(K, \sigma, K, \Delta t, r)$, which places the net holding value below the intrinsic value. Thus, for all installments greater than $c(K, \sigma, K, \Delta t, r)$, the holding region vanishes, and the remaining possibilities are exit or exercise at the next decision date.

Proposition 3 *Let s and σ be the price and the volatility of the underlying asset, r the risk-free rate, K, $T = t_n = n\Delta t$, and $t_{n^*} = t_1 = \Delta t$ the strike, the maturity, and the first exercise date of the installment call option with an installment vector $\pi = (\pi_1, \ldots, \pi_{n-1})$ to be paid at $t_1, \ldots, t_{n-1}$. If the π_m's are all greater than $c(K, \sigma, K, \Delta t, r)$, one has*

$$v_0(s) = c(s, \sigma, K, \Delta t, r), \quad \textit{for all } s > 0.$$

Proof. By Lemma 2, we get

$$\begin{aligned} v_0(s) &= v_0^h(s) \\ &= E\left[e^{-r\Delta t} v_1(S_{t_1}) \mid S_0 = s\right] \\ &= E\left[e^{-r\Delta t} v_1^e(S_{t_1}) \mid S_0 = s\right] \\ &= c(s, \sigma, K, \Delta t, r). \end{aligned}$$

This ends the proof. ■

5 Simulation analysis

5.1 Convergence speed and accuracy

Table 1 displays the main pricing properties of our approach. First, convergence to the "true" price is rather fast. A fairly good approximation of the IO price can be obtained almost instantaneously with a 125-point grid. A two-digit accuracy is achieved with a 250-point grid, which involves a computational time of a tenth of a second (CPU times are reported with a 933 MHz Windows PC). A four-digit accuracy can be obtained with a 1000-point grid, which implies a computational time that does not exceed two seconds. Second, note that the number of installments in the contract increase computational time only slightly. For a given grid size, computational time is divided in two components, a fixed cost to pre-compute the transition matrices, and a variable cost roughly linear in the number of installments. In particular, computational time increases by around 20% as the number of installments goes from 0 to 4. Thus, complex IOs can still be priced with a satisfactory trade-off between accuracy and computational time. Third and most importantly, convergence to the "true" price is monotonic. This allows for extrapolation methods that can significantly reduce computational time for a desired accuracy. In addition, note that our approach, like any other numerical method, may be implemented in conjunction with variance-reducing techniques, such as control variate methods for example.

Number of installments	Number of grid points 125	250	500	1000	2000
0	13.34809	13.34664	13.34658	13.34650	13.34648
	(0.02)	(0.09)	(0.39)	(1.53)	(6.14)
1	11.49561	11.49268	11.49236	11.49221	11.49218
	(0.02)	(0.11)	(0.41)	(1.61)	(6.45)
2	9.86059	9.85653	9.85595	9.85575	9.85571
	(0.03)	(0.11)	(0.42)	(1.69)	(6.78)
3	8.65862	8.65312	8.65243	8.65217	8.65211
	(0.03)	(0.11)	(0.44)	(1.77)	(7.08)
4	7.80531	7.79948	7.79856	7.79828	7.79822
	(0.03)	(0.11)	(0.47)	(1.84)	(7.39)

Table 1: IO prices and computational time

Table 1 reports IO upfront payments for various grid sizes with the corresponding CPU time in seconds (in parentheses). The code line is written in C and compiled with GCC. CPU times are obtained with a 933 MHz Windows PC. The IO is a call with equal installments ($\pi = 2$) and the following characteristics: $S = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.05$, $\delta = 0$, and $T = 1$. The number of installments varies from 0 to 4. In the case of zero installment, the call is European and its theoretical price is 13.34647.

5.2 Non-redundant IO contracts

Table 2 reports prices of installment calls for various levels of constant installments. Clearly, these prices are decreasing with the level of installment.

They reach the minimum $c(s, \sigma, K, \Delta t, r)$ for installments greater than $\pi = c(K, \sigma, K, \Delta t, r)$, as shown in Proposition 3. For example, when $K = 110$, we have $c(100, 0.2, 110, 0.25, 0.05) = 1.191$ and $c(110, 0.2, 110, 0.25, 0.05) = 5.076$. For any installment greater than 5.076, the holding region vanishes, and the installment call is worth the European call expiring at the next decision date.

Installment	$K = 90$	$K = 100$	$K = 110$
0	16.699	10.451	6.040
0.5	15.262	9.072	4.785
1	13.857	7.787	3.738
1.5	12.779	6.660	2.886
2	12.206	5.840	2.266
2.5	11.910	5.286	1.833
3	11.763	4.943	1.547
3.5	11.695	4.748	1.368
4	11.671	4.650	1.264
4.5	11.670	4.616	1.210
5	11.670	4.615	1.191
5.5	11.670	4.615	1.191

Table 2: IO prices and installment level

Table 2 reports installment call upfront payments for various levels of installment and strikes. Parameters are the following: $S = 100$, $\sigma = 0.2$, $r = 0.05$, $\delta = 0$, and $T = 1$. Exercise rights are quarterly and the IO has three installments.

A direct implication for IO design is that contracts with installments that eliminate the holding region are simply redundant with European options. Within the range of possible installment levels, various hedging properties may be designed. We now investigate these properties.

5.3 IO Greeks

In this subsection, we examine the hedging properties of the installment call option with respect to the level of installments, the installment schedule, and the option moneyness. Figures 3 and 4 report our findings. Unless otherwise indicated, parameters are the following

S	r	δ	σ	T	n
100	0.05	0	0.2	1	4

Figures 3a, 3b, and 3c respectively report the installment call delta, gamma and vega as a function of the *constant* installment π_1. For each figure, out-of-the-money installment calls are plotted with triangles ($K = 110$), at-the-money installment calls are plotted with squares, and in-the-money installment calls are plotted with crosses ($K = 90$).

Figure 3a reports that installment call delta decreases (increases) with π_1 for out-of-the-money (in-the-money) options. Indeed, if the constant installment increases, it gets more and more likely that the IO will be exercised or forsaken at the first date. Thus, if the IO is currently in the money, its price becomes even more sensitive to price variations of the underlying. By contrast, a currently out-of-the-money IO with a high π_1 has little chance of future exercise, so its delta remains low. A direct implication of this property is that the higher π_1, the more volatile delta is with respect to S. In other words, IOs with high installments are more difficult to hedge.

Figure 3b confirms this latter finding as IO gammas are increasing with π_1. Hedging IOs with high installments requires more frequent rebalancing of the replicating portfolio. This effect is more pronounced for at-the-money IOs. For these options indeed, moneyness is uncertain so that delta could rapidly shift to very low or very high values. That is why Figure 3b indicates that, as π_1 increases, gamma becomes a more humped function of moneyness.

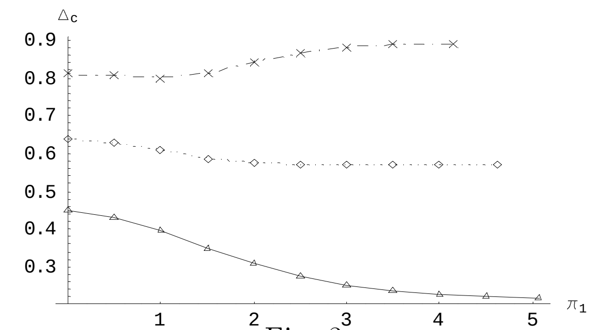

Fig. 3a

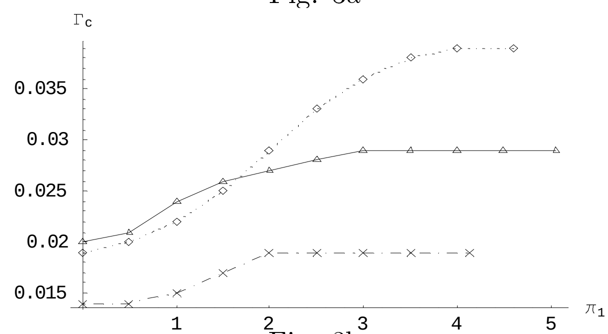

Fig. 3b

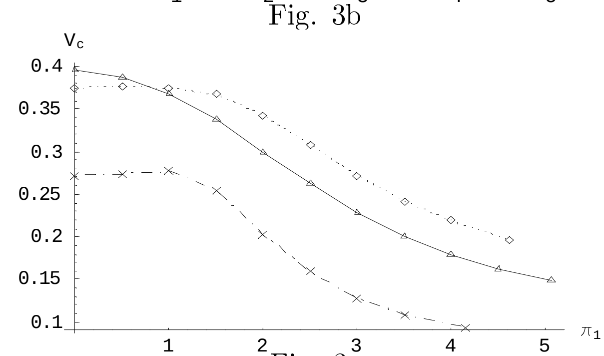

Fig. 3c

Figure 3: IO Greeks, installment level, and moneyness

Figures 3a, 3b, and 3c respectively report the installment call delta, gamma and vega as a function of the constant installment π_1. Parameters are the following: $S = 100$, $r = 0.05$, $\delta = 0$, $\sigma = 0.2$, and $T = 1$. Exercise rights are quarterly and the IO has three installments. For each figure, out-of-the-money installment calls are plotted with triangles $(K = 110)$, at-the-money installment calls are plotted with squares, and in-the-money installment calls are plotted with crosses $(K = 90)$.

A similar effect applies for vega as illustrated by Figure 3c. Since a high installment reduces the likelihood of future exercise and therefore lowers the IO speculative value, vega decreases with π_1. Interestingly, this cutoff in speculative value is less pronounced for at-the-money installment calls. Thus, as π_1 increases, vega also becomes a more humped function of moneyness.

Figures 4a, 4b, and 4c respectively report the installment call delta, gamma and vega as a function of the first installment π_1. In this case however, the installment schedule may be increasing or decreasing. Specifically, installment calls with installments decreasing at rate 0.8 are plotted with triangles, installment calls with constant installments are plotted with squares, and installment calls with installments increasing at rate 1.2 are plotted with crosses. Simulations are reported for at-the-money calls.

As illustrated by Figures 4a to 4c, non-constant installments introduce an additional degree of freedom in the IO design. Specifically, if installments are increasing, then, all else being equal, call delta and vega are reduced. As shown in Proposition 3, higher installments tighten up the holding region and reduces the option speculative value. Therefore the installment call value is less sensitive to underlying price or volatility variations (see Figures 4a and 4c). By contrast, since the net holding value tends to mimic the intrinsic value, the installment call with high installments exhibits a high gamma and requires a more frequent hedging (see Figure 4b).

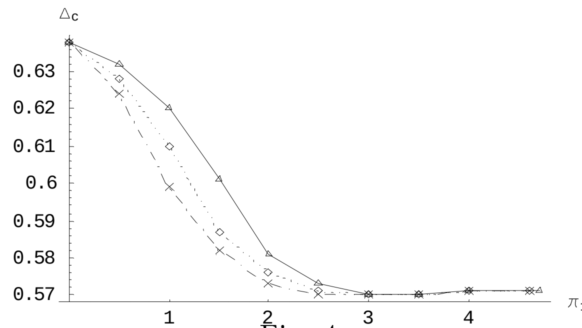

Fig. 4a

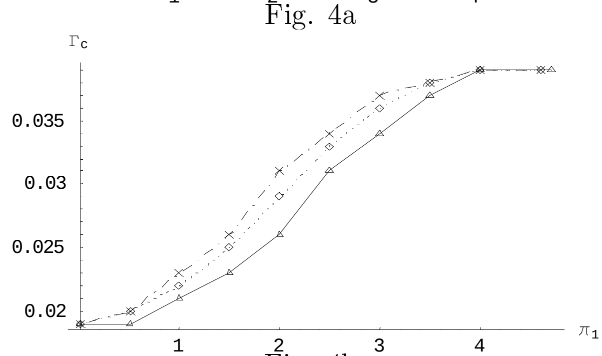

Fig. 4b

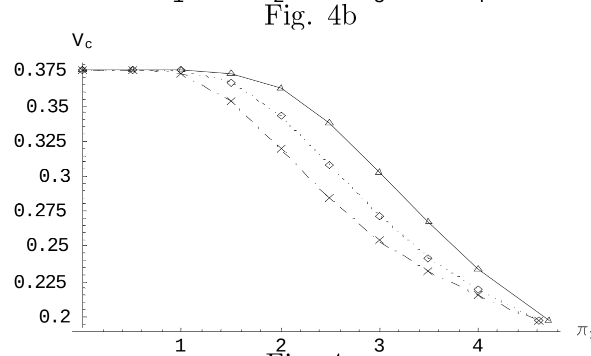

Fig. 4c

Figure 4: IO Greeks, installment schedule, and moneyness

Figures 4a, 4b, and 4c respectively report the installment call delta, gamma and vega as a function of the first installment π_1. Parameters are the following: $S = 100$, $K = 100$, $r = 0.05$, $\delta = 0$, $\sigma = 0.2$, and $T = 1$. Exercise rights are quarterly and the IO has three installments. For each figure, installment calls with installments decreasing at rate 0.8 are plotted with triangles, installment calls with constant installments are plotted with squares, and installment calls with installments increasing at rate 1.2 are plotted with crosses.

6 Application to ASX installment warrants

One of the most actively traded installment options throughout the world are currently the installment warrants on Australian stocks. These warrants were launched on the Australian Stock Exchange (ASX) in January 1997. Since then, both the number of listed installment warrants and the trading volume have grown exponentially, as documented by Figure 5 (obtained from the ASX website).

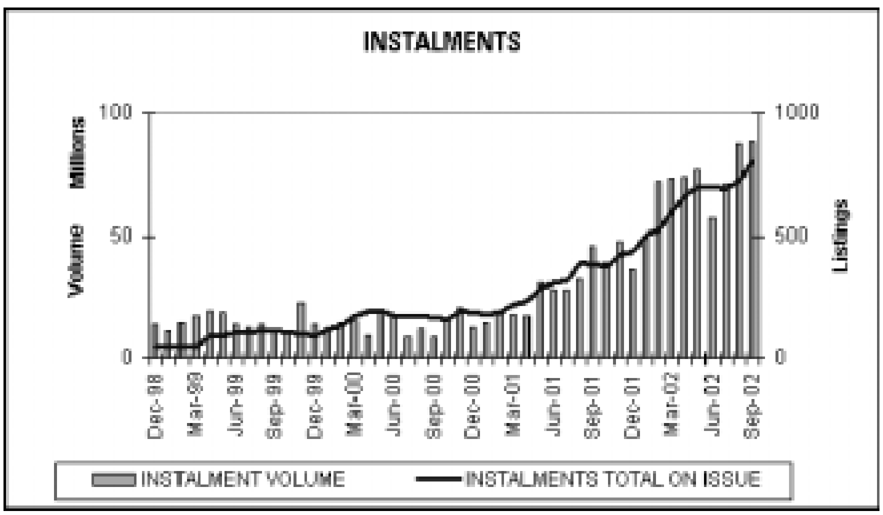

Figure 5: Installment warrants listings and volume

Some of the ASX installment warrants (called rolling installment warrants) have several installments and their expiry date may be up to 10 years. However, most ASX warrants have only one installment with maturities ranging from 1 to 3 years. The single installment is usually set equal to the upfront payment. This clearly puts a restriction on the strike price of the warrant.

In this section, we apply our model to the pricing of installment warrants. By contrast to call options, warrants have a dilution effect on the issuer's stocks. Black and Scholes (1973) suggest to price warrants as an option on the issuer's equity (i.e. stocks plus warrants). For so doing, the valuation formula must be adjusted for dilution.[3] Specifically, let M, N, and γ respectively denote the number of outstanding warrants, the number of outstanding shares, and the conversion ratio. Extending the approach by Lauterbach and Schultz (1990), the installment warrant in this context is interpreted as – a fraction of – an IO issued by the firm. Its payoff process is

$$Y_t = \frac{N\gamma}{N + M\gamma}\left(\frac{V_t}{N} - K\right)^+, \quad \text{for } t \in \{t_0 = 0, \ldots, t_m = T\},$$

where $\{V\} = \{NS + MW\}$ is the value of the firm's equity, $\{V/N\} = \{S + MW/N\}$ is the asset underlying the warrant, $\{S\}$ is the stock price of the firm within the warrant life, and $\{W\}$ is the value of the installment warrant.

The DP algorithm described in Section 3 may be easily modified to the pricing of warrants in the context of IOs. The exercise value in (2) is now the payoff of the warrant if exercised optimally

$$v_t^e(x) = \frac{N\gamma}{N + M\gamma}(x - K)^+.$$

To compute W_0, one should solve

$$v_0(S_0 + MW_0/N) = W_0.$$

[3] Another possibility, first explored by Galai and Schneller (1978), is to price warrants as an option on the issuer's underlying stock. Handley (2002) points out that if the warrant is priced after its announcement date, then the efficient market hypothesis implies that the dilution effect is already reflected in the stock price. Consequently, no adjustment for dilution is required. However, to be consistent with the assumption of a geometric Brownian motion for the firm's equity, this approach requires a time-varying volatility for the underlying stock returns.

This is easy to implement as the procedure gives[4]

$$v_0(x), \quad \text{for all } x > 0.$$

Table 3 reports installment warrant prices for degrees of dilution and numbers of installments.

Number of	Number of outstanding warrants				
installments	$M = 0$	$M = 10$	$M = 50$	$M = 100$	$M = 200$
0	13.346	13.006	11.988	11.184	10.322
	(13.346)	(13.006)	(11.989)	(11.185)	(10.324)
1	11.492	11.011	9.567	8.557	7.726
2	9.855	9.364	8.054	7.289	6.790
3	8.652	8.215	7.177	6.658	6.315
4	7.798	7.445	6.666	6.296	6.030

Table 3: Installment warrant prices and the dilution effect

Table 3 reports installment warrant upfront payments for various degrees of dilution. The installment warrant has equal installments ($\pi = 2$) and the following characteristics: $S = 100$, $K = 95$, $\sigma = 0.2$, $r = 0.05$, $\delta = 0$, $N = 100$, $\gamma = 1$, and $T = 1$. Grid size is 500 points. The number of outstanding warrants varies from 0 to 200, and the number of installments varies from 0 to 4. In the case of zero installment, the warrant is European and its theoretical price (below in parentheses) is given by Lauterbach and Schultz (1990). In the case of $M = 0$, the installment warrant is fully diluted and its price equals that of the installment call option (see Table 1).

[4] As a special case, we get the procedure by Lauterbach and Schultz (1990) for pricing European warrants, namely the price w of the European warrant is obtained using the Black-Scholes formula where: (1) The underlying S is replaced with $S + \frac{M}{N}w$, (2) Volatility σ is that of equity returns, and (3) The whole formula is multiplied by the dilution factor $\frac{N\gamma}{N+M\gamma}$. These adjustments result in an equation of the type $w = f(w)$ which must be solved numerically.

As can be seen from Table 3, installment warrants prices decrease with the installment and are therefore lower than prices of otherwise identical European warrants. Thus, installment warrants have a weaker dilution effect than European warrants, i.e. the wealth transfer from stockholders to warrantholders is less pronounced. The reason for this is that the presence of installments implies that warrants may be abandoned and simply not exercised. As a consequence, the design of installment warrants gives the firm some flexibility in controlling capital dilution when raising funds.

7 Conclusion

In this paper, we have developed a pricing methodology for installment options using dynamic programming. This numerical procedure is particularly well suited for IOs because these options are Bermudan-style and involve a limited number of distant exercise dates. Simulations indicate that prices converge monotonically and quickly reach good levels of accuracy. In addition, we have shown that IOs installment schedule may be designed with a great flexibility. Various hedging properties can thus be tailored.

We have adapted our model to the pricing of installment warrants that are actively traded on the Australian Stock Exchange. Numerical investigation shows the various capital dilution effects resulting from different installment warrant designs.

Our approach is flexible enough to be extended to many other pricing issues. For instance, levered equity may be seen as a compound call on asset value when debt bears discrete coupons (see Geske (1977)). Consider now the coupon-bearing debt is callable. At each coupon date, shareholders decide whether or not to call the debt. If they do not call, they decide whether or not to pay the coupon to keep their claim on firm asset value. Consequently, levered equity may be priced as an installment call on firm asset value.

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