Estimation via stochastic filtering

in financial market models

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Main points

- Financial models may not be completely specified, or specified in terms of unknown quantities (unknown parameters, hidden processes,..)

  → *Instead of a fixed model, an entire family of models.*

- Identify the model within the class by estimating unobserved quantities possibly in a dynamic/recursive way and this is where stochastic filtering comes in.

  → *Estimation obtained by filtering the information coming from observing over time prices not only of underlying primary assets, but also of derivatives.*

- Filtered information useful for portfolio optimization, but also for pricing of illiquid derivatives.

→ **Problem :** *Derivative prices are specified as expectations under a martingale measured (MM); observations occur under the real world measure.*

OUTLINE

- **The case when the underlyings have a market and their prices are Markovian**

- **Underlyings have a market but are not Markovian** (*factor models*)

- **Filtering for derivative pricing under partial observation** (*a general setup*)

- **Filtering for pricing under partial observation in factor models** (*a specific setup*)

1. Underlying asset prices are Markovian

- *Standard B & S market model for underlyings*

$$dS_t = diag(S_t)A_t dt + diag(S_t)\Sigma_t dw_t$$

$S_t = [S_t^1, \cdots, S_t^N]'$

$w_t = [w_t^1, \cdots, w_t^M]'$ a $(P, \mathcal{F}_t)-$Wiener, $M \geq N$.

- Given a claim $H(S_T)$, its $t-$price ($t \leq T$) is

$$\begin{aligned} \Pi(t, S_t) &= e^{-\int_t^T r_s ds} E^Q\{H(S_T) \,|\, \mathcal{F}_t\} \\ &= e^{-\int_t^T r_s ds} E^Q\{H(S_T) \,|\, S_t\} \end{aligned}$$

where r_t is assumed to be deterministically given and, under Q,

$$dS_t = diag(S_t) r_t \underline{1} dt + diag(S_t)\Sigma_t dw_t^Q$$

with

$$dw_t^Q = dw_t + \theta_t dt$$

and $\theta_t = [\theta_t^1, \cdots, \theta_t^M]'$ s.t.

$$A_t - r_t \underline{1} = \Sigma_t \theta_t \quad (\textit{market price of risk})$$

- If the market is complete then, for the only purpose of derivative pricing, the knowledge of Σ_t suffices, A_t is not needed. Σ_t estimated either as implied or historical volatility. For other purposes, e.g. portfolio optimization, need also knowledge of A_t. Given Σ_t, A_t follows from θ_t

- *If A_t and Σ_t and thus also θ_t are unobserved then, by borrowing ideas from Bayesian statistics, one may assume them to be stochastic processes that could also be adapted to a filtration larger than $\mathcal{F}_t^w$ (exogenous randomness). The market is then incomplete and estimation of θ becomes important also for derivative pricing.*

→ Estimation should be based on observations not only of S_t, but also of their derivatives.

Filtering approach (for estimation of θ)

- Assume on the market one can observe, in addition to S_t, also K derivative prices $\Pi_i^*(t)$, $i = 1, \cdots, K$ so that the observation filtration becomes

$$\mathcal{F}_t^O = \sigma\{S_u,\, \Pi_i^*(u)\,;\, u \le t\,;\, i = 1, \cdots, K\}$$

  Problem : Given a a-priori dynamics for θ_t, determine recursively

$$\pi_t\left(\theta_t \,|\, \mathcal{F}_t^O\right)$$

  starting from a given $\pi_0(\theta_0)$.

$\rightarrow$ *Gives not only a point estimate but an entire, continuously updated distribution.*

Filter model

- Having assumed θ_t to be possibly affected by exogenous randomness, we model it under P as

$$d\theta_t = \kappa(\bar{\theta} - \theta_t)dt + \rho^w dw_t + \rho^v dv_t$$

κ : diagonal matrix;
$\bar{\theta} = [\bar{\theta}^1, \cdots, \bar{\theta}^M]'$;
ρ^w, ρ^v : matrices;
v_t (multivariate) Wiener independent of w_t;
$\pi_0(\theta_0)$ a given Gaussian initial distribution.

$\rightarrow$ *Reasonable to assume the evolution of θ_t to be affected by that of the underlyings (driving noise w_t) and also of exogenous factors (noise v_t).*

$\rightarrow$ *Need next the dynamics, under P, of the observed prices*

- Consider the $(Q, \mathcal{F}_t)-$martingales

$$
\begin{aligned}
Y_t^i = F^i(t, S_t) &:= e^{\int_t^T r_s ds} \Pi^i(t, S_t) \\
&= E^Q\{H^i(S_T) \,|\, \mathcal{F}_t\}\,; \; i=1,\cdots,K
\end{aligned}
$$

By Itô's formula and the martingality of Y_t^i

$$
\begin{aligned}
dY_t^i &= dF^i(t, S_t) \\
&= \Big[F_t^i(\cdot) + F_S^i(\cdot) diag(S_t) r_t \underline{1} \\
&\quad + \tfrac{1}{2} tr[\Sigma_t' \, diag(s) F_{SS}^i(\cdot) diag(s) \Sigma_t] \Big] \, dt \\
&+ F_S^i(\cdot) diag(S_t) \Sigma_t dw_t^Q \\
&= F_S^i(\cdot) diag(S_t) \Sigma_t dw_t^Q
\end{aligned}
$$

Observation dynamics under the real world measure P

$$\begin{cases} dS_t = diag(S_t)[r_t\underline{1} + \Sigma_t\theta_t]dt \\ \qquad\qquad +diag(S_t)\Sigma_t dw_t \\ dY_t^i = F_S^i(t,S_t)diag(S_t)\Sigma_t\theta_t \\ \qquad\qquad +F_S^i(t,S_t)diag(S_t)\Sigma_t dw_t \end{cases}$$

Synthesizing :

$$\begin{cases} d\theta_t & = & \kappa(\bar{\theta}-\theta_t)dt+\rho^w dw_t+\rho^v dv_t \\ dS_t & = & diag(S_t)[r_t\underline{1}+\Sigma_t\theta_t]dt+diag(S_t)\Sigma_t dw_t \\ dY_t^i & = & F_S^i(t,S_t)diag(S_t)\Sigma_t\theta_t dt+F_S^i(t,S_t)diag(S_t)\Sigma_t dw_t \end{cases}$$

θ_t : unobservable ($\pi_0(\theta_0)$ given Gaussian);

$S_t, Y_t^i \, (i = 1, \cdots, K)$: observable;

r_t : supposed given;

Σ_t : given/observable (either through quadratic variation or as implied volatility $\hat{\Sigma}_t = \Sigma(S_t, Y_t^i, t)$).

→ *A model of the conditionally Gaussian type to which the Kalman filter can be applied (the parameters* $(\kappa, \bar{\theta}, \rho^w, \rho^v)$ *may be estimated by maximizing the likelihood of the innovations)*

→ *See [BCR, 2002].*

2.Underlyings not Markovian themselves

(*Factor models*)

- Consider the (combined Markovian) model

$$\begin{cases} dS_t = diag(S_t)A_t(Z_t)dt + diag(S_t)\Sigma_t(Z_t)dw_t \\ \\ dZ_t = b_t(Z_t)dt + \gamma_t(Z_t)dv_t \end{cases}$$

w_t, v_t : independent multivariate Wiener (v_t : "exogenous randomness");

S_t : observed asset price vector;

Z_t : multivariate factor process (hidden with known $\pi_0(Z_0)$)

$A_t(Z_t), \Sigma_t(Z_t)$: may possess additional randomness adapted to $\mathcal{F}_t^w$.

→ *The market is incomplete and estimation of Z_t is important also for derivative pricing*

→ *Again, estimate Z_t on the basis of observations of S_t, but also of their derivatives.*

- Given a claim $H(S_T)$, its $t-$price ($t \leq T$) is now

$$\Pi(t, S_t, Z_t) = e^{-\int_t^T r_s ds} E^Q\{H(S_T) \,|\, \mathcal{F}_t\}$$

where r_t is again deterministically given and, under Q,

$$dS_t = diag(S_t) r_t \underline{1} dt + diag(S_t) \Sigma_t(Z_t) dw_t^Q$$

with

$$dw_t^Q = dw_t + \theta_t dt$$

and θ_t the (unitary) market price of risk satisfying

$$A_t(Z_t) - r_t \underline{1} = \Sigma_t(Z_t)\theta_t$$

→ *The process θ_t may thus be considered as a function*

$$\theta_t = \theta(t, Z_t)$$

but, as for $A_t(Z_t)$ and $\Sigma_t(Z_t)$, we shall assume that it possesses also additional randomness adapted to $\mathcal{F}_t^w$.

→ *The market is incomplete → different possible MM.*

- Applying Itô's rule one has

$$d\theta_t = d\theta(t, Z_t)$$

$$= \Big[\tfrac{\partial}{\partial t}\theta(t, Z_t) + \tfrac{\partial}{\partial Z}\theta(t, Z_t) b_t(Z_t)$$
$$+ \tfrac{1}{2} tr \left(\gamma_t'(Z_t) \frac{\partial^2}{(\partial Z)^2} \theta(t, Z_t) \gamma_t(Z_t) \right) \Big] dt$$
$$+ \tfrac{\partial}{\partial Z}\theta(t, Z_t)\gamma_t(Z_t) dv_t$$

$$:= \Theta_t(Z_t)dt + \Psi_t(Z_t)dv_t$$

→ *Due to the additionally assumed randomness adapted to $\mathcal{F}_t^w$, we shall postulate for θ_t the following dynamics under P*

$$d\theta_t = \Theta_t(Z_t)dt + \rho^w dw_t + \Psi_t(Z_t)dv_t$$

→ *Inferring θ_t on the basis of market data allows to infer also the prevailing MM.*

Filtering approach

- *Given $\mathcal{F}_t^O$, determine recursively*

$$\pi_t(Z_t, \theta_t | \mathcal{F}_t^O)$$

starting from a given $\pi_0(Z_0, \theta_0)$.

- Analogously as before, put

$$Y_t^i = F^i(t, S_t, Z_t) := e^{\int_t^T r_s ds} \Pi^i(t, S_t, Z_t)$$

and the martingality of Y_t^i under Q implies

$$\begin{cases} F_t^i(t,s,z) + F_S^i(t,s,z) diag(s) r_t \underline{1} \\ \quad + \frac{1}{2} tr[\Sigma_t'(z)\, diag(s) F_{SS}^i(t,s,z) diag(s) \Sigma_t(z)] \\ \quad + F_Z^i(t,s,z) b_t(z) + \frac{1}{2} tr[\gamma_t'(z) F_{ZZ}(t,s,z) \gamma_t(z)] = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, (t,s,z);\, i=1,\cdots,K \\ F^i(T,s,z) = H^i(s) \end{cases}$$

- **Synthesizing,** we obtain the following filtering model (under P)

$$\begin{cases} d\theta_t &= \Theta_t(Z_t)dt + \rho^w dw_t + \Psi_t(Z_t)dv_t \\ dZ_t &= b_t(Z_t)dt + \gamma_t(Z_t)dv_t \\ dS_t &= diag(S_t)[r_t\underline{1} + \Sigma_t(Z_t)\theta_t]dt + diag(S_t)\Sigma_t(Z_t)dw_t \\ dY_t^i &= [F_S^i(t, S_t, Z_t)diag(S_t)\Sigma_t(Z_t)\theta_t]dt \\ &+ F_S^i(t, S_t, Z_t)diag(S_t)\Sigma_t(Z_t)dw_t \\ & \qquad + F_Z^i(t, S_t, Z_t)\gamma_t(Z_t)dv_t \\ & \qquad\qquad i = 1, \cdots, K \end{cases}$$

$\rightarrow$ *The only parameter in the model is now ρ^w. One may thus either search for the complete filter solution*

$$\pi_t(Z_t, \theta_t, \rho^w | \mathcal{F}_t^O)$$

or calibrate ρ^w by matching theoretical with observed prices.

→ The filtering problem is now **nonlinear** and the observation **diffusion coefficients depend on unobserved quantities**

→ *To overcome the latter difficulty : consider noisy observations of the observable quantities (justified by bid-ask spread, mispricing, a-synchronicity, etc...) with sufficiently small observation noise to prevent arbitrage opportunities.*

- Putting

$$\bar{Y}_t^i = \begin{cases} S_t^i & , \quad i = 1, \cdots, N \\ Y_t^{i-N} & , \quad i = N+1, \cdots, N+K \end{cases}$$

and denoting by η_t^i the (cumulative) noisy observations let

$$d\eta_t^i = \bar{Y}_t^i dt + d\beta_t^i \quad (i = 1, \cdots, N+K)$$

where $\beta_t = (\beta_t^1, \cdots, \beta_t^{N+K})'$ and $\bar{Y}_0^i$ is supposed to be observed without noise.

→ *A specific application of this approach to equity markets is in [BCR, 2002], to bond markets in [CPR, 2001].*

3.Filtering for derivative pricing under partial information

(general setup)

- *For the case of the previous factor model let $\mathcal{F}_t = \sigma\{w_s, v_s;\, s \leq t\}$ so that $\mathcal{F}_t^O \subset \mathcal{F}$ and assume r_t deterministically given.*

- Considering a claim $H(S_T)$ and a martingale measure Q, define the $t-$price ($t \leq T$) of $H(S_T)$ with respect to the information $\mathcal{F}_t^O$ as

$$\tilde{\Pi}(t) = E^Q\left\{e^{-\int_t^T r_s ds} H(S_T) \,|\, \mathcal{F}_t^O\right\}$$

$\rightarrow$ It is an arbitrage-free price with respect to the information represented by $\mathcal{F}_t^O$ in the sense that

$$\frac{\tilde{\Pi}(t)}{B_t} = E^Q\left\{\frac{\tilde{\Pi}(T)}{B_T} \mid \mathcal{F}_t^O\right\}$$

with $B_t = B_0 \exp\left\{\int_0^t r_s ds\right\}$

$\rightarrow$ *To perform pricing of (illiquid OTC) derivatives on the basis of the information $\mathcal{F}_t^O$, one has thus to compute expectations of the form*

$$E^Q\{H(S_T) \,|\, \mathcal{F}_t^O\}$$

- Taking as Q the minimal MM, only w_t is translated, i.e.

$$dw_t^Q = dw_t + \theta_t dt$$

and one has

$$\frac{dQ}{dP}_{|\mathcal{F}_T} = L_T \quad \text{with} \quad dL_t = -L_t \theta_t dw_t\,;\ L_0 = 1$$

- By Bayes' rule

$$E^Q\{H(S_T)\,|\,\mathcal{F}_t^O\} = \frac{E^P\{L_T H(S_T)\,|\,\mathcal{F}_t^O\}}{E^P\{L_T\,|\,\mathcal{F}_t^O\}}$$

For the given setup, the tuple $(S_t, Z_t, \theta_t, L_t)$ is Markov under P, thus

$$E^P\{L_T H(S_T)\,|\,\mathcal{F}_t^O\} = E^P\{E^P\{L_T H(S_T)\,|\,\mathcal{F}_t\}\,|\,\mathcal{F}_t^O\}$$

$$= E^P\{\Psi^H(t, S_t, Z_t, \theta_t, L_t)\,|\,\mathcal{F}_t^O\}$$

for a suitable $\Psi^H(\cdot)$.

→ *To compute the quantities of interest one needs the filter distribution*

$$\pi_t(Z_t, \theta_t, L_t|\,\mathcal{F}_t^O)$$

→ *The practical implementability of this approach depends on the specific problem at hand.*

The filter model

For the given setup the filter model is, under P, synthesized as

$$
\begin{cases}
d\theta_t &= \Theta_t(Z_t)dt + \rho^w dw_t + \Psi_t(Z_t)dv_t \\
dL_t &= -L_t\theta_t dw_t \\
dZ_t &= b_t(Z_t)dt + \gamma_t(Z_t)dv_t \\
dS_t &= diag(S_t)[r_t\underline{1} + \Sigma_t(Z_t)\theta_t]dt + diag(S_t)\Sigma_t(Z_t)dw_t \\
dY_t^i &= [F_S^i(t,S_t,Z_t)diag(S_t)\Sigma_t(Z_t)\theta_t]dt \\
&+ F_S^i(t,S_t,Z_t)diag(S_t)\Sigma_t(Z_t)dw_t \\
&\qquad +F_Z^i(t,S_t,Z_t)\gamma_t(Z_t)dv_t \\
&\qquad\qquad (i = 1,\cdots,K) \\
d\eta_t^i &= \bar{Y}_t^i dt + d\beta_t^i \;;\; i = 1,\cdots,N+K
\end{cases}
$$

with $\quad \bar{Y}_t^i = \begin{cases} S_t^i & , \quad i = 1,\cdots,N \\ Y_t^{i-N} & , \quad i = N+1,\cdots,N+K \end{cases}$

and the initial distribution of $(\theta_0, L_0, Z_0, S_0, Y_0^i)$ is characterized by $\pi_0(\theta_0, Z_0)$, $L_0 = 1$ and S_0, Y_0^i deterministically given (observed without noise).

4. Filtering for pricing in general factor models

- Z_t : a generic Markovian factor process (some components of Z_t may be unobservable, some may be observable asset prices)

$\rightarrow$ *Z_t is globally Markov : the evolution of each component may depend on the entire vector Z.*

- Assume $\exists\, T > 0$ (w.l.of g. the same for all assets) at which the price of any asset can be expressed as a known function of Z; i.e. for each asset $\exists\, H(\cdot)$ s.t.

$$\Pi^H(T) = H(T, Z_T)$$

$\rightarrow$ *for the components of Z that are asset prices themselves the function $H(\cdot)$ is simply the projection onto the corresponding component.*

- At $t \neq T$ assume

$$\Pi^H(t) = F^H(t; Z_t)$$

- Assume also the short rate stochastic and a known function of Z_t, i.e.

$$r_t = r(t, Z_t)$$

→ On $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ consider then the model

$$\begin{cases} dZ_t = b_t(Z_t)dt + \gamma_t(Z_t)dw_t \\ \\ \Pi^H(t) = F^H(t, Z_t) \text{ with } \Pi^H(T) = H(T, Z_T) \\ \qquad\qquad\qquad\qquad \text{and } H(\cdot) \text{ a known function} \end{cases}$$

→ *To prevent arbitrage, the function $F^H(t, z)$ cannot be arbitrary. What are the conditions on $F^H(\cdot)$ to have absence of arbitrage and, in particular, so that P itself becomes a martingale measure ?*

Note : for derivative pricing one computes expectations under a martingale measure; for filtering the dynamics have to be under the real world measure.

$\rightarrow$ *Have to impose that the discounted values of* $\Pi^H(t) = F^H(t, Z_t)$ *are* $(P, \mathcal{F}_t)-$*martingales*

- By Itô's rule and putting the finite variation terms equal to zero:

$$\begin{cases} F_t^H(t,z) + F_Z^H(t,z)b_t(z) \\ \quad + \frac{1}{2}tr[\gamma_t'(z)F_{ZZ}^H(t,z)\gamma_t(z)] - r_t(z)F^H(t,z) = 0 \,, \forall(t,z) \\ F^H(T,z) = H(T,z) \end{cases}$$

$\rightarrow$ *For particular families of* $H(\cdot)$ *this condition may take on more specific forms (e.g. for exponentially affine models of the bond market it becomes a system of ODE's).*

$\rightarrow$ *By Feynman-Kac also*

$$F^H(t,z) = E_{t,z}\left\{ e^{-\int_t^T r_s(Z_s)ds} H(T, Z_T) \right\}$$

- *Let*

$$\mathcal{F}_t^O = \sigma\left\{\Pi^{H_i}(s)\,;\; s \leq t\,;\; i = 1, \cdots, K\right\}$$

represent the information coming from market data.

- **Problem.** Pricing of illiquid (OTC) derivatives : for a claim $\Phi(F^H(\tau, Z_\tau))$ with maturity τ on an underlying with price $\Pi^H(\tau) = F^H(\tau, Z_\tau)$ compute, for $t \leq \tau$,

$$E\left\{e^{-\int_t^T r_s(Z_s)ds}\Phi(F^H(\tau, Z_\tau))\,|\,\mathcal{F}_t^O\right\}$$

$$= E\left\{E\left\{e^{-\int_t^T r_s(Z_s)ds}\Phi(F^H(\tau, Z_\tau))\,|\,\mathcal{F}_t\right\}\,|\,\mathcal{F}_t^O\right\}$$

$$= E\{\Psi(t, Z_t)\,|\,\mathcal{F}_t^O\}$$

for a suitable $\Psi^H(\cdot)$.

→ Need the filter distribution

$$\pi_t(Z_t | \mathcal{F}_t^O)$$

for the model

$$\begin{cases} dZ_t = b_t(Z_t)dt + \gamma_t(Z_t)dw_t \\ \\ dY_t^i = r_t(Z_t)F^{H_i}(t,Z_t)dt + F_Z^{H_i}(t,Z_t)\gamma_t(Z_t)dw_t \\ \qquad\qquad\qquad\qquad\qquad\qquad i=1,\cdots,K \end{cases}$$

with

Z_t : unobserved (at least some of its components);
$Y_t^i = \Pi^{H_i}(t) = F^{H_i}(t, Z_t)$: observed.

→ *Again, the observation diffusion term depends in general on Z_t. Introduce therefore a further observation noise $\beta_t = (\beta_t^1, \cdots, \beta_t^K)$ considering also Z_t and Y_t^i as only partially observed with observations η_t^i satisfying*

$$d\eta_t^i = Y_t^i dt + d\beta_t^i \ ; \quad i = 1, \cdots, K$$

→ In the context of bond markets this approach has been explicitly implemented in [GR, 2001] without the need of a further observation noise and with the use of the Kalman filter.

An alternative approach

- *The previous approach was based on the assumption that* $\Pi^H(t) = F^H(t, Z_t)$ *and on the conditions imposed on* $F^H(\cdot)$ *for* P *to be a martingale measure.*

- An alternative for having P as a MM is based on a change of numeraire :

  Question : is there a numeraire (portfolio) for which the real world measure becomes a MM ?

  Answer : Yes ! The GOP (*growth optimal portfolio*) which is a self financing portfolio that achieves maximum expected logarithmic utility from terminal wealth.

- **Pricing with GOP as numeraire :** for complete markets the prices coincide with those computed as expectations w.r.to the unique MM. In incomplete markets it corresponds to pricing under the minimal MM.

- *In this way the pricing measure and the measure for filtering are the same and given by the real world probability measure. It allows, in particular, to avoid delicate issues resulting from measure transformation under different information structures (see [PR, 2003])*.