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EQUILIBRIUM ASSET PRICING WITH TIME-VARYING PESSIMISM

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MOTIVATIONS

- The market equity premium is hard to explain

- Anderson & Hansen & Sargent 1998 introduce robust control as a behavior that may explain the puzzle

- Robustness is pessimism

- Chen & Epstein 2002 give recursive-preference-based foundation to pessimism (Recursive Multiple Priors Utility, RMPU)

- We study prices in a tractable and calibrable RMPU equilibrium

RESULTS

ASSUMPTIONS: The agent has RMPU; Her alternative models have Locally Constrained Entropy (LCE) from the reference model; the opportunity set is stochastic; markets can be incomplete

THEORETICAL RESULTS: LCE-RMPU pessimism can be ¯xed in realistic amounts; With log utility, equilibrium prices are analytic; Equity premia are a®ected with a ¯rst-order-in-volatility e®ect; Equity returns are not a®ected

CALIBRATION RESULTS: With log utility and an 11% of pessimism, the unconditional equity premium becomes at least ¯ve times bigger than its non-pessimistic level

THE ECONOMY

- There are two assets, a riskfree asset with rate r and a risky equity with ex-dividend price P

- Equity is a claim on dividends, e

- The opportunity set is governed by the process X

- The model risk process $f(X)$ drives agent's distrust about that reference model

- The distrust $f(X)$ depends on the state of the economy

THE REFERENCE MODEL AND THE ALTERNATIVES AROUND IT

- The reference model on the total returns to equity is

$$\frac{dP + edt}{P} = \alpha_P dt + \sigma_P \left(\begin{array}{c} \rho_P , \sqrt{1 - \rho_P^2} \end{array} \right)' \left(\begin{array}{c} \text{Brownian news for } X \\ \text{Brownian news for } e \end{array} \right)$$

- The alternatives around the reference model are due to a drift-contaminating vector, $h = \left(\begin{array}{c} h^X \\ h^P \end{array} \right)$:

$$E_t^h \left[\frac{dP + edt}{P} \right] = \alpha_P dt + \sigma_P \left(\begin{array}{c} \rho_P , \sqrt{1 - \rho_P^2} \end{array} \right)' \left(\begin{array}{c} h^X \\ h^P \end{array} \right) dt$$

LOCALLY CONSTRAINED ENTROPY (LCE)

- ´ (t) ´ E_t [d (alternative law) =d (reference law)]

- Time-t Relative Entropy for time t + ² = E_t · $\frac{1}{´(t)}$ ´ (t + ²) ¢ ln ´ (t + ²) ¸

- Given a small time increment ²,

$$\frac{\text{Relative Entropy for } t + ² \quad ¡ \quad \text{Relative Entropy for } t}{²} = \frac{1}{2} h^{>} h$$

- We impose LCE, that is, a local bound on the the time growth rate of relative entropy,

$$\frac{1}{2} h^{>} h \;\cdot\; ' f^{2}(X):$$

Local means for any t and state of the world

LCE VERSUS OTHER AMBIGUITY SET CHOICES

- LCE expresses genuine misspecification risk (all the misspecification directions have the same Euclidean distance from the reference belief):

$$\left(h^{\top} h\right)^{\frac{1}{2}} \leq 2\varphi\,|f(X)|.$$

- κ-ignorance (Chen & Epstein 2002) does not (one sets constraints component by component; some misspecification directions have lesser Eucledian distance from the reference model):

$$|h| = \begin{pmatrix} \left|h^{X}\right| \\ \left|h^{P}\right| \end{pmatrix} \leq \kappa.$$

LCE-RMPU

- w is the fraction of wealth, W, allocated to equity. The current consumption rate is cW. Agent's value function is

$$J = \max_{c;w} \min_{h} E_t^h \left[u(cW)\,dt + \exp(-\delta dt)(J + dJ) \right];$$

subject to
$$\frac{dW}{W} = (1 - w) \cdot r \cdot dt + w \cdot \frac{dP + e dt}{P}$$

and to
$$\frac{1}{2} h^{\top} h \le \varphi f^2(X); \quad \text{with } u(cW) = \frac{(cW)^{\gamma} - 1}{\gamma}; \quad \gamma < 1$$

LCE–RMPU VALUE FUNCTION

- The value function is $J(X;W) = \frac{1}{\delta}\frac{\left(e^{g(\gamma;\varphi;X)}W\right)^{\gamma} - 1}{\gamma}$.

- The function g expresses how the agent's welfare is affected by the X-driven stochastic evolution of the opportunity set under the reference model as well as by time variation of her confidence in such model:

$$\frac{\partial}{\partial X} g(\gamma;\varphi;X) = g' \quad ; \quad \frac{\partial^2}{\partial X^2} g(\gamma;\varphi;X) = g''$$

- The closed-form Arrow-Pratt measure of relative risk aversion is

$$-\frac{W J_{WW}}{J_W} = 1 - \gamma:$$

MODEL DETECTION AND AMOUNT OF LCE-RMPU PESSIMISM

- The probability of confusing the reference model ($h = 0$) with the worst case model (h^*) is

$$\frac{1}{2}\exp\left(-N\frac{h^{*\prime}h^{*}}{8}\right) = \frac{1}{2}\exp\left(-N\frac{\varphi}{4}\right); \qquad \frac{1}{2}\exp\left(-400 \times \frac{0.015}{4}\right) = 0.11$$

- After 400 quarterly century-long observations, the residual pessimism is associated to $\varphi = 0.015$.

OPTIMAL POLICIES UNDER LCE-RMPU

- ξ is X's instantaneous volatility. The optimal policies are

  * (consumption)
$$c^* = \left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma - 1}};$$

  * (equity investment)
$$w^* = \left[\frac{1}{1 - \gamma - \left(\frac{2\varphi}{G(w^*)}\right)^{.5}|f|}\right] \times$$
$$\left(\frac{\alpha_P - r}{\sigma_P^2} + \left(\gamma - \left(\frac{2\varphi}{G(w^*)}\right)^{.5}|f|\right) g' \frac{\rho_P \xi}{\sigma_P}\right);$$

$$G(w^*) = \sigma_P^2 w^{*2} + \xi^2 \left(g'\right)^2 + 2w^* \rho_P \sigma_P \xi g'$$

LCE-RMPU: FIRST-ORDER-IN-VOLATILITY EFFECT

- The `hat' symbol denotes equilibrium quantities (market clearing: $\hat{w} = 1$; $\hat{W}\hat{c} = \hat{e}$)

- The log-utility conditional equity premium is

$$\hat{\alpha}_P - \hat{\ } = \hat{\sigma}_P^2 + \left(\frac{2\varphi}{\hat{\sigma}_P^2 + 2\hat{\rho}_P \hat{\sigma}_P \xi \hat{g}' + (\xi \hat{g}')^2} \right)^{.5} |f(X)| \left(\hat{\sigma}_P^2 + \hat{\rho}_P \xi \hat{\sigma}_P \hat{g}' \right).$$

A COX-INGERSOLL-ROSS (CIR) EXAMPLE

CIR dynamics for dividend growth volatility and pessimistic maximal distrust function proportional to dividend growth volatility:

$$
\begin{aligned}
dX &= -\kappa\left(X - \overline{X}\right)dt + \xi(X)^{.5}dZ^X ;\\
\frac{de}{e} &= \alpha_e dt + \sigma_e(X)^{.5}\left[\rho_e dZ^X + \left(1 - \rho_e^2\right)^{.5} dZ^e\right];\\
f(X) &= \left(X/\overline{X}\right)^{.5}.
\end{aligned}
$$

CALIBRATION TO US CONSUMPTION GROWTH DATA

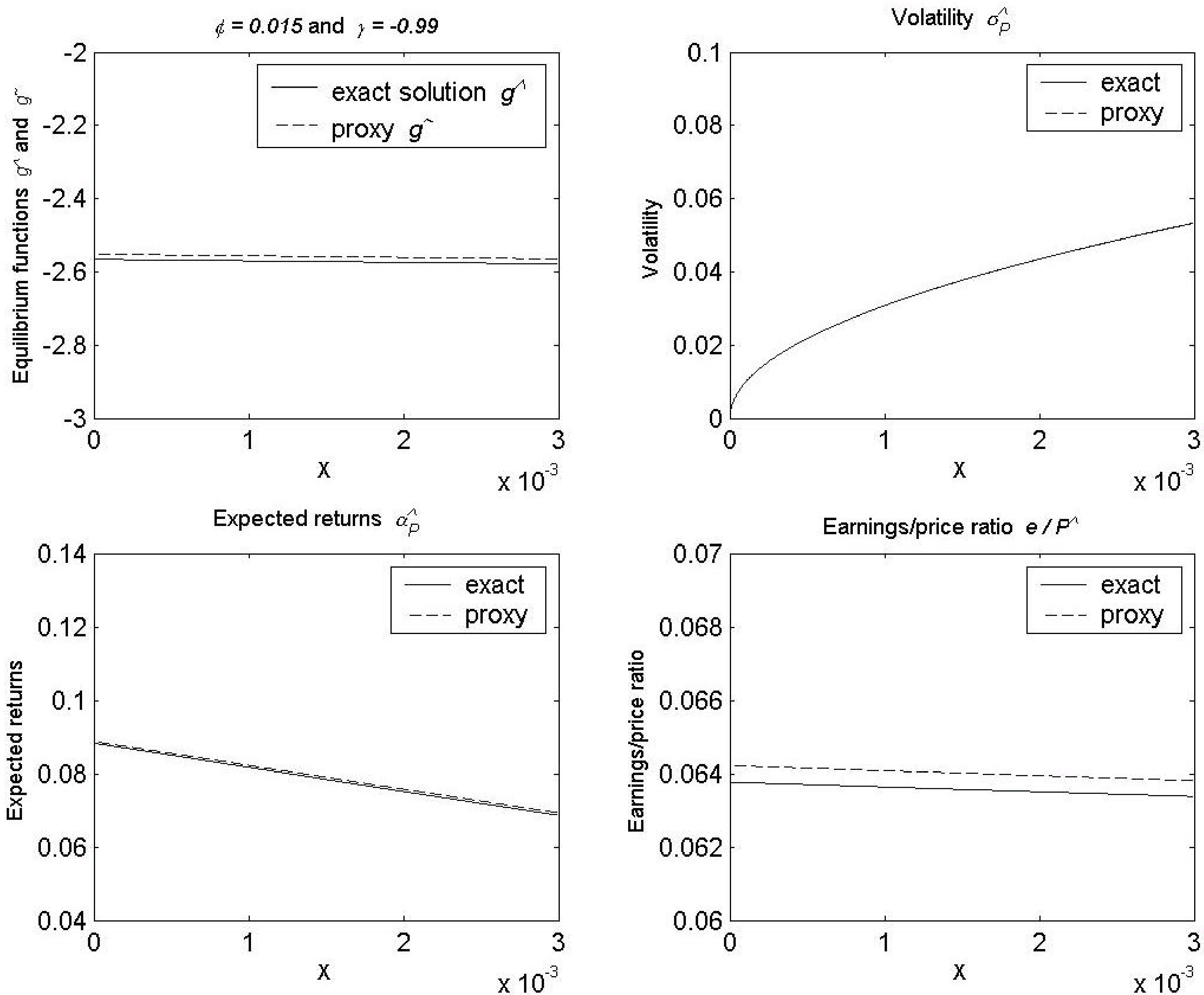
Variable	Mean	Standard deviation
Consumption growth	0.0172	0.0328

$$\Delta X = -3 \cdot \left(X - (0.0328)^2 \right) \cdot \Delta t + 0.0783 \cdot X^{.5} \cdot \text{news for } X;$$

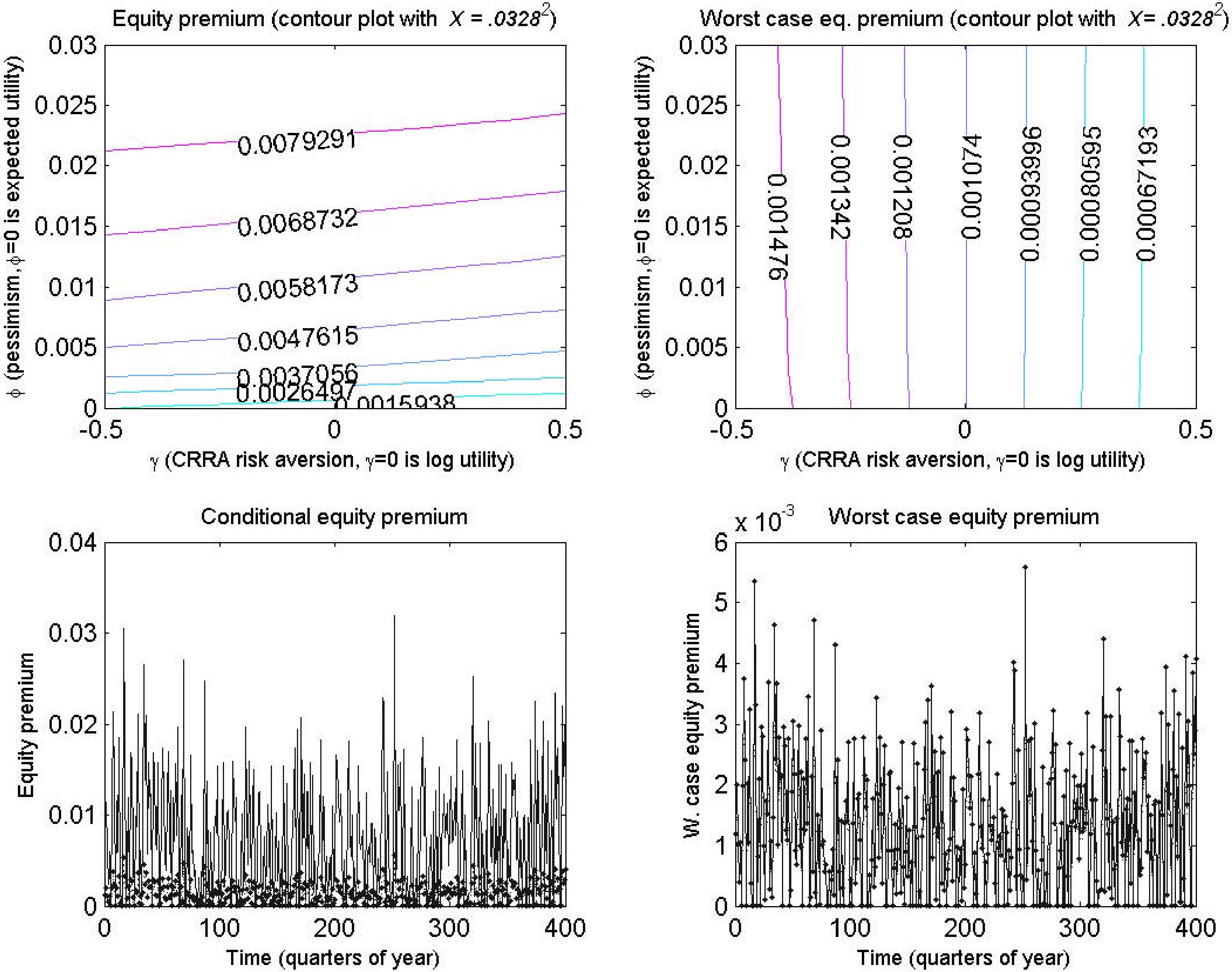
(dividend growth variance)

$$\Delta \ln e = (0.0177 - 0.5 \cdot X) \cdot \Delta t + X^{.5} \cdot \left(-0.25 \cdot \text{news for } X + \left(1 - (-0.25)^2 \right)^{.5} \cdot \text{other news} \right)$$

$$f(X) = \left(X / (0.0328)^2 \right)^{.5}$$



ϕ = 0.015 and γ = -0.99
Equilibrium functions g^ and g~
exact solution g^
proxy g~
-2
-2.2
-2.4
-2.6
-2.8
-3
0
1
2
3
x
x 10^-3
Volatility σ^P
Volatility
exact
proxy
0.1
0.08
0.06
0.04
0.02
0
Expected returns α^P
Expected returns
exact
proxy
0.14
0.12
0.1
0.08
0.06
0.04
Earnings/price ratio e / P^
Earnings/price ratio
exact
proxy
0.07
0.068
0.066
0.064
0.062
0.06



Equity premium (contour plot with X = .0328^2)
φ (pessimism, φ=0 is expected utility)
γ (CRRA risk aversion, γ=0 is log utility)
0.0079291
0.0068732
0.0058173
0.0047615
0.0037056
0.0026497
0.0015938
Worst case eq. premium (contour plot with X= .0328^2)
φ (pessimism, φ=0 is expected utility)
γ (CRRA risk aversion, γ=0 is log utility)
0.001476
0.001342
0.001208
0.001074
0.00093996
0.00080595
0.00067193
Conditional equity premium
Equity premium
Time (quarters of year)
x 10^-3
Worst case equity premium
W. case equity premium
Time (quarters of year)

CONCLUSIONS

- What does a good story of pessimism say on the historical cost of capital?

- A good story of pessimism is LCE-RMPU because

  * it is preference-based

  * it is tractable

  * it ¯xes the amount of pessimism

- LCE-RMPU does have impact on equity premia