INTRODUCTION TO STOCHASTIC CONTROL OF MARKOV DIFFUSIONS

Nizar TOUZI, CREST, Paris, touzi@ensae.fr

Lunteren, January 24-26, 2005

- Conditional expectations and linear parabolic PDE's

- Standard formulation of stochastic control problems

- Dynamic programming principle and HJB equation

- Viscosity solutions

CONDITIONAL EXPECTATIONS AND LINEAR PARABOLIC PDE's

Consider the function :

$$V(t,x) \;:=\; \mathbb{E}_{t,x}\left[\int_t^T f(X_u)\beta(t,u)du + \beta(t,T)g(X_T)\right]$$

where

$$dX_t \;=\; \mu(X_t)dt + \sigma(X_t)dW_t\,, \qquad \beta(t,u) \;:=\; e^{-\int_t^u k(X_v)dv}$$

and

$$\mu \;:\; \mathbb{R}^n \longrightarrow \mathbb{R}^n\,, \qquad \sigma \;:\; \mathbb{R}^n \longrightarrow \mathcal{S}^n_{\mathbb{R}}\,,$$

$$f,\; g,\; k \;:\quad \mathbb{R}^n \longrightarrow \mathbb{R}$$

Second order PDE :

$$\text{(E)} \quad \frac{\partial v}{\partial t}(t,x) + F\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) = 0\,, \qquad t<T,\ x\in\mathcal{O}\subset\mathbb{R}$$

• (E) is parabolic if $F(x,r,p,A)$ is non-increasing in A

• (E) is linear if $F(x,r,p,A)$ is linear in (r,p,A)

• v is a classical super-solution (resp. subsolution) of (E) is $v\in C^{1,2}$ and
$\frac{\partial v}{\partial t} + F\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) \geq 0$ (resp. ≤ 0) on $[0,t)\times\mathbb{R}^n$

Maximum Principle. *Let $\mathcal{O}$ bounded and $F(t,x,r,p,A)$ parabolic strictly increasing in r. Let u (resp. v) be a classical subsolution (resp. super-solution) of (E), with $u\leq v$ on $\partial\{(0,T)\times\mathcal{O}\}$. Then $u\leq v$ on $[0,T]\times\overline{\mathcal{O}}$.*

Dynkin operator

$$\mathcal{L}V(t,x) \;:=\; V_t(t,x) + \mu(x)\cdot DV(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^*(x)D^2V(t,x)\right]$$

$\Longrightarrow$ Tower property : for any $h > 0$

$$\beta(0,t)V(t,x) \;=\; \mathbb{E}_{t,x}\left[\int_t^{t+h}\beta(0,u)f(X_u)du + \beta(0,t+h)V(t+h,X_{t+h})\right]$$

$\Longrightarrow$ if V is smooth, then it follows from Itô's lemma

$$\begin{aligned} 0 \;&=\; \frac{1}{h}\mathbb{E}_{t,x}\left[\int_t^{t+h}\beta(t,u)\left(kV-\mathcal{L}V-f\right)(u,X_u)du + \int_t^T DV(u,X_u)\cdot\sigma(X_u)dW \right. \\ &=\; \frac{1}{h}\mathbb{E}_{t,x}\left[\int_t^{t+h}\beta(t,u)\left\{k(X_u)V(u,X_u)-\mathcal{L}V(u,X_u)-f(X_u)\right\}du\right] \qquad \textbf{!} \end{aligned}$$

send h to zero $\Longrightarrow$ V solves the parabolic linear PDE

$$-\mathcal{L}V(t,x) + k(x)V(t,x) - f(x) \;=\; 0$$

Feynman-Kac representation formula

Cauchy problem

$$-\mathcal{L}v(t,x) + k(x)v(t,x) - f(x) \; = \; 0 \quad \text{and} \quad v(T,x) \; = \; g(x)$$

Theorem *Let v be a classical solution of the above Cauchy problem with $|v(t,x)| \leq C(1+|x|^p)$. Then*

$$v(t,x) \; = \; V(t,x) \; = \; \mathbb{E}_{t,x}\left[\int_t^T f(X_u)\beta(t,u)du + \beta(t,T)g(X_T)\right]$$

- Uniqueness
- Important implication for numerical approximation

Cauchy problem can be solved

by means of Monte Carlo method

STANDARD FORMULATION OF STOCHASTIC CONTROL PROBLEMS

- *Control process* $\nu \in \mathcal{U}_0$

ν_t v.a. $\mathcal{F}_t -$ measurable with values in $U \subset \mathbb{R}^k$

- *Controlled process* For $\nu \in \mathcal{U}_0$, define X^ν by

EDS(ν) $dX_t^\nu = b(X_t^\nu, \nu_t)dt + \sigma(X_t^\nu, \nu_t)dW_t$ and X_0^ν given

where

$b : \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n, \quad \sigma : \mathbb{R}^n \times U \longrightarrow \mathcal{M}_{\mathbb{R}}^{n,d}$ Lip in x unif. in u

- *Admissible control process* $\nu \in \mathcal{U}$ if

EDS(ν) has a unique solution in L^2 for every initial data $X_0 = x$

REWARD CHARACTERISTICS

$$f, k \;:\; \mathbb{R}^n \times U \longrightarrow \mathbb{R} \quad \text{and} \quad g \;:\; \mathbb{R}^n \longrightarrow \mathbb{R}$$

with

$$k \geq 0 \quad \text{and} \quad |f(t,x,u)| + |g(x)| \;\leq\; C(1 + |x|^2)$$

- f : cont. reward rate

- g : terminal reward

- k : discount rate

STOCHASTIC CONTROL PROBLEM

$$V(t,x) \;:=\; \sup_{\nu\in\mathcal{U}} J(t,x,\nu)$$

where

$$J(t,x,\nu) \;:=\; \mathbb{E}_{t,x}\left[\int_t^T \beta^\nu(t,s) f\left(X_s^\nu, \nu_s\right) ds + \beta^\nu(t,T) g\left(X_T^\nu\right)\right]$$

with the discount factor

$$\beta^\nu(t,s) \;:=\; e^{-\int_t^s k(X_r^\nu,\nu_r)dr}$$

Goal : caracterize the local behavior of V by means of

the *Hamilton-Jacobi-Bellman* equation

SOME VOCABULARY

- $\widehat{\nu}$ is an *optimal control* if

$$\widehat{\nu} \in \mathcal{U} \quad \text{and} \quad V(t,x) \;=\; J(t,x,\widehat{\nu}_{t,x})$$

- $\nu \in \mathcal{U}$ is a *feedback control*

$$\nu \text{ is adapted to } \mathbb{F}^X$$

- ν is a *Markov control* if

$$\nu_s \;=\; \tilde{u}(s, X_s) \quad \text{for some measurable function} \quad u$$

- ν is an *open-loop control* if

$$\nu \quad \text{is deterministic}$$

DYNAMIC PROGRAMMING PRINCIPLE

Theorem *For any stopping time τ with values in $[t,T]$:*

$$V(t,x) \;=\; \sup_{\nu\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^\tau \beta^\nu(t,s) f\left(s,X_s^\nu,\nu_s\right) ds + \beta^\nu(t,\tau) V\left(\tau,X_\tau^\nu\right)\right]$$

• Basic tool of stochastic control / compare with tower property

• *Main ingredient* : concatenation of control processes

• In finite discrete time

$$V(t,x) \;=\; \sup_{u\in U} \mathbb{E}_{t,x}\left[f\left(X_t^\nu,u\right) + e^{-k(X_t^\nu,\nu_t)} V\left(t+1,X_{t+1}^\nu\right)\right]$$

$\Longrightarrow$ Reduction to a (backward) sequence of finite-dimensional optimization problems

REDUCTION TO MAYER FORM ($f = k \equiv 0$)

Consider new controlled processes (Y, Z) :

$$dY^{\nu_s} = Z_s\, f(X_s, \nu_s)ds \quad \text{and} \quad dZ^{\nu_s} = -Z_s\, k(X_s, \nu_s)ds$$

$\Longrightarrow$ Augmented controlled process

$$\bar{X} := (X, Y, Z)$$

Then $V(t, x) = \bar{V}(t, x, 0, 1)$, where

$$\bar{V}(t, \bar{x}) := \sup_{\nu \in \mathcal{U}} \mathbb{E}_{t,x}\left[\bar{g}\left(\bar{X}_T^{t,x}\right)\right] \quad \text{and} \quad \bar{g}(x, y, z) := y + g(x)z$$

HAMILTON-JACOBI-BELLMAN EQUATION

Denote

$$\mathcal{L}^u v(t,x) \;:=\; b(x,u)\cdot Dv(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^*(x,u)D^2 v(t,x)\right]$$

$$H(x,r,p,A) \;:=\; \sup_{u\in U}\left\{-k(x,u)r + b(x,u)\cdot p + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^*(x,u)A] + f(x,u)\right\}$$

Proposition *If* $V \in C^{1,2}([0,T),\mathbb{R}^n)$ *:*

$$-\frac{\partial V}{\partial t}(t,x) - H\left(x, V(t,x), DV(t,x), D^2V(t,x)\right) \;\geq\; 0$$

i.e. V *is a super-solution of the associated HJB equation*

Proof of super-solution property. $(t,x) \in [0,T) \times \mathbb{R}^n$, $u \in U$ fixed, constant control $\nu_s = u$, controlled process X^u, and

$$\tau_h \;:=\; (t+h) \wedge \inf\{s > t \;:\; |X_s^u - x| \geq 1\}$$

Dynamic programming and Itô's lemma :

$$\begin{aligned}
0 \;&\leq\; \frac{1}{h}\,\mathbb{E}_{t,x}\left[\beta(0,t)V(t,x) - \beta(0,\tau_h)V(\tau_h, X_{\tau_h}) - \int_t^{\tau_h} \beta(0,r) f(r, X_r, \nu_r) dr\right] \\
&=\; -\frac{1}{h}\,\mathbb{E}_{t,x}\left[\int_t^{\tau_h} \beta(0,r)(-kV + V_t + \mathcal{L}^u V + f)(r, X_r, u) dr\right] \\
&\qquad\qquad - \frac{1}{h}\,\mathbb{E}_{t,x}\left[\int_t^{\tau_h} \beta(0,r) DV(r, X_r)^* \sigma(X_r, u) dW_r\right] \\
&=\; -\frac{1}{h}\,\mathbb{E}_{t,x}\left[\int_t^{\tau_h} \beta(0,r)(-kV + V_t + \mathcal{L}^u V + f)(r, X_r, u) dr\right]
\end{aligned}$$

Finally, send h to zero, and use the dominated convergence theorem

Proposition *If $V \in C^{1,2}([0,T), \mathbb{R}^n)$, and H is continuous, then :*

$$-\frac{\partial V}{\partial t}(t,x) - H\left(x, V(t,x), DV(t,x), D^2V(t,x)\right) \;=\; 0$$

$\Longrightarrow$ Proof... more technical

In order to complete the characterization of V :

(i) Terminal condition

(ii) Uniqueness result

VERIFICATION RESULT

Theorem $v \in C^{1,2}([0,T), \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *with* $|v(t,x)| \leq C(1+|x|^2)$

(i) *If* $v(T,\cdot) \geq g$ *and* $-v_t(t,x) - H\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) \geq 0$.

Then $v \geq V$

(ii) *Assume further that*

- $v(T,.) = g$ *and* $0 = v_t(t,x) + \mathcal{L}^{\hat{u}(t,x)}v(t,x) + f(t,x,u)$
- *there is a unique solution for the SDE*

$$dX_s = b\left(X_s, \hat{u}(s,X_s)\right)ds + \sigma\left(X_s, \hat{u}(s,X_s)\right)dW_s \quad \text{for any} \quad X_0 = x$$

- $\hat{\nu} \in \mathcal{U}$, *where* $\hat{\nu}_s := \hat{u}(s,X_s)$

Then $v = V$, *et* $\hat{\nu}$ *is a (Markov) optimal control*

Sketch of the proof

(i) Let $\nu \in \mathcal{U}$, $X = X^\nu$, $X_t = x \Longrightarrow$ Itô's lemma :

$$\begin{aligned} v(t,x) &= \beta(t,T)v\left(T, X_T^\nu\right) \\ &\quad - \int_t^T \beta^\nu(t,r)(-kv + v_t + \mathcal{L}^{\nu(r)}v)(r, X_r^\nu)dr \\ &\quad - \int_t^T \beta^\nu(t,r)Dv(r, X_r^\nu)\cdot\sigma(r, X_r^\nu, \nu_r)dW_r \end{aligned}$$

Since $-v_t + kv - \mathcal{L}^u v - f(\cdot,u) \geq -v_t - H(\cdot, v, Dv, D^2 v) \geq 0$:

$$\begin{aligned} v(t,x) &\geq \mathbb{E}_{t,x}\left[\beta^\nu(t,T)v\left(T, X_T^\nu\right) + \int_t^T \beta^\nu(t,r)f(X_r^\nu, \nu_r)dr\right] \\ &\geq \mathbb{E}_{t,x}\left[\beta^\nu(t,T)g\left(X_T^\nu\right) + \int_t^T \beta^\nu(t,r)f(X_r^\nu, \nu_r)dr\right] \end{aligned}$$

(ii) inequalities are in fact equalities with the control $\hat{\nu}$

ON THE REGULARITY OF THE VALUE FUNCTION

$f = k \equiv 0$ (Mayer's formulation)

Proposition (i) *g Lipschitz, then $V(t, \cdot)$ is Lipschitz-continuous*

(ii) *U bounded, then $V(\cdot, x)$ is* $(1/2)-$*Hölder-continuous*

Example. Let $U = \mathbb{R}$, $\mathcal{U} := \{$bounded predictable processes valued in$U\}$,

$$dX_t^\nu \;=\; X_t^\nu \, \nu_t \, dW_t \quad \text{and} \quad V(t,x) \;:=\; \sup_{\nu \in \mathcal{U}} \mathbb{E}_{t,x}\left[g\left(X_T^\nu\right)\right]$$

where g is l.s.c. and bounded from below. Then

$$V(t,x) \;=\; g^{conc}(x) \qquad g^{conc} \text{ is the concave envelope of } g$$

V not continuous at $t = T$ and not C^1 in x, in general.

VISCOSITY SOLUTIONS

Consider the elliptic PDE

$$\text{(E)} \quad F\left(z, v(z), Dv(z), D^2v(z)\right) = 0 \quad \text{for} \quad z \in \mathcal{O} \text{ open subset of } \mathbb{R}^d$$

($F(z,r,p,A)$ non-increasing in A)

• $v : \mathcal{O} \longrightarrow \mathbb{R}$ l.s.c. is a viscosity super-solution of (E) if, for every $(z_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$:

$$(v-\varphi)(z_0) = \min_{\mathcal{O}}(v-\varphi) \implies F\left(z_0, v(z_0), D\varphi(z_0), D^2\varphi(z_0)\right) \geq 0$$

• $v : \mathcal{O} \longrightarrow \mathbb{R}$ u.s.c. is a viscosity sub-solution of (E) if, for every $(z_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$:

$$(v-\varphi)(z_0) = \max_{\mathcal{O}}(v-\varphi) \implies F\left(z_0, v(z_0), D\varphi(z_0), D^2\varphi(z_0)\right) \leq 0$$

Semi-continuous envelopes :

$$v_*(z) \;:=\; \liminf_{z' \to z} \; v(z') \quad \text{and} \quad v_*(z) \;:=\; \limsup_{z' \to z} \; v(z')$$

finite for locally bounded $v \; : \; \mathbb{R}^d \longrightarrow \mathbb{R}$

Proposition (i) *If V is locally bounded, then*

$$-\frac{\partial V_*}{\partial t}(t,x) - H\left(x, V_*(t,x), DV_*(t,x), D^2V_*(t,x)\right) \;\geq\; 0$$

i.e. V_* is a super-solution of the associated HJB equation

(ii) *If in addition H is continuous, then*

$$-\frac{\partial V^*}{\partial t}(t,x) - H\left(x, V^*(t,x), DV^*(t,x), D^2V^*(t,x)\right) \;\leq\; 0$$

i.e. V^* is a sub-solution of the associated HJB equation

UNIQUE CHARACTERIZATION AND CONTINUITY

• Boundary condition :

$V_*(T,x)$ and $V^*(T,x)$ might not be given by the natural BC $g(x)$

(Recall our example)

• If we can prove that $V_*(T,x) \geq V^*(T,x)$ and that Maximum principle in the viscosity sense holds, then :

$$V_* = V^* \text{ on } [0,T] \times \mathbb{R}^n$$

$\Longrightarrow V$ is the unique continuous viscosity solution in a certain class.

SUPER-HEDGING UNDER PORTFOLIO CONSTRAINTS

Nizar TOUZI, CREST, Paris, touzi@ensae.fr

Lunteren, January 24-26, 2005

- Problem formulation

- Dual formulation

- Geometric dynamic programming and HJB equation

- Boundary condition : *face-lifting*

- Explicit solution in the Black-Scholes model

PROBLEM FORMULATION : the financial market

• 1 non-risky asset $S^0 \equiv 1$ ($r = 0$, *change of numéraire*)

• d risky assets S :

$$\frac{dS_t^i}{S_t^i} = \mu_t^i \, dt + \sum_{j=1}^{d} \sigma_t^{ij} \, dW_t^j \,, \qquad i = 1, \ldots, d$$

μ, σ and σ^{-1} bounded $\mathbb{F}-$adapted with values respectively in $\mathbb{R}^d$ and $\mathcal{S}_{\mathbb{R}}^d$

• *Wealth process* $X^{x,\pi}$, under *self-financing* condition, defined by

$$dX_0^{x,\pi} = x \quad \text{and} \quad dX_t^{x,\pi} = \sum_{i=1}^{d} X_t^{x,\pi} \pi_u^i \frac{dS_u^i}{S_u^i} = X_t^{x,\pi} \pi_u \cdot (\mu_u du + \sigma_u dW_u)$$

• $\pi \in \mathcal{A}$: admissible portfolio if

$$\int_0^T |\sigma_u^* \pi_u|^2 du < \infty$$

PROBLEM FORMULATION, portfolio constraints

Let K be a closed convex (!) subset of $\mathbb{R}^d$ containing 0

• $K-$ admissible portfolio : $\pi \in \mathcal{A}_K$ if

$$\pi \in \mathcal{A} \text{ and } \pi_u \in K \text{ Leb} \otimes \mathbb{P} - \text{a.s.}$$

Example 1 *No short-selling* : $K = \left\{ x \in \mathbb{R}^d : x^i \geq 0 \right\}$

Example 2 *Incomplete market* : $K = \left\{ x \in \mathbb{R}^d : x^{i_0} = 0 \right\}$

Example 3 *No borrowing* : $K = \left\{ x \in \mathbb{R}^d : \sum_i x^i \leq 1 \right\}$

Example 4 *Rectangular constraints* : $K = \left\{ x \in \mathbb{R}^d : m^i \leq x^i \leq M^i \right\}$

Example 5 *Finite capitalizations* : change model expressing portfolios in terms of number of shares...

PROBLEM FORMULATION, the super-replication problem

• Contingent claim G : $\mathcal{F}_T-$measurable random variable, we will mainly consider $G = g(S_T)$ with

$$g \ : \ [0,\infty) \longrightarrow \mathbb{R} \quad \text{l.s.c. and bounded from below}$$

• Super-replication problem

$$V(0,S_0) \ := \ \inf\left\{x \ : \ X_T^{x,\pi} \geq G \text{ a.s. for some } \pi \in \mathcal{A}_K\right\}$$

$\Longrightarrow$ Stochastic control problem in non-standard form !

$\Longrightarrow$ Connection with backward stochastic differential equations

$\Longrightarrow$ Very difficult to reach any *a-priori* regularity result

$\Longrightarrow$ 1^{st} idea : *reduce to the classical setting,* i.e. standard formulation

DUAL FORMULATION : dual characterization of the constraints

Support function of K :

$$\delta(y) \ := \ \sup_{x \in K} \ x \cdot y$$

Effective domain of δ :

$$\tilde{K} \ := \ \left\{ y \in \mathbb{R}^d \ : \ \delta(y) < \infty \right\}$$

Lemma Let K be a closed convex subset of $\mathbb{R}^n$. Then

$$x \ \in \ K \ \Longleftrightarrow \ \delta(y) - x \cdot y \ \geq \ 0 \text{ for all } y \in \tilde{K}$$

DUAL FORMULATION : dual variables

Let $\mathcal{D} := \left\{ \text{bounded } \mathbb{F}-\text{adapted processes with values in } \tilde{K} \right\}$

$$\left.\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\right|_{\mathcal{F}_T} := \exp\left[\int_0^T \sigma_u^{-1}\left(\nu_u - \mu_u\right) \cdot dW_u - \frac{1}{2}\int_0^T \left|\sigma_u^{-1}(\nu_u - \mu_u)\right|^2 du\right]$$

By Girsanov's Theorem, the process

$$W_u^\nu := W_u - \int_0^u \sigma_u^{-1}\left(\nu_u - \mu_u\right) du \qquad 0 \le u \le T$$

is a Brownian motion under P^ν, and

$$d\left(X_t^{x,\pi} e^{-\int_0^t \delta(\nu_u) du}\right) = X_t^{x,\pi} e^{-\int_0^t \delta(\nu_r) dr}\left[-\left(\delta(\nu_t) - \pi_t \cdot \nu_t\right) dt + \sigma_u dW_u^\nu\right]$$

$$\Longrightarrow \text{The process } \left\{X_t^{x,\pi} e^{-\int_0^t \delta(\nu_u) du},\ 0 \le t \le T\right\}$$

is a $\mathbb{P}^\nu-$super-martingale for every $\pi \in \mathcal{A}_K$ and $\nu \in \mathcal{D}$

DUAL FORMULATION :

reducing to a standard stochastic control problem

Theorem $$V(0,S_0) \;=\; \tilde{V}(0,S_0) \;:=\; \sup_{\nu\in\mathcal{D}} \;\mathbb{E}^{\mathbb{P}^\nu}\left[G\; e^{-\int_0^T \delta(\nu_u)du}\right]$$

<ElKaroui-Quenez 1995, Cvitanić-Karatzas 1993, Föllmer-Kramkov 1999>

$G = g\left(S_T\right)$ and S is a Markov diffusion $\Longrightarrow$ Girsanov's Theorem

$$V(0,S_0) \;=\; \tilde{V}(0,S_0) \;:=\; \sup_{\nu\in\mathcal{D}} \;\mathbb{E}\left[g\left(S_T^\nu\right)\; e^{-\int_0^T \delta(\nu_u)du}\right]$$

where

$$S_0^\nu = S_0 \;\text{ and }\; dS_t^\nu \;=\; \text{diag}\left[S_t^\nu\right]\left(\nu_t dt + \sigma\left(S_t^\nu\right)dW_t\right)$$

Stochastic control problem in standard form

DUAL FORMULATION : the HJB equation

From general theory, if V is locally bounded, then

$$-(V_*)_t - \frac{1}{2}\mathrm{Tr}\left[\overline{\sigma\sigma}^*(s)D^2V_*\right] - \mathrm{diag}[s]y \cdot DV_* + \delta(y)V_* \ \geq \ 0 \quad \text{for all} \quad u \in \tilde{K}$$

in the viscosity sense (super-solution property), where $\overline{\sigma}(s) := \mathrm{diag}[s]\sigma(s)$

Since $\tilde{K}$ is a cone, this is equivalent to

$$\min\left\{-(V_*)_t - \frac{1}{2}\mathrm{Tr}\left[\overline{\sigma\sigma}^*(s)D^2V_*\right] \, , \, \inf_{y\in\tilde{K}_1} \left(\delta(y)V_* - \mathrm{diag}[s]y \cdot DV_*\right)\right\} \ \geq \ 0$$

where $\tilde{K}_1 := \left\{y \in \tilde{K} \ : \ |y| = 1\right\}$

We will see later that this is the HJB equation for our problem

FROM NOW ON : MARKOV MODEL

• Risky assets dynamics :

$$\frac{dS_t^i}{S_t^i} = \mu^i(t, S_t)\, dt + \sum_{j=1}^{d} \sigma^{ij}(t, S_t)\, dW_t^j\,, \qquad i = 1, \ldots, d$$

μ and σ Lipschitz, linearly growing, and we will usually forget about the dependence upon t.

• Contingent claim

$$G = g(S_T)$$

for some

$$g \;:\; [0, \infty)^d \longrightarrow \mathbb{R} \text{ l.s.c. and bounded from below}$$

GEOMETRIC DYNAMIC PROGRAMMING PRINCIPLE

- *Trivial claim :* Let (t,s), x, $\pi \in \mathcal{A}_K$ be such that $X_T^{x,\pi} \geq g\left(S_T^{t,s}\right)$. Then

$$X_\tau^{x,\pi} \;\geq\; V\left(\tau, S_\tau\right) \;\; \text{for every stopping time} \;\; \tau \in [t,T] \text{ a.s.}$$

In fact, we have the following *geometric dynamic programming principle* (without dual formulation)

Theorem. *For all $(t,s) \in [0,T) \times \mathbb{R}_+$, and stopping time $\tau \in [t,T]$ a.s.*

$$V(t,s) \;=\; \inf\left\{x \;:\; X_\tau^{x,\pi} \geq V\left(\tau, S_\tau\right) \text{ a.s. for some } \pi \in \mathcal{A}_K\right\}$$

$\Longrightarrow$ Super-solution property

Proposition $-\dfrac{\partial V_*}{\partial t} - \dfrac{1}{2}\mathrm{Tr}\left[\overline{\sigma\sigma}^* D^2 V_*\right] \geq 0$ *and* $\dfrac{\mathrm{diag}[s] D V_*}{V_*} \in K$

Sketch of proof (super-solution property)

For simplicity, assume V is smooth and

$$V(t,s) \;:=\; \underline{\min}\left\{x \;:\; X_T^{x,\pi} \geq g\left(S_T\right) \text{ for some } \pi \in \mathcal{A}_K\right\}$$

Then, starting from initial wealth $\widehat{x} := V(t,s)$:

$$X_T^{\widehat{x},\widehat{\pi}} \;\geq\; g\left(S_T^{t,s}\right) \quad \text{for some} \quad \widehat{\pi} \in \mathcal{A}_K$$

$\Longrightarrow$ Geometric dynamic programming

$$X_\tau^{\widehat{x},\widehat{\pi}} \;=\; V(t,s) + \int_t^\tau X_u^{\widehat{x},\widehat{\pi}} \widehat{\pi}_u \left[\mu_u du + \sigma_u dW_u\right] \;\geq\; V\left(\tau, S_\tau^{t,s}\right)$$

$\Longrightarrow$ Itô's lemma

$$0 \;\leq\; -\int_t^\tau \mathcal{L}V\left(u, S_u^{t,s}\right) du + \int_t^\tau \sigma_u \left(X_u^{\widehat{x},\widehat{\pi}} \widehat{\pi}_u - \text{diag}[S_u] DV(u, S_u)\right) dW_u^0$$

Sketch of proof (super-solution property), continued

$$0 \;\leq\; -\int_t^\tau \mathcal{L}V\left(u, S_u^{t,s}\right) du + \int_t^\tau \sigma_u \left(X_u^{\widehat{x},\widehat{\pi}} \widehat{\pi}_u - \mathrm{diag}[S_u] DV(u, S_u)\right) dW_u^0$$

1. Set $\tau_h := (t+h) \wedge \inf\{u > t \;:\; |\ln S_u - \ln s| \geq 1\}$, and take expected values $\Longrightarrow$ $-\mathcal{L}V \;\geq\; 0$

2. **Lemma.** (Loc. behavior of stoch. int.) *Let b be a predictable $W-$integrable process satisfying $\int_0^t b_s \cdot dW_s \geq -C\,t$, $0 \leq t \leq \tau$, for some $C > 0$ and positive stopping time τ. Then $\liminf_{t \searrow 0} \frac{1}{t} \int_0^t |b_s| ds = 0$ $\mathbb{P}-$a.s.*

$\Longrightarrow$ $\dfrac{\mathrm{diag}[s] DV}{V} \in K$, or equivalently $\inf_{y \in \tilde{K}_1} \left(\delta(y) - \dfrac{\mathrm{diag}[s] y \cdot DV}{V}\right) \geq 0$

CHARACTERIZING THE TERMINAL CONDITION :

implications from the HJB equation

We have of course $V(T,s) = g(s)$, by definition. Let

$$\overline{V}(s) := \liminf_{(t',s')\to(T,s)} V(t,s) \qquad [= V_*(T,s)]$$

Lemma *We have* $\overline{V} \geq g$ *and* $\dfrac{\text{diag}[s] D\overline{V}}{\overline{V}} \in K$.

The latter condition might not be satisfied by g*. Then*

$$\overline{V} \neq g \quad \textit{in general}$$

Sketch of proof (implications from HJB)

• $g \geq C$ *and l.s.c.* $\Longrightarrow V \geq g$ *(Fatou's lemma)*

• *For* $t < T$,

$$\delta(y)V(t,s) - y \cdot \text{diag}[s]DV(t,s) \ \geq \ 0 \quad \textit{for every} \quad y \in \tilde{K}$$

or equivalently,

$$\alpha \longmapsto \ln \overline{V}\,(se^{\alpha y}) - \delta(y)\alpha \qquad \textit{is non-decreasing}$$

$\Longrightarrow$ *send* t *to* T...

CHARACTERIZING THE TERMINAL CONDITION :

face-lifting

Lemma $\overline{V}(s) \geq \hat{g}(s) := \sup_{y \in \tilde{K}} g\left(se^{y}\right) e^{-\delta(y)}$

Proof *For every* $y \in \tilde{K}$ *:* $0 \leq \delta(y)\overline{V}(s) - y \cdot \mathrm{diag}[s] D\overline{V}$

$$\Longrightarrow \quad 0 \;\leq\; \delta(y) - \frac{\partial}{\partial \alpha} \ln \overline{V}\left(se^{\alpha y}\right)$$

integrate between $\alpha = 0$ *and* $\alpha = 1$, *and recall* $\overline{V} \geq g$ *:*

$$\overline{V}(s) \;\geq\; \overline{V}\left(se^{y}\right) e^{-\delta(y)} \;\geq\; g\left(se^{y}\right) e^{-\delta(y)}$$

y *is arbitrary in* $\tilde{K}$...

CHARACTERIZING THE TERMINAL CONDITION :

properties of the face-lifting operator

- $\widehat{g} \geq g$ (*$\widehat{g}$ majorant of g*)

- $\dfrac{\mathrm{diag}[s]D\widehat{g}}{\widehat{g}} \in K$ (*satisfies the constraints*)

- $\widehat{\widehat{g}} = \widehat{g}$ (*"projection" property*)

- *If h is such that $h \geq g$ and* $\dfrac{\mathrm{diag}[s]Dh}{h} \in K$, *then* $h \geq \widehat{g}$ (*minimality*)

$\widehat{g}$ is the smallest majorant of g

which satisfies the constraints

CHARACTERIZING THE TERMINAL CONDITION :

Examples for $d = 1$, $K = [-\ell, u] \ni 0$

European call option $\quad g(s) = (s - \kappa)^+$

$$\hat{g}(s) = \begin{cases} (s-\kappa) & \textit{pour } s \geq \frac{\kappa u}{u-1} \\ \frac{\kappa}{u-1}\left(\frac{(u-1)s}{\kappa u}\right)^u & \textit{pour } s \leq \frac{\kappa u}{u-1} \end{cases}$$

European put option $\quad g(s) = (\kappa - s)^+$

$$\hat{g}(s) = \begin{cases} (\kappa - s) & \textit{pour } s \leq \frac{\kappa \ell}{\ell+1} \\ \frac{s}{\ell+1}\left(\frac{\kappa \ell}{(\ell+1)s}\right)^\ell & \textit{pour } s \leq \frac{\kappa u}{u-1} \end{cases}$$

EXPLICIT RESULT IN THE BLACK-SCHOLES MODEL

<Broadie-Cvitanić-Soner 1998>

Theorem *For constant* σ, *we have* $V(t,s) = \mathbb{E}^{\mathbb{P}^0}_{t,s}\left[\widehat{g}\left(S_T\right)\right]$, *and the optimal hedging strategy is the classical Black-Scholes hedging strategy of the face-lifted contingent claim* $\widehat{g}(S_T)$

In the more general local volatility model $\sigma(t,s)$ *:*

Theorem *Under some conditions,* V *is the unique (in a certain class) continuous viscosity solution of the associated HJB equation*

$$\min\left\{-V_t - \frac{1}{2}\mathrm{Tr}\left[\overline{\sigma\sigma}^*(s)D^2V\right]\ ,\ \inf_{y\in\tilde{K}_1}\left(\delta(y)V - \mathrm{diag}[s]y\cdot DV\right)\right\} \ = \ 0$$

Proof of Broadie-Cvitanić-Soner's result

Denote $w(t,s) := \mathbb{E}^{\mathbb{P}^0}_{t,s}[\widehat{g}(S_T)]$

(i) $w(t,s) - V(t,s) \leq \mathbb{E}^{\mathbb{P}^0}_{t,s}\left[\overline{V}(S_T) - V_*(t,s)\right] = \mathbb{E}^{\mathbb{P}^0}_{t,s}\left[\int_t^T \mathcal{L}V_*(u,S_u)\right] \leq 0$

(iia) $\delta(y)w(t,s) - y \cdot \text{diag}[s]Dw(t,s) = \mathbb{E}^{\mathbb{P}^0}_{t,s}[\delta(y)\widehat{g}(S_T) - y \cdot \text{diag}[S_T]D\widehat{g}(S_T)]$

≥ 0 *for all* $y \in \tilde{K}$

(iib) $\mathcal{L}w = 0 \Longrightarrow$ *set* $\widehat{\pi}_u := \dfrac{\text{diag}[s]Dw(u,S_u)}{w(u,S_u)}$, *and apply Itô's lemma :*

$$\begin{aligned} \widehat{g}(S_T) &= w(T,S_T) \\ &= w(t,s) + \int_t^T \mathcal{L}w(u,S_u)du + \int_t^T w(u,S_u)\widehat{\pi}_u \cdot \text{diag}[S_u]^{-1}dS_u \\ &= w(t,s) + \int_t^T w(u,S_u)\widehat{\pi}_u \cdot \text{diag}[S_u]^{-1}dS_u \;=\; X_T^{w(t,s),\widehat{\pi}} \end{aligned}$$

Since $\widehat{g} \geq g$, *this implies that* $w(t,s) \geq V(t,s)$

PROOF OF SUBSOLUTION PROPERTY

IN THE LOCAL VOLATILITY MODEL

Consider the simple case $\mathrm{int}(K) \neq \emptyset$, *and show that*

$$\min\left\{-V_t^* - \frac{1}{2}\mathrm{Tr}\left[\overline{\sigma\sigma}^*(s)D^2V^*\right]\ ,\ \inf_{y\in\tilde{K}_1}\left(\delta(y)V^* - \mathrm{diag}[s]y\cdot DV^*\right)\right\} \ \leq\ 0$$

in the viscosity sense. Let $(t_0, s_0) \in [0,T) \times \mathbb{R}^d_+$, $\varphi \in C^2$ *be such that*

$$0 \ = \ (V^* - \varphi)(t_0, s_0) \ = \ \max\,(\text{strict}) \ (V^* - \varphi)$$

and assume to the contrary that

$$f(t_0, s_0) \ := \ \left(-\varphi_t - \frac{1}{2}\mathrm{Tr}\left[\overline{\sigma\sigma}^* D^2\varphi\right]\right)(t_0, s_0) \ > \ 0$$

$$\textit{and} \qquad \hat{\pi}(t_0, s_0) \ := \ \frac{\mathrm{diag}[s_0]D\varphi(t_0, s_0)}{\varphi(t_0, s_0)} \ \in \ \mathrm{int}(K)$$

PROOF OF SUBSOLUTION PROPERTY, continued (2)

Define the open neighborhood of (t_0, s_0) *:*

$$\mathcal{N} \;:=\; \{(t,s) \;:\; |(t, \ln s) - (t_0, \ln s_0)| \le 1,\; f(t,s) \ge 0 \text{ and } \widehat{\pi}(t,s) \in K\}$$

Since (t_0, s_0) *is a point of strict maximum of* $V^* - \varphi$*, we have*

$$\max_{\partial\mathcal{N}} \left(\ln V^* - \ln \varphi\right) \;=:\; -3\,\eta \;<\; 0$$

Choose $(t_1, s_1) \in \text{int}(\mathcal{N})$ *so that*

$$|\ln V(t_1, s_1) - \ln \varphi(t_1, s_1)| \;\le\; \eta$$

Take (t_1, s_1) *as initial data for the process* S*, and define*

$$\tau \;:=\; \inf\{u > t_1 \;:\; (u, S_u) \notin \mathcal{N}\}$$

PROOF OF SUBSOLUTION PROPERTY, continued (3)

Consider the initial capital $\widehat{x} := V(t_1, s_1)e^{-\eta}$, *and compute that*

$$\begin{aligned}
\ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln V(\tau, S_\tau) &\geq \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln V^*(\tau, S_\tau) \\
&\geq \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln \varphi(\tau, S_\tau) + 3\eta \\
&\geq \ln X_\tau^{\varphi(t_1,s_1),\widehat{\pi}} - \ln \varphi(\tau, S_\tau) + \eta
\end{aligned}$$

Next observe that

$$\begin{aligned}
\frac{d\varphi(t, S_t)}{\varphi(t, S_t)} &= \frac{\mathcal{L}\varphi(t, S_t)}{\varphi(t, S_t)}\, dt + \widehat{\pi}(t, S_t) \cdot \mathrm{diag}[S_t]^{-1} dS_t \\
&= \frac{\mathcal{L}\varphi(t, S_t)}{\varphi(t, S_t)}\, dt + \frac{dX_t^{\varphi(t_1,s_1),\widehat{\pi}}}{X_t^{\varphi(t_1,s_1),\widehat{\pi}}}
\end{aligned}$$

PROOF OF SUBSOLUTION PROPERTY, continued (4)

Since $\mathcal{L}\varphi \leq 0$ *for* $t \in [t_1, \tau]$*, and* $X_{t_1}^{\varphi(t_1,s_1),\widehat{\pi}} = \varphi(t_1, s_1)$*, this implies that*

$$X_\tau^{\varphi(t_1,s_1),\widehat{\pi}} \;\geq\; V(\tau, S_\tau)$$

Hence, starting from the initial capital $\widehat{x} := V(t_1, s_1)e^{-\eta}$*, we have*

$$\begin{aligned} \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln V(\tau, S_\tau) \;&\geq\; \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln V^*(\tau, S_\tau) \\ &\geq\; \ln X_\tau^{\widehat{x},\widehat{\pi}} - \ln \varphi(\tau, S_\tau) + 3\eta \\ &\geq\; \ln X_\tau^{\varphi(t_1,s_1),\widehat{\pi}} - \ln \varphi(\theta, S_\tau) + \eta \geq \eta \end{aligned}$$

thus contradicting the geometric dynamic programming

HEDGING UNDER GAMMA CONSTRAINTS

Nizar TOUZI, CREST, Paris, touzi@ensae.fr

Lunteren, January 24-26, 2005

Soner M. and Touzi N. SIAM J. Cont. Opt. 2000

Cheridito P., Soner M. and Touzi N. A.A.P., to appear

Cheridito P., Soner M. and Touzi N. A. I.H.P. Analyse non Linéaire, to appear

Soner M. and Touzi N. Preprint 2004

1. INTRODUCTION : THE BLACK-SCHOLES MODEL

1. The financial market : $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, W *Brownian motion valued in* $\mathbb{R}^1$

- *1 non-risky asset* $S^0 \equiv 1$ *(change of numéraire)*

- *1 risky asset* S *:* $dS_t = S_t\,[\mu\,dt + \sigma\,dW_t]$

- *Contingent claim :* $g(S_T)$, *where* $g \;:\; \mathbb{R}_+ \longrightarrow \mathbb{R}$ *l.s.c. and bounded from below (not necessarily continuous)*

Main problem *Valuation of the option* $g(S_T)$

1. INTRODUCTION, Continued

2. Superhedging : under the self-financing condition, wealth process

$$X_t^{x,\theta} \;:=\; x + \int_0^t \theta_u \left(\mu du + \sigma dW_u\right)$$

$\theta \in \mathcal{A}$ *: set of admissible strategies*

$$\int_0^T |\theta_u|^2 du \;<\; \infty \;\; \textit{and} \;\; X^{x,\theta} \textit{ bounded from below}$$

• Super-replication problem

$$v_0 \;:=\; \inf\left\{x \;:\; X_T^{x,\theta} \geq g(S_T) \textit{ a.s. for some } \theta \in \mathcal{A}\right\}$$

$\Longrightarrow$ *Reduction : Change of measure* $\Longrightarrow$ *assume* $\mu = 0$ *wlog*

1. INTRODUCTION, Continued

3. *Explicit solution in complete market* :

$$v_t \;=\; V(t, S_t) \;:=\; \mathbb{E}\left[g(S_T)|S_t\right]$$

PDE characterization

$$-\mathcal{L}V \;:=\; -\frac{\partial V}{\partial t} - \frac{1}{2}\,\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} \;=\; 0 \quad \text{and} \quad V(T, s) \;=\; g(s)$$

$\Longrightarrow$ *Differentiate w.r.t.* σ :

$$-\frac{\partial V_\sigma}{\partial t} - \frac{1}{2}\,\sigma^2 s^2 \frac{\partial^2 V_\sigma}{\partial s^2} \;=\; \sigma s^2 V_{ss} \quad \text{and} \quad V_\sigma(T, s) \;=\; 0$$

1. INTRODUCTION, Continued

4. Greeks

- $\Delta_t := \frac{\partial V}{\partial s}(t, S_t)$: *Hedging portfolio*

- $\Gamma_t := \frac{\partial^2 V}{\partial s^2}(t, S_t)$: *variation of the hedging portfolio*

- *Vega*$_t := \frac{\partial V}{\partial \sigma}(t, S_t)$: *sensitivity to volatility*

- *Classical connection between* Γ *and Vega*

$$\mathrm{Vega}_t \;=\; \mathbb{E}\left[\int_t^T \sigma S_u^2 \Gamma_u du \middle| S_t \right]$$

2. SUPER-REPLICATION UNDER PORTFOLIO CONSTRAINTS : Formulation

$K = [\ell, u] \ni 0$ *(K closed convex subset of $\mathbb{R}^d \ni 0$)*

• *Set of admissible portfolios*

$$\mathcal{A}_K \ := \ \{\theta \in \mathcal{A} \ : \ \theta \text{ valued in } K\}$$

• *Super-replication problem*

$$V(t,s) \ := \ \inf\left\{x \ : \ X_T^{x,\theta} \geq g\left(S_T^{t,s}\right) \text{ for some } \theta \in \mathcal{A}_K\right\}$$

2. PORTFOLIO CONSTRAINTS : Main result

Face-lifting : *introduce Support function of* K : $\delta(y) := \sup_{x \in K} x \cdot y$

Face-lifting operator : $\widehat{g}(s) := \sup_{y \in \mathbb{R}^n} g\left(se^y\right) - \delta(y)$

1. **Theorem** *<Broadie, Cvitanić and Soner 98>* $V(t,s) = \mathbb{E}_{t,s}\left[\widehat{g}\left(S_T\right)\right]$

i.e. The problem of hedging $g(S_T)$ *under constraints is solved by the classical Black-Scholes hedging of* $\widehat{g}(S_T)$

2. **Local volatility model** $\sigma(t,s) \Longrightarrow$ *Free boundary problem*

$$\min\left\{-V_t - \frac{1}{2}\sigma(t,s)^2 s^2 V_{ss}, sV_s - \ell, u - sV_s\right\} = 0 \quad \textit{and} \quad V(T,.) = \widehat{g}$$

2. PORTFOLIO CONSTRAINTS : Duality

<*Cvitanić and Karatzas 93, Föllmer-Kramkov 95*>

$$V(t,s) \;=\; \sup_{\nu \in \mathcal{D}} \mathbb{E}^{P^\nu}_{t,s}\left[e^{-\int_t^T \delta(\nu_u)du} g(S_T)\right]$$

where $\mathcal{D} = \left\{\text{predictable bounded processes valued in } \tilde{K}\right\}$ *and*

$$P^\nu \;\sim\; P\,, \quad \left\{S_t e^{-\int_0^t \delta(\nu_u)du}\right\} \text{ is } P^\nu \text{ supermartingale}$$

i.e. penalization of the drift of price processes

$\Longrightarrow$ *Standard stochastic control problem...*

3. HEDGING UNDER GAMMA CONSTRAINTS

Recall the Black-Scholes model, optimal wealth process $X_t^* := V(t, S_t)$

By Itô's lemma, twice

$$\begin{aligned} X_t^* &= X_0^* + \int_0^t \mathcal{L}V(u, S_u)du + \int_0^t \Delta_u^* dS_u \\ &= V(0, S_0) + \int_0^t \Delta_u^* dS_u \end{aligned}$$

and

$$\Delta_t^* = V_s(0, S_0) + \int_0^t \mathcal{L}V_s(u, S_u)du + \int_0^t \Gamma_u^* dS_u$$

3. GAMMA CONSTRAINTS : Motivation

Goal : *Hedge under constraints on the gamma of the portfolio* Γ_t

$\Longrightarrow$ *Control on the portfolio re-balancement*

$$\Delta_{t+dt} - \Delta_t \;=\; V_s\left(t+dt, S_{t+dt}\right) - V_s\left(t, S_t\right)$$

$\Longrightarrow$ *Controlling the Vega risk*

$$\mathrm{Vega}_t \;=\; E\left[\int_t^T \sigma S_u^2 \Gamma_u du \,\middle|\, S_t\right]$$

- *large investor problem*
- *transaction costs*
- *The digital option example*

3. GAMMA CONSTRAINTS : Model formulation

- *Non-risky asset* S^0 *normalized to* 1

- *Risky asset* S : $dS_t = S_t \sigma \, dW_t$

- *European option* $g(S_T)$,

$$g \ : \ \mathbb{R}_+ \longrightarrow \mathbb{R} \ \text{ l.s.c. and } \ -C \le g(s) \le C(1+s)$$

- *Wealth process* $X_t = x + \int_0^t Y_u dS_u = x + \int_0^t Y_u \sigma S_u dW_u$

- *Portfolio process* $Y_t = y + \int_0^t \alpha_u du + \int_0^t \Gamma_u dS_u$

3. GAMMA CONSTRAINTS : Problem formulation

• *Admissible portfolio :* $\nu = (y, \alpha, \Gamma) \in \mathcal{G}$ *if*

$$y \in \mathbb{R},\ \alpha,\ \gamma \text{ bounded predictable processes,}$$

$$\text{and} \quad -\underline{\Gamma} \le \Gamma_u\, S_u^2 \le \overline{\Gamma}$$

• *Super-replication problem*

$$V(t,s) := \inf\left\{x \ : \ X_T^{t,x,\nu} \ge g\left(S_T^{t,s}\right) \text{ for some } \nu \in \mathcal{G}\right\}$$

where

$$X_T^{t,x,\nu} = x + \int_t^T Y_u^{t,\nu} dS_u \ \text{ and } \ Y_u^{t,\nu} = y + \int_t^u \alpha_u du + \int_t^u \Gamma_u dS_u$$

3. GAMMA CONSTRAINTS : First intuitions

We formally expect that V solves the free boundary problem

$$F\left(V_t, s^2 V_{ss}\right) \;:=\; \min\left\{-V_t - \frac{1}{2}\sigma^2 s^2 V_{ss}\,,\ \overline{\Gamma} - s^2 V_{ss}\,,\ \underline{\Gamma} + s^2 V_{ss}\right\} \;=\; 0$$

- *Correct if* $\underline{\Gamma} = +\infty$ *<Soner-Touzi 2000>*

- *Can not be true if* $\underline{\Gamma} > +\infty$ *:* F *is not elliptic*

- *Example :* $g(s) := s \wedge 1$, $\underline{\Gamma} = 0$, $\overline{\Gamma} = \infty$. *Then* $V = g$ *(is not convex !)*

$\Longrightarrow$ *Hedging by buy-and-hold strategies*

$\Longrightarrow$ *if* **jumps** *are allowed in the* Y *process, then non-uniqueness of hedging strategy...*

3. GAMMA CONSTRAINTS : Warnings (1)

Lemma *For all predictable* $W-$*integrable cadlag process* ϕ, *and all* $\varepsilon > 0$ *:*

$$\sup_{0 \le t \le 1} \left| \int_0^t \phi_r dW_r - \int_0^t \phi_r^\varepsilon dW_r \right| \;\le\; \varepsilon$$

$\longrightarrow$ *for some predictable step process* ϕ^ε *<Levental-Skorohod AAP95>*

$\longrightarrow$ *for some absolutely continuous predictable process* $\phi_t^\varepsilon = \phi_0^\varepsilon + \int_0^t \alpha_r dr$, $\int_0^1 |\alpha_r| dr < \infty$ *a.s.* *<Bank-Baum 04>*

3. GAMMA CONSTRAINTS : Warnings (2)

$\Longrightarrow$ *Usual control relaxation in stochastic control problems does not hold here :*

- *Allow for arbitrary jumps in* $Y \Longrightarrow V =$ *BS price*

- *Allow for arbitrary absolutely continuous* $\int_0^t \alpha_u du \Longrightarrow V =$ *BS price*

(with $\gamma = 0$ *in both cases)*

- $V >$*BS price, in general, for bounded* α *and bounded number of jumps*

3. GAMMA CONSTRAINTS : The dynamic programming PDE

Theorem 3 *V is the unique viscosity solution of*

$$\text{(DPE)} \quad \widehat{F}\left(V_t, s^2 V_{ss}\right) \;=\; 0 \;\; \textit{and} \;\; V(T-,.) = \widehat{g}$$

with $|V - \widehat{g}|_\infty < \infty$, where $\widehat{F}(p, A) := \sup_{\beta \geq 0} F(p, A + \beta)$ is the elliptic envelope of F, and

$$\widehat{g}(s) \;:=\; h^{\text{conc}}(s) - \overline{\Gamma} \ln s\,, \qquad h(s) \;:=; g(s) + \overline{\Gamma} \ln s$$

- *If $\overline{\Gamma} = +\infty \Longrightarrow$ No Face-lifting ! !*

- *For $\underline{\Gamma} = +\infty$: $\widehat{F} = F$ (Agree with intuition)*

- *Example : $g(s) := s \wedge 1$, $\underline{\Gamma} = 0$, $\overline{\Gamma} = \infty$, we find $V = g$*

Sketch of proof of the super-solution property

For simplicity, assume V *smooth and*

$$V(t,s) \;:=\; \underline{\text{min}}\left\{x \;:\; X_T^{t,x,\nu} \geq g\left(S_T^{t,s}\right) \text{ for some } \nu \in \mathcal{G}\right\}$$

Then, with $\hat{x} := V(t,s) \Longrightarrow X_T^{t,\hat{x},\hat{\nu}} \;\geq\; g\left(S_T^{t,s}\right)$ *for some* $\hat{\nu} \;\in\; \mathcal{G}$

- *Geometric Dynamic programming (trivial inequality) :*

$$X_{\theta_h}^{t,\hat{x},\hat{\nu}} \;\geq\; V\left(\theta_h, S_{\theta_h}^{t,s}\right)$$

- *Apply Itô's lemma twice :*

$$\begin{aligned} 0 \;&\leq\; V(t,s) - V\left(\theta_h, S_{\theta_h}^{t,s}\right) + \int_t^{\theta_h} Y_u^{t,\hat{y}} dS_u \\ &=\; -\int_t^{\theta_h} \mathcal{L}V\left(u, S_u^{t,s}\right) du + \int_t^{\theta_h}\left(c + \int_t^u a_v dv + \int_t^u b_v dS_v\right) dS_u \end{aligned}$$

where $c \;:=\; y - V_s(t,s)$, $a_u \;:=\; \alpha_u - \mathcal{L}V_s(u,S_u)$, $b_u \;:=\; \gamma_u - V_{ss}(u,S_u)$

Sketch of proof of the super-solution property, continued

- *Compare orders of the different terms*

$$0 \;\le\; -\int_t^{\theta_h} \mathcal{L}V\left(u, S_u^{t,s}\right) du + \int_t^{\theta_h} \left(c + \int_t^u a_v dv + \int_t^u b_v dS_v\right) dS_u$$

$\Longrightarrow$ $c \;=\; y - V_s(t,s) \;=\; 0$, *and* *forget the term* $\int\int dt dW_t$

- *Analysis of the term* $\int\int b_u dW_u dW_v$ *requires fine results on the local path behavior of double stochastic integrals*

$\longrightarrow$ *Intuition : if* $b_u \equiv \beta$ *constant, then*

$$0 \;\le\; -\int_t^{\theta_h} \mathcal{L}V\left(u, S_u^{t,s}\right) du + \frac{\beta}{2}\left(\left(S_{\theta_h} - s\right)^2 - \int_t^{\theta_h} \sigma^2 S_u^2 du\right)$$

Divide by h *and send* h *to zero :*

$$\limsup \;\Longrightarrow\; \beta \ge 0 \quad \textit{and} \quad \liminf \;\Longrightarrow\; 0 \;\le\; -\mathcal{L}V(t,s) - \frac{1}{2}\beta\sigma^2 s^2$$

3. GAMMA CONSTRAINTS : Main result

Theorem *The function v has the stochastic representation*

$$V(t,s) \;=\; \sup_{\theta \in \mathcal{T}_t^T} \mathbb{E}_{t,s}\left[\hat{g}\left(S_\theta\right) - \frac{1}{2}\underline{\Gamma}\,\sigma^2(T-\theta)\right]$$

where $\mathcal{T}_t^T$ is the collection of all $\mathbb{F}-$stopping times with values in $[t,T]$.

$\Longrightarrow$ *Upper bound on gamma* $\Longrightarrow$ *Face-lifting*

$\Longrightarrow$ *Lower bound on gamma* $\Longrightarrow$ *American option/optimal stopping*

• *For general local volatility models $\sigma(t,s)$:*

Treatment of both bounds does not separate, in general

3. GAMMA CONSTRAINTS : Hedging strategy

• *Pass from* g *to* $\hat{g}$, *and forget about upper bound* $\overline{\Gamma}$

• *For simplicity, consider the case* $\underline{\Gamma} = 0$

• *smoothfit holds, i.e.* V *is* C^1

• *Buy-and-hold* $\equiv$ *"keep going along the tangent"*

$\Longrightarrow$ *Hedge by* **succession** *of*

Standard Black-Scholes and buy-and-hold strategy

3. GAMMA CONSTRAINTS : Duality

$$V(t,s) \;=\; \sup_{\mathcal{X}\times\mathcal{Y}} \mathbb{E}_{t,s}\left[g\left(\widehat{S}_T^{x,y}\right) - \frac{1}{2}\,\sigma^2 \int_t^T \left(\underline{\Gamma}\; x_r + \overline{\Gamma}\; y_r\right) dr\right]$$

where

$$d\widehat{S}_r^{x,y} \;=\; \widehat{S}_r^{x,y}\,\sigma\sqrt{1 - x_r + y_r}\;\; dW_r$$

and

$$\mathcal{X} \;=\; \{\textit{predictable processes with values in } [0,1]\}$$

$$\mathcal{Y} \;=\; \{\textit{predictable processes with values in } \mathbb{R}_+\}$$

$\Longrightarrow$ *Dual problem by penalizing the volatility !*

4. BACKWARD SDE's AND SEMI-LINEAR PDE's

Consider the Backward Stochastic Differential Equation :

$$Y_t \;=\; g(X_T) + \int_t^T f(X_r, Y_r, Z_r)dr - \int_t^T Z_r \cdot \sigma(X_r)dW_r$$

where $X_.$ is defined by the (forward) SDE

$$dX_t \;=\; \sigma(X_t)dW_t$$

Then $Y_t = V(t, X_t)$, and V satisfies the semi-linear PDE

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T(x)D^2V(t,x)\right] - f\left(x, V(t,x), DV(t,x)\right) \;=\; 0$$

(Easy application of Itô's lemma)

4. BSDE's AND SEMI-LINEAR PDE's :

Stochastic representation

- *Any semi-linear PDE has a representation in terms of a BSDE*

- *Consider a stochastic control problem with no control on volatility.*

Then, the associated HJB equation is semi-linear :

$$-V_t(t,x) - \frac{1}{2}\text{Tr}\left[\sigma\sigma^*(x)D^2V(t,x)\right] - \sup_{u\in U} b(x,u)\cdot DV(t,x) \;=\; 0$$

So any stochastic control problem with no control on volatility has a representation in terms of a Backward SDE

4. BSDE's AND SEMI-LINEAR PDE's : Numerical issue

Numerical solution of a semi-linear PDE by simulating the associated backward sde by means of Monte Carlo methods

Start from Euler discretization : $Y^\pi_{t_n} = g\left(X^\pi_{t_n}\right)$ *is given, and*

$$Y^\pi_{t_{i+1}} - Y^\pi_{t_i} = -f\left(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}\right)\Delta t_i + Z^\pi_{t_i} \cdot \sigma\left(X^\pi_{t_i}\right)\Delta W_{t_{i+1}}$$

$\Longrightarrow$ *Discrete-time approximation :*

$$Y^\pi_{t_n} = g\left(X^\pi_{t_n}\right)$$

4. BSDE's AND SEMI-LINEAR PDE's : Numerical issue

Numerical solution of a semi-linear PDE by simulating the associated backward sde by means of Monte Carlo methods

Start from Euler discretization : $Y^\pi_{t_n} = g\left(X^\pi_{t_n}\right)$ *is given, and*

$$\mathbb{E}^\pi_i[\qquad Y^\pi_{t_{i+1}} - Y^\pi_{t_i} = -f\left(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}\right)\Delta t_i + Z^\pi_{t_i} \cdot \sigma\left(X^\pi_{t_i}\right)\Delta W_{t_{i+1}}$$

$\Longrightarrow$ *Discrete-time approximation :*

$$\begin{aligned} Y^\pi_{t_n} &= g\left(X^\pi_{t_n}\right) \\ Y^\pi_{t_i} &= \mathbb{E}^\pi_i\left[Y^\pi_{t_{i+1}}\right] + f\left(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}\right)\Delta t_i \quad 0 \le i \le n-1\ , \end{aligned}$$

4. BSDE's AND SEMI-LINEAR PDE's : Numerical issue

Numerical solution of a semi-linear PDE by simulating the associated backward sde by means of Monte Carlo methods

Start from Euler discretization : $Y^\pi_{t_n} = g\left(X^\pi_{t_n}\right)$ is given, and

$$\mathbb{E}^\pi_i[\Delta W_{t_{i+1}} \quad Y^\pi_{t_{i+1}} - Y^\pi_{t_i} = -f\left(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}\right)\Delta t_i + Z^\pi_{t_i}\cdot\sigma\left(X^\pi_{t_i}\right)\Delta W_{t_{i+1}}$$

$\Longrightarrow$ Discrete-time approximation :

$$\begin{aligned}
Y^\pi_{t_n} &= g\left(X^\pi_{t_n}\right)\\
Y^\pi_{t_i} &= \mathbb{E}^\pi_i\left[Y^\pi_{t_{i+1}}\right] + f\left(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}\right)\Delta t_i \quad 0 \le i \le n-1\\
Z^\pi_{t_i} &= \frac{1}{\sigma\left(X^\pi_{t_i}\right)\Delta t_i}\,\mathbb{E}^\pi_i\left[Y^\pi_{t_{i+1}}\Delta W_{t_{i+1}}\right]
\end{aligned}$$

4. BSDE's AND SEMI-LINEAR PDE's : Numerical issue

Numerical solution of a semi-linear PDE by simulating the associated backward sde by means of Monte Carlo methods

Start from Euler discretization : $Y^\pi_{t_n} = g\left(X^\pi_{t_n}\right)$ is given, and

$$\mathbb{E}^\pi_i[\Delta W_{t_{i+1}} \quad Y^\pi_{t_{i+1}} - Y^\pi_{t_i} = -f\left(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}\right)\Delta t_i + Z^\pi_{t_i} \cdot \sigma\left(X^\pi_{t_i}\right)\Delta W_{t_{i+1}}$$

$\Longrightarrow$ Discrete-time approximation :

$$\begin{aligned}
Y^\pi_{t_n} &= g\left(X^\pi_{t_n}\right) \\
Y^\pi_{t_i} &= \mathbb{E}^\pi_i\left[Y^\pi_{t_{i+1}}\right] + f\left(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}\right)\Delta t_i \quad 0 \le i \le n-1 \\
Z^\pi_{t_i} &= \frac{1}{\sigma\left(X^\pi_{t_i}\right)\Delta t_i}\mathbb{E}^\pi_i\left[Y^\pi_{t_{i+1}}\Delta W_{t_i}\right]
\end{aligned}$$

$\equiv$ Pricing of Bermudan options [Bally-Pagès 01, Bouchard-Touzi 04]

5. SECOND ORDER BSDE's and FULLY NONLINEAR PDE's

Let $f(x,y,z,\gamma)+\frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(x)\gamma]$ non-decreasing in γ

Consider the 2nd order BSDE :

$$\text{(2BSDE)}\qquad \begin{aligned} dX_t &= \sigma(X_t)dW_t \\ dY_t &= -f(X_t,Y_t,Z_t,\Gamma_t)dt + Z_t\cdot\sigma(X_t)dW_t\,,\quad Y_T = g(X_T) \\ dZ_t &= \alpha_t dt + \Gamma_t dW_t \end{aligned}$$

A solution of (2BSDE) is

a process (Y,Z,α,Γ) with values in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{S}^n$

Question : existence ? uniqueness ?

5. 2nd ORDER BSDE's : Main result

Set $\mathcal{L}v(t,x) := \frac{\partial v}{\partial t}(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T(x)D^2v(t,x)\right]$

Theorem *Assume that there is a unique smooth solution v of the fully-nonlinear PDE*

$$-\mathcal{L}v(t,x) - f\left(x, v(t,x), Dv(t,x), D^2v(t,x)\right) = 0\,, \quad v(T,x) = g(x)\,.$$

Then

$$Y_t := v(t,X_t),\; Z_t := Dv(t,X_t),\; \alpha_t := \mathcal{L}Dv(t,X_t),\; \Gamma_t := D^2v(t,X_t)$$

is the unique solution of (2BSDE)

<Cheridito, Soner, Touzi and Victoir 05>

5. 2nd ORDER BSDE's : Numerical implication

- *Any fully nonlinear PDE has a representation in terms of a 2BSDE*
- *In particular, any stochastic control problem has a representation in terms of a Backward SDE (the associated HJB equation is a fully nonlinear PDE)*

$\Longrightarrow$ *Numerical solution by Monte Carlo methods* *(future project)*

$$
\begin{aligned}
Y^\pi_{t_n} &= g\left(X^\pi_{t_n}\right) , \\
Y^\pi_{t_{i-1}} &= \mathbb{E}^\pi_{i-1}\left[Y^\pi_{t_i}\right] + f\left(X^\pi_{t_{i-1}}, Y^\pi_{t_{i-1}}, Z^\pi_{t_{i-1}}, \Gamma^\pi_{t_{i-1}}\right)\Delta t_i \ , \quad 1 \le i \le n \ , \\
Z^\pi_{t_{i-1}} &= \frac{1}{\sigma\left(X^\pi_{t_{i-1}}\right)\Delta t_i}\, \mathbb{E}^\pi_{i-1}\left[Y^\pi_{t_i}\Delta W_{t_i}\right] \\
\Gamma^\pi_{t_{i-1}} &= \frac{1}{\sigma\left(X^\pi_{t_{i-1}}\right)\Delta t_i}\, \mathbb{E}^\pi_{i-1}\left[Z^\pi_{t_i}\Delta W_{t_i}\right]
\end{aligned}
$$

Sketch of the proof :

Existence of solution for (2BSDE)

1. (Easy part) Let v *be the unique solution of*

$$-\mathcal{L}v(t,x) - f\left(x, v(t,x), Dv(t,x), D^2v(t,x)\right) = 0 \,, \qquad v(T,x) = g(x)\,.$$

Then

$$Y_t = v(t, X_t)\,, \qquad Z_t = Dv(t, X_t)$$

$$z = Dv(t,x)\,, \;\; \alpha_t = \mathcal{L}Dv(t,X_t)\,, \;\; \Gamma_t = D^2v(t,X_t)$$

is a solution of (2BSDE)

Sketch of the proof :

Uniqueness of solution for (2BSDE)

2. *Given a control* $\nu := (z, \alpha, \Gamma)$*, define the controlled process*

$$dY_t^\nu = -f(X_t, Y_t^\nu, Z_t, \Gamma_t)\, dt + Z_t \cdot \sigma(X_t)\, dW_t$$
$$dZ_t = \alpha_t dt + \Gamma_t \cdot \sigma(X_t)\, dW_t$$

together with the "super-hedging" problems (Seller / Buyer)

$$V(t,x) := \inf\{y \ : \ Y_T^\nu \geq g(X_T) \text{ a.s. for some } \nu \in \mathcal{G}\}$$
$$U(t,x) := -\inf\{y \ : \ Y_T^\nu \geq -g(X_T) \text{ a.s. for some } \nu \in \mathcal{G}\}$$

- *Any solution of (2BSDE) satisfies* $V(t,X_t) \leq Y_t \leq U(t,X_t)$
- V *and* U *are both solution of the fully nonlinear PDE* $\Longrightarrow U = V$