PRICING AND TRADING CREDIT DEFAULT SWAPS*

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Contents

Introduction

The topic of this work is a detailed study of stylized credit default swaps within the framework of a generic *reduced-form* credit risk model. By a reduced-form model we mean any model of a single default or several dependent defaults in which we can explicitly identify the distribution of default times. Therefore, the set-up presented in this work covers in fact various alternative approaches, which are usually classified as, for instance, value-of-the-firm approach, intensity-based approach, copula-based approach, etc. Note that such a classification refers to a particular way in which default times are constructed, rather than to the question whether the distribution (conditional distribution, joint distribution, etc.) of default can be found explicitly.

The main goal is to develop general results dealing with the relative valuation of defaultable claims (credit derivatives) with respect to market values of traded credit-risk sensitive securities. As expected, we have chosen stylized credit default swaps as liquidly traded assets, so that other credit derivatives are valued with respect to CDS spreads as a benchmark. The tool used to is pretty standard. We simply show that a generic defaultable claim (or a generic basket claim, in the case of several underlying credit names) can be replicated by a dynamical trading in single-name CDSs.

This work is organized as follows.

We start, in Section 1, by dealing with the valuation and trading of a generic defaultable claim. The presentation in this section, although largely based on Section 2.1 in Bielecki and Rutkowski [3], is adapted to our current purposes, and the notation is modified accordingly. We believe that it is more convenient to deal with a generic dividend-paying asset, rather than with any specific examples of credit derivatives, since the fundamental properties of arbitrage prices of defaultable assets, and of related trading strategies, are already apparent in a general set-up.

In Section 2, we first provide results concerning the valuation and trading of credit default swaps under the assumption that the default intensity is deterministic and the interest rate is zero. Subsequently, we derive a closed-form solution for replicating strategy for an arbitrary non-dividend paying defaultable claim on a single credit name, in a market in which a bond and a credit default swap are traded. Also, we examine the completeness of such a security market model.

Section 3, deals with the hedging of basket credit derivatives using single-name CDSs. In Section 3.3, we present results dealing with the case of a first-to-default claim. Subsequently, in Section 3.5, we show that these results can be adapted to cover the case of a generic basket claim, which can be formally seen as a sequence of conditional first-to-default claim, where the condition encompasses dates of the past defaults and identities of defaulting names, and a suitably defined recovery payoff occurs at the moment of the next default.

In Section 4, we extend some of previously established results to the case of stochastic default intensity. Let us note that hedging under stochastic default intensity covers both default and spread risks. For more general results concerning various techniques of replication of defaultable claims, the interested reader is referred to Bielecki et al. [4].

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1 Pricing and Trading Defaultable Claims

This section gives an overview of basic results concerning the valuation and trading of defaultable claims.

1.1 Generic Defaultable Claims

A strictly positive random variable τ, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, is termed a *random time*. In view of its interpretation, it will be later referred to as a *default time*. We introduce the jump process $H_t = \mathbb{1}_{\{\tau \le t\}}$ associated with τ, and we denote by $\mathbb{H}$ the filtration generated by this process. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$, and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that we have $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$.

Definition 1.1 By a *defaultable claim maturing at* T we mean the quadruple (X, A, Z, τ), where X is an $\mathcal{F}_T$-measurable random variable, A is an $\mathbb{F}$-adapted process of finite variation, Z is an $\mathbb{F}$-predictable process, and τ is a random time.

The financial interpretation of the components of a defaultable claim becomes clear from the following definition of the dividend process D, which describes all cash flows associated with a defaultable claim over the lifespan $]0, T]$, that is, after the contract was initiated at time 0. Of course, the choice of 0 as the date of inception is arbitrary.

Definition 1.2 The *dividend process* D of a defaultable claim maturing at T equals, for every $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u) \, dA_u + \int_{]0,t]} Z_u \, dH_u.$$

The financial interpretation of the definition above justifies the following terminology: X is the *promised payoff*, A represents the process of *promised dividends*, and the process Z, termed the *recovery process*, specifies the recovery payoff at default. It is worth stressing that, according to our convention, the cash payment (premium) at time 0 is not included in the dividend process D associated with a defaultable claim.

When dealing with a credit default swap, it is natural to assume that the premium paid at time 0 equals zero, and the process A represents the fee (annuity) paid in instalments up to maturity date or default, whichever comes first. For instance, if $A_t = -\kappa t$ for some constant $\kappa > 0$, then the 'price' of a stylized credit default swap is formally represented by this constant, referred to as the continuously paid *credit default rate* or *premium* (see Section 2.1 for details).

If the other covenants of the contract are known (i.e., the payoffs X and Z are given), the valuation of a swap is equivalent to finding the level of the rate κ that makes the swap valueless at inception. Typically, in a credit default swap we have $X = 0$, and Z is determined in reference to recovery rate of a reference credit-risky entity. In a more realistic approach, the process A is discontinuous, with jumps occurring at the premium payment dates. In this note, we shall only deal with a stylized CDS with a continuously paid premium.

Let us return to the general set-up. It is clear that the dividend process D follows a process of finite variation on $[0, T]$. Since

$$\int_{]0,t]} (1 - H_u) \, dA_u = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}} \, dA_u = A_{\tau-} \mathbb{1}_{\{\tau \le t\}} + A_t \mathbb{1}_{\{\tau > t\}},$$

it is also apparent that if default occurs at some date t, the 'promised dividend' $A_t - A_{t-}$ that is due to be received or paid at this date is disregarded. If we denote $\tau \wedge t = \min(\tau, t)$ then we have

$$\int_{]0,t]} Z_u \, dH_u = Z_{\tau \wedge t} \mathbb{1}_{\{\tau \le t\}} = Z_\tau \mathbb{1}_{\{\tau \le t\}}.$$

Let us stress that the process $D_u - D_t$, $u \in [t,T]$, represents all cash flows from a defaultable claim received by an investor who purchases it at time t. Of course, the process $D_u - D_t$ may depend on the past behavior of the claim (e.g., through some intrinsic parameters, such as credit spreads) as well as on the history of the market prior to t. The past dividends are not valued by the market, however, so that the current market value at time t of a claim (i.e., the price at which it trades at time t) depends only on future dividends to be paid or received over the time interval $]t,T]$.

Suppose that our underlying financial market model is arbitrage-free, in the sense that there exists a *spot martingale measure* $\mathbb{Q}^*$ (also referred to as a *risk-neutral probability*), meaning that $\mathbb{Q}^*$ is equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$, and the price process of any tradeable security, paying no coupons or dividends, follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$, when discounted by the *savings account* B, given by

$$B_t = \exp\left(\int_0^t r_u \, du\right), \quad \forall\, t \in \mathbb{R}_+. \tag{1}$$

1.2 Buy-and-hold Strategy

We write S^i, $i = 1, \ldots, k$ to denote the price processes of k primary securities in an arbitrage-free financial model. We make the standard assumption that the processes S^i, $i = 1, \ldots, k-1$ follow semimartingales. In addition, we set $S^k_t = B_t$ so that S^k represents the value process of the savings account. The last assumption is not necessary, however. We can assume, for instance, that S^k is the price of a T-maturity risk-free zero-coupon bond, or choose any other strictly positive price process as as numéraire.

For the sake of convenience, we assume that S^i, $i = 1, \ldots, k-1$ are non-dividend-paying assets, and we introduce the discounted price processes S^{i*} by setting $S^{i*}_t = S^i_t / B_t$. All processes are assumed to be given on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{Q})$, where $\mathbb{Q}$ is interpreted as the real-life (i.e., statistical) probability measure.

Let us now assume that we have an additional traded security that pays dividends during its lifespan, assumed to be the time interval $[0,T]$, according to a process of finite variation D, with $D_0 = 0$. Let S denote a (yet unspecified) price process of this security. In particular, we do not postulate a priori that S follows a semimartingale. It is not necessary to interpret S as a price process of a defaultable claim, though we have here this particular interpretation in mind.

Let a $\mathbb{G}$-predictable, $\mathbb{R}^{k+1}$-valued process $\phi = (\phi^0, \phi^1, \ldots, \phi^k)$ represent a generic trading strategy, where ϕ^j_t represents the number of shares of the j^{th} asset held at time t. We identify here S^0 with S, so that S is the 0^{th} asset. In order to derive a pricing formula for this asset, it suffices to examine a simple trading strategy involving S, namely, the buy-and-hold strategy.

Suppose that one unit of the 0^{th} asset was purchased at time 0, at the initial price S_0, and it was hold until time T. We assume all the proceeds from dividends are re-invested in the savings account B. More specifically, we consider a *buy-and-hold* strategy $\psi = (1, 0, \ldots, 0, \psi^k)$, where ψ^k is a $\mathbb{G}$-predictable process. The associated *wealth process* $V(\psi)$ equals

$$V_t(\psi) = S_t + \psi^k_t B_t, \quad \forall\, t \in [0,T], \tag{2}$$

so that its initial value equals $V_0(\psi) = S_0 + \psi^k_0$.

Definition 1.3 We say that a strategy $\psi = (1, 0, \ldots, 0, \psi^k)$ is *self-financing* if

$$dV_t(\psi) = dS_t + dD_t + \psi^k_t \, dB_t,$$

or more explicitly, for every $t \in [0,T]$,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi^k_u \, dB_u. \tag{3}$$

We assume from now on that the process ψ^k is chosen in such a way (with respect to S, D and B) that a buy-and-hold strategy ψ is self-financing. Also, we make a standing assumption that the random variable $Y = \int_{]0,T]} B_u^{-1} dD_u$ is $\mathbb{Q}^*$-integrable.

Lemma 1.1 *The discounted wealth $V_t^*(\psi) = B_t^{-1} V_t(\psi)$ of any self-financing buy-and-hold trading strategy ψ satisfies, for every $t \in [0,T]$,*

$$V_t^*(\psi) = V_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} B_u^{-1} dD_u. \tag{4}$$

Hence we have, for every $t \in [0,T]$,

$$V_T^*(\psi) - V_t^*(\psi) = S_T^* - S_t^* + \int_{]t,T]} B_u^{-1}\, dD_u. \tag{5}$$

Proof. We define an auxiliary process $\widehat{V}(\psi)$ by setting $\widehat{V}_t(\psi) = V_t(\psi) - S_t = \psi_t^k B_t$ for $t \in [0,T]$. In view of (3), we have

$$\widehat{V}_t(\psi) = \widehat{V}_0(\psi) + D_t + \int_{]0,t]} \psi_u^k \, dB_u,$$

and so the process $\widehat{V}(\psi)$ follows a semimartingale. An application of Itô's product rule yields

$$\begin{aligned} d\big(B_t^{-1}\widehat{V}_t(\psi)\big) &= B_t^{-1} d\widehat{V}_t(\psi) + \widehat{V}_t(\psi)\, dB_t^{-1} \\ &= B_t^{-1} dD_t + \psi_t^k B_t^{-1} dB_t + \psi_t^k B_t \, dB_t^{-1} \\ &= B_t^{-1} dD_t, \end{aligned}$$

where we have used the obvious identity: $B_t^{-1} dB_t + B_t\, dB_t^{-1} = 0$. Integrating the last equality, we obtain

$$B_t^{-1}\big(V_t(\psi) - S_t\big) = B_0^{-1}\big(V_0(\psi) - S_0\big) + \int_{]0,t]} B_u^{-1} dD_u,$$

and this immediately yields (4). □

It is worth noting that Lemma 1.1 remains valid if the assumption that S^k represents the savings account B is relaxed. It suffices to assume that the price process S^k is a *numéraire*, that is, a strictly positive continuous semimartingale. For the sake of brevity, let us write $S^k = \beta$. We say that $\psi = (1, 0, \ldots, 0, \psi^k)$ is self-financing it the wealth process

$$V_t(\psi) = S_t + \psi_t^k \beta_t, \quad \forall\, t \in [0,T],$$

satisfies, for every $t \in [0,T]$,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_{]0,t]} \psi_u^k \, d\beta_u.$$

Lemma 1.2 *The relative wealth $V_t^*(\psi) = \beta_t^{-1} V_t(\psi)$ of a self-financing trading strategy ψ satisfies, for every $t \in [0,T]$,*

$$V_t^*(\psi) = V_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} \beta_u^{-1} dD_u,$$

where $S^* = \beta_t^{-1} S_t$.

Proof. The proof proceeds along the same lines as before, noting that $\beta^1 d\beta + \beta d\beta^1 + d\langle \beta, \beta^1 \rangle = 0$. □

1.3 Spot Martingale Measure

Our next goal is to derive the risk-neutral valuation formula for the ex-dividend price S_t. To this end, we assume that our market model is arbitrage-free, meaning that it admits a (not necessarily unique) martingale measure $\mathbb{Q}^*$, equivalent to $\mathbb{Q}$, which is associated with the choice of B as a numéraire.

Definition 1.4 We say that $\mathbb{Q}^*$ is a *spot martingale measure* if the discounted price S^{i*} of any non-dividend paying traded security follows a $\mathbb{Q}^*$-martingale with respect to $\mathbb{G}$.

It is well known that the discounted wealth process $V^*(\phi)$ of any self-financing trading strategy $\phi = (0, \phi^1, \phi^2, \dots, \phi^k)$ is a local martingale under $\mathbb{Q}^*$. In what follows, we shall only consider *admissible* trading strategies, that is, strategies for which the discounted wealth process $V^*(\phi)$ is a martingale under $\mathbb{Q}^*$. A market model in which only admissible trading strategies are allowed is *arbitrage-free*, that is, there are no arbitrage opportunities in this model.

Following this line of arguments, we postulate that the trading strategy ψ introduced in Section 1.2 is also *admissible*, so that its discounted wealth process $V^*(\psi)$ follows a martingale under $\mathbb{Q}^*$ with respect to $\mathbb{G}$. This assumption is quite natural if we wish to prevent arbitrage opportunities to appear in the extended model of the financial market. Indeed, since we postulate that S is traded, the wealth process $V(\psi)$ can be formally seen as an additional non-dividend paying tradeable security.

To derive a pricing formula for a defaultable claim, we make a natural assumption that the market value at time t of the 0^{th} security comes exclusively from the future dividends stream, that is, from the cash flows occurring in the open interval $]t,T[$. Since the lifespan of S is $[0,T]$, this amounts to postulate that $S_T = S^*_T = 0$. To emphasize this property, we shall refer to S as the *ex-dividend price* of the 0^{th} asset.

Definition 1.5 A process S with $S_T = 0$ is the *ex-dividend price* of the 0^{th} asset if the discounted wealth process $V^*(\psi)$ of any self-financing buy-and-hold strategy ψ follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$.

As a special case, we obtain the ex-dividend price a defaultable claim with maturity T.

Proposition 1.1 *The ex-dividend price process S associated with the dividend process D satisfies, for every $t \in [0,T]$,*

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big). \tag{6}$$

Proof. The postulated martingale property of the discounted wealth process $V^*(\psi)$ yields, for every $t \in [0,T]$,

$$\mathbb{E}_{\mathbb{Q}^*}\big(V^*_T(\psi) - V^*_t(\psi) \,\big|\, \mathcal{G}_t \big) = 0.$$

Taking into account (5), we thus obtain

$$S^*_t = \mathbb{E}_{\mathbb{Q}^*}\Big(S^*_T + \int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big).$$

Since, by virtue of the definition of the ex-dividend price we have $S_T = S^*_T = 0$, the last formula yields (6). □

It is not difficult to show that the ex-dividend price S satisfies, for every $t \in [0,T]$,

$$S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t, \tag{7}$$

where the process $\widetilde{S}$ represents the *ex-dividend pre-default price* of a defaultable claim.

The *cum-dividend price* process $\bar{S}$ associated with the dividend process D is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$, given by the formula, for every $t \in [0,T]$,

$$\bar{S}_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big). \tag{8}$$

The savings account B can be replaced by an arbitrary numéraire β. The corresponding valuation formula becomes, for every $t \in [0,T]$,

$$S_t = \beta_t \, \mathbb{E}_{\mathbb{Q}^\beta}\Big(\int_{]t,T]} \beta_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \tag{9}$$

where $\mathbb{Q}^\beta$ is a martingale measure on $(\Omega, \mathcal{G}_T)$ associated with a numéraire β, that is, a probability measure on $(\Omega, \mathcal{G}_T)$ given by the formula

$$\frac{d\mathbb{Q}^\beta}{d\mathbb{Q}^*} = \frac{\beta_T}{\beta_0 B_T}, \quad \mathbb{Q}^*\text{-a.s.}$$

1.4 Self-Financing Trading Strategies

Let us now examine a general trading strategy $\phi = (\phi^0, \phi^1, \dots, \phi^k)$ with $\mathbb{G}$-predictable components. The associated *wealth process* $V(\phi)$ equals $V_t(\phi) = \sum_{i=0}^k \phi_t^i S_t^i$, where, as before $S^0 = S$. A strategy ϕ is said to be *self-financing* if $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0,T]$, where the *gains process* $G(\phi)$ is defined as follows:

$$G_t(\phi) = \int_{]0,t]} \phi_u^0 \, dD_u + \sum_{i=0}^k \int_{]0,t]} \phi_u^i \, dS_u^i.$$

Corollary 1.1 *Let $S^k = B$. Then for any self-financing trading strategy ϕ, the discounted wealth process $V^*(\phi) = B_t^{-1} V_t(\phi)$ follows a martingale under $\mathbb{Q}^*$.*

Proof. Since B is a continuous process of finite variation, Itô's product rule gives

$$dS_t^{i*} = S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i$$

for $i = 0, 1, \dots, k$, and so

$$\begin{aligned}
dV_t^*(\phi) &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} dV_t(\phi) \\
&= V_t(\phi) \, dB_t^{-1} + B_t^{-1} \Big(\sum_{i=0}^k \phi_t^i \, dS_t^i + \phi_t^0 \, dD_t \Big) \\
&= \sum_{i=0}^k \phi_t^i \big(S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i \big) + \phi_t^0 B_t^{-1} dD_t \\
&= \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \big(dS_t^* + B_t^{-1} dD_t \big) = \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \, d\widehat{S}_t,
\end{aligned}$$

where the auxiliary process $\widehat{S}$ is given by the following expression:

$$\widehat{S}_t = S_t^* + \int_{]0,t]} B_u^{-1} \, dD_u.$$

To conclude, it suffices to observe that in view of (6) the process $\widehat{S}$ satisfies

$$\widehat{S}_t = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \tag{10}$$

and thus it follows a martingale under $\mathbb{Q}^*$. □

It is worth noting that $\widehat{S}_t$, given by formula (10), represents the discounted *cum-dividend price* at time t of the 0^{th} asset, that is, the arbitrage price at time t of all past and future dividends associated with the 0^{th} asset over its lifespan. To check this, let us consider a buy-and-hold strategy such that $\psi_0^k = 0$. Then, in view of (5), the terminal wealth at time T of this strategy equals

$$V_T(\psi) = B_T \int_{]0,T]} B_u^{-1}\, dD_u. \tag{11}$$

It is clear that $V_T(\psi)$ represents all dividends from S in the form of a single payoff at time T. The arbitrage price $\pi_t(\widehat{Y})$ at time $t < T$ of a claim $\widehat{Y} = V_T(\psi)$ equals (under the assumption that this claim is attainable)

$$\pi_t(\widehat{Y}) = B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big)$$

and thus $\widehat{S}_t = B_t^{-1}\pi_t(\widehat{Y})$. It is clear that discounted cum-dividend price follows a martingale under $\mathbb{Q}^*$ (under the standard integrability assumption).

Remarks. (i) Under the assumption of uniqueness of a spot martingale measure $\mathbb{Q}^*$, any $\mathbb{Q}^*$-integrable contingent claim is attainable, and the valuation formula established above can be justified by means of replication.
(ii) Otherwise – that is, when a martingale probability measure $\mathbb{Q}^*$ is not uniquely determined by the model $(S^1, S^2, \ldots, S^k)$ – the right-hand side of (6) may depend on the choice of a particular martingale probability, in general. In this case, a process defined by (6) for an arbitrarily chosen spot martingale measure $\mathbb{Q}^*$ can be taken as the no-arbitrage price process of a defaultable claim. In some cases, a market model can be completed by postulating that S is also a traded asset.

1.5 Martingale Properties of Prices of a Defaultable Claim

In the next result, we summarize the martingale properties of prices of a generic defaultable claim.

Corollary 1.2 *The discounted cum-dividend price $\widehat{S}_t$, $t \in [0,T]$, of a defaultable claim is a $\mathbb{Q}^*$-martingale with respect to $\mathbb{G}$. The discounted ex-dividend price S_t^*, $t \in [0,T]$, satisfies*

$$S_t^* = \widehat{S}_t - \int_{]0,t]} B_u^{-1}\, dD_u, \quad \forall\, t \in [0,T],$$

and thus it follows a supermartingale under $\mathbb{Q}^*$ if and only if the dividend process D is increasing.

In an application considered in Section 2, the finite variation process A is interpreted as the positive premium paid in instalments by the claimholder to the counterparty in exchange for a positive recovery (received by the claimholder either at maturity or at default). It is thus natural to assume that A is a decreasing process, and all other components of the dividend process are increasing processes (that is, we postulate that $X \geq 0$, and $Z \geq 0$). It is rather clear that, under these assumptions, the discounted ex-dividend price S^* is neither a super- or submartingale under $\mathbb{Q}^*$, in general.

Assume now that $A \equiv 0$, so that the premium for a defaultable claim is paid upfront at time 0, and it is not accounted for in the dividend process D. We postulate, as before, that $X \geq 0$, and $Z \geq 0$. In this case, the dividend process D is manifestly increasing, and thus the discounted ex-dividend price S^* is a supermartingale under $\mathbb{Q}^*$. This feature is quite natural since the discounted expected value of future dividends decreases when time elapses.

The final conclusion is that the martingale properties of the price of a defaultable claim depend on the specification of a claim and conventions regarding the prices (ex-dividend price or cum-dividend price). This point will be illustrated below by means of a detailed analysis of prices of credit default swaps.

2 Pricing and Trading a CDS under Deterministic Intensity

We are now in the position to apply the general theory to the case of a particular class contracts, specifically, credit default swaps. We work throughout under a spot martingale measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{G}_T)$. In the first step, we shall work under additional assumptions that the auxiliary filtration $\mathbb{F}$ is trivial, so that $\mathbb{G} = \mathbb{H}$ and the interest rate $r = 0$. Subsequently, these restrictions will be relaxed.

2.1 Valuation of a Credit Default Swap

A stylized credit default swap is formally introduced through the following definition.

Definition 2.1 A *credit default swap* with a constant rate κ and *recovery at default* is a defaultable claim $(0, A, Z, \tau)$, where $Z_t \equiv \delta(t)$ and $A_t = -\kappa t$ for every $t \in [0, T]$. An RCLL function $\delta : [0, T] \to \mathbb{R}$ represents the *default protection*, and a constant $\kappa \in \mathbb{R}$ represents the *CDS rate* (also termed the *spread, premium* or *annuity* of a CDS).

We shall first analyze the valuation and trading credit default swaps in a simple model of default risk with the filtration $\mathbb{G} = \mathbb{H}$ generated by the process $H_t = \mathbb{1}_{\{\tau \le t\}}$. We denote by F the cumulative distribution function of the default time τ under $\mathbb{Q}^*$, and we assume that F is a continuous function, with $F(0) = 0$ and $F(T) < 1$ for some fixed date $T > 0$. Also, we write $G = 1 - F$ to denote the *survival probability function* of τ, so that $G(t) > 0$ for every $t \in [0, T]$. For simplicity of exposition, we assume in this section that the interest rate $r = 0$, so that the price of a savings account $B_t = 1$ for every t. This assumption is relaxed in Section 4, in which we deal with random interest rate and default intensity. Note also that we have only one tradeable asset in our model (a savings account), and we wish to value a defaultable claim within this model. It is clear that any probability measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{H}_T)$, equivalent to $\mathbb{Q}$, can be chosen as a spot martingale measure for our model. The choice of $\mathbb{Q}^*$ is reflected in the cumulative distribution function F (in particular, in the default intensity if F is absolutely continuous).

2.1.1 Ex-dividend Price of a CDS

Consider a CDS with the rate κ, which was initiated at time 0 (or indeed at any date prior to the current date t). Its market value at time t does not depend on the past otherwise than through the level of the rate κ. Unless explicitly stated otherwise, we assume that κ is an arbitrary constant.

Unless explicitly stated otherwise, we assume that the default protection payment is received at the time of default, and it is equal $\delta(t)$ if default occurs at time t, prior to or at maturity date T.

In view of (6), the ex-dividend price of a CDS maturing at T with rate κ is given by the formula

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\le T\}}\delta(\tau)\,\Big|\,\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\}}\kappa\big((\tau \wedge T) - t\big)\,\Big|\,\mathcal{H}_t\Big), \tag{12}$$

where the first conditional expectation represents the current value of the *default protection stream* (or the *protection leg*), and the second is the value of the *survival annuity stream* (or the *fee leg*).

Note that in Lemma 2.1, we do not need to specify the inception date s of a CDS. We only assume that the maturity date T, the rate κ, and the protection payment δ are given.

Lemma 2.1 *The ex-dividend price at time $t \in [s, T]$ of a credit default swap started at s, with rate κ and protection payment $\delta(\tau)$ at default, equals*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\,\frac{1}{G(t)}\left(-\int_t^T \delta(u)\,dG(u) - \kappa\int_t^T G(u)\,du\right). \tag{13}$$

Proof. We have, on the set $\{t < \tau\}$,

$$\begin{aligned} S_t(\kappa) &= -\frac{\int_t^T \delta(u)\, dG(u)}{G(t)} - \kappa \left(\frac{-\int_t^T u\, dG(u) + TG(T)}{G(t)} - t \right) \\ &= \frac{1}{G(t)} \left(-\int_t^T \delta(u)\, dG(u) - \kappa \Big(TG(T) - tG(t) - \int_t^T u\, dG(u) \Big) \right). \end{aligned}$$

Since

$$\int_t^T G(u)\, du = TG(T) - tG(t) - \int_t^T u\, dG(u), \tag{14}$$

we conclude that (13) holds. □

The ex-dividend price of a CDS can also be represented as follows (see (7))

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa), \quad \forall\, t \in [0, T], \tag{15}$$

where $\widetilde{S}_t(\kappa)$ stands for the *ex-dividend pre-default price* of a CDS. It is useful to note that formula (13) yields an explicit expression for $\widetilde{S}_t(\kappa)$, and that $\widetilde{S}(\kappa)$ follows a continuous function, provided that G is continuous.

2.2 Market CDS Rate

Assume now that a CDS was initiated at some date $s \le t$ and its initial price was equal to zero. Since a CDS with this property plays an important role, we introduce a formal definition. In Definition 2.2, it is implicitly assumed that a recovery function δ is given.

Definition 2.2 A *market CDS started at* s is a CDS initiated at time s whose initial value is equal to zero. A T-maturity *market CDS rate* (also known as the *fair CDS spread*) at time s is the level of the rate $\kappa = \kappa(s, T)$ that makes a T-maturity CDS started at s valueless at its inception. A market CDS rate at time s is thus determined by the equation $S_s(\kappa(s, T)) = 0$, where S is defined by (12). By assumption, $\kappa(s, T)$ is an $\mathcal{F}_s$-measurable random variable (hence, a constant if the reference filtration is trivial).

Under the present assumptions, by virtue of Lemma 2.1, the T-maturity market CDS rate $\kappa(s, T)$ solves the following equation

$$\int_s^T \delta(u)\, dG(u) + \kappa(s, T) \int_s^T G(u)\, du = 0,$$

and thus we have, for every $s \in [0, T]$,

$$\kappa(s, T) = -\frac{\int_s^T \delta(u)\, dG(u)}{\int_s^T G(u)\, du}. \tag{16}$$

Remarks. Let us comment briefly on a model calibration. Suppose that at time 0 the market gives the premium of a CDS for any maturity T. In this way, the market chooses the risk-neutral probability measure $\mathbb{Q}^*$. Specifically, if $\kappa(0, T)$ is the T-maturity market CDS rate for a given recovery function δ then we have

$$\kappa(0, T) = -\frac{\int_0^T \delta(u)\, dG(u)}{\int_0^T G(u)\, du}.$$

Hence, if credit default swaps with the same recovery function δ and various maturities are traded at time 0, it is possible to find the implied risk-neutral c.d.f. F (and thus the default intensity γ under $\mathbb{Q}^*$) from the *term structure of CDS rates* $\kappa(0,T)$ by solving an ordinary differential equation.

Standing assumptions. We fix a maturity date T, and we write briefly $\kappa(s)$ instead of $\kappa(s,T)$. In addition, we assume that all credit default swaps have a common recovery function δ.

Note that the ex-dividend pre-default value at time $t \in [0,T]$ of a CDS with any fixed rate κ can be easily related to the market rate $\kappa(t)$. We have the following result, in which the quantity $\nu(t,s) = \kappa(t) - \kappa(s)$ represents the *calendar CDS market spread* (for a given maturity T).

Proposition 2.1 *The ex-dividend price of a market CDS started at s with recovery δ at default and maturity T equals, for every $t \in [s,T]$,*

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}}\,(\kappa(t) - \kappa(s))\,\frac{\int_t^T G(u)\,du}{G(t)} = \mathbb{1}_{\{t<\tau\}}\,\nu(t,s)\,\frac{\int_t^T G(u)\,du}{G(t)}, \tag{17}$$

or more explicitly,

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}}\,\frac{\int_t^T G(u)\,du}{G(t)}\left(\frac{\int_s^T \delta(u)\,dG(u)}{\int_s^T G(u)\,du} - \frac{\int_t^T \delta(u)\,dG(u)}{\int_t^T G(u)\,du}\right). \tag{18}$$

Proof. To establish equality (18), it suffices to observe that $S_t(\kappa(s)) = S_t(\kappa(s)) - S_t(\kappa(t))$, and to use (13) and (16). □

Remark. Note that the price of a CDS can take negative values.

2.2.1 Forward Start CDS

A representation of the value of a swap in terms of the market swap rate, similar to (17), is well known to hold for default-free interest rate swaps. It is particularly useful if the calendar spread is modeled as a stochastic process. In particular, it leads to the Black swaption formula within the framework of Jamshidian's [11] model of co-terminal forward swap rates.

In the present context, it is convenient to consider a *forward start CDS* initiated at time $s \in [0,U]$ and giving default protection over the future time interval $[U,T]$. If the reference entity defaults prior to the start date U the contract is terminated and no payments are made. The price of this contract at any date $t \in [s,U]$ equals

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\le T\}}\delta(\tau)\,\Big|\,\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\}}\kappa\big((\tau\wedge T) - U\big)\,\Big|\,\mathcal{H}_t\Big). \tag{19}$$

Since a forward start CDS does not pays any dividends prior to the start date U, the price $S_t(\kappa)$, $t \in [s,U]$, can be considered here as either the cum-dividend price or the ex-dividend price. Note that since G is continuous, the probability of default occurring at time U equals zero, and thus for $t = U$ the last formula coincides with (12). This is by no means surprising, since at time T a forward start CDS becomes a standard (i.e., spot) CDS.

If G is continuous, representation (19) can be made more explicit, namely,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\,\frac{1}{G(t)}\left(-\int_U^T \delta(u)\,dG(u) - \kappa\int_U^T G(u)\,du\right).$$

A *forward start market CDS* at time $t \in [0,U]$ is a forward CDS in which κ is chosen at time t in such a way that the contract is valueless at time t. The corresponding (pre-default) *forward CDS rate* $\kappa(t,U,T)$ is thus determined by the the following equation

$$S_t(\kappa(t,U,T)) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\le T\}}\delta(\tau)\,\Big|\,\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\}}\kappa(t,U,T)\big((\tau\wedge T) - U\big)\,\Big|\,\mathcal{H}_t\Big) = 0,$$

which yields, for every $t \in [0, U]$,

$$\kappa(t, U, T) = -\frac{\int_U^T \delta(u)\, dG(u)}{\int_U^T G(u)\, du}.$$

The price of an arbitrary forward CDS can be easily expressed in terms of κ and $\kappa(t, U, T)$. We have, for every $t \in [0, U]$,

$$S_t(\kappa) = S_t(\kappa) - S_t(\kappa(t, U, T)) = (\kappa(t, U, T) - \kappa)\, \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\}}\big((\tau \wedge T) - U\big) \,\Big|\, \mathcal{H}_t\Big),$$

or more explicitly,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\kappa(t, U, T) - \kappa)\, \frac{\int_U^T G(u)\, du}{G(t)}.$$

Under the assumption of a deterministic default intensity, the formulae above are of rather limited interest. Let us stress, however, that similar representations are also valid in the case of a stochastic default intensity (see Section 4.5 below), where they prove useful in pricing of options on a forward start CDS (equivalently, options on a forward CDS rate).

2.2.2 Case of a Constant Default Intensity

Assume that $\delta(t) = \delta$ is independent of t, and $F(t) = 1 - e^{-\gamma t}$ for a constant default intensity $\gamma > 0$ under $\mathbb{Q}^*$. In this case, the valuation formulae for a CDS can be further simplified. In view of Lemma 2.1, the ex-dividend price of a (spot) CDS with rate κ equals, for every $t \in [0, T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\delta\gamma - \kappa)\gamma^{-1}\Big(1 - e^{-\gamma(T-t)}\Big).$$

The last formula (or the general formula (16)) yields $\kappa(s) = \delta\gamma$ for every $s < T$, so that the market rate $\kappa(s)$ is independent of s. As a consequence, the ex-dividend price of a market CDS started at s equals zero not only at the inception date s, but indeed at any time $t \in [s, T]$, both prior to and after default). Hence, this process follows a trivial martingale under $\mathbb{Q}^*$. As we shall see in what follows, this martingale property the ex-dividend price of a market CDS is an exception, rather than a rule, so that it no longer holds if default intensity is not constant.

2.3 Price Dynamics of a CDS

Unless explicitly stated otherwise, we consider a spot CDS and we assume that

$$G(t) = \mathbb{Q}^*(\tau > t) = \exp\left(-\int_0^t \gamma(u)\, du\right),$$

where the default intensity $\gamma(t)$ under $\mathbb{Q}^*$ is a strictly positive deterministic function. We first focus on the dynamics of the ex-dividend price of a CDS with rate κ started at some date $s < T$.

Lemma 2.2 *The dynamics of the ex-dividend price $S_t(\kappa)$ on $[s, T]$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt, \tag{20}$$

where the $\mathbb{H}$-martingale M under $\mathbb{Q}^*$ is given by the formula

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du, \quad \forall\, t \in \mathbb{R}_+. \tag{21}$$

Hence, the process $\bar{S}_t(\kappa)$, $t \in [s, T]$, given by the expression

$$\bar{S}_t(\kappa) = S_t(\kappa) + \int_s^t \delta(u)\, dH_u - \kappa \int_s^t (1 - H_u)\, du \tag{22}$$

is a $\mathbb{Q}^*$-martingale for $t \in [s,T]$. Specifically,

$$d\bar{S}_t(\kappa) = \big(\delta(t) - S_{t-}(\kappa)\big)\, dM_t. \tag{23}$$

Proof. It suffices to recall that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa) = (1 - H_t)\widetilde{S}_t(\kappa)$$

so that

$$dS_t(\kappa) = (1 - H_t)\, d\widetilde{S}_t(\kappa) - \widetilde{S}_{t-}(\kappa)\, dH_t.$$

Using formula (13), we find easily that we have

$$d\widetilde{S}_t(\kappa) = \gamma(t)\widetilde{S}_t(\kappa)\, dt + (\kappa - \delta(t)\gamma(t))\, dt. \tag{24}$$

In view of (21), the proof of (20) is complete. To prove the second statement, it suffices to observe that the process N given by

$$N_t = S_t(\kappa) - \int_s^t (1 - H_u)(\kappa - \delta(u)\gamma(u))\, du = -\int_s^t S_{u-}(\kappa)\, dM_u$$

is an $\mathbb{H}$-martingale under $\mathbb{Q}^*$. But for every $t \in [s,T]$

$$\bar{S}_t(\kappa) = N_t + \int_s^t \delta(u)\, M_u,$$

so that $\bar{S}(\kappa)$ also follows an $\mathbb{H}$-martingale under $\mathbb{Q}^*$. Note that the process $\bar{S}(\kappa)$ given by (22) represents the cum-dividend price of a CDS, so that the martingale property $\bar{S}(\kappa)$ is expected. □

Equality (20) emphasizes the fact that a single cash flow of $\delta(\tau)$ occurring at time τ can be formally treated as a dividend stream at the rate $\delta(t)\gamma(t)$ paid continuously prior to default. It is clear that we also have

$$dS_t(\kappa) = -\widetilde{S}_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt. \tag{25}$$

In some instances, it can be useful to reformulate the dynamics of a market CDS in terms of market observables, such as CDS spreads.

Corollary 2.1 *The dynamics of the ex-dividend price $S_t(\kappa(s))$ on $[s,T]$ are also given as*

$$dS_t(\kappa(s)) = -S_{t-}(\kappa(s))\, dM_t + (1 - H_t)\left(\frac{\int_t^T G(u)\, du}{G(t)}\, d_t\nu(t,s) - \nu(t,s)\, dt \right). \tag{26}$$

Proof. Under the present assumptions, for any fixed s, the calendar spread $\nu(t,s)$, $t \in [s,T]$ is a continuous function of bounded variation. In view of (20), it suffices to check that

$$\frac{\int_t^T G(u)\, du}{G(t)}\, d_t\nu(t,s) - \nu(t,s)\, dt = (\kappa(s) - \delta(t)\gamma(t))\, dt, \tag{27}$$

where $d_t\nu(t,s) = d_t(\kappa(t) - \kappa(s)) = d\kappa(t)$. Equality (27) follows by elementary computations. □

2.3.1 Trading a Credit Default Swap

We shall show that, in the present set-up, in order to replicate an arbitrary contingent claim Y settling at time T and satisfying the usual integrability condition, it suffices to deal with two traded assets: a CDS with maturity $U \geq T$ and a constant savings account $B = 1$. Since one can always work with discounted values, the last assumption is not restrictive.

According to Section 1.4, a strategy $\phi_t = (\phi^0_t, \phi^1_t)$, $t \in [0, T]$, is self-financing if the wealth process $V(\phi)$, defined as

$$V_t(\phi) = \phi^0_t + \phi^1_t S_t(\kappa), \tag{28}$$

satisfies

$$dV_t(\phi) = \phi^1_t \big(dS_t(\kappa) + dD_t \big), \tag{29}$$

where $S(\kappa)$ is the ex-dividend price of a CDS with the dividend stream D. As usual, we say that a strategy ϕ replicates a contingent claim Y if $V_T(\phi) = Y$. On the set $\{\tau \leq t \leq T\}$ the ex-dividend price $S(\kappa)$ equals zero, and thus the total wealth is necessarily invested in B, so that it is constant. This means that ϕ replicates Y if and only if $V_{\tau \wedge T}(\phi) = Y$.

Lemma 2.3 *For any self-financing strategy ϕ we have, on the set $\{\tau \leq T\}$,*

$$\Delta_\tau V(\phi) := V_\tau(\phi) - V_{\tau-}(\phi) = \phi^1_\tau (\delta(\tau) - \widetilde{S}_\tau(\kappa)). \tag{30}$$

Proof. In general, the process ϕ^1 is $\mathbb{G}$-predictable. In our model, ϕ^1 is assumed to be an RCLL function. The jump of the wealth process $V(\phi)$ at time τ equals, on the set $\{\tau \leq T\}$,

$$\Delta_\tau V(\phi) = \phi^1_\tau \Delta_\tau S + \phi^1_\tau \Delta_\tau D$$

where $\Delta_\tau S(\kappa) = S_\tau(\kappa) - S_{\tau-}(\kappa) = -\widetilde{S}_\tau(\kappa)$ (recall that the ex-dividend price $S(\kappa)$ drops to zero at default time) and manifestly $\Delta_\tau D = \delta(\tau)$. □

2.4 Hedging of Defaultable Claims

An $\mathcal{H}_T$-measurable random variable Y is known to admit the following representation

$$Y = \mathbb{1}_{\{T \geq \tau\}} h(\tau) + \mathbb{1}_{\{T < \tau\}} c(T), \tag{31}$$

where $h : [0, T] \to \mathbb{R}$ is a Borel measurable function, and $c(T)$ is a constant. For definiteness, we shall deal with claims Y such that h is an RCLL function, but this formal restriction is not essential.

We first recall a suitable version of the predictable representation theorem. Subsequently, we derive closed-form solution for the replicating strategy for a claim Y given by (31) and settling at time T. As tradeable assets, we shall use a CDS started at time 0 and maturing at T, and a savings account.

2.4.1 Representation Theorem

For any RCLL function $\widehat{h} : \mathbb{R}_+ \to \mathbb{R}$ such that the random variable $\widehat{h}(\tau)$ is integrable, we set $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(\widehat{h}(\tau) \,|\, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. It is clear that $\widehat{M}$ is an $\mathbb{H}$-martingale under $\mathbb{Q}^*$. The following version of the martingale representation theorem is well known (see, for instance, Blanchet-Scalliet and Jeanblanc [5], Jeanblanc and Rutkowski [13] or Proposition 4.3.2 in Bielecki and Rutkowski [3]).

Proposition 2.2 *Assume that G is continuous and $\widehat{h}$ is an RCLL function such that the random variable $\widehat{h}(\tau)$ is $\mathbb{Q}^*$-integrable. Then the $\mathbb{H}$-martingale $\widehat{M}$ admits the following integral representation*

$$\widehat{M}_t = \widehat{M}_0 + \int_{]0,t]} (\widehat{h}(u) - \widehat{g}(u))\, dM_u, \tag{32}$$

where the continuous function $\widehat{g} : \mathbb{R}_+ \to \mathbb{R}$ is given by the formula

$$\widehat{g}(t) = \frac{1}{G(t)}\, \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{\tau > t\}} \widehat{h}(\tau)\big) = -\frac{1}{G(t)} \int_t^\infty \widehat{h}(u)\, dG(u). \tag{33}$$

Remark. It is easily seen that on the set $\{t \le \tau\}$ we have $\widehat{g}(t) = \widehat{M}_{t-}$. Therefore, formula (32) can also be rewritten as follows

$$\widehat{M}_t = \widehat{M}_0 + \int_{]0,t]} \big(\widehat{h}(u) - \widehat{M}_{u-}\big)\, dM_u = \widehat{M}_0 + \int_{]0,t]} \big(\widehat{h}(u) - \widetilde{M}(u-)\big)\, dM_u, \tag{34}$$

where $\widetilde{M} = \widehat{g}$ is the unique function such that $\widehat{M}_t \mathbb{1}_{\{\tau>t\}} = \widetilde{M}(t)\mathbb{1}_{\{\tau>t\}}$ for every $t \in \mathbb{R}_+$.

2.4.2 Replication of a Defaultable Claim

Assume now that a random variable Y given (31) represents a contingent claim settling at T. Formally, we deal with a defaultable claim of the form $(X, 0, Z, \tau)$, where $X = c(T)$ and $Z_t = h(t)$.

To deal with such a claim, we shall apply Proposition 2.2 to the function $\widehat{h}$, where $\widehat{h}(t) = h(t)$ for $t < T$ and $\widehat{h}(t) = c(T)$ for $t \ge T$ (recall that $\mathbb{Q}^*(\tau = T) = 0$). In this case, we obtain

$$\widehat{g}(t) = \frac{1}{G(t)} \left(- \int_t^T h(u)\, dG(u) + c(T)G(T) \right), \tag{35}$$

and thus for the process $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{H}_t)$, $t \in [0,T]$, we have

$$\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y) + \int_{]0,t]} (h(u) - \widehat{g}(u))\, dM_u \tag{36}$$

with $\widehat{g}$ given by (35). Recall that $\widetilde{S}(\kappa)$ is the pre-default ex-dividend price process of a CDS with rate κ and maturity T. We know that $\widetilde{S}(\kappa)$ is a continuous function of t if G is continuous.

Proposition 2.3 *Assume that the inequality $\widetilde{S}_t(\kappa) \neq \delta(t)$ holds for every $t \in [0,T]$. Let ϕ^1 be an RCLL function given by the formula*

$$\phi^1_t = \frac{h(t) - \widehat{g}(t)}{\delta(t) - \widetilde{S}_t(\kappa)}, \tag{37}$$

and let $\phi^0_t = V_t(\phi) - \phi^1_t S_t(\kappa)$, where the process $V(\phi)$ is given by (29) with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$, where Y is given by (31). Then the self-financing trading strategy $\phi = (\phi^0, \phi^1)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$, where $X = c(T)$ and $Z_t = h(t)$.

Proof. The idea of the proof is based on the observation that it is enough to concentrate on the formula for trading strategy prior to default. In view of Lemma 2.2, the dynamics of the price $S(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt.$$

and thus we have, on the set $\{\tau > t\}$,

$$dS_t(\kappa) = d\widetilde{S}_t(\kappa) = \big(\gamma(t)\widetilde{S}_t(\kappa) + \kappa - \delta(t)\gamma(t)\big)dt. \tag{38}$$

¿From Corollary 1.1, we know that the wealth $V(\phi)$ of any admissible self-financing strategy is an $\mathbb{H}$-martingale under $\mathbb{Q}^*$. Since under the present assumptions $dB_t = 0$, for the wealth process $V(\phi)$ we obtain, on the set $\{\tau > t\}$,

$$dV_t(\phi) = \phi^1_t \, (d\widetilde{S}_t(\kappa) - \kappa\, dt) = -\phi^1_t \gamma(t)\big(\delta(t) - \widetilde{S}_t(\kappa)\big)\, dt. \tag{39}$$

For the martingale $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{H}_t)$ associated with Y, in view of (36) we obtain, on the set $\{\tau > t\}$,

$$d\widehat{M}_t = -\gamma(t)(h(t) - \widehat{g}(t))\, dt. \tag{40}$$

We wish to find ϕ^1 such that $V_t(\phi) = \widehat{M}_t$ for every $t \in [0,T]$. To this end, we first focus on the equality $\mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widehat{M}_t$ for pre-default values. A comparison of (39) with (40) yields

$$\phi_t^1 = \frac{h(t) - \widehat{g}(t)}{\delta(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0,T]. \tag{41}$$

We thus see that if $V_0(\phi) = \widehat{M}_0$ then also $\mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widehat{M}_t$ for every $t \in [0,T]$. As usual, the second component of a self-financing strategy ϕ is given by (28), that is, $\phi_t^0 = V_t(\phi) - \phi_t^1 S_t(\kappa)$, where $V(\phi)$ is given by (29) with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$. In particular, we have that $\phi_0^0 = \mathbb{E}_{\mathbb{Q}^*}(Y) - \phi_0^1 S_0(\kappa)$.

To complete the proof, that is, to show that $V_t(\phi) = \widehat{M}_t$ for every $t \in [0,T]$, it suffices to compare the jumps of both processes at time τ (both martingales are stopped at τ). It is clear from (36) that the jump of $\widehat{M}$ equals $\Delta_\tau \widehat{M} = h(\tau) - \widehat{g}(\tau)$. Using (30), we get for the jump of the wealth process

$$\Delta_\tau V(\phi) = \phi_\tau^1(\delta(\tau) - \widetilde{S}_\tau(\kappa)) = h(\tau) - \widehat{g}(\tau),$$

and thus we conclude that $V_t(\phi) = \widehat{M}_t$ for every $t \in [0,T]$. In particular, ϕ is admissible and $V_T(\phi) = V_{\tau\wedge T}(\phi) = h(\tau \wedge T) = Y$, so that ϕ replicates a claim Y. Note that if $\kappa = \kappa(0)$ then $S_0(\kappa(0)) = 0$, so that $\phi_0^0 = V_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$. □

Let us now analyze the condition $\widetilde{S}_t(\kappa) \neq \delta(t)$ for every $t \in [0,T]$. It ensures, in particular, that the wealth process $V(\phi)$ has a non-zero jump at default time for any the self-financing trading strategy such that $\phi_t^1 \neq 0$ for every $t \in [0,T]$. It appears that this condition is not restrictive, since it is satisfied under mild assumptions.

Indeed, if $\kappa > 0$ and δ is a non-increasing function then the inequality $\widetilde{S}_t(\kappa) < \delta(t)$ is valid for every $t \in [0,T]$ (this follows easily from (12)). For instance, if $\gamma(t) > 0$ and the protection payment $\delta > 0$ is constant then it is clear from (16) that the market rate $\kappa(0)$ is strictly positive. Consequently, formula (12) implies that $\widetilde{S}_t(\kappa(0)) < \delta$ for every $t \in [0,T]$, as was required. To summarize, when a tradeable asset is a market CDS with a constant $\delta > 0$ and the default intensity is strictly positive then the inequality holds. Let us finally observe that if the default intensity vanishes on some set then we do not need to impose the inequality $\widetilde{S}_t(\kappa) \neq \delta(t)$ on this set in order to equate (39) with (40), since the desired equality holds anyway.

It is useful to note that the proof of Proposition 2.3 was implicitly based on the following observation. In our case, Lemma 2.4 can be applied to the following $\mathbb{H}$-martingales under $\mathbb{Q}^*$: $M^1 = V(\phi)$, that is, the wealth process of an admissible self-financing strategy ϕ and $M^2 = \widehat{M}$, that is, the conjectured price of a claim Y, as given by the risk-neutral valuation formula.

Lemma 2.4 *Let M^1 and M^2 be arbitrary two $\mathbb{H}$-martingales under $\mathbb{Q}^*$. If for every $t \in [0,T]$ we have $\mathbb{1}_{\{t<\tau\}} M_t^1 = \mathbb{1}_{\{t<\tau\}} M_t^2$ then $M_t^1 = M_t^2$ for every $t \in [0,T]$.*

Proof. We have $M_t^i = \mathbb{E}_{\mathbb{Q}^*}(h_i(\tau)\,|\,\mathcal{H}_t)$ for some functions $h_i : \mathbb{R}_+ \to \mathbb{R}$ such that $h_i(\tau)$ is $\mathbb{Q}^*$-integrable. Using the well known formula for the conditional expectation

$$\mathbb{E}_{\mathbb{Q}^*}(h_i(\tau)\,|\,\mathcal{H}_t) = \mathbb{1}_{\{t\geq\tau\}} h_i(\tau) - \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \int_t^\infty h_i(u)\, dG(u) = \mathbb{1}_{\{t\geq\tau\}} h_i(\tau) + \mathbb{1}_{\{t<\tau\}} \widehat{g}_i(t),$$

and the assumption that $\mathbb{1}_{\{t<\tau\}} M_t^1 = \mathbb{1}_{\{t<\tau\}} M_t^2$, we obtain the equality $\widehat{g}_1(t) = \widehat{g}_2(t)$ for every $t \in [0,T]$ (recall that $\mathbb{Q}^*(\tau > t) > 0$ for every $t \in [0,T]$). Therefore, we have

$$\int_t^\infty h_1(u)\, dG(u) = \int_t^\infty h_2(u)\, dG(u), \quad \forall\, t \in [0,T].$$

This immediately implies that $h_1(t) = h_2(t)$ on $[0,T]$, almost everywhere with respect to the distribution of τ, and thus we have $h_1(\tau \wedge T) = h_2(\tau \wedge T)$, $\mathbb{Q}^*$-a.s. Consequently, $M_t^1 = M_t^2$ for every $t \in [0,T]$. □

The method presented above can be extended to replicate a defaultable claim (X, A, Z, τ), where $X = c(T)$, $A_t = \int_0^t a(u)\,du$ and $Z_t = h(t)$ for some RCLL functions a and h. In this case, it is natural to expect that the cum-dividend price process π_t associated with a defaultable claim (X, A, Z, τ), is given by the formula, for every $t \in [0, T]$,

$$\pi_t = \widehat{M}_t + \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \int_t^T a(u)G(u)\,du + \mathbb{1}_{\{t\geq\tau\}} \int_0^\tau a(u)\,du, \tag{42}$$

where $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{H}_t)$ with Y is given by (31). Hence, the pre-default dynamics of this process are

$$d\pi_t = d\widehat{M}_t + \gamma(t)\widehat{a}(t)\,dt = -\gamma(t)\big(h(t) - \widehat{g}(t) - \widehat{a}(t)\big)\,dt,$$

where we set $\widehat{a}(t) = (G(t))^{-1} \int_t^T a(u)G(u)\,du$. Note that $\widehat{a}(t)$ represents the pre-default value of the future promised dividends associated with A. Therefore, arguing as in the proof of Proposition 2.3, we find the following expression for the component ϕ^1 of a replicating strategy for a defaultable claim (X, A, Z, τ)

$$\phi^1_t = \frac{h(t) - \widehat{g}(t) - \widehat{a}(t)}{\delta(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0, T]. \tag{43}$$

It is easy to see that the jump condition at time τ, mentioned in the second part of the proof of Proposition 2.3, is also satisfied in this case.

Remark. Of course, if we take as (X, A, Z, τ) a CDS with rate κ and recovery function δ, then we have $h(t) = \delta(t)$ and $\widehat{g}(t) + \widehat{a}(t) = \widetilde{S}_t(\kappa)$, so that clearly $\phi^1_t = 1$ for every $t \in [0, T]$.

The following immediate corollary to Proposition 2.3 is worth stating.

Corollary 2.2 *Assume that $\widetilde{S}_t(\kappa) \neq \delta(t)$ for every $t \in [0, T]$. Then the market is complete, in the sense, that any defaultable claim (X, A, Z, τ), where $X = c(T)$, $A_t = \int_0^t a(u)\,du$ and $Z_t = h(t)$ for some constant $c(T)$ and RCLL functions a and h, is attainable through continuous trading in a CDS and a bond. The cum-dividend arbitrage price π_t of such defaultable claim satisfies, for every $t \in [0, T]$,*

$$\pi_t = V_t(\phi) = \pi_0 + \int_{]0,t]} (h(u) - \Pi_{u-})\,dM_u,$$

where

$$\pi_0 = \mathbb{E}_{\mathbb{Q}^*}(Y) + \int_0^T a(t)G(t)\,dt,$$

with Y given by (31) . Its pre-default price is $\widetilde{\pi}(t) = \widehat{g}(t) + \widehat{a}(t) + A_t$, so that we have, for every $t \in [0, T]$

$$\pi_t = \mathbb{1}_{\{t<\tau\}}(\widehat{g}(t) + \widehat{a}(t) + A_t) + \mathbb{1}_{\{t\geq\tau\}}(h(\tau) + A_\tau) = \mathbb{1}_{\{t<\tau\}}\widetilde{\pi}(t) + \mathbb{1}_{\{t\geq\tau\}}\pi_\tau.$$

2.4.3 Case of a Constant Default Intensity

As a partial check of the calculations above, we shall consider once again the case of constant default intensity and constant protection payment. In this case, $\kappa(0) = \delta\gamma$ and $S_t(\kappa(0)) = 0$ for every $t \in [0, T]$, so that

$$dV_t(\phi) = -\phi^1_t \delta\gamma\,dt = -\phi^1_t \kappa(0)\,dt. \tag{44}$$

Furthermore, for any RCLL function h, formula (41) yields

$$\phi^1_t = \delta^{-1}\Big(h(t) + e^{\gamma t} \int_t^T h(u)\,d\big(e^{-\gamma u}\big) - c(T)e^{-\gamma T}\Big). \tag{45}$$

Assume, for instance, that $h(t) = \delta$ for $t \in [0, T[$ and $c(T) = 0$. Then (45) gives $\phi_t^1 = e^{-\gamma(T-t)}$. Since $S_0(\kappa(0)) = 0$, we have $\phi_0^0 = \pi_0(Y) = V_0(\phi) = \delta(1 - e^{-\gamma T})$. In view of (44), the gains/losses from positions in market CDSs over the time interval $[0, t]$ equal, on the set $\{\tau > t\}$,

$$V_t(\phi) - V_0(\phi) = -\delta\gamma \int_0^t \phi_u^1 \, du = -\delta\gamma \int_0^t e^{-\gamma(T-u)} \, du = -\delta e^{-\gamma T} \big(e^{\gamma t} - 1 \big) < 0.$$

Suppose that default occurs at some date $t \in [0, T]$. Then the protection payments is collected, and the wealth at time t becomes

$$V_t(\phi) = V_{t-}(\phi) + \phi_t^1 \delta = \delta(1 - e^{-\gamma T}) - \delta e^{-\gamma T} \big(e^{\gamma t} - 1 \big) + \delta e^{-\gamma(T-t)} = \delta.$$

The last equality shows that the strategy is indeed replicating on the set $\{\tau \leq T\}$. On the set $\{\tau > T\}$, the wealth at time T equals

$$V_T(\phi) = \delta(1 - e^{-\gamma T}) - \delta e^{-\gamma T} \big(e^{\gamma T} - 1 \big) = 0.$$

Since $S_t(\kappa(0)) = 0$ for every $t \in [0, T]$, we have that $\phi_t^0 = V_t(\phi)$ for every $t \in [0, T]$.

2.4.4 Short Sale of a CDS

As usual, we assume that the maturity T of a CDS is fixed and we consider the situation where the default has not yet occurred.

1. Long position. We say that an agent has a long position at time t in a CDS if he owns at time t a CDS contract that had been created (initiated) at time s_0 by some two parties and was sold to the agent (by means of assignment for example) at time s. If $s_0 = s$ then the agent is an original counter-party to the contract, that is the agent owns the contract from initiation. If an agent owns a CDS contract, the agent is entitled to receive the protection payment for which the agent pays the premium. The long position in a contract may be liquidated at any time $s < t < T$ by means of assignment or offsetting.

2. Short position. We stress that the short position, namely, selling a CDS contract to a dealer, can only be created for a newly initiated contract. It is not possible to sell to a dealer at time t a CDS contract initiated at time $s_0 < t$.

3. Offsetting a long position. If an agent has purchased at time $s_0 \leq s < T$ a CDS contract initiated at s_0, he can offset his long position by creating a short position at time t. A new contract is initiated at time t, with the initial price $S_t(\kappa(s_0))$, possibly with a new dealer. This short position offsets the long position outstanding, so that the agent effectively has a zero position in the contract at time t and thereafter.

4. Market constraints. The above taxonomy of positions may have some bearing on portfolios involving short positions in CDS contracts. It should be stressed that not all trades involving a CDS are feasible in practice. Let us consider the CDS contract initiated at time t_0 and maturing at time T. Recall that the ex-dividend price of this contract for any $t \in [t_0, \tau \wedge T[$ is $S_t(\kappa(t_0))$. This is the theoretical price at which the contract should trade so to avoid arbitrage. This price also provides substance for the P&L analysis as it really marks-to-market positions in the CDS contract.

Let us denote the time-t position in the CDS contract of an agent as ϕ_t^1, where $t \in [t_0, \tau \wedge T]$. The strategy is subject to the following constraints: $\phi_t^1 \geq 0$ if $\phi_{t_0}^1 \geq 0$ and $\phi_t^1 \geq \phi_{t_0}^1$ if $\phi_{t_0}^1 \leq 0$. It is clear that both restrictions are related to short sale of a CDS. The next result shows that under some assumptions a replicating strategy for a claim Y does not require a short sale of a CDS.

Corollary 2.3 *Assume that $\widetilde{S}_t(\kappa) < \delta(t)$ for every $t \in [0, T]$. Let h be a non-increasing function and let $c(T) \leq h(T)$. Then $\phi_t^1 \geq 0$ for every $t \in [0, T]$.*

Proof. It is enough to observe that if h be a non-increasing function and $c(T) \leq h(T)$ then it follows easily from the first equality in (33) that for the function $\widehat{g}$ given by (35) we have that $h(t) \geq \widehat{g}(t)$ for every $t \in [0, T]$. In view of (37), this shows that $\phi_t^1 \geq 0$ for every $t \in [0, T]$. □

3 Hedging of Basket Credit Derivatives with Single-Name CDSs

Our goal is to examine hedging of basket credit derivatives with single-name credit default swaps on the underlying n credit names, denoted as $1, 2, \ldots, n$. For the clarity of exposition, the assumption that $B(t) = 1$ for every $t \in \mathbb{R}_+$ is maintained throughout this section. In Section 4, we shall show that this assumption is not restrictive.

Let $\tau_1, \tau_2, \ldots, \tau_n$ be the default times associated with n names, respectively. Let

$$F(t_1, t_2, \ldots, t_n) = \mathbb{Q}^*(\tau_1 \le t_1, \tau_2 \le t_2, \ldots, \tau_n \le t_2)$$

denote the joint distribution function of the default times associated with the n names. We shall frequently assume that the probability distribution of default times is jointly continuous, and we shall write $f(t_1, t_2, \ldots, t_n)$ to denote the joint probability density function. Also, let

$$G(t_1, t_2, \ldots, t_n) = \mathbb{Q}^*(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_n > t_n)$$

stand for the joint probability that the names $1, 2, \ldots, n$ have survived up to times $t_1, t_2, \ldots, t_n$, respectively.

For each $i = 1, 2, \ldots, n$ we define the default indicator process for the ith firm as $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$ and the corresponding σ-field $\mathbb{H}_t^i = \sigma(H_u^i : u \le t)$. We write

$$\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2 \vee \ldots \vee \mathbb{H}^n$$

and

$$\mathbb{G}^i = \mathbb{H}^1 \vee \ldots \vee \mathbb{H}^{i-1} \vee \mathbb{H}^{i+1} \vee \ldots \vee \mathbb{H}^n,$$

so that $\mathbb{G} = \mathbb{G}^i \vee \mathbb{H}^i$ for $i = 1, 2, \ldots, n$. The main tool in the analysis of joint defaults is the concept of the $\mathbb{G}^i$-hazard process.

Definition 3.1 Assume that $\mathbb{Q}^*(\tau_i > t \,|\, \mathcal{G}_t^i) < 1$ for $t \in \mathbb{R}_+$. Then the $\mathbb{G}^i$-adapted process Γ^i defined through the formula

$$\mathbb{Q}^*(\tau_i > t \,|\, \mathcal{G}_t^i) = G_t^i = e^{-\Gamma_t^i}, \quad \forall\, t \in \mathbb{R}_+.$$

is called the $\mathbb{G}^i$-*hazard process* Γ^i of τ_i,

For the properties of an $\mathbb{G}^i$-hazard process, we refer to Chapter 5 in Bielecki and Rutkowski [3].

The process G^i is a $\mathbb{G}^i$-supermartingale and admits a Doob Meyer decomposition as $G^i = Z^i - A^i$, where Z is a martingale and A^i a predictable increasing process. We assume that A^i is absolutely continuous, so that $A_t^i = \int_0^t a_u^i \, du$ for some non negative process a^i. Then, the process

$$H_t^i - \int_0^t (1 - H_s^i) \lambda_s^i ds$$

is a $\mathbb{G}$-martingale, with $\lambda_i^t = \frac{a_s^i}{G_s^i}$. In the case of two names, one has: the process

$$H_t^1 - \int_0^{t \wedge \tau_1} \frac{a_s^{(1)}}{1 - F_{s^-}^{1*}} ds$$

where $F_s^{1*} = \mathbb{P}(\tau_1 \le t | \mathcal{H}_t^2)$ $a_t^{(1)} = -H_t^2 \partial_1 h^{(1)}(t, \tau_2) - (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(t,t)}$ and

$$h^{(1)}(t, s) = \frac{\partial_2 G(t, s)}{\partial_2 G(0, s)}\,.$$

is a $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$-martingale, and

$$dF_t^{1*} = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,\tau_2)}{\partial_2 G(0,\tau_2)}\right) dH_t^2 + \left(-H_t^2 \partial_1 h(t,\tau_2) + (1-H_t^2)\psi'(t)\right) dt$$

One can prove that the $\mathbb{G}^i$-intensity process λ^i is given as the $\mathbb{Q}^*$-almost sure limit

$$\lambda_t^i = \lim_{h\downarrow 0} \frac{1}{h} \frac{\mathbb{Q}^*(t < \tau_i \le t+h \,|\, \mathcal{G}_t^i)}{\mathbb{Q}^*(\tau_i > t \,|\, \mathcal{G}_t^i)}. \tag{46}$$

3.1 First-to-Default Intensities and Martingales

In this section, we introduce the so-called first-to-default intensities (or pre-default intensities). This concept will prove useful in the valuation and hedging of first-to-default claims, and thus also in the case of general kth-to-default claims.

Let $\tau_{(1)} = \tau_1 \wedge \ldots \wedge \tau_n = \min(\tau_1, \ldots, \tau_n)$ and, more generally, let $\tau_{(k)}$ be the kth order statistics of the collection $\{\tau_1, \tau_2, \ldots, \tau_n\}$ of default times. In the financial interpretation, the random time $\tau_{(k)}$ represents the moment of the kth default out of n names. In particular, $\tau_{(1)}$ is the moment of the first default, and thus no defaults are observed on the event $\{\tau_{(1)} > t\}$.

It is clear that, for any $i = 1, 2, \ldots, n$ and $t, h \in \mathbb{R}_+$,

$$\mathbb{1}_{\{\tau_{(1)}>t\}} \mathbb{Q}^*(\tau_i > t \,|\, \mathcal{G}_t^i) = \mathbb{1}_{\{\tau_{(1)}>t\}} \mathbb{Q}^*(\tau_i > t \,|\, \tau_1 > t, \ldots, \tau_{i-1} > t, \tau_{i+1} > t, \ldots, \tau_n > t)$$

and

$$\mathbb{1}_{\{\tau_{(1)}>t\}} \mathbb{Q}^*(t < \tau_i \le t+h \,|\, \mathcal{G}_t^i) = \mathbb{1}_{\{\tau_{(1)}>t\}} \mathbb{Q}^*(t < \tau_i \le t+h \,|\, \tau_1 > t, \ldots, \tau_{i-1} > t, \tau_{i+1} > t, \ldots, \tau_n > t).$$

Consequently, formula (**??**) yields $\mathbb{1}_{\{\tau_{(1)}>t\}} \gamma_t^i = \mathbb{1}_{\{\tau_{(1)}>t\}} \widetilde{\gamma}_i(t)$, where the function $\widetilde{\gamma}_i : \mathbb{R}_+ \to \mathbb{R}_+$ is given as

$$\begin{aligned} \widetilde{\gamma}_i(t) &= \lim_{h\downarrow 0} \frac{1}{h} \frac{\mathbb{Q}^*(t < \tau_i \le t+h \,|\, \tau_1 > t, \ldots, \tau_{i-1} > t, \tau_{i+1} > t, \ldots, \tau_n > t)}{\mathbb{Q}^*(\tau_i > t \,|\, \tau_1 > t, \ldots, \tau_{i-1} > t, \tau_{i+1} > t, \ldots, \tau_n > t)} \\ &= \lim_{h\downarrow 0} \frac{1}{h} \mathbb{Q}^*(t < \tau_i \le t+h \,|\, \tau_{(1)} > t). \end{aligned} \tag{47}$$

This observation and the interpretation of γ^i lead to the following definition.

Definition 3.2 The function $\widetilde{\gamma}_i$ given by formula (47) is called the *ith first-to-default intensity*. The *first-to-default intensity* $\widetilde{\gamma}$ is defined as the sum $\widetilde{\gamma} = \sum_{i=1}^n \widetilde{\gamma}_i$, or equivalently, as we prove later, as the intensity function of the random time $\tau_{(1)}$ modeling the moment of the first default.

In view of the definition of $\widetilde{\gamma}_i$ we have that

$$\begin{aligned} \widetilde{\gamma}_i(t) &= \frac{\int_t^\infty \int_t^\infty \ldots \int_t^\infty f(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_n)\, du_1 \ldots du_{i-1} du_{i+1} \ldots du_n}{\mathbb{Q}^*(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_n > t_n)} \\ &= \frac{\int_t^\infty \int_t^\infty \ldots \int_t^\infty dF(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_n)}{G(t, \ldots, t)} = -\frac{\partial_i G(t, \ldots, t)}{G(t, \ldots, t)}. \end{aligned}$$

In particular, for the case of two credit names, the first-to-default intensities $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are given as

$$\widetilde{\gamma}_1(t) = \lim_{h\downarrow 0} \frac{\mathbb{Q}^*(t < \tau_1 \le t+h \,|\, \tau_2 > t)}{h\, \mathbb{Q}^*(\tau_1 > t \,|\, \tau_2 > t)} = \frac{\int_t^\infty f(t,u)\, du}{G(t,t)} = -\frac{\partial_1 G(t,t)}{G(t,t)} \tag{48}$$

and

$$\widetilde{\gamma}_2(t) = \lim_{\Delta t\downarrow 0} \frac{\mathbb{Q}^*(t < \tau_2 \le t+h \,|\, \tau_1 > t)}{h\, \mathbb{Q}^*(\tau_2 > t \,|\, \tau_1 > t)} = \frac{\int_t^\infty f(u,t)\, du}{G(t,t)} = -\frac{\partial_2 G(t,t)}{G(t,t)}. \tag{49}$$

Let us denote $G_{(1)}(t) = G(t, \ldots, t) = \mathbb{Q}^*(\tau_{(1)} > t)$. Then we have the following elementary lemma, which summarizes the above considerations.

Lemma 3.1 *For any $i = 1, 2, \dots, n$ we have*

$$\partial_i G(t, \dots, t) := \frac{\partial G(t_1, \dots, t_n)}{\partial t_i}\Big|_{t_1 = \dots = t_n = t} = -G_{(1)}(t)\widetilde{\gamma}_i(t). \tag{50}$$

and

$$\frac{dG(t, \dots, t)}{dt} = \frac{dG_{(1)}(t)}{dt} = -G_{(1)}(t)\sum_{i=1}^{n}\widetilde{\gamma}_i(t) = -G_{(1)}(t)\widetilde{\gamma}(t). \tag{51}$$

The ith first-to-default intensity function $\widetilde{\gamma}_i$ should not be confused with the marginal intensity function γ_i of τ_i, which is defined as

$$\gamma_i(t) = \frac{f_i(t)}{G_i(t)}, \quad \forall\, t \in \mathbb{R}_+,$$

where f_i is the marginal probability density function of τ_i, that is,

$$f_i(t) = \int_0^\infty \int_0^\infty \dots \int_0^\infty f(u_1, \dots, u_{i-1}, t, u_{i+1}, \dots, u_n)\, du_1 \dots du_{i-1} du_{i+1} \dots du_n,$$

and $G_i(t) = 1 - F_i(t) = \int_t^\infty f_i(u)\, du$. It is worth noting that we have $\widetilde{\gamma}_i \neq \gamma_i$, in general (see Example 3.1). However, if $\tau_1, \dots, \tau_n$ are mutually independent under $\mathbb{Q}^*$ then $\widetilde{\gamma}_i = \gamma_i$, so that first-to-default and marginal default intensities coincide.

¿From Definition 3.2 (see also (51)), it follows that

$$\widetilde{\gamma}(t) = \lim_{h \downarrow 0} \frac{1}{h} \sum_{i=1}^{n} \mathbb{Q}^*(t < \tau_i \leq t + h \,|\, \tau_{(1)} > t) = \lim_{h \downarrow 0} \frac{1}{h}\, \mathbb{Q}^*(t < \tau_{(1)} \leq t + h \,|\, \tau_{(1)} > t),$$

and it is rather clear that the first-to-default intensity $\widetilde{\gamma}$ is not equal to the sum of marginal default intensities, so that $\widetilde{\gamma} \neq \sum_{i=1}^n \gamma_i$, in general.

Example 3.1 Let τ_1 be a random time, and let $\tau_2 = \tau_1 + a$ where a is a strictly positive constant. Then $F_2(t) := \mathbb{Q}^*(\tau_2 \leq t) = F_1(t - a)$ where $F_1(t) := \mathbb{Q}^*(\tau_1 \leq t)$. Then we have, for any $t \in \mathbb{R}_+$,

$$\gamma_1(t) = \frac{f_1(t)}{1 - F_1(t)}, \quad \gamma_2(t) = \mathbb{1}_{\{t > a\}} \frac{f_1(t-a)}{1 - F_1(t-a)}$$

whereas $\widetilde{\gamma}_1(t) = \gamma(t)$ and $\widetilde{\gamma}_2(t) = 0$ for $t \in \mathbb{R}_+$.

Let us now define the basic jump $\mathbb{G}$-martingales corresponding to each of the n names. Specifically, for each $i = 1, 2, \dots, n$, the process M^i given by the formula (cf. (21))

$$M_t^i = H_t^i - \int_0^t (1 - H_u^i)\gamma_u^i\, du, \quad \forall\, t \in \mathbb{R}_+, \tag{52}$$

is known to be a $\mathbb{G}$-martingale under $\mathbb{Q}^*$ (see Proposition 5.1.3 in Bielecki and Rutkowski [3]).

A random time $\tau_{(1)}$ is manifestly a $\mathbb{G}$-stopping time. Therefore, for each $i = 1, 2, \dots, n$, using that $\mathbb{1}_{\{\tau_{(1)} > t\}}\gamma_t^i = \mathbb{1}_{\{\tau_{(1)} > t\}}\widetilde{\gamma}_t^i$, the process $\widehat{M}^i$, given by the formula (cf. (52))

$$\widehat{M}_t^i := M_{t \wedge \tau_{(1)}}^i = H_{t \wedge \tau_{(1)}}^i - \int_0^t \mathbb{1}_{\{\tau_{(1)} > u\}}\widetilde{\gamma}_i(u)\, du, \quad \forall\, t \in \mathbb{R}_+, \tag{53}$$

also follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$. Processes $\widehat{M}^i$ are referred to as the *basic first-to-default martingales*. They will play an essential role in a multivariate version of a martingale representation theorem (see Proposition 3.1 below).

Remark. It should be noted that the postulated knowledge of the joint default distribution and the possibility of explicit computations first-to-default intensities are rather strong assumptions, especially, when dependent defaults are modelled using the value-of-the-firm or the copula-based approaches. However, in certain intensity-based models of dependent defaults, the first-to-default (and other conditional) intensities are in fact given a priori as inputs, so that in that case the abovementioned issues do not arise at all.

3.2 First-to-Default Martingale Representation Theorem

A suitable version of a martingale representation theorem (see Proposition 2.2) played a key role in the replication of claims in the single-name set-up. In the multi-name setup, this theorem will also form the keystone of the replication argument. We now state an integral representation theorem for a $\mathbb{G}$-martingale stopped at $\tau_{(1)}$ with respect to basic first-to-default martingales, which also, by definition, are stopped at $\tau_{(1)}$. To simplify notation, we provide the proof of this result in a bivariate setting. In that case, $\tau_{(1)} = \tau_1 \wedge \tau_2$ and $\mathcal{G}_t = \mathcal{H}^1_t \vee \mathcal{H}^2_t$.

Proposition 3.1 *Assume that the joint distribution of default times τ_1 and τ_2 admits the probability density function $f(u,v)$. Let $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{G}_t)$, $t \in [0,T]$, be a $\mathbb{G}$-martingale, where*

$$Y = Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\le T\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\le T\}} + c(T)\mathbb{1}_{\{\tau_{(1)}>T\}} \tag{54}$$

for some functions $Z_i : [0,T] \to \mathbb{R}$, $i = 1,2$, and some constant $c(T)$. Then the $\mathbb{Q}^*$-martingale $\widehat{M}$ admits the following representation, for $t \in [0,T]$,

$$\widehat{M}_t = \widehat{M}_0 + \int_{]0,t]} h_1(u)\, d\widehat{M}^1_u + \int_{]0,t]} h_2(u)\, d\widehat{M}^2_u \tag{55}$$

where $\widehat{M}^i_t := M^i_{t\wedge\tau_{(1)}}$ for $i = 1,2$ are defined in (53), and the functions h_i, $i = 1,2$, are given by

$$h_i(t) = Z_i(t) - \widehat{M}_{t-} = Z_i(t) - \widetilde{M}(t-), \quad \forall\, t \in [0,T], \tag{56}$$

where $\widetilde{M}$ is the unique function such that $\widehat{M}_t\mathbb{1}_{\{\tau_{(1)}>t\}} = \widetilde{M}(t)\mathbb{1}_{\{\tau_{(1)}>t\}}$ for every $t \in [0,T]$. The function $\widetilde{M}$ satisfies $\widetilde{M}_0 = \widehat{M}_0$ and

$$d\widetilde{M}(t) = \widetilde{\gamma}(t)\widetilde{M}(t)\, dt - \big(Z_1(t)\widetilde{\gamma}_1(t) + Z_2(t)\widetilde{\gamma}_2(t)\big)\, dt, \tag{57}$$

or equivalently,

$$d\widetilde{M}(t) = \widetilde{\gamma}_1(t)\big(\widetilde{M}(t) - Z_1(t)\big)\, dt + \widetilde{\gamma}_2(t)\big(\widetilde{M}(t) - Z_2(t)\big)\, dt. \tag{58}$$

Proof. It is clear that

$$\begin{aligned}\widehat{M}_t &= \mathbb{E}_{\mathbb{Q}^*}(Z_1(\tau_1)\mathbb{1}_{\{\tau_1\le T,\,\tau_2>\tau_1\}} \,|\, \mathcal{G}_t) + \mathbb{E}_{\mathbb{Q}^*}(Z_2(\tau_2)\mathbb{1}_{\{\tau_2\le T,\,\tau_1>\tau_2\}} \,|\, \mathcal{G}_t) + \mathbb{E}_{\mathbb{Q}^*}(c(T)\mathbb{1}_{\{\tau_{(1)}>T\}} \,|\, \mathcal{G}_t)\\ &= Y^1_t + Y^2_t + Y^3_t.\end{aligned}$$

We have

$$\mathbb{1}_{\{\tau_{(1)}>t\}}\widehat{M}_t = \mathbb{1}_{\{\tau_{(1)}>t\}}\widetilde{M}_t = \mathbb{1}_{\{\tau_{(1)}>t\}}(\widetilde{Y}^1(t) + \widetilde{Y}^2(t) + \widetilde{Y}^3(t)),$$

where the functions $\widetilde{Y}^i$, $i = 1,2,3$, are given by (by convention, $0/0 = 0$)

$$\widetilde{Y}^1(t) = \frac{\int_t^T du Z_1(u) \int_u^\infty dv f(u,v)}{G(t,t)}, \quad \widetilde{Y}^2(t) = \frac{\int_t^T dv Z_2(v) \int_v^\infty du f(u,v)}{G(t,t)}, \quad \widetilde{Y}^3(t) = \frac{c(T)G(T,T)}{G(t,t)}.$$

¿From (48), it follows that

$$\begin{aligned}\frac{d}{dt}\widetilde{Y}^1(t) &= -\frac{Z_1(t)\int_t^\infty dvf(t,v)}{G(t,t)} - \frac{\int_t^T duZ_1(u)\int_u^\infty dvf(u,v)}{G^2(t,t)}\Big(\frac{d}{dt}G(t,t)\Big)\\ &= -Z_1(t)\frac{\int_t^\infty dvf(t,v)}{G(t,t)} - \widetilde{Y}^1(t)\frac{1}{G(t,t)}\Big(\frac{d}{dt}G(t,t)\Big)\\ &= -Z_1(t)\widetilde{\gamma}_1(t) + \widetilde{Y}^1(t)(\widetilde{\gamma}_1(t)+\widetilde{\gamma}_2(t)) \qquad (59)\end{aligned}$$

and thus, by symmetry,

$$\frac{d\widetilde{Y}^2(t)}{dt} = \widetilde{Y}^2(t)(\widetilde{\gamma}_1(t)+\widetilde{\gamma}_2(t)) - \widetilde{\gamma}_2(t)Z_2(t). \qquad (60)$$

For $\widetilde{Y}^3(t)$, we obtain

$$\frac{d\widetilde{Y}^3(t)}{dt} = \frac{C(T)G(T,T)}{G(t,t)}\Big(\frac{-1}{G(t,t)}\Big)\Big(\frac{d}{dt}G(t,t)\Big) = \widetilde{Y}^3(t)(\widetilde{\gamma}_1(t)+\widetilde{\gamma}_2(t)).$$

Hence, taking $\widetilde{M}(t) = \widetilde{Y}^1(t)+\widetilde{Y}^2(t)+\widetilde{Y}^3(t)$, we obtain

$$\begin{aligned}d\widetilde{M}(t) &= d\widetilde{Y}^1(t)+d\widetilde{Y}^2(t)+d\widetilde{Y}^3(t)\\ &= -\widetilde{\gamma}_1(t)\Big(Z_1(t)-\sum_{i=1}^3\widetilde{Y}^i(t)\Big)\,dt - \widetilde{\gamma}_2(t)\Big(Z_2(t)-\sum_{i=1}^3\widetilde{Y}^i(t)\Big)\,dt\\ &= -\widetilde{\gamma}_1(t)\big(Z_1(t)-\widetilde{M}(t)\big)\,dt - \widetilde{\gamma}_2(t)\big(Z_2(t)-\widetilde{M}(t)\big)\,dt\\ &= -\widetilde{\gamma}_1(t)\big(Z_1(t)-\widetilde{M}(t-)\big)\,dt - \widetilde{\gamma}_2(t)\big(Z_2(t)-\widetilde{M}(t-)\big)\,dt.\end{aligned}$$

Consequently, on the event $\{\tau_{(1)} > t\}$,

$$d\widehat{M}_t = -\widetilde{\gamma}_1(t)\big(Z_1(t)-\widehat{M}_{t-}\big)\,dt - \widetilde{\gamma}_2(t)\big(Z_2(t)-\widehat{M}_{t-}\big)\,dt.$$

This shows that (55) is valid on the event $\{\tau_{(1)} > t\}$ for every $t\in[0,T]$. To conclude the proof, it suffices to show that both sides of equality (55) coincide at time $\tau_{(1)}$ on the event $\{\tau_{(1)} \le T\}$. To this end, we observe that we have, on $\{\tau_{(1)} \le T\}$,

$$\widehat{M}_{\tau_{(1)}} = Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}}.$$

The right-hand side in (55) is equal to, on the set $\{\tau_{(1)} \le T\}$,

$$\begin{aligned}\widehat{M}_0 &+ \int_{]0,\tau_{(1)}[} h_1(u)\,d\widehat{M}^1_u + \int_{]0,\tau_{(1)}[} h_2(u)\,d\widehat{M}^2_u\\ &+ \mathbb{1}_{\{\tau_{(1)}=\tau_1\}}\int_{[\tau_{(1)}]} h_1(u)\,dH^1_u + \mathbb{1}_{\{\tau_{(1)}=\tau_2\}}\int_{[\tau_{(1)}]} h_2(u)\,dH^2_u\\ &= \widetilde{M}(\tau_{(1)}-) + \Big(Z_1(\tau_1)-\widetilde{M}(\tau_{(1)}-)\Big)\,\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + \Big(Z_2(\tau_2)-\widetilde{M}(\tau_{(1)}-)\Big)\,\mathbb{1}_{\{\tau_{(1)}=\tau_2\}}\\ &= Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}}\end{aligned}$$

as $\widetilde{M}(\tau_{(1)}-) = \widehat{M}_{\tau_{(1)}-}$. Since the processes on both sides of equality (55) are stopped at $\tau_{(1)}$, we conclude that equality (55) is valid for every $t\in[0,T]$. Uniqueness of function $\widetilde{M}$ is obvious, as $\tau_{(1)}$ can take any value in the interval $[0,T]$. Formulae (57)-(58) are easy consequences of (55). □

The proof of a first-to-default martingale representation theorem for the bivariate version can be easily extended to a more general multivariate setting. Hence, we are in a position to state without proof the following extension of Proposition 3.1. Recall that $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2 \vee \ldots \vee \mathbb{H}^n$.

Proposition 3.2 *Assume that the joint distribution of default times* $\tau_1, \tau_2, \ldots, \tau_n$ *has the density function* $f(t_1, t_2, \ldots, t_n)$. *Let* $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y|\mathcal{G}_t)$, $t \in [0,T]$, *be the* $\mathbb{G}$*-martingale under* $\mathbb{Q}^*$, *where*

$$Y = \sum_{i=1}^{n} Z_i(\tau_i) \mathbb{1}_{\{\tau_i \leq T,\, \tau_i = \tau_{(1)}\}} + c(T) \mathbb{1}_{\{\tau_{(1)} > T\}} \tag{61}$$

for some functions $Z_i : [0,T] \to \mathbb{R}$, $i = 1, 2, \ldots, n$, *and some constant* $c(T)$. *Then the process* $\widehat{M}$ *admits the following representation*

$$\widehat{M}_t = \widehat{M}_0 + \sum_{i=1}^{n} \int_{]0,t]} h_i(u)\, d\widehat{M}^i_u$$

where $\widehat{M}^i_t = M^i_{t \wedge \tau_{(1)}}$ *for* $i = 1, 2, \ldots, n$, *and the functions* h_i, $i = 1, 2, \ldots, n$ *are given by*

$$h_i(t) = Z_i(t) - \widehat{M}_{t-} = Z_i(t) - \widetilde{M}(t-), \quad \forall\, t \in [0,T],$$

where $\widetilde{M}$ *is the unique function such that* $\widehat{M}_t \mathbb{1}_{\{\tau_{(1)} > t\}} = \widetilde{M}(t) \mathbb{1}_{\{\tau_{(1)} > t\}}$ *for every* $t \in [0,T]$. *The function* $\widetilde{M}$ *satisfies* $\widetilde{M}_0 = \widehat{M}_0$ *and*

$$d\widetilde{M}(t) = \widetilde{\gamma}(t) \widetilde{M}(t)\, dt - \sum_{i=1}^{n} Z_i(t) \widetilde{\gamma}_i(t)\, dt, \tag{62}$$

or equivalently,

$$d\widetilde{M}(t) = \sum_{i=1}^{n} \widetilde{\gamma}_i(t) \big(\widetilde{M}(t) - Z_i(t) \big)\, dt. \tag{63}$$

3.3 First-to-Default (FtD) Claims

We first analyze the case of a first-to-default claim. In the next section, we shall argue that the general kth to default claim can be dealt with as a family of (conditional) first-to-default claims.

Definition 3.3 A *first-to-default claim* (an *FtD claim*, for short) on a basket of n credit names is a defaultable claim $(X, 0, Z, \tau_{(1)})$, where X is a constant amount payable at maturity if no default occurs, and $Z = (Z_1, Z_2, \ldots, Z_n)$, where a function $Z_i : [0,T] \to \mathbb{R}$ specifies the recovery payment made at the time τ_i if the ith firm was the first defaulted firm, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.

3.3.1 Pricing of a First-to-Default Claim

The next result deals with the valuation of a first-to-default claim in a multivariate set-up. Let us stress that the concept of the tentative risk-neutral price will be later supported by strict replication arguments (see Propositions 3.4 and 3.5). In this section, we simply define the *price process* π a first-to-default claim by setting

$$\pi_t = \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}^*}(Z_i(\tau_i) \mathbb{1}_{\{\tau_{(1)} = \tau_i \leq T\}} | \mathcal{G}_t) + \mathbb{E}_{\mathbb{Q}^*}(c(T) \mathbb{1}_{\{\tau_{(1)} > T\}} | \mathcal{G}_t).$$

By a *pre-default price* associated with a $\mathbb{G}$-adapted price process π, we mean here the function $\widetilde{\pi}$ such that $\pi_t \mathbb{1}_{\{\tau_{(1)} > t\}} = \widetilde{\pi}(t) \mathbb{1}_{\{\tau_{(1)} > t\}}$ for every $t \in [0,T]$. Manifestly, the pre-default pricing function $\widetilde{\pi}$ and the price process π coincide prior to the first default only.

The following result is an easy consequence of Proposition 3.2.

Proposition 3.3 *The pre-default price of a FtD claim* $(X, 0, Z, \tau_{(1)})$*, where* $Z = (Z_1, \ldots, Z_n)$ *and* $X = c(T)$*, equals*

$$\begin{aligned}\widetilde{\pi}(t) &= \sum_{i=1}^{n} \frac{\int_{t_i}^{\infty}\int_{t_i}^{\infty}\cdots\int_{t_i}^{\infty}\int_{t}^{T}\int_{t_i}^{\infty}\cdots\int_{t_i}^{\infty} Z_i(t_i)\, f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n)dt_1\cdots dt_n}{G(t,\ldots,t)} \\ &\quad + c(T)\frac{G(T,\ldots,T)}{G(t,\ldots,t)}.\end{aligned}$$

Moreover, $\widehat{g}(0) = \pi_0$ *and*

$$d\widetilde{\pi}(t) = \widetilde{\gamma}(t)\widetilde{\pi}(t)\,dt - \sum_{i=1}^{n} Z_i(t)\widetilde{\gamma}_i(t)\,dt, \tag{64}$$

or equivalently,

$$d\widetilde{\pi}(t) = \sum_{i=1}^{n} \widetilde{\gamma}_i(t)\big(\widetilde{\pi}(t) - Z_i(t)\big)\,dt. \tag{65}$$

Example 3.2 To provide a simple illustration of Proposition 3.3, we evaluate the pre-default price of the FtD claim $Y = (X, 0, Z, \tau_{(1)})$, where $Z = (Z_1, Z_2)$ for some constants Z_1, Z_2 and $X = c(T)$. We assume that the default times τ_1 and τ_2 are modelled as independent exponentially distributed with constant parameters $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, respectively. Then the joint survival function equals

$$G(u, v) = e^{-\widetilde{\gamma}_1 u} e^{-\widetilde{\gamma}_2 v},$$

so that

$$G(du, dv) = \widetilde{\gamma}_1\widetilde{\gamma}_2 e^{-\widetilde{\gamma}_1 u} e^{-\widetilde{\gamma}_2 v}\, dudv = F(du, dv).$$

We thus have

$$\begin{aligned}\widetilde{\pi}(t) &= \frac{Z_1\int_t^T\int_u^{\infty} G(du,dv)}{G(t,t)} + \frac{Z_1\int_t^T\int_v^{\infty} G(du,dv)}{G(t,t)} + c(T)\frac{G(T,T)}{G(t,t)} \\ &= \frac{\widetilde{\gamma}_1\widetilde{\gamma}_2 Z_1\int_t^T e^{-\widetilde{\gamma}_1 u}\left[\int_u^{\infty} e^{-\widetilde{\gamma}_2 v}\,dv\right]du}{e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)t}} + \frac{\widetilde{\gamma}_1\widetilde{\gamma}_2 Z_2\int_t^T e^{-\widetilde{\gamma}_2 v}\left[\int_v^{\infty} e^{-\widetilde{\gamma}_1 u}\,du\right]dv}{e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)t}} + c(T)\frac{G(T,T)}{G(t,t)} \\ &= \frac{\widetilde{\gamma}_1 Z_1\int_t^T e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)u}\,du}{e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)t}} + \frac{\widetilde{\gamma}_2 Z_2\int_t^T e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)v}\,dv}{e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)t}} + c(T)\frac{G(T,T)}{G(t,t)} \\ &= \frac{\widetilde{\gamma}_1 Z_1}{\widetilde{\gamma}_1+\widetilde{\gamma}_2}\left[1 - e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)(T-t)}\right] + \frac{\widetilde{\gamma}_2 Z_2}{\widetilde{\gamma}_1+\widetilde{\gamma}_2}\left[1 - e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)(T-t)}\right] + c(T)\frac{G(T,T)}{G(t,t)} \\ &= \frac{\widetilde{\gamma}_1 Z_1 + \widetilde{\gamma}_2 Z_2}{\widetilde{\gamma}_1+\widetilde{\gamma}_2}\left[1 - e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)(T-t)}\right] + c(T)e^{-(\widetilde{\gamma}_1+\widetilde{\gamma}_2)(T-t)}.\end{aligned}$$

Of course, due to the independence assumption, the first-to-default intensities $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ coincide here with the marginal default intensities γ_1 and γ_2, respectively.

3.3.2 Replication of a First-to-Default Claim

Let us consider a family of single-name CDSs with default protections δ_i and rates κ_i. For convenience, we assume that they have the same maturity T, but this assumption can be easily relaxed. We say that a strategy $\phi_t = (\phi_t^0, \phi_t^1, \ldots, \phi_t^n)$, $t \in [0, T]$, in traded assets $(B, S^1(\kappa_1), \ldots, S^n(\kappa_n))$ is self-financing if its wealth process $V(\phi)$, defined as

$$V_t(\phi) = \phi_t^0 + \sum_{i=1}^{n} \phi_t^i S_t^i(\kappa_i), \tag{66}$$

satisfies

$$dV_t(\phi) = \sum_{i=1}^{n} \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i \big), \tag{67}$$

where $S^i(\kappa_i)$ is the ex-dividend price of the ith CDS, that is, a single-name CDS with a fixed rate κ_i and a protection payment function δ_i, which insures against the default of the ith credit name. As expected, the ith CDS is formally defined by the dividend process $D_t^i = \delta_i(t)\mathbb{1}_{t\geq\tau_i} - \kappa_i(t\wedge\tau_i)$ for $t\in[0,T]$. Consequently, the ex-dividend price at time t of the i-th CDS is

$$S_t^i(\kappa_i) = \mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{t<\tau_i\leq T}\delta_i(\tau_i)|\mathcal{G}_t) - \kappa_i\,\mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau_i\}}\big((\tau_i\wedge T)-t\big)\,\big|\,\mathcal{G}_t\big). \tag{68}$$

The crucial observation is that when examining dynamic replication of a first-to-default claim, we will only need to deal with the dynamics of each CDS on the interval $[0,\tau_{(1)}\wedge T[$, as well as the value of each CDS at the moment $\tau_{(1)}\wedge T$.

We first note that the ex-dividend price $S_t^i(\kappa_i)$ can be represented as follows

$$\begin{aligned} S_t^i(\kappa_i) &= \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\leq T\}}\delta_i(\tau_{(1)}) + \sum_{j\neq i}\mathbb{1}_{\{t<\tau_{(1)}=\tau_j\leq T\}}(\delta_i(\tau_i)+\kappa_i(\tau_i-\tau_{(1)}))\,\Big|\,\mathcal{G}_t\Big) \\ &\quad - \kappa_i\,\mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau_{(1)}\}}(\tau_{(1)}\wedge T - t)\,\big|\,\mathcal{G}_t\big). \end{aligned}$$

Hence, by conditioning on the σ-field $\mathcal{G}_{\tau_{(1)}}$, we obtain, on $\{\tau_{(1)}>t\}$,

$$\begin{aligned} S_t^i(\kappa_i) &= \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_1\leq T\}}\delta_i(\tau_{(1)}) + \sum_{j\neq i}\mathbb{1}_{\{t<\tau_{(1)}=\tau_j\leq T\}}S^i_{\tau_{(1)}}(\kappa_i)\,\Big|\,\mathcal{G}_t\Big) \\ &\quad - \kappa_i\,\mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau_{(1)}\}}(\tau_{(1)}\wedge T - t)\,\big|\,\mathcal{G}_t\big). \end{aligned}$$

The last representation by no means surprising, since it only shows that in order to compute the price prior to the first default (i.e., on the event $\{\tau_{(1)}>t\}$), we can either do the computations in one shot by considering the cash flows occurring on $]t,\tau_1\wedge T]$, or we can compute first the ex-dividend price at time $\tau_{(1)}\wedge T$ and then price all resulting cash flows on $]t,\tau_{(1)}\wedge T]$. Note also that $S^i_{\tau_{(1)}}(\kappa_i)$ is equal to the conditional expectation with respect to σ-field $\mathcal{G}_{\tau_{(1)}}$ of the cash flows of the ith CDS on $]\tau_{(1)}\vee\tau_i,\tau_i\wedge T]$.

We find it convenient to introduce the following formal definition, in which we do not specify a new contract, but we simply describe the cash flows from the ith CDS on the random interval $[0,\tau_{(1)}\wedge T]$. Nevertheless, for the ease of future reference, we give a new name to this specific description of the ith CDS.

Definition 3.4 The ith *embedded first-to-default CDS* (briefly, the ith *embedded FtD CDS*) is described as follows:
(i) it pays $\delta_i(t)$ at time t on the event $\{\tau_{(1)}=\tau_i=t\leq T\}$,
(ii) it pays the ex-dividend price $S_t^i(\kappa_i)$ of the ith CDS at time t on the event $\{\tau_{(1)}=\tau_j=t\leq T\}$ for any $j\neq i$,
(iii) the fee payments are made at the rate κ_i until the moment of the first default, i.e., until $\tau_{(1)}\wedge T$.

As mentioned above, in Definition 3.4, we do not define a new product, but we adapt the description of the standard single-name ith CDS to the current purposes of replication of a first-to-default claim. If someone wishes to use single-name CDSs to replicate a first-to-default claim with maturity T, he will have to liquidate the portfolio of CDSs at time $\tau_{(1)}$, if the first default occurs prior to or at T (otherwise, the portfolio will be liquidated at T). Hence, we will assume in this subsection that any particular single-name CDS can be formally treated as an embedded first-to-default CDS.

Remark. An embedded first-to-default CDS differs slightly from the standard first-to-default CDS, since in a first-to-default CDS the protection payments (or functions) are specified a priori in a contract. In case of the ith embedded first-to-default CDS, only the protection payment δ_i is given a priori. Other 'protection payments' are not real protection payments, but merely embedded protection payments, and thus they need to be computed using the joint distribution of default times $\tau_1, \ldots, \tau_n$. Nevertheless, it is convenient to look at a single-name CDS from the perspective of the moment of the first default, since such an approach has a strong intuitive appeal, and it largely simplifies purely mathematical considerations, especially in Section 3.5 below, where a simple inductive algorithm for the valuation and replication of a generic basket claim is provided.

In view of Definition 3.4, we find it convenient to introduce the following notation and auxiliary terminology. For any $j \neq i$, we introduce a function $S^i_{t|j}(\kappa_i) : [0,T] \to \mathbb{R}$, which represents the ex-dividend price of the ith CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$. Formally, this quantity is defined as the unique function satisfying

$$\mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}} S^i_{\tau_{(1)}|j}(\kappa_i) = \mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}} S^i_{\tau_{(1)}}(\kappa_i)$$

so that

$$\mathbb{1}_{\{\tau_{(1)}\le T\}} S^i_{\tau_{(1)}}(\kappa_i) = \mathbb{1}_{\{\tau_{(1)}=\tau_i\le T\}} \delta_i(\tau_{(1)}) + \sum_{j\neq i} \mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}} S^i_{\tau_{(1)}|j}(\kappa_i).$$

We shall argue later that the value $S^i_{t|j}(\kappa_i)$ can be easily computed using the conditional distribution of random times $(\tau_1, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_n)$ given that $\tau_{(1)} = \tau_j = t$ (the knowledge of the joint density of $(\tau_1, \ldots, \tau_n)$ is, of course, sufficient for these computations).

By the (embedded) *protection leg* of the ith embedded first-to-default CDS, we formally mean a first-to-default claim with recovery processes Z^i_j such that $Z^i_i(t) = \delta_i(t)$ and $Z^i_j(t) = S^i_{t|j}(\kappa_i)$. The (embedded) *fee leg* of the ith embedded first-to-default CDS is given as $A_t = -\kappa_i t$, so that it is the same as for the ith CDS, but it is only paid till the first default, i.e., on the random interval $[0, \tau_{(1)}]$.

Definition 3.5 We write $\widehat{g}_i(t)$ to denote the *pre-default value* at time t of the protection leg of the ith embedded first-to-default CDS. Formally, $\widehat{g}_i$ is defined as a function $\widehat{g}_i : [0,T] \to \mathbb{R}$ such that $\widehat{\pi}^i_t \mathbb{1}_{\{\tau_{(1)}>t\}} = \widehat{g}_i(t)\mathbb{1}_{\{\tau_{(1)}>t\}}$, where $\widehat{\pi}^i$ is the ex-dividend price process of the first-to-default claim $(0, 0, Z^i, \tau_{(1)})$ where $Z^i = (Z^i_1, Z^i_2, \ldots, Z^i_n)$.

Case of two credit names. For the reader's convenience, we shall first examine a dynamic replication of a first-to-default claim on a basket of two credit names. As hedging instruments, we use a savings account and the single-name CDSs on the underlying names.

In the next result, we extend formula (24) to the present set-up. According to the notation introduced above, the function $S^1_{t|2}(\kappa_1)$, $t \in [0,T]$, is aimed to represent the ex-dividend price of the first CDS at time t on the event $\{\tau_{(1)} = \tau_2 = t\}$ (this function is computed Lemma 3.3, but in Lemma 3.2 an explicit representation for $S^1_{t|2}(\kappa_1)$ is immaterial). In particular, we shall see that if τ_1 and τ_2 are independent, then $S^1_{t|2}(\kappa_1) = \widetilde{S}^1_t(\kappa_1)$.

Lemma 3.2 *Let $\widehat{g}_1(t)$ be the pre-default value at time t of the 'protection leg' of the first embedded FtD CDS. The following equality holds*

$$\widehat{g}_1(t) = \frac{\int_t^T du\, \delta_1(u) \int_u^\infty dv f(u,v)}{G(t,t)} + \frac{\int_t^T dv\, S^1_{v|2}(\kappa_1) \int_v^\infty du f(u,v)}{G(t,t)}, \tag{69}$$

where $G(t,t) = \int_t^\infty \int_t^\infty f(u,v)\, dudv$. Consequently, the dynamics of the pre-default price $\widetilde{S}^1_t(\kappa_1)$ are

$$d\widetilde{S}^1_t(\kappa_1) = (\widetilde{\gamma}_1(t) + \widetilde{\gamma}_2(t))\widetilde{S}^1_t(\kappa_1)\, dt + \big(\kappa_1 - \delta_1(t)\widetilde{\gamma}_1(t) - S^1_{t|2}(\kappa_1)\widetilde{\gamma}_2(t)\big)\, dt, \tag{70}$$

or equivalently,

$$d\widetilde{S}^1_t(\kappa_1) = \widetilde{\gamma}_1(t)\big(\widetilde{S}^1_t(\kappa_1) - \delta_1(t)\big)\, dt + \widetilde{\gamma}_2(t)\big(\widetilde{S}^1_t(\kappa_1) - S^1_{t|2}(\kappa_1)\big)\, dt + \kappa_1 dt. \tag{71}$$

Proof. The validity of equality (69) is rather clear, since this representation for $\widehat{g}_1$ can be deduced directly from Definitions 3.4-3.5, using also Proposition 3.3.

To derive the dynamics of $\widehat{g}_1(t)$, it suffices to adapt equalities (59)-(60). In the present context, they yield

$$d\widehat{g}_1(t) = \Big((\widetilde{\gamma}_1(t)+\widetilde{\gamma}_2(t))\widehat{g}_1(t) - \widetilde{\gamma}_1(t)\delta_1(t) - \widetilde{\gamma}_2(t)S^1_{t|2}(\kappa_1)\Big)dt. \tag{72}$$

To complete the proof of the lemma, we need to examine the fee leg of the first embedded FtD. Its price at time $t \in [0,T]$ equals (cf. (12))

$$h_1(t) := \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau_{(1)}\}}\kappa_1\big((\tau_{(1)}\wedge T)-t\big)\,\Big|\,\mathcal{G}_t\Big),$$

To compute this conditional expectation, it suffices to use the c.d.f. $F_{(1)}$ of the random time $\tau_{(1)}$. As in Section 2.1 (see the proof of Lemma 2.1), we obtain

$$h_1(t) = \mathbb{1}_{\{t<\tau_{(1)}\}}\frac{\kappa_1}{G_{(1)}(t)}\int_t^T G_{(1)}(u)\,du. \tag{73}$$

where $G_{(1)}(t) = 1 - F_{(1)}(t) = G(t,t)$. Hence, the pre-default price of the fee leg of the first FtD CDS, denoted as $\widehat{h}_1(t)$, satisfies $d\widehat{h}_1(t) = \kappa_1 dt + (\widetilde{\gamma}_1(t)+\widetilde{\gamma}_2(t))\widehat{h}_1(t)\,dt$. Since $\widetilde{S}^1_t(\kappa_1) = \widehat{g}_1(t) - \widehat{h}_1(t)$, the formulae (70)-(71) easily follow. □

Remark. Let us now make few comments regarding representation (69). It shows, in particular, that on the event $\{\tau_{(1)} > t\}$ the first CDS can be formally seen as a first-to-default claim with the following recovery functions $Z_1(t) = \delta_1(t)$ and $Z_2(t) = S^1_{v|2}(\kappa_1)$ and with $A_t = -\kappa_1 t$. As already argued in this section, this representation is quite natural, since it refers to the behavior of the price of the first CDS on the random interval $[0,\tau_{(1)}]$. Specifically, if $\tau_{(1)} = \tau_1 = t \le T$ then the CDS pays $\delta_1(t)$, $\tau_{(1)} = \tau_2 = t \le T$ then the CDS is worth (i.e., formally "pays") $S^1_{t|2}(\kappa_1)$. Additionally, we need, of course, to pay fees at the rate κ_1 until the moment of the first default.

Lemma 3.3 *The function $S^1_{v|2}(\kappa_1)$, $v \in [0,T]$, equals*

$$S^1_{v|2}(\kappa_1) = \frac{\int_v^T \delta_1(u)f(u,v)du}{\int_v^\infty f(u,v)\,du} - \kappa_1\frac{\int_v^T du\int_u^\infty dz f(z,v)}{\int_v^\infty f(u,v)\,du}. \tag{74}$$

Proof. Note that the conditional c.d.f. of τ_1 given that $\tau_1 > \tau_2 = v$ equals

$$\mathbb{Q}^*(\tau_1 \le u|\tau_1 > \tau_2 = v) = F_{\tau_1|\tau_1>\tau_2=v}(u) = \frac{\int_v^u f(z,v)\,dz}{\int_v^\infty f(z,v)\,dz}, \quad \forall\, u \in [v,\infty],$$

so that the conditional tail equals

$$G_{\tau_1|\tau_1>\tau_2=v}(u) = 1 - F_{\tau_1|\tau_1>\tau_2=v}(u) = \frac{\int_u^\infty f(z,v)\,dz}{\int_v^\infty f(z,v)\,dz}, \quad \forall\, u \in [v,\infty]. \tag{75}$$

Let A be the right-hand side of (74). Combining with (75), we obtain

$$A = -\int_v^T \delta_1(u)\,dG_{\tau_1|\tau_1>\tau_2=v}(u) - \kappa_1\int_v^T G_{\tau_1|\tau_1>\tau_2=v}(u)\,du.$$

Combining Lemma 2.1 with the fact that $S^1_{\tau_{(1)}}(\kappa_i)$ is equal to the conditional expectation with respect to σ-field $\mathcal{G}_{\tau_{(1)}}$ of the cash flows of the ith CDS on $]\tau_{(1)}\vee\tau_i, \tau_i\wedge T]$, we conclude that A coincides with $S^1_{v|2}(\kappa_1)$, the ex-dividend price of the first CDS on the event $\{\tau_{(1)} = \tau_2 = v\}$. □

The following result, based on formula (70), extends Lemma 2.2.

Lemma 3.4 *The dynamics of the ex-dividend price* $S^1_t(\kappa_i)$ *on the random interval* $[0, \tau_{(1)} \wedge T]$ *are*

$$dS^1_t(\kappa_1) = -S^1_{t-}(\kappa_1)\, d\widehat{M}^1_t - S^1_{t-}(\kappa_1)\, d\widehat{M}^2_t + \big(\kappa_1 - \delta_1(t)\widetilde{\gamma}_1(t) - S^1_{t|2}(\kappa_1)\widetilde{\gamma}_2(t)\big)\, dt. \tag{76}$$

Let $\bar{S}^1(\kappa_1)$ *stand for the cum-dividend price of the first CDS stopped at* $\tau_{(1)}$. *Then*

$$\bar{S}^1_t(\kappa_1) = S^1_t(\kappa_1) + \int_0^t \delta_1(u)\, dH^1_{u\wedge\tau_{(1)}} + \int_0^t S^1_{u|2}(\kappa_1)\, dH^2_{u\wedge\tau_{(1)}} - \kappa_1(t \wedge \tau_{(1)}), \tag{77}$$

and thus

$$d\bar{S}^1_t(\kappa_1) = \big(\delta_1(t) - \widetilde{S}^1_{t-}(\kappa_1)\big)\, d\widehat{M}^1_t + \big(S^1_{t|2}(\kappa_1) - \widetilde{S}^1_{t-}(\kappa_1)\big)\, d\widehat{M}^2_t. \tag{78}$$

Of course, analogous formulae can be derived for the second CDS. In particular,

$$d\bar{S}^2_t(\kappa_2) = \big(\delta_2(t) - \widetilde{S}^2_{t-}(\kappa_2)\big)\, d\widehat{M}^2_t + \big(S^2_{t|1}(\kappa_2) - \widetilde{S}^2_{t-}(\kappa_2)\big)\, d\widehat{M}^1_t. \tag{79}$$

where $S^2_{t|1}(\kappa_2)$ is the ex-dividend price of the second CDS at time t on the event $\{\tau_{(1)} = \tau_1 = t\}$.

Remark. Representation (78) can be seen as an example supporting Proposition 3.1. Let us recall that we decided to work with ex-dividend prices of CDSs, since they are more suitable for dealing with continuous dividend streams. However, when analyzing the wealth process of a self-financing strategy, we effectively deal with cum-dividend values. Therefore, representation (78) will appear to be useful in this context, as the following result shows.

Proposition 3.4 *Assume that the inequality* $\det C(t) \neq 0$ *holds for every* $t \in [0, T]$, *where the deterministic matrix* $C(t)$ *equals*

$$C(t) = \begin{bmatrix} \delta_1(t) - \widetilde{S}^1_t(\kappa_1) & \widetilde{S}^2_t(\kappa_2) - S^2_{t|1}(\kappa_2) \\ \widetilde{S}^1_t(\kappa_1) - S^1_{t|2}(\kappa_1) & \delta_2(t) - \widetilde{S}^2_t(\kappa_2) \end{bmatrix}$$

Let $\phi_t = (\phi^1_t, \phi^2_t)$ *be the unique solution to the system of linear equations*

$$\phi^1_t\big(\delta_1(t) - \widetilde{S}^1_t(\kappa_1)\big) + \phi^2_t\big(\widetilde{S}^2_t(\kappa_2) - S^2_{t|1}(\kappa_2)\big) = h_1(t), \tag{80}$$

$$\phi^2_t\big(\delta_2(t) - \widetilde{S}^2_t(\kappa_2)\big) + \phi^1_t\big(\widetilde{S}^1_t(\kappa_1) - S^1_{t|2}(\kappa_1)\big) = h_2(t), \tag{81}$$

where $h_i(t) = Z_i(t) - \widehat{g}(t)$, *and let*

$$\phi^0_t = V_t(\phi) - \phi^1_t S^1_t(\kappa_1) - \phi^2_t S^2_t(\kappa_2)$$

where the wealth process is $V_t(\phi)$ *satisfies (67) with the initial condition* $V_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$, *where* Y *is given by (54). Then the self-financing trading strategy* ϕ *replicates the first-to-default claim* $(X, 0, Z, \tau_{(1)})$.

Proof. As in the single name case, we start by analyzing the pre-default dynamics. Combining (67) with (70), we obtain

$$\begin{aligned} d\widetilde{V}_t(\phi) &= \phi^1_t(d\widetilde{S}^1_t(\kappa_1) - \kappa_1 dt) + \phi^2_t(d\widetilde{S}^2_t(\kappa_2) - \kappa_2 dt) \\ &= \phi^1_t\big(\widetilde{\gamma}(t)\widetilde{S}^1_t(\kappa_1) - \delta_1(t)\widetilde{\gamma}_1(t) - S^1_{t|2}(\kappa_1)\widetilde{\gamma}_2(t)\big)\, dt \\ &+ \phi^2_t\big(\widetilde{\gamma}(t)\widetilde{S}^2_t(\kappa_2) - \delta_2(t)\widetilde{\gamma}_2(t) - S^2_{t|1}(\kappa_2)\widetilde{\gamma}_1(t)\big)\, dt \end{aligned}$$

where we write $\widetilde{\gamma}(t) = \widetilde{\gamma}_1(t) + \widetilde{\gamma}_2(t)$. The wealth process $V_t(\phi)$ of a replicating strategy satisfies $V_t(\phi) = \widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y|\mathcal{G}_t)$, and thus

$$\mathbb{1}_{\{\tau_{(1)}>t\}} d\widetilde{V}_t(\phi) = \mathbb{1}_{\{\tau_{(1)}>t\}} d\widetilde{M}(t)$$

so that

$$\phi^1_t\big(\widetilde\gamma(t)\widetilde S^1_t(\kappa_1)-\delta_1(t)\widetilde\gamma_1(t)-S^1_{t|2}(\kappa_1)\widetilde\gamma_2(t)\big)\,dt+\phi^2_t\big(\widetilde\gamma(t)\widetilde S^2_t(\kappa_2)-\delta_2(t)\widetilde\gamma_2(t)-S^2_{t|1}(\kappa_2)\widetilde\gamma_1(t)\big)\,dt$$
$$=\widetilde\gamma_1(t)h_1(t)\,dt+\widetilde\gamma_2(t)h_2(t)\,dt,$$

where $h_i(t)=Z_i(t)-\widehat g(t)$. After rearranging, we obtain

$$\begin{aligned}&\widetilde\gamma_1(t)\Big(\phi^1_t\big(\delta_1(t)-\widetilde S^1_t(\kappa_1)\big)+\phi^2_t\big(\widetilde S^2_t(\kappa_2)-S^2_{t|1}(\kappa_2)\big)\Big)\\&+\widetilde\gamma_2(t)\Big(\phi^2_t\big(\delta_2(t)-\widetilde S^2_t(\kappa_2)\big)+\phi^1_t\big(\widetilde S^1_t(\kappa_1)-S^1_{t|2}(\kappa_1)\big)\Big)\\&=\widetilde\gamma_1(t)h_1(t)\,+\widetilde\gamma_2(t)h_2(t).\end{aligned}$$

It is thus natural to conjecture that (ϕ^1,ϕ^2) solve the system of linear equations (80)-(81).

To complete the proof, it suffices to check that $V_{\tau_{(1)}}(\phi)=\widehat M_{\tau_{(1)}}$. It is clear that

$$\Delta_{\tau_{(1)}}\widehat M=h_1(\tau_{(1)})\mathbb{1}_{\{\tau_1<T,\tau_1<\tau_2\}}+h_2(\tau_{(1)})\mathbb{1}_{\{\tau_2<T,\tau_2<\tau_1\}}$$

and

$$\begin{aligned}\Delta_{\tau_{(1)}}V(\phi)&=\phi^1_{\tau_{(1)}}\big(\delta_1(\tau_{(1)})-\widetilde S^1_{\tau_{(1)}}(\kappa_1)\big)\mathbb{1}_{\{\tau_1\le T,\tau_1<\tau_2\}}+\phi^2_{\tau_{(1)}}\big(S^2_{\tau_{(1)}}(\kappa_2)-\widetilde S^2_{\tau_{(1)}}(\kappa_2)\big)\mathbb{1}_{\{\tau_1\le T,\tau_1<\tau_2\}}\\&+\phi^2_{\tau_{(1)}}\big(\delta_2(\tau_{(1)})-\widetilde S^2_{\tau_{(1)}}(\kappa_2)\big)\mathbb{1}_{\{\tau_2\le T,\tau_2<\tau_1\}}+\phi^1_{\tau_{(1)}}\big(S^1_{\tau_{(1)}}(\kappa_1)-\widetilde S^1_{\tau_{(1)}}(\kappa_1)\big)\mathbb{1}_{\{\tau_2\le T,\tau_2<\tau_1\}}.\end{aligned}$$

If we use the formula for ϕ^1 and ϕ^2, we find easily that $\Delta_{\tau_{(1)}}\widehat M=\Delta_{\tau_{(1)}}V(\phi)$.

Alternatively, we may observe that the equality $\Delta_{\tau_{(1)}}\widehat M=\Delta_{\tau_{(1)}}V(\phi)$ is equivalent to equations (80)-(81). It is thus clear that this is the unique choice of (ϕ^1,ϕ^2) given by is the unique strategy satisfying the above jump condition. □

Example 3.3 We now consider an example of hedging an FtD claim that pays a constant amount Z_i on the first default on a basket of two names. CDSs on the individual names that pay a constant amount δ_1 and δ_2 respectively will be used as hedging assets. As in Example 3.2, we assume that defaults are mutually independent and are exponentially distributed with constant pre-default intensities $\widetilde\gamma_1\neq 0$ and $\widetilde\gamma_2\neq 0$, respectively. Hence, the pre-default value of the FtD claim equals (cf. Example 3.2)

$$\widetilde\pi(t)=\frac{\widetilde\gamma_1Z_1+\widetilde\gamma_2Z_2}{\widetilde\gamma}\left(1-e^{-\widetilde\gamma(T-t)}\right)+c(T)e^{-\widetilde\gamma(T-t)},$$

where $\widetilde\gamma=\widetilde\gamma_1+\widetilde\gamma_2$.

We postulate that $\kappa_i=\widetilde\gamma_i\delta_i$, that is, we deal with single-name market CDSs. It is important to note that, under the present assumptions of independence of default times, we have $\widetilde S^2_t(\kappa_2)=S^2_{t|1}(\kappa_2)$ and since $\kappa=\widetilde\gamma_i\delta_i$, $\widetilde S^2_t(\kappa_2)=0$ for $t\in[0,T]$ (this can be shown along the same lines as in Sections 2.2.2 and 2.4.3).

Consequently, the matrix $C(t)$ reduces to (cf. (80)-(81))

$$C(t)=\begin{bmatrix}\delta_1 & 0\\ 0 & \delta_2\end{bmatrix},$$

and the hedging strategy satisfies (cf. (37))

$$\phi^i_t=\frac{Z_i-\widetilde\pi(t)}{\delta_i},\quad \forall\, t\in[0,T].$$

To check directly that ϕ is indeed a hedging strategy for this claim, we will consider the special case of $Z_1\neq 0$ and $Z_2=0$. The initial wealth equals

$$V_0(\phi)=\widetilde\pi(0)=\frac{\widetilde\gamma_1Z_1}{\widetilde\gamma}\left(1-e^{-\widetilde\gamma T}\right)+c(T)e^{-\widetilde\gamma T}.$$

Since

$$\begin{aligned} d\widetilde{V}_t(\phi) &= -\phi^1_t\widetilde{\gamma}_1\delta_1\,dt - \phi^2_t\widetilde{\gamma}_2\delta_2\,dt = \Big(-\widetilde{\gamma}_1 Z_1 e^{-\widetilde{\gamma}(T-t)} + c(T)\widetilde{\gamma}e^{-\widetilde{\gamma}(T-t)}\Big)\,dt \\ &= \left(c(T) - Z_1\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}}\right)\widetilde{\gamma}e^{-\widetilde{\gamma}(T-t)}\,dt, \end{aligned}$$

we have

$$\widetilde{V}_t(\phi) - \widetilde{V}_0(\phi) = \left(c(T) - Z_1\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}}\right)e^{-\widetilde{\gamma}T}\Big(e^{\widetilde{\gamma}t} - 1\Big).$$

Suppose first that no default occurs prior to T. Then we have, on the event $\{\tau_{(1)} > T\}$,

$$V_T(\phi) = V_0(\phi) + \left(c(T) - Z_1\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}}\right)\Big(1 - e^{-\widetilde{\gamma}T}\Big) = c(T).$$

On the event $\{\tau_{(1)} \le T\}$, we obtain

$$\begin{aligned} V_{\tau_{(1)}}(\phi) &= V_0(\phi) + \widetilde{V}_{\tau_{(1)}-}(\phi) + \Delta_{\tau_{(1)}}V(\phi) \\ &= \frac{\widetilde{\gamma}_1 Z_1}{\widetilde{\gamma}}\Big(1 - e^{-\widetilde{\gamma}T}\Big) + c(T)e^{-\widetilde{\gamma}T}\left(c(T) - Z_1\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}}\right)e^{-\widetilde{\gamma}T}\Big(e^{\widetilde{\gamma}\tau_{(1)}} - 1\Big) \\ &\quad + \phi^1_{\tau_{(1)}}\delta_1\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + \phi^2_{\tau_{(1)}}\delta_2\mathbb{1}_{\{\tau_{(1)}=\tau_2\}} = Z_1\mathbb{1}_{\{\tau_{(1)}=\tau_1\}}, \end{aligned}$$

as expected.

Of course, this example is by far too simple to demonstrate the full strength of Proposition 3.4. The goal of this exercise is rather to indicate that a direct extension of Proposition 2.3 to the multivariate case can only be expected under rather stringent assumptions on default times.

Case of n credit names. We shall now consider the case of a first-to-default claim on a basket n credit names. Let the function $S^i_{t|j}(\kappa_j)$, $t \in [0,T]$, stand for the ex-dividend price of the ith CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$, where $j \ne i$ represents the identity of the first defaulted name. The following result extends Lemmas 3.2 and 3.4.

Lemma 3.5 *The dynamics of the pre-default price $\widetilde{S}^i_t(\kappa_i)$ are*

$$d\widetilde{S}^i_t(\kappa_i) = \widetilde{\gamma}(t)\widetilde{S}^i_t(\kappa_i)\,dt + \Big(\kappa_i - \delta_i(t)\widetilde{\gamma}_i(t) - \sum_{j\ne i}^n S^i_{t|j}(\kappa_i)\widetilde{\gamma}_i(t)\Big)dt, \tag{82}$$

where $\widetilde{\gamma}(t) = \sum_{i=1}^n \widetilde{\gamma}_i(t)$, or equivalently,

$$d\widetilde{S}^i_t(\kappa_i) = \widetilde{\gamma}_i(t)\big(\widetilde{S}^i_t(\kappa_i) - \delta_i(t)\big)\,dt + \sum_{j\ne i}\widetilde{\gamma}_j(t)\big(\widetilde{S}^i_t(\kappa_i) - S^i_{t|j}(\kappa_i)\big)\,dt + \kappa_i dt. \tag{83}$$

The cum-dividend price of the ith CDS stopped at $\tau_{(1)}$ satisfies

$$\bar{S}^i_t(\kappa_i) = S^i_t(\kappa_i) + \int_0^t \delta_i(u)\,dH^i_{u\wedge\tau_{(1)}} + \sum_{j\ne i}\int_0^t S^i_{u|j}(\kappa_i)\,dH^j_{u\wedge\tau_{(1)}} - \kappa_i(t\wedge\tau_{(1)}), \tag{84}$$

and thus

$$d\bar{S}^i_t(\kappa_i) = \big(\delta_i(t) - \widetilde{S}^i_{t-}(\kappa_i)\big)\,d\widehat{M}^i_t + \sum_{j\ne i}\big(S^i_{t|j}(\kappa_i) - \widetilde{S}^i_{t-}(\kappa_i)\big)\,d\widehat{M}^j_t. \tag{85}$$

We are now in a position to state a generalization of Proposition 3.4 to first-to-default claims on a basket of n credit names. Since we assume that $B = 1$, a generic FtD claim can be seen as a random payoff Y settling at T, where

$$Y = \sum_{i=1}^n Z_i(\tau_i)\mathbb{1}_{\{\tau_{(1)}=\tau_i\le T\}} + c(T)\mathbb{1}_{\{\tau_{(1)}>T\}} = \sum_{i=1}^n Z_i(\tau_{(1)})\mathbb{1}_{\{\tau_{(1)}=\tau_i\le T\}} + c(T)\mathbb{1}_{\{\tau_{(1)}>T\}}. \tag{86}$$

Proposition 3.5 *Assume that* $\det C(t) \neq 0$ *for every* $t \in [0, T]$, *where*

$$C(t) = \begin{bmatrix} \delta_1(t) - \widetilde{S}^1_t(\kappa_1) & \widetilde{S}^2_t(\kappa_2) - S^2_{t|1}(\kappa_2) & . & \widetilde{S}^n_t(\kappa_n) - S^n_{t|1}(\kappa_n) \\ \widetilde{S}^1_t(\kappa_1) - S^1_{t|2}(\kappa_1) & \delta_2(t) - \widetilde{S}^2_t(\kappa_2) & . & \widetilde{S}^n_t(\kappa_n) - S^n_{t|2}(\kappa_n) \\ . & . & . & . \\ \widetilde{S}^1_t(\kappa_1) - S^1_{t|n}(\kappa_1) & \widetilde{S}^2_t(\kappa_1) - S^2_{t|n}(\kappa_1) & . & \delta_n(t) - \widetilde{S}^n_t(\kappa_n) \end{bmatrix}$$

Let $\phi_t = (\phi^1_t, \phi^2_t, \ldots, \phi^n_t)$ *be the unique solution to the system of linear equations* $C(t)\phi_t = \bar{h}(t)$, *where* $\bar{h} = (h_1, \ldots, h_n)$ *with* $h_i(t) = Z_i(t) - \widehat{g}(t)$ *where* $\widehat{g}$ *is given by Proposition 3.3. More explicitly, the processes* $\phi^1_t, \phi^2_t, \ldots, \phi^n_t$ *satisfy*

$$\phi^i_t\big(\delta_i(t) - \widetilde{S}^i_t(\kappa_i)\big) + \sum_{j \neq i} \phi^j_t\big(\widetilde{S}^j_t(\kappa_j) - S^j_{t|i}(\kappa_j)\big) = h_i(t). \tag{87}$$

Let

$$\phi^0_t = V_t(\phi) - \sum_{i=1}^{n} \phi^i_t S^i_t(\kappa_i)$$

where the wealth process $V(\phi)$ *satisfies (67) with the initial condition* $V_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$, *where* Y *is given by (86). Then the trading strategy* $\phi = (\phi^0, \phi^1, \ldots, \phi^n)$ *is admissible and it is a replicating strategy for the first-to-default claim* $(X, 0, Z, \tau_{(1)})$, *where* $X = c(T)$ *and* $Z = (Z_1, \ldots, Z_n)$.

3.4 Conditional Default Distributions

In the case of first-to-default claims, it was enough to consider the unconditional distribution of default times. However, since we are now going to analyze successive default times, the information structure we deal with becomes more complicated, so that it will be convenient to introduce the identifiers for all names that have defaulted prior to any given moment $t \in [0, T]$.

In the foregoing definitions, we adopt the convention that k names out of a total of n names have already defaulted. The $n-k$ names that have not yet defaulted are in their 'natural' order $j_1 < \ldots < j_{n-k}$, and the k defaulted names are denoted by their identifiers $i_1, \ldots, i_k$, and they are ordered according their corresponding default times $u_1 < u_2 < \ldots < u_k$. In other words, the defaulted names are ordered in increasing order of the respective default times.

Definition 3.6 The *joint conditional distribution function* of default times $\tau_{j_1}, \ldots, \tau_{j_{n-k}}$ equals, for every $t_1, \ldots, t_{n-k} > u_k$,

$$F(t_1, \ldots, t_{n-k} \,|\, i_1, \ldots, i_k; u_1, \ldots, u_k) = \mathbb{Q}^*\left(\tau_{j_1} \leq t_1, \ldots, \tau_{j_{n-k}} \leq t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\right).$$

Consequently, the *joint conditional density function* of default times $\tau_{j_1}, \ldots, \tau_{j_{n-k}}$ is denoted as $f(t_1, \ldots, t_{n-k} \,|\, i_1, \ldots, i_k; u_1, \ldots, u_k)$, and the *joint conditional survival function* of default times is given by the expression

$$G(t_1, \ldots, t_{n-k} \,|\, i_1, \ldots, i_k; u_1, \ldots, u_k) = \mathbb{Q}^*\left(\tau_{j_1} > t_1, \ldots, \tau_{j_{n-k}} > t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\right)$$

where, as before, $t_1, \ldots, t_{n-k} > u_k$.

Example 3.4 Suppose we have are working with a basket of five names. Let us assume that names 2 and 4 have survived until $t = 4$, and names 1, 3 and 5 defaulted at times 3, 4 and 2, respectively. Then the joint conditional distribution function for default times τ_2 and τ_4 is denoted as $F(t_2, t_4|5, 1, 3; 2, 3, 4)$.

3.4.1 Conditional Intensity

As expected, the conditional default intensities are defined using the conditional joint distributions, instead of the unconditional distribution. For brevity, we write $D_k = (i_1, \ldots, i_k; u_1, \ldots, u_k)$ to denote the *information structure* of the past k defaults.

Definition 3.7 For any $j_l = j_1, \ldots, j_{n-k}$, the *conditional first-to-default intensity* of a surviving name j_l is denoted by $\widetilde{\gamma}^{j_l}(t\,|i_1, \ldots, i_k; u_1, \ldots, u_k)$, or briefly, $\widetilde{\gamma}^{j_l}(t\,|D_k)$, and is given by the formula

$$\widetilde{\gamma}^{j_l}(t\,|D_k) = \frac{\int_t^\infty \int_t^\infty \ldots \int_t^\infty dF(t_1, t_2, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{n-k}|D_k)}{G(t, \ldots, t\,|D_k)}, \quad \forall\, t \in [u_k, T].$$

Note that we are integrating $n - k - 1$ times, and that $G(t, \ldots, t\,|D_k)$ has $n - k$ variables t.

3.4.2 Conditional Values

In Section 3.3.2, we have introduced the processes $S^i_{t|j}(\kappa_j)$, which were aimed to represent the value of the ith CDS on the event at time t on the event $\{\tau_{(1)} = \tau_j = t\}$. According to the notation introduced above, we deal with the conditional value of the ith CDS where the condition $D_1 = (j; t)$ (in words, the jth name was the first defaulted name, and its default occurred at time t).

As expected, the valuation of a CDS for each surviving firm is exactly the same as the valuation prior to the first default, except that now we should use the conditional distribution

$$F(t_1, \ldots, t_{n-1}\,|\,j; t) = F(t_1, \ldots, t_{n-1}\,|\,D_1), \quad \forall\, t_1, \ldots, t_{n-1} \in [t, T],$$

rather than the unconditional distribution $F(t_1, \ldots, t_n)$, as was done in Proposition 3.3. The corresponding conditional version of this proposition is rather easy to formulate and prove, so that there is no need of providing an explicit conditional pricing formula.

3.4.3 Conditional Martingales

The conditional first-to-default intensities introduced in Definition 3.7 will allow us to construct the conditional first-to-default martingales in a much the same way as the we have constructed the first-to-default martingale M^i associated with the first-to-default intensity $\widetilde{\gamma}^i$. However, since any name can default at any time, we need to introduce an entire family of basic conditional martingales, whose compensators are based on intensities conditioned on the information structure of past defaults.

Definition 3.8 Given the information structure $D_k = (i_1, \ldots, i_k; u_1, \ldots, u_k)$ of the past k defaults, for each surviving name $j_l = j_1, \ldots, j_{n-k}$, we define the *basic conditional first-to-default martingale* $\widehat{M}^{j_l}_{t|D_k}$ by setting

$$\widehat{M}^{j_l}_{t|D_k} = H^{j_l}_{t\wedge\tau_{(k+1)}} - \int_{u_k}^t \mathbb{1}_{\{\tau_{(k+1)} > u\}} \widetilde{\gamma}^{j_l}(u\,|D_k)\, du, \quad \forall\, t \in [u_k, T]. \tag{88}$$

The process $M^{j_l}_{t|D_k}$, $t \in [u_k, T]$, follows a martingale with respect to the filtration generated by default processes of the surviving names, that is, the filtration $\mathcal{G}^{D_k}_t := \mathcal{H}^{j_1}_t \vee \ldots \vee \mathcal{H}^{j_{n-k}}_t$ for $t \in [u_k, T]$, with respect to the probability measure $\mathbb{Q}^*$ conditioned on the event D_k.

Note that since we condition on the event D_k, we have that $\tau_{(k+1)} = \tau_{j_1} \wedge \ldots \wedge \tau_{j_{n-k}}$. Formula (88) is a rather straightforward generalization of formula (53). In particular, for $k = 0$ we obtain $\widehat{M}^i_{t|D_0} = \widehat{M}^i_t$, $t \in [0, T]$, for $i = 1, 2, \ldots, n$.

3.4.4 Conditional Martingale Representation Theorem

We now state a 'conditional' version of the martingale representation theorem for first-to-default claims. The main reason we require such a version is that in a multi-name setup, once the first name defaults, the value process of a basket claim becomes a specific conditional martingale. To hedge a claim, we thus need to represent any such conditional martingale in terms of the corresponding *basic conditional first-to-default martingales.* It is worth stressing that, mathematically speaking, the conditional first-to-default martingale representation result is nothing else than a restatement of the martingale first-to-default representation formula of Proposition 3.1 in terms of conditional first-to-default intensities and basic conditional first-to-default martingales.

Proposition 3.6 *Let $D_k = (i_1, \ldots, i_k; u_1, \ldots, u_k)$ and let $\mathbb{G}^{D_k} = \mathbb{H}^{j_1} \vee \ldots \vee \mathbb{H}^{j_{n-k}}$. Let Y be a random variable given by the formula*

$$Y = \sum_{l=1}^{n-k} \widehat{Z}_{i|D_k}(\tau_{j_l}) \mathbb{1}_{\{\tau_{j_l} \leq T,\, \tau_{j_l} = \tau_{(k+1)}\}} + c(T) \mathbb{1}_{\{\tau_{(k+1)} > T\}} \tag{89}$$

for some continuous functions $\widehat{Z}_{i|D_k} : [u_k, T] \to \mathbb{R}$, $i = 1, 2, \ldots, n$, and some constant $c(T)$. We define

$$\bar{M}_{t|D_k} = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{G}_t^{D_k} \vee D_k), \quad \forall\, t \in [u_k, T], \tag{90}$$

Then the process $\bar{M}_{t|D_k}$, $t \in [u_k, T]$ follows a $\mathbb{G}^{D_k}$-martingale with respect to the probability measure $\mathbb{Q}^*$ conditioned on the event D_k. Moreover, it admits the following representation, for $t \in [u_k, T]$,

$$\bar{M}_{t|D_k} = \bar{M}_{0|D_k} + \sum_{l=1}^{n-k} \int_{]u_k, t]} h_{j_l}(u|D_k)\, d\widehat{M}^{j_l}_{u|D_k},$$

where the processes h_{j_l} are given by

$$h_{j_l}(t\,|D_k) = \widehat{Z}_{j_l|D_k}(t) - \widehat{M}_{t-|D_k}, \quad \forall\, t \in [u_k, T].$$

Proof. The proof relies on a direct extension of arguments used in the proof of Proposition 3.1 to the context of conditional default distributions. □

3.5 General Basket Claims

We are ready extend the results developed in the context of first-to-default claims to value and hedge general basket claims. A generic basket claim is any contingent claim that pays a specified amount on each default from a basket of n credit names and a constant amount at maturity T if no defaults have occurred prior to or at T.

Definition 3.9 A *basket claim* associated with a family of n credit names is given as $(X, 0, \bar{Z}, \bar{\tau})$, where X is a constant amount payable at maturity only if no defaults occur, a random vector $\bar{\tau} = (\tau_1, \ldots, \tau_n)$ represents default times, and a time-dependent matrix $\bar{Z}$ represents the payoffs at defaults, specifically,

$$\bar{Z} = \begin{bmatrix} Z_1(t\,|D_0) & Z_2(t\,|D_0) & . & Z_n(t\,|D_0) \\ Z_1(t\,|D_1) & Z_2(t\,|D_1) & . & Z_n(t\,|D_1) \\ . & . & . & . \\ Z_1(t\,|D_{n-1}) & Z_2(t\,|D_{n-1}) & . & Z_n(t\,|D_{n-1}) \end{bmatrix}$$

Note that the above matrix $\bar{Z}$ is presented in a shorthand notation. In fact, in each row we need to specify, for an arbitrary choice of the set $D_k = (i_1, \ldots, i_k; u_1, \ldots, u_k)$ and any $j_l \neq i_1, \ldots, i_k$, the function

$$Z_{j_l}(t\,|D_{n-1}) = Z_{j_l}(t\,|i_1, \ldots, i_k; u_1, \ldots, u_k), \quad \forall\, t \in [u_k, T],$$

If the financial interpretation, this function determines the recovery payment at the default of the j_lth name, conditional on the set $D_k = (i_1, \dots, i_k; u_1, \dots, u_k)$ representing the information structure of the previous k defaults. In particular, $Z_i(t \,|D_0) := Z_i(t)$ represents the recovery payment at the default of the ith name at time t, given that no defaults have occurred prior to t (not that the symbol D_0 means merely that we consider a situation of no defaults prior to t). Also, we shall frequently use the shorthand notation $Z(t \,|D_k)$ for the vector $(Z_1(t \,|D_k), \dots, Z_n(t \,|D_k))$.

Example 3.5 Let us consider the kth-to-default claim for some fixed $k = 1, 2, \dots, n$. Assume that the payoff at the kth default depends on the timing of k-th default and the identity of the kth defaulting name, so that it only depends on the moment of the kth default and the identity of the kth-to-default name. Then all rows of the matrix $\bar{Z}$ are equal to zero, except for the kth row, which is $[Z_1(t|k-1), Z_2(t|k-1), \dots, Z_n(t|k-1)]$. We write here $k-1$ rather than D_{k-1} to emphasize that the knowledge of exact moments and identities of the k defaulted names is inessential.

More generally, for a generic basket claim in which the payoff at default depends on the time and identity of defaulting name, the recovery matrix $\bar{Z}$ reads

$$\bar{Z} = \begin{bmatrix} Z_1(t) & Z_2(t) & . & Z_n(t) \\ Z_1(t \,|1) & Z_2(t \,|1) & . & Z_n(t \,|1) \\ . & . & . & . \\ Z_1(t \,|n-1) & Z_2(t \,|n-1) & . & Z_n(t \,|n-1) \end{bmatrix}$$

This shows that for several practical examples of basket credit derivatives, the matrix $\bar{Z}$ will have a reasonably simple structure.

Remark. It is clear that any basket claim can be represented as a static portfolio of kth-to-default claims for $k = 1, 2, \dots, n$. However, such a decomposition does not seem to be advantageous, in general. In what follows, we shall represent a basket claim as an iterative sequence of conditional first-to-default claims. By proceeding in this way, we will be able to directly apply results developed for the case of first-to-default claims, and in consequence, to produce simple iterative algorithms for the valuation and hedging of basket claims of any type.

3.5.1 Valuation of a Basket Claim

Instead of stating formal results, using rather heavy notation, we prefer to present first the basic idea of valuation and hedging of basket claims. The important concept is the *conditional pre-default price*

$$\widetilde{Z}(t \,|D_k) = \widetilde{Z}(t \,|\, i_1, \dots, i_k; u_1, \dots, u_k), \quad \forall\, t \in [u_k, T],$$

of a conditional first-to-default claim. Each value function $\widetilde{Z}(t \,|D_k)$, $t \in [u_k, T]$ is formally defined as the arbitrage price of a conditional first-to-default claim on $n-k$ surviving names, with the following recovery payoffs

$$\widehat{Z}_{j_l}(t \,|\, i_1, \dots, i_k; u_1, \dots, u_k) = Z_{j_l}(t \,|\, i_1, \dots, i_k; u_1, \dots, u_k) + \widetilde{Z}(t \,|\, i_1, \dots, i_k, j_l; u_1, \dots, u_k, t). \quad (91)$$

Assuming that the conditional payoffs $\widehat{Z}_{j_l}(t \,|\, i_1, \dots, i_k, j_l; u_1, \dots, u_k, u_{k+1})$ are known, the function

$$\widetilde{Z}(t \,|\, i_1, \dots, i_k; u_1, \dots, u_k), \quad \forall\, t \in [u_{k+1}, T],$$

can be computed as in Proposition 3.3, using conditional default distribution

The assumption that the conditional payoffs are known is not restrictive, since the functions appearing in right-hand side of (91) are known from the previous step in the pricing algorithm backward induction. To present this algorithm, let us abbreviate (91) as follows

$$\widehat{Z}(t \,|D_k) = Z(t \,|D_k) + \widetilde{Z}(t \,|D_{k+1}).$$

Under this notational convention, we obtain the following recurrent pricing scheme for a generic basket claim $(X, 0, \bar{Z}, \bar{\tau})$ in which the time variables $u_1, \ldots, u_n$ denote the moments of successive defaults.

Time	Claim Payoff at Time u_i	Pre-Default Price on $t \in [u_i, T]$
$u_0 = 0$	—	$\widetilde{Z}(t \mid D_0)$
u_1	$\widehat{Z}(t \mid D_0) = Z(t \mid D_0) + \widetilde{Z}(t \mid D_1)$	$\widetilde{Z}(t \mid D_1)$
u_2	$\widehat{Z}(t \mid D_1) = Z(t \mid D_1) + \widetilde{Z}(t \mid D_2)$	$\widetilde{Z}(t \mid D_2)$
...	...	...
u_k	$\widehat{Z}(t \mid D_{k-1}) = Z(t \mid D_{k-1}) + \widetilde{Z}(t \mid D_k)$	$\widetilde{Z}(t \mid D_k)$
...	...	...
u_{n-2}	$\widehat{Z}(t \mid D_{n-3}) = Z(t \mid D_{n-3}) + \widetilde{Z}(t \mid D_{n-2})$	$\widetilde{Z}(t \mid D_{n-2})$
u_{n-1}	$\widehat{Z}(t \mid D_{n-2}) = Z(t \mid D_{n-2}) + \widetilde{Z}(t \mid D_{n-1})$	$\widetilde{Z}(t \mid D_{n-1})$
u_n	$\widehat{Z}(t \mid D_{n-1}) = Z(t \mid D_{n-1})$	0
T	0	0

Let us note that the constant payoff $X = c(T)$ appears implicitly in the pricing formula for $\widetilde{Z}(t \mid D_0)$ only.

To better appreciate the above scheme, let us analyze how we should proceed when we split the time interval $[0, T]$ into sub-intervals according to the moments of occurrence of successive default times $\tau_{(1)}, \ldots, \tau_{(n)}$. For simplicity of exposition, we shall focus on the case that all n names have defaulted at prior to maturity T, i.e., $\omega \in \Omega$ such that $\tau_{(n)}(\omega) = u_n \leq T$. Hence, we first consider $D_n = (i_1, \ldots, i_n; u_1, \ldots, u_n)$ for some sequence $0 < u_1 < \ldots < u_n \leq T$. Between each of the times $u_1, \ldots, u_n$, a basket claim is treated as a conditional FtD claim with the condition D_k given as $D_k = (i_1, \ldots, i_k; u_1, \ldots, u_k)$.

We start at the terminal date T and move backwards in time. Since all our names have already defaulted, our claim has a zero pre-default price after u_n. At time u_n, we deal with the payoff

$$\widehat{Z}_{i_n}(u_n \mid D_{n-1}) = Z_{i_n}(u_n \mid D_{n-1}) = Z_{i_n}(u_n \mid i_1, \ldots, i_n; u_1, \ldots, u_n),$$

where the recovery payment function $Z_{i_n}(u_n \mid D_{n-1}) = Z_{i_n}(t \mid i_1, \ldots, i_n; u_1, \ldots, u_n)$, $t \in [u_{n-1}, T]$, is given by the specification of the basket claim. Hence, we can evaluate the pre-default value $\widetilde{Z}(t \mid D_{n-1})$ at any time $t \in [u_{n-1}, T]$, as a value of a conditional first-to-default claim with the said payoff. This in fact means that we temporarily fix the date u_{n-1}, but we no longer assume that the date u_n is fixed. We deal instead with the conditional distribution of the random time τ_{i_n} on the interval $[u_{n-1}, T]$, where we condition on the event that $\{\tau_{i_1} = u_1, \ldots, \tau_{i_{n-1}} = u_{n-1}\}$.

To compute the price of the conditional first-to-default claim, we use tools developed in Section 3.4. Now, it is clear that the payoff $\widehat{Z}_{i_{n-1}}(u_{n-1} \mid D_{n-2})$ of a basket claim at time $\tau_{(n-1)} = u_{n-1}$ (that is, upon the default of the name i_{n-1}) comprises the recovery payoff from the name i_{n-1} which is $Z_{i_{n-1}}(u_{n-1} \mid D_{n-2})$ and the pre-default value $\widetilde{Z}(u_{n-1} \mid D_{n-1})$. Of course, if only D_{n-2} is given, we may use this procedure to compute the functions $\widehat{Z}_{j_1}(t \mid D_{n-2})$ and $\widehat{Z}_{j_2}(t \mid D_{n-2})$ for $t \in [u_{n-2}, T]$, where $j_1, j_2 \notin \{i_1, \ldots, i_{n-2}\}$ are surviving names.

In the next step, we wish to find the value of a basket claim between the $(n-2)$th and $(n-1)$th default. Assuming that $D_{n-2} = (i_1, \ldots, i_{n-2}; u_1, \ldots, u_{n-2})$ is given, we deal with the conditional first-to-default claim associated with the two surviving names, $j_1, j_2 \notin \{i_1, \ldots, i_{n-2}\}$. Since from the previous step we know the conditional payoffs $\widehat{Z}_{j_1}(t \mid D_{n-2})$, we may apply the standard technique to find the pre-default value this claim on $[u_{n-2}, T]$.

In this way we can proceed backwards up to time 0 considering FtD payoffs and pre-default values (conditional on the most recent D set) at each possible default time and finally obtain the pre-default value of a basket claim at time 0.

The above procedure leads to the following algorithm based on the backward induction. Let $\widetilde{Z}(t \mid D_j)$ denote the pre-default value at time $t \in [u_j, u_{j+1})$ of a conditional FtD claim on a basket of $n-j$ surviving names with the payoff $\widehat{Z}(t\,|D_j)$:

$$\begin{aligned}
\text{First Step:} \quad & \widetilde{Z}(t\,|D_n) = 0 \\
\text{Induction Step:} \quad & \text{for} j = n-1, n-2, \dots, 0, \\
& \text{Set } \widehat{Z}(t\,|D_j) = Z(t\,|D_j) + \widetilde{Z}(t\,|D_{j+1}) \\
& \text{Compute } \widetilde{Z}(t\,|D_j)
\end{aligned}$$

Proposition 3.7 *The pre-default price at time $t \in [0,T]$ of a basket claim $(X, 0, \bar{Z}, \bar{\tau})$ is given by the expression $\widetilde{Z}(t\,|D_k)$ where k denotes number of names that have defaulted prior to time t for a given $\omega \in \Omega$. Consequently, the arbitrage price at time $t \in [0,T]$ of a basket claim equals*

$$\pi_t = \sum_{k=0}^{n} \widetilde{Z}(t\,|D_k) \mathbb{1}_{[\tau_{(k)} \wedge T, \tau_{(k+1)} \wedge T[}(t), \quad \forall\, t \in [0,T],$$

where $D_k = D_k(\omega) = (i_1(\omega), \dots, i_n(\omega); \tau_{(1)}(\omega), \dots, \tau_{(n)}(\omega))$ for $k = 1, 2, \dots, n$, and D_0 means that no defaults have yet occurred.

3.5.2 Replication of a Basket Claim

¿From the previous section it is clear that a basket claim can be interpreted as a specific sequence of conditional first-to-default claims. Hence, it is easy to guess that the replication of a basket claim should follow the same hedging mechanism as the underlying sequence of conditional first-to-default claims. Let us write $\tau_{(0)} = 0$.

Proposition 3.8 *For each $k = 0, 1, \dots, n$, the replicating strategy ϕ for a basket claim $(X, 0, \bar{Z}, \bar{\tau})$ on the time interval $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ coincides with the hedging strategy of the conditional FtD claim with payoffs $\widehat{Z}(t\,|D_k)$ given the information structure D_k.*

The hedging strategy $\phi = (\phi^0, \phi^{j1}, \dots, \phi^{j_{n-k}})$, corresponding to the units of savings account and units of CDS of each surviving name at time t, for $\widetilde{Z}(t\,|D_i)$ is based on the wealth process:

$$V_t(\phi) = \phi^0_t + \sum_{l=1}^{n-i} \phi^{j_l} S^{j_l}_t(\kappa_{j_l})$$

where each component ϕ^{j_l}, $l = 1, \dots, n-k$, is given by the expression that can be found as in Proposition 3.5.

Proof. We know that the basket claim can be decomposed into a series of conditional FtD claims. So at any given point of time $t \in [0,T]$, assuming that k defaults have already occurred, our basket claim is equivalent to the conditional FtD claim with pre-default value $\widetilde{Z}(t\,|D_i)$. This FtD claim is 'alive' up to the next default $\tau_{(i+1)}$ or maturity T, whichever comes first. Hence, it is clear that the hedging strategy over the interval $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ is the hedging strategy for this conditional first-to-default claim, and thus it can be found along the same lines as in Proposition 3.5, using a suitable conditional distribution of default for surviving names. □

4 Pricing and Trading a CDS under Stochastic Intensity

In this section, the standing assumption that the default intensity is deterministic is relaxed. This means that we shall deal here with a more realistic model, in which we address hedging of not only the default (jump) risk, but also the spread (volatility) risk. As expected, the results derived in this section are more general, but less explicit than in the previous section. Unless explicitly stated otherwise, we set $r = 0$ so that $B_t = 1$ for every $t \in \mathbb{R}_+$.

4.1 Hazard Process

We now assume that some *reference filtration* $\mathbb{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ is given. We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration.* It is clear that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time (but not necessarily an $\mathbb{F}$-stopping time). The concept of the hazard process of a random time τ is closely related to the process F defined through the formula

$$F_t = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}, \quad \forall\, t \in \mathbb{R}_+.$$

Let us denote by $G_t = 1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\}$ the *survival process* with respect to the filtration $\mathbb{F}$, and let us assume that $G_0 = 1$ and $G_t > 0$ for every $t \in \mathbb{R}_+$ (hence, we exclude the case where τ is an $\mathbb{F}$-stopping time). We assume throughout that G follows a continuous process.

The process $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by the formula

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t, \quad \forall\, t \in \mathbb{R}_+,$$

is termed the *hazard process* of a random time τ with respect to the filtration $\mathbb{F}$, or briefly the $\mathbb{F}$-*hazard process* of τ. The interpretation of the hazard process becomes more transparent from the following well-known equality

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{T<\tau\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t}\, \mathbb{E}_{\mathbb{Q}^*}(G_T Y \,|\, \mathcal{F}_t) = \mathbb{1}_{\{t<\tau\}}\, \mathbb{E}_{\mathbb{Q}^*}(e^{\Gamma_t - \Gamma_T} Y \,|\, \mathcal{F}_t), \tag{92}$$

which holds for any two dates $0 \le t \le T$, and for an arbitrary $\mathcal{F}_T$-measurable, $\mathbb{Q}^*$-integrable random variable Y.

4.2 Market CDS Rate

We first focus on the valuation of a CDS and the derivation of a general formula for market CDS rate. We maintain Definition 2.1 of a stylized credit default swap; the default protection stream is now represented by an $\mathbb{F}$-predictable process δ, however. As before, we assume that the default protection payment is received at the time of default, and it is equal δ_t if default occurs at time t, prior to or at maturity date T.

In the present set-up, the ex-dividend price of a CDS maturing at T with rate κ is given by the formula

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\le T\}}\delta_\tau \,\Big|\, \mathcal{G}_t\Big) - \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\}}\kappa\big((\tau\wedge T) - t\big) \,\Big|\, \mathcal{G}_t\Big), \tag{93}$$

where, as in formula (12), the two conditional expectations represent the current values of two legs of a CDS, namely, the default protection stream and the survival annuity stream.

For our further purposes, it will be convenient to assume that the CDS rate κ follows an $\mathbb{F}$-predictable stochastic process, so that (93) becomes

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\le T\}}\delta_\tau \,\Big|\, \mathcal{G}_t\Big) - \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\}} \int_t^{\tau\wedge T} \kappa_u \, du \,\Big|\, \mathcal{G}_t\Big). \tag{94}$$

In the next result, we do not need to assume that G is increasing. We make the standing assumption that $\mathbb{E}_{\mathbb{Q}^*}|\delta_\tau| < \infty$ and $\mathbb{E}_{\mathbb{Q}^*}\Big(\int_0^{\tau\wedge T} |\kappa_u| \, du\Big) < \infty$. The following result is a counterpart of Lemma 2.1.

Lemma 4.1 *The ex-dividend price of a credit default swap started at s, with a rate process κ and a protection payment δ_τ at default, equals, for every $t \in [s,T]$,*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(-\int_t^T \delta_u \, dG_u + \int_t^\infty \int_t^{u\wedge T} \kappa_v \, dv \, dG_u \,\Big|\, \mathcal{F}_t \right). \tag{95}$$

Proof. The proof is based on similar arguments as the proof of Lemma 2.1, combined with the following well-known formula

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{t<\tau\le T\}} Z_\tau \,|\, \mathcal{G}_t) = -\mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(\int_t^T Z_u \, dG_u \,\Big|\, \mathcal{F}_t \right), \tag{96}$$

which holds for any $\mathbb{F}$-predictable process such that $\mathbb{E}_{\mathbb{Q}^*}|Z_\tau| < \infty$. Let us fix t and let us set $\widetilde{\kappa}_u = \int_t^u \kappa_v \, dv$. To derive (95) from (94), it suffices to apply (96) to the process $Z_u = \delta_u - \widetilde{\kappa}_u$ for $u \in [t,T]$, and to use formula (92) to compute the conditional expectation $\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{T<\tau\}}\widetilde{\kappa}_T \,|\, \mathcal{G}_t)$. □

4.2.1 Case of a Constant CDS Rate

Under the standard assumption of a constant CDS rate κ, formula (95) simplifies to (note that $G_\infty = 0$)

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(-\int_t^T \delta_u \, dG_u + \kappa\Big(tG_t - TG_T + \int_t^T u \, dG_u\Big) \,\Big|\, \mathcal{F}_t \right).$$

Since G is a continuous process, the Itô integration by parts formula (14) is still valid, and thus straightforward calculations yield

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(-\int_t^T \delta_u \, dG_u - \kappa \int_t^T G_u \, du \,\Big|\, \mathcal{F}_t \right). \tag{97}$$

This shows that the pricing formula (13) extends in a natural way to the case of stochastic default intensity. It follows immediately from (97) that the T-maturity CDS market rate $\kappa(s,T)$, that is, the level of the CDS rate that makes the values of the two legs of a CDS equal to each other at time s, admits a generic representation analogous to (16). Specifically, we have that (recall that $\kappa(s,T)$ is $\mathcal{F}_s$-measurable)

$$\kappa(s,T) = -\frac{\mathbb{E}_{\mathbb{Q}^*}\left(\int_s^T \delta_u \, dG_u \,|\, \mathcal{F}_s\right)}{\mathbb{E}_{\mathbb{Q}^*}\left(\int_s^T G_u \, du \,|\, \mathcal{F}_s\right)}, \quad \forall\, s \in [0,T]. \tag{98}$$

A more explicit expression for $\kappa(s,T)$ can be derived under additional assumptions on δ and γ.

4.2.2 Case of a Non-zero Interest Rate

In what follows, we postulate that the $\mathbb{F}$-hazard process Γ is increasing and satisfies $\Gamma_t = \int_0^t \gamma_u \, du$ for some *intensity process* γ under $\mathbb{Q}^*$, which follows an $\mathbb{F}$-predictable, non-negative process. In this case, G is an absolutely continuous, decreasing process given by

$$G_t = e^{-\Gamma_t} = \exp\left(-\int_0^t \gamma_u \, du \right). \tag{99}$$

Under this assumption, equality (92) can be rewritten as follows

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}^*} \left(\exp\Big(-\int_t^T \gamma_u \, du\Big) \,\Big|\, \mathcal{F}_t \right). \tag{100}$$

Also, we assume from now on that the interest rate is not null, so that the savings account B is given by expression (1) for a certain short-term rate process r. Then formula (94) generalizes as follows

$$S_t(\kappa) = B_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\le T\}}B_\tau^{-1}\delta_\tau\,\Big|\,\mathcal{G}_t\Big) - B_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\}}\int_t^{\tau\wedge T} B_u^{-1}\kappa_u\,du\,\Big|\,\mathcal{G}_t\Big). \tag{101}$$

Consequently, under the assumptions that $\mathbb{E}_{\mathbb{Q}^*}|B_\tau^{-1}\delta_\tau| < \infty$ and $\mathbb{E}_{\mathbb{Q}^*}\Big(\int_0^{\tau\wedge T}|B_u^{-1}\kappa_u|\,du\Big) < \infty$, Lemma 4.1 yields the following pricing formula

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\frac{B_t}{G_t}\,\mathbb{E}_{\mathbb{Q}^*}\bigg(-\int_t^T B_u^{-1}\delta_u\,dG_u + \int_t^\infty\int_t^{u\wedge T} B_v^{-1}\kappa_v\,dv\,dG_u\,\bigg|\,\mathcal{F}_t\bigg). \tag{102}$$

This also means that the market CDS rate is given by the formula

$$\kappa(s,T) = \frac{\mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^T B_u^{-1}\delta_u\,dG_u\,\big|\,\mathcal{F}_s\Big)}{\mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^\infty\int_s^{u\wedge T} B_v^{-1}dv\,dG_u\,\big|\,\mathcal{F}_s\Big)}, \quad \forall\, s\in[0,T]. \tag{103}$$

The last two formulae can be further simplified, as the next result shows.

Proposition 4.1 *Assume that κ is constant. Then we have*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\frac{B_t}{G_t}\bigg(\int_t^T \mathbb{E}_{\mathbb{Q}^*}\big(B_u^{-1}\delta_u G_u\gamma_u\big|\,\mathcal{F}_t\big)\,du - \kappa\int_t^T \mathbb{E}_{\mathbb{Q}^*}\big(B_u^{-1}G_u\,\big|\,\mathcal{F}_t\big)\,du\bigg)$$

and thus

$$\kappa(s,T) = \frac{\int_s^T \mathbb{E}_{\mathbb{Q}^*}\big(B_u^{-1}\delta_u G_u\gamma_u\big|\,\mathcal{F}_s\big)\,du}{\int_s^T \mathbb{E}_{\mathbb{Q}^*}\big(B_u^{-1}G_u\,\big|\,\mathcal{F}_s\big)\,du}, \quad \forall\, s\in[0,T].$$

Proof. Let us fix t and let us assume that κ is constant. Then (102) can be represented as follows

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\frac{B_t}{G_t}\,\mathbb{E}_{\mathbb{Q}^*}\bigg(-\int_t^T B_u^{-1}\delta_u\,dG_u + \kappa\int_t^\infty X_{u\wedge T}\,dG_u\,\bigg|\,\mathcal{F}_t\bigg),$$

where we set $X_u = \int_t^u B_v^{-1}dv$. Since

$$G_T X_T = \int_t^T X_u\,dG_u + \int_t^T G_u\,dX_u,$$

we also obtain

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\frac{B_t}{G_t}\,\mathbb{E}_{\mathbb{Q}^*}\bigg(-\int_t^T B_u^{-1}\delta_u\,dG_u - \kappa\int_t^T B_u^{-1}G_u\,du\,\bigg|\,\mathcal{F}_t\bigg). \tag{104}$$

To complete the proof of the proposition, it suffices to note that $dG_t = -\gamma_t G_t\,dt$. □

Note that if G is not assumed to be increasing, $G = Z - A$ where Z is a martingale and A an increasing process assumed to be absolutely continuous with respect to the Lebesgue measure, and that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\frac{B_t}{G_t}\bigg(\int_t^T \mathbb{E}_{\mathbb{Q}^*}(B_u^{-1}\delta_u\gamma_u G_u\Big|\,\mathcal{F}_t)\,du - \kappa\int_t^T \mathbb{E}_{\mathbb{Q}^*}(B_u^{-1}A_u\Big|\,\mathcal{F}_t)\,du\, - \mathbb{E}_{\mathbb{Q}^*}(X_T Z_T\Big|\,\mathcal{F}_t)\bigg).$$

where $\gamma_u = a_u/G_u$.

4.3 Price Dynamics of a CDS

Under the assumption of a stochastic default intensity, it is natural to conjecture that the dynamics of a CDS will have an additional term, related to an uncertain behavior of the credit spread prior to default. The next result, which extends Lemma 2.2 to the case of a stochastic intensity, shows that this is indeed the case.

Proposition 4.2 *The dynamics of the ex-dividend price $S_t(\kappa)$ on $[s,T]$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1-H_t)B_tG_t^{-1}\, d\widehat{m}_t + (1-H_t)(r_tS_t(\kappa) + \kappa - \delta_t\gamma_t)\, dt, \tag{105}$$

where the $\mathbb{G}$-martingale M under $\mathbb{Q}^*$ equals

$$M_t = H_t - \int_0^{t\wedge\tau} \gamma_u\, du, \quad \forall\, t \in \mathbb{R}_+, \tag{106}$$

and the $\mathbb{F}$-martingale $\widehat{m}$ under $\mathbb{Q}^*$ is given by the formula

$$\widehat{m}_t = \mathbb{E}_{\mathbb{Q}^*}\left(\int_0^T B_u^{-1}\delta_u G_u\gamma_u\, du - \kappa \int_0^T B_u^{-1}G_u\, du \,\Big|\, \mathcal{F}_t \right). \tag{107}$$

Proof. Observe that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\frac{B_t}{G_t}Y_t = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa),$$

where the auxiliary process Y equals (see (104))

$$Y_t = \widehat{m}_t - \int_0^t \delta_u B_u^{-1}G_u\gamma_u\, du + \kappa\int_0^t B_u^{-1}G_u\, du,$$

where in turn $\widehat{m}_t$ is given by (107). Let us set $Z_t = G_t^{-1}Y_t$. Standard Itô's calculus leads to

$$dZ_t = \big(-\delta_tB_t^{-1}\gamma_t + \kappa B_t^{-1} + Z_t\gamma_t\big)dt + \frac{1}{G_t}\, d\widehat{m}_t.$$

Therefore, we also have

$$d\widetilde{S}_t(\kappa) = \big(-\delta_t\gamma_t + \kappa + \widetilde{S}_t(\kappa)(\gamma_t + r_t)\big)dt + \frac{B_t}{G_t}\, d\widehat{m}_t.$$

Finally, for the process $S_t(\kappa) = (1-H_t)\widetilde{S}_t(\kappa)$ we obtain

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1-H_t)\Big(B_tG_t^{-1}\, d\widehat{m}_t + \big(r_tS_t(\kappa) + \kappa - \delta_t\gamma_t\big)dt\Big),$$

as was expected. □

In what follows, we shall also make use of the dynamics of the process $\widetilde{S}(\kappa)$. Recall that $\widetilde{S}(\kappa)$ is the pre-default ex-dividend price of a CDS, so that $S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa)$ for every $t \in [0,T]$. It is easy to see that prior to default, that is, on the set $\{t < \tau\}$, we have

$$dS_t(\kappa) = d\widetilde{S}_t(\kappa) = \big((r_t + \gamma_t)\widetilde{S}_t(\kappa) + \kappa - \delta_t\gamma_t\big)dt + B_tG_t^{-1}\, d\widehat{m}_t, \tag{108}$$

and manifestly $\widetilde{S}_0(\kappa) = S_0(\kappa)$. The formula above is a natural extension of (38). It shows, in particular, that the pre-default ex-dividend price $\widetilde{S}(\kappa)$ is an $\mathbb{F}$-semimartingale.

Remark. One can expect that, under suitable technical assumptions, the filtration generated by $\widetilde{S}(\kappa)$ will coincide with $\mathbb{F}$. When this property does not hold, so that the filtration generated by $\widetilde{S}(\kappa)$ is strictly smaller than $\mathbb{F}$, we need to examine whether prices and hedging strategies of attainable claims can be expressed in terms of prices of traded assets.

4.4 Replicating Strategies with CDSs

We shall now assume that $k \geq 1$ credit default swaps, with maturities $T^i \geq T$, rates κ_i and protection payments δ_i for $i = 0, \ldots, k-1$, are traded. The kth asset is the savings account B. We consider hedging of a defaultable claim $(X, 0, Z, \tau)$ such that $\mathbb{E}_{\mathbb{Q}^*}|B_T^{-1}X| < \infty$ and $\mathbb{E}_{\mathbb{Q}^*}|B_\tau^{-1}Z_\tau| < \infty$. It is natural to define the replication of such a claim in the following way.

Definition 4.1 We say that a self-financing strategy $\phi = (\phi^0, \ldots, \phi^k)$ replicates a defaultable claim $(X, 0, Z, \tau)$ if its wealth process $V(\phi)$ satisfies the following equalities: $V_T(\phi)\mathbb{1}_{\{T<\tau\}} = X\mathbb{1}_{\{T<\tau\}}$ and $V_\tau(\phi)\mathbb{1}_{\{T\geq\tau\}} = Z_\tau\mathbb{1}_{\{T\geq\tau\}}$.

When dealing with replicating strategies in the sense of the definition above, we may and do assume, without loss of generality, that the components of the process ϕ are $\mathbb{F}$-predictable processes.

Since all processes considered here are stopped at time τ we find it convenient to adapt slightly the definition of admissibility. Namely, we say that a self-financing trading strategy ϕ is *admissible* if the discounted stopped wealth process $V^*_{t\wedge\tau}(\phi) = B^{-1}_{t\wedge\tau}V_{t\wedge\tau}(\phi)$, $t \in [0, T]$, is a $\mathbb{Q}^*$-martingale.

We assume from now on that the filtration $\mathbb{F}$ is generated by a Brownian motion W under $\mathbb{Q}^*$, and the so-called hypothesis (H) holds, so that W is also a Brownian motion with respect to $\mathbb{G}$. Recall that all (local) martingales with respect to a Brownian filtration are continuous.

In the statement and the proof of the next result, we deal in fact with discounted price processes. However, for the sake of notational simplicity, we do not account for this feature in the notation.

Proposition 4.3 *Assume that there exist $\mathbb{F}$-predictable processes $\phi^0, \ldots, \phi^{k-1}$ such that*

$$\sum_{i=0}^{k-1} \phi^i_t\big(\delta^i_t - \widetilde{S}^i_t(\kappa_i)\big) = Z_t - \widehat{g}_t, \quad \sum_{i=0}^{k-1} \phi^i_t\zeta^i_t = \zeta_t, \tag{109}$$

where the $\mathbb{F}$-predictable processes ζ^i, $i = 0, \ldots, k-1$ and ζ are given by (112) *and* (116)*, respectively, and the continuous, $\mathbb{F}$-adapted process $\widehat{g}$ is given by* (115)*. Let $\phi^k_t = V_t(\phi) - \sum_{i=0}^{k-1}\phi^i_t S^i_t(\kappa_i)$, where the process $V(\phi)$ is given by*

$$dV_t(\phi) = \sum_{i=0}^{k-1} \phi^i_t\big(dS^i_t(\kappa_i) + dD^i_t\big) \tag{110}$$

with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{Q}^*}(Y)$, where Y is given by

$$Y = \mathbb{1}_{\{T\geq\tau\}}Z_\tau + \mathbb{1}_{\{T<\tau\}}X. \tag{111}$$

Then the self-financing trading strategy $\phi = (\phi^0, \ldots, \phi^k)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.

In the proof of Proposition 4.3, we shall use the following version of a predictable representation theorem (see, for instance, Blanchet-Scalliet and Jeanblanc [5]).

Proposition 4.4 *Assume that G is a continuous, decreasing process. Let $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Z_\tau \,|\, \mathcal{G}_t)$, where Z is an arbitrary $\mathbb{F}$-predictable process such that $\mathbb{E}_{\mathbb{Q}^*}|Z_\tau| < \infty$. Then we have, for every $t \in \mathbb{R}_+$,*

$$\widehat{M}_t = \widehat{M}_0 + \int_{]0,t]} (Z_u - \widehat{g}_u)\, dM_u + \int_0^t (1 - H_u)G_u^{-1}\, d\widehat{m}_u,$$

where the continuous $\mathbb{F}$-martingale (and $\mathbb{G}$-martingale) $\widehat{m}$ is given by the formula

$$\widehat{m}_t = \mathbb{E}_{\mathbb{Q}^*}\left(\int_0^\infty Z_u\, dF_u \,\Big|\, \mathcal{F}_t\right) = -\mathbb{E}_{\mathbb{Q}^*}\left(\int_0^\infty Z_u\, dG_u \,\Big|\, \mathcal{F}_t\right)$$

and the continuous, $\mathbb{F}$-adapted process $\widehat{g}$ equals

$$\widehat{g}_t = e^{\Gamma_t}\left(\widehat{m}_t - \int_0^t Z_u\,dF_u\right) = -\frac{1}{G_t}\,\mathbb{E}_{\mathbb{Q}^*}\left(\int_t^\infty Z_u\,dG_u \,\Big|\, \mathcal{F}_t\right).$$

Moreover, we have that $\widehat{M}_t = \widehat{g}_t$ on the set $\{t < \tau\}$.

Proof of Proposition 4.3. As in the proof of Proposition 2.3, we first seek a replicating strategy prior to default. Since $dB_t = 0$, for the wealth process $V(\phi)$ we obtain, on the set $\{\tau > t\}$,

$$dV_t(\phi) = \sum_{i=0}^{k-1} \phi^i_t\,(d\widetilde{S}^i_t(\kappa_i) - \kappa_i\,dt) = \sum_{i=0}^{k-1} \phi^i_t\Big(\gamma_t\big(\widetilde{S}^i_t(\kappa_i) - \delta^i_t\big)\,dt + G_t^{-1}\,d\widehat{m}^i_t\Big),$$

where the second equality follows from (108), and $\widehat{m}^i$, $i = 0,\dots,k-1$ are given by (107) with δ and κ replaced by δ^i and κ_i. In view of the predictable representation property of a Brownian motion, we also have

$$dV_t(\phi) = \sum_{i=0}^{k-1} \phi^i_t\Big(\gamma_t\big(\widetilde{S}^i_t(\kappa_i) - \delta^i_t\big)\,dt + G_t^{-1}\zeta^i_t\,dW_t\Big), \tag{112}$$

for some $\mathbb{F}$-predictable processes ζ^i, $i = 0,\dots,k-1$ such that $d\widehat{m}^i_t = \zeta^i_t\,dW_t$.

To deal with a defaultable claim $(X, 0, Z, \tau)$, we shall apply Proposition 4.4 to the process $\bar{Z}$ given by the formula $\bar{Z}_t = Z_t\mathbb{1}_{[0,T[}(t) + X\mathbb{1}_{[T,\infty[}(t)$ (recall that $\mathbb{Q}^*(\tau = T) = 0$). We obtain

$$\widehat{M}_t = \widehat{M}_0 + \int_{]0,t]} (Z_u - \widehat{g}_u)\,dM_u + \int_0^t (1 - H_u)G_u^{-1}\,d\widehat{m}_u, \tag{113}$$

where the continuous $\mathbb{F}$-martingale $\widehat{m}$ is given by the formula

$$\widehat{m}_t = \mathbb{E}_{\mathbb{Q}^*}\left(-\int_0^T Z_u\,dG_u + G_T X \,\Big|\, \mathcal{F}_t\right), \tag{114}$$

and the process $\widehat{g}$ equals

$$\widehat{g}_t = \frac{1}{G_t}\,\mathbb{E}_{\mathbb{Q}^*}\left(-\int_t^T Z_u\,dG_u + G_T X \,\Big|\, \mathcal{F}_t\right). \tag{115}$$

Consequently, on the set $\{t < \tau\}$,

$$d\widehat{M}_t = -\gamma_t(Z_t - \widehat{g}_t)\,dt + G_t^{-1}\,d\widehat{m}_t = -\gamma_t(Z_t - \widehat{g}_t)\,dt + G_t^{-1}\zeta_t\,dW_t \tag{116}$$

for some $\mathbb{F}$-predictable processes ζ such that $d\widehat{m}_t = \zeta_t\,dW_t$ (the existence of ζ follows from the predictable representation property of W).

It is clear that strategy $\phi = (\phi^0,\dots,\phi^k)$ replicates a claim $(X, 0, Z, \tau)$ prior to default, provided that its initial value $V_0(\phi)$ is equal to $\mathbb{E}_{\mathbb{Q}^*}(Y)$, and the components $\phi^0,\dots,\phi^{k-1}$ are judiciously chosen so that the equality $dV_t(\phi) = d\widehat{M}_t$ holds on $\{t < \tau\}$. More explicitly, the $\mathbb{F}$-predictable processes $\phi^0,\dots,\phi^{k-1}$ are bound to satisfy

$$\sum_{i=0}^{k-1} \phi^i_t\big(\delta^i_t - \widetilde{S}^i_t(\kappa_i)\big) = Z_t - \widehat{g}_t, \quad \sum_{i=0}^{k-1} \phi^i_t\zeta^i_t = \zeta_t, \quad \forall\, t \in [0,T], \tag{117}$$

where the first condition is essential only for those values of $t \in [0,T]$ for which $\gamma_t \neq 0$.

To complete the proof, it suffices to compare the jumps of $\widehat{M}$ and $V(\phi)$ at time τ. On the one hand, it is obvious that $\Delta_\tau\widehat{M} = Z_\tau - \widehat{g}_\tau$. On the other hand, for the wealth process of ϕ, we obtain

$$\Delta_\tau V(\phi) = \sum_{i=0}^{k-1} \phi^i_\tau(\delta^i_\tau - \widetilde{S}^i_\tau(\kappa_i)) = Z_\tau - \widehat{g}_\tau,$$

where the last equality follows from (117). We conclude that $V_{t\wedge\tau}(\phi) = \widehat{M}_{t\wedge\tau}$ for every $t \in [0,T]$. In particular, ϕ is admissible in the sense that the stopped wealth process $V_{t\wedge\tau}(\phi)$, $t\in[0,T]$, is a $\mathbb{Q}^*$-martingale, and we have that $V_{\tau\wedge T}(\phi) = Y$, where Y is given by (111).

This means that ϕ replicates a defaultable claim $(X,0,Z,\tau)$. Hence, the stopped $\mathbb{Q}^*$-martingale $\widehat{M}_{t\wedge\tau}$, where $\widehat{M}$ is given by (113)-(115), represents the arbitrage price of this claim on $[\![0,\tau\wedge T]\!]$. □

In general, the existence of a solution $(\phi^0,\dots,\phi^{k-1})$ to (109) is not ensured, and in fact it is easy to give an example when a solution fails to exist. In Example 4.1 below, we deal with a (somewhat artificial) situation when the jump risk can be perfectly hedged, but the traded assets are deterministic prior to default, so that the volatility risk of a defaultable claim is unhedgeable. In general, solvability of (109) depends on such factors as: the number of traded assets, the dimension of the driving Brownian motion, a random (or non-random) character of default intensity γ and recovery payoffs δ^i, and, last but not least, the specific features of a defaultable claim we wish to hedge.

Example 4.1 Assume assume that $k=2$, $\kappa_0 \neq \kappa_1$ are non-zero constants, and let $\delta^0=\delta^1=Z=0$. Assume also that the default intensity $\gamma(t)>0$ is deterministic, and the promised payoff X is a non-constant $\mathcal{F}_T$-measurable random variable. We thus have (cf. (114))

$$\widehat{m}_t = \mathbb{E}_{\mathbb{Q}^*}(G_T X \,|\, \mathcal{F}_t) = G_T\, \mathbb{E}_{\mathbb{Q}^*}(X \,|\, \mathcal{F}_t) = G_T \left(\mathbb{E}_{\mathbb{Q}^*}(X) + \int_0^t \zeta_u \, dW_u \right)$$

for some non-vanishing process ζ. Since γ is deterministic, we deduce easily from (107) that $\zeta^i_t = 0$ for every $t\in[0,T]$. The first condition in (109) reads $\sum_{i=0}^1 \phi^i_t \widetilde{S}^i_t(\kappa_i) = \widehat{g}_t$, and since $\widetilde{S}^i_t(\kappa_i) \neq 0$ for every $t\in[0,T]$, this condition poses no problems. However, the second equality, $\sum_{i=0}^1 \phi^i_t \zeta^i_t = \zeta_t$, is never satisfied, since the left-hand side equals zero for every $t\in[0,T]$. To improve the situation, it is enough to assume that $\gamma>0$ is random, so that the processes ζ^0 and ζ^2 no longer vanish. It is then possible to establish the existence of a unique solution (ϕ^0,ϕ^1) to (109) for any X.

Before concluding this section, let us formulate an auxiliary result that underpins the validity of Proposition 4.3 (note that Lemma 4.2 was not explicitly used in the proof given above, however).

Lemma 4.2 *Let $M^l_t = \mathbb{E}_{\mathbb{Q}^*}(Z^l_\tau \,|\, \mathcal{G}_t)$, be two $\mathbb{G}$-martingales under $\mathbb{Q}^*$, where Z^l, $l=1,2$ are $\mathbb{F}$-predictable process such that $\mathbb{E}_{\mathbb{Q}^*}|Z^l_\tau| < \infty$. If the equality $\mathbb{1}_{\{t<\tau\}} M^1_t = \mathbb{1}_{\{t<\tau\}} M^2_t$ holds for every $t\in[0,T]$ then $M^1_{t\wedge\tau} = M^2_{t\wedge\tau}$ for every $t\in[0,T]$.*

Proof. We shall merely sketch the proof, which is again based on an application of Proposition 4.4. Under the assumptions of the lemma, we have that $\widehat{g}^1_t = M^1_t = M^2_t = \widehat{g}^2_t$ on the set $\{t<\tau\}$ for every $t\in[0,T]$. Since the processes $\widehat{g}^1$ and $\widehat{g}^2$ are $\mathbb{F}$-predictable and $\mathbb{Q}^*(\tau>t \,|\, \mathcal{F}_t) > 0$ for every $t\in\mathbb{R}_+$, this implies that the equality $\widehat{g}^1_t = \widehat{g}^2_t$ is in fact valid for every $t\in[0,T]$. Consequently,

$$\mathbb{E}_{\mathbb{Q}^*}\left(\int_t^\infty Z^1_u \, dG_u \,\Big|\, \mathcal{F}_t \right) = \mathbb{E}_{\mathbb{Q}^*}\left(\int_t^\infty Z^2_u \, dG_u \,\Big|\, \mathcal{F}_t \right).$$

The last equality can be used to show that $Z^1_t = Z^2_t$ on $\Omega\times[0,T]$, almost everywhere with respect to the random measure generated by a continuous, decreasing process G. This in turn implies that $Z^1_{t\wedge\tau} = Z^2_{t\wedge\tau}$, $\mathbb{Q}^*$-a.s. Consequently, we find that $M^1_{t\wedge\tau} = M^2_{t\wedge\tau}$ for every $t\in[0,T]$.

4.5 Forward Start CDS

Recall that a *forward start CDS* initiated at some date $s\in[0,U]$ gives the default protection over the future time interval $[U,T]$. The price of this contract at any date $t\in[s,U]$ equals

$$S_t(\kappa) = B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\le T\}} B^{-1}_\tau \delta_\tau \,\Big|\, \mathcal{G}_t \Big) - B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\}} \kappa \int_U^{\tau\wedge T} B^{-1}_u \, du \,\Big|\, \mathcal{G}_t \Big), \tag{118}$$

or more explicitly,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(- \int_U^T B_u^{-1} \delta_u \, dG_u - \kappa \int_U^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right). \tag{119}$$

A *forward start market CDS* at time $t \in [0, U]$ is a forward CDS, which is valueless at time t. The corresponding (pre-default) *forward CDS rate* $\kappa(t, U, T)$ is thus an $\mathcal{F}_t$-measurable random variable implicitly determined by the equation

$$S_t(\kappa(t,U,T)) = B_t \, \mathbb{E}_{\mathbb{Q}^*} \left(\mathbb{1}_{\{U<\tau \le T\}} B_\tau^{-1} \delta_\tau \,\Big|\, \mathcal{G}_t \right) - B_t \, \mathbb{E}_{\mathbb{Q}^*} \left(\mathbb{1}_{\{U<\tau\}} \kappa(t,U,T) \int_U^{\tau \wedge T} B_u^{-1} \, du \,\Big|\, \mathcal{G}_t \right) = 0.$$

We thus we have, for every $t \in [0, U]$,

$$\kappa(t,U,T) = - \frac{\mathbb{E}_{\mathbb{Q}^*} \left(\int_U^T B_u^{-1} \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \right)}{\mathbb{E}_{\mathbb{Q}^*} \left(\int_U^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right)}. \tag{120}$$

For an arbitrary forward CDS with rate κ we have, for every $t \in [0, U]$,

$$S_t(\kappa) = S_t(\kappa) - S_t(\kappa(t,U,T)) = (\kappa(t,U,T) - \kappa) \, \mathbb{E}_{\mathbb{Q}^*} \left(\mathbb{1}_{\{U<\tau\}} \int_U^{\tau \wedge T} B_u^{-1} \, du \,\Big|\, \mathcal{G}_t \right), \tag{121}$$

or more explicitly,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} (\kappa(t,U,T) - \kappa) \, \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(\int_U^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right).$$

4.5.1 CDS Options

The last representation proves useful in the valuation and hedging of options on a forward start CDS (equivalently, options on a forward CDS rate). Indeed, it shows that

$$(S_t(\kappa))^+ = \mathbb{1}_{\{t<\tau\}} (\kappa(t,U,T) - \kappa)^+ \, \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(\int_U^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right),$$

so that a call option on the value of a forward CDS with rate κ appears to be equivalent to a call option on a forward CDS rate. Note that here the date $t \in]0, U]$ is interpreted as the exercise date of the option, and we are interested in the value of this claim at time $s \in [0, t]$. Moreover, the role of the *swap annuity* (or *level process*), which serves as a convenient numéraire when dealing with default-free interest rate swaps, is now played by the the (pre-default) *CDS annuity* α_t, which is given by the formula, for every $t \in [0, U]$,

$$\alpha_t = \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(\int_U^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right).$$

Of course, A_t is nothing else as the (pre-default) value at time t of the survival annuity stream per unit of the rate κ (recall that κ is usually expressed in basis points). It is clear that A follows a strictly positive process, so that it may serve as a natural numéraire in the study of CDS options.

Similar results can be derived for real-world, rather then stylized, forward CDS rates and the associated option contracts. In the case of traded CDS contracts, the fee payments occur at some pre-determined settlement dates, referred to as the *tenor structure* of a CDS. In the case of default, the protection payment is either done immediately, or it is postponed to the next settlement date. This means, of course, that some of the integrals appearing in our formulae should be replaced by the corresponding finite sums. Such a modification is indeed crucial if one wishes to develop a practical approach to modeling of forward CDS rates, and hedging of credit derivatives with CDS contracts.

4.5.2 Modeling of Forward CDS Rates

An important issue that should be addressed in the context of CDS rates is a construction of a model in which a family of credit default options are valued through a suitable version of Black's formula (or some other widely accepted pricing formula). For an option on a particular credit default swap, such pricing formula was derived by Schönbucher [15]-[16] and Jamshidian [12], who formally used the risk-neutral valuation formula in an intensity-based credit risk model, which was not fully specified. The derivations of a version of Black's formula for CDS options presented in these papers are based on rather abstract approximation arguments for a positive martingale, as opposed to an explicit construction of a (lognormal) model for a family of CDS rates associated with a given tenor structure, in which the pricing of a CDS option could be supported by strict replication arguments.

In recent works by Brigo and Cousot [9] and Ben-Ameur et al. [2], the authors deal with the valuation of European and Bermudan CDS options within the set-up of the so-called SSRD (Shifted Square Root Diffusion) model, which was introduced previously by Brigo and Alfonsi [8].

In important papers by Brigo [6]-[7], the author analyzes the joint dynamics of certain forward CDS rates under judiciously chosen martingale measures. He shows that in some cases (especially, for one-period and two-period CDS), it is possible to develop a change of a numéraire approach, which is analogous to arbitrage-free modeling of forward LIBOR and (constant maturity) swap rates. He also briefly discusses the difficulties arising in the context of modeling of a family of co-terminal forward CDS rates. However, neither of the above-mentioned papers addresses the issue of hedging of credit derivatives with the use of (forward) CDS contracts. It is thus worth mentioning in this regard, that Kurtz and Riboulet [14] examine replicating strategies for basket credit derivatives, under a simplifying assumption that the reference filtration is trivial. Their approach is indeed quite similar to the one presented in this note.

5 Concluding Remarks

In the financial literature, it is common to split the risk of trading a defaultable security into the following components: the *default risk* (that is, the jump risk associated with the default event), and the *spread risk* (that is, the risk due to the volatile character of the pre-default price of a defaultable claim). The pertinent issue is thus to find a way of dealing simultaneously with both kinds of risks in a simple, but efficient, way. In our main result, Proposition 4.3, we show that, in a generic intensity-based model driven by a Brownian motion, it is possible to deal in a unified way with both kinds of credit risk.

Specifically, the first equality in formula (109) permits to perfectly hedge the default risk, while the second allows us to effectively hedge the spread risk. It is clear from these formulae that hedging of the default risk relies on keeping under control the unexpected jumps that may come as a surprise at any moment. By contrast, hedging of the spread risk hinges on more standard techniques related to the volatilities and correlations of underlying stochastic processes. As shown in Bielecki et al. [4], in the case of a *survival claim* (that is, a defaultable claim with $Z = 0$), it is enough to concentrate on hedging of the spread risk, provided that hedging instruments are also subject to zero recovery.

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