On the Tail Probability for Discounted Sums of Heavy-tailed Losses

Roger J.A. Laeven
University of Amsterdam
www.fee.uva.nl/ke/laeven

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Main references

This talk is mainly based on

[1] Laeven, Roger J.A., Marc J. Goovaerts & Tom Hoedemakers (2005). "Some asymptotic results for sums of dependent random variables, with actuarial applications," *Insurance: Mathematics and Economics* 37, 154-172.

[2] Goovaerts, Marc J., Rob Kaas, Roger J.A. Laeven, Qihe Tang & Raluca Vernic (2005). "The tail probability of discounted sums of Pareto-like losses in insurance," *Scandinavian Actuarial Journal* 6, 446-461.

Outline

- The general problem;
- Classes of heavy-tailed distributions;
- The main results;
- Examples.

The general problem

Consider the randomly weighted sum $\sum_{i=1}^{n} \theta_i X_i$, with

- $\{X_n, n = 1, 2, \ldots\}$ a sequence of i.i.d. r.v.'s;

- $\{\theta_n, n = 1, 2, \ldots\}$ a sequence of non-negative **dependent** r.v.'s;

- the sequences $\{X_n, n = 1, 2, \ldots\}$ and $\{\theta_n, n = 1, 2, \ldots\}$ being independent.

We want to investigate its tail probability and functionals (risk measures) thereof.

The general problem: interpretation

- X_n: represents the net loss or payoff of an insurance or financial product (or portfolio, line of business, conglomerate,...) in (development) year n.
  - Is assumed to be independent across time.
  - In insurance, typically heavy-tailed.

- θ_n: represents the stochastic discount factor for year n.
  - Case 1: $\theta_n = Y_1 \cdots Y_n$, with $\{Y_n, n = 1, 2, \ldots\}$ a sequence of non-negative i.i.d. r.v.'s.
  - Case 2: no assumption on the dependence structure. Includes e.g., GARCH models.

The general problem: possible solutions

- Monte Carlo simulation;
- Easy-computable bounds or approximations à la Roger & Shi (1995);
- Asymptotics.

Classes of heavy-tailed distributions [1]

- Class $\mathcal{S}$:

$$\lim_{x \to +\infty} \overline{F^{*n}}(x)/\overline{F}(x) = n,$$

  for any (or equivalently, for some) $n \geq 2$.

- Class $\mathcal{L}$:

$$\lim_{x \to +\infty} \overline{F}\,(x+y)\,/\overline{F}(x) = 1,$$

  for any real number y (or equivalently, for $y = 1$).

Classes of heavy-tailed distributions [2]

- Class $\mathcal{D}$:

$$\limsup_{x \to +\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < +\infty,$$

  for any $0 < y < 1$ (or equivalently for some $0 < y < 1$).

- $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$; see e.g., Embrechts, Klüppelberg & Mikosch (1997).

Classes of heavy-tailed distributions [3]

- Class $\mathcal{R}_{-\alpha}$:

$$\lim_{x \to +\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha},$$

for any $y > 0$.

- Class $\mathcal{R}_{-\infty}$:

$$\lim_{x \to +\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \begin{cases} 0, & y > 1; \\ +\infty, & 0 < y < 1. \end{cases}$$

Asymptotic results [1]

Let

- $\{Y_n, n = 1, 2, \ldots\}$ i.i.d. supported on $(0, +\infty)$;

- $Z_n := Y_1 Y_2 \cdots Y_n$;

- $0 < a_n < +\infty$, $n = 1, 2, \ldots$

If $F_Y \in \mathcal{S} \cap \mathcal{R}_{-\infty}$, then it holds for each $n = 1, 2, \ldots$ that

$$\mathbb{P}\left(\sum_{i=1}^{n} a_i Z_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(a_i Z_i > x\right).$$

Asymptotic results [2]

Let

- $\{X_n, n = 1, 2, \ldots\}$ i.i.d. supported on $(-\infty, +\infty)$.

If $F_X \in \mathcal{D} \cap \mathcal{L}$ and $F_Y \in \mathcal{R}_{-\infty}$, then it holds for each $n = 1, 2, \ldots$ that

$$\mathbb{P}\left(\sum_{i=1}^{n} (a_i + X_i) Z_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left((a_i + X) Z_i > x\right)$$

and that

$$\mathbb{P}\left(\sum_{i=1}^{n} (a_i X_i) Z_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left((a_i X) Z_i > x\right).$$

Asymptotic results [3]

If X and Y follow a lognormal law with $\sigma_Y < \sigma_X$, then it holds for each $n = 1, 2, \ldots$ that

$$\mathbb{P}\left(\sum_{i=1}^{n}(a_i + X_i)Z_i > x\right) \sim \sum_{i=1}^{n}\mathbb{P}\left((a_i + X)Z_i > x\right)$$

and that

$$\mathbb{P}\left(\sum_{i=1}^{n}(a_i X_i)Z_i > x\right) \sim \sum_{i=1}^{n}\mathbb{P}\left((a_i X)Z_i > x\right).$$

Asymptotic results [4]

Let

- $\{\theta_n, n = 1, 2, \ldots\}$ non-negative and **dependent**.

If $F_X \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ and there exists some $\delta > 0$ such that $\mathbb{E}[\theta_i^{\alpha+\delta}] < +\infty$ for each $1 \leq i \leq n$, then it holds for each $n = 1, 2, \ldots$ that

$$\mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\theta_i X > x\right) \sim \overline{F}(x) \sum_{i=1}^{n} \mathbb{E}[\theta_i^{\alpha}].$$

Holds even uniformly for $n = 1, 2, \ldots$; see Wang (2005).

Example: Stop-loss premium and Value-at-Risk [1]

Let $\widetilde{S}_n = \sum_{i=1}^n \theta_i X_i$. Then

- Stop-loss premium:

$$\mathbb{E}[(\widetilde{S}_n - d)_+] \approx \sum_{i=1}^n \mathbb{E}[(\theta_i X - d)_+].$$

- VaR:

$$\text{inf}\{s : F_{\widetilde{S}_n}(s) \geq p\} \approx$$

$$\text{inf}\left\{s : \sum_{i=1}^n \overline{F}_{\theta_i X}(s) \leq 1 - p\right\}.$$

Example: Stop-loss premium and Value-at-Risk [2]

Furthermore, let $F_X \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. Then

- Stop-loss premium:

$$\mathbb{E}[(\widetilde{S}_n - d)_+] \approx \mathbb{E}[(X - d)_+] \sum_{i=1}^{n} \mathbb{E}[\theta_i^\alpha].$$

- VaR:

$$\text{inf}\{s : F_{\widetilde{S}_n}(s) \geq p\} \approx$$

$$\text{inf}\left\{s : \overline{F}_X(s) \sum_{i=1}^{n} \mathbb{E}[\theta_i^\alpha] \leq 1 - p\right\}.$$

Example: Stop-loss premium and Value-at-Risk [3]

- $\theta_n = Y_1 \cdots Y_n$, i.i.d.: $\mathbb{E}[\theta_n^\alpha] = \mathbb{E}[Y^\alpha]^n$.

- $(\theta_1, \ldots, \theta_n) =_d \mathsf{LE}_n(\mu_n, \Sigma_n, \phi)$: $\mathbb{E}[\theta_n^\alpha]$ is explicit; see e.g., Fang, Kotz & Ng (1990) and Owen & Rabinovitch (1983).

- $(\theta_1, \ldots, \theta_n) =_d \mathsf{LNVMM}_n(\mu_n, \beta_n, \Sigma_n, G)$: $\mathbb{E}[\theta_n^\alpha]$ is explicit; see e.g., Barndorff-Nielsen (1997).

A numerical illustration

d	"Real"	Appr.	Diff.	Rdiff.	p	"Real"	Appr.	Diff.	Rdiff.
30	1.75	1.56	0.19	11%	0.975	36	30	6	17%
40	1.48	1.35	0.13	9%	0.99	63	57	6	10%
60	1.18	1.11	0.07	6%	0.995	96	90	6	6%
80	1.01	0.96	0.05	5%	0.999	274	265	9	3%
100	0.90	0.86	0.04	4%					
150	0.72	0.70	0.02	3%					
200	0.62	0.61	0.01	2%					
250	0.56	0.55	0.01	2%					
300	0.51	0.50	0.01	2%					

Notes: "Real" versus approximate values of stop-loss premiums and quantiles for Pareto losses and i.i.d. lognormal stochastic discount factors. Fixed parameter values: $n = 5$, $\alpha = 1.5$, $\mu = -0.04$, $\sigma = 0.10$ and 5,000,000 simulations.

Analytic approximations!

References [1]

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