PRICING INFLATION-INDEXED OPTIONS WITH STOCHASTIC VOLATILITY

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Stylized facts

- Inflation-indexed bonds have been issued since the 80's, but it is only in the very last years that these bonds, and inflation-indexed derivatives in general, have become quite popular.

- Inflation is defined as the percentage increment of a reference index, the Consumer Price Index (CPI), which is a basket of good and services.

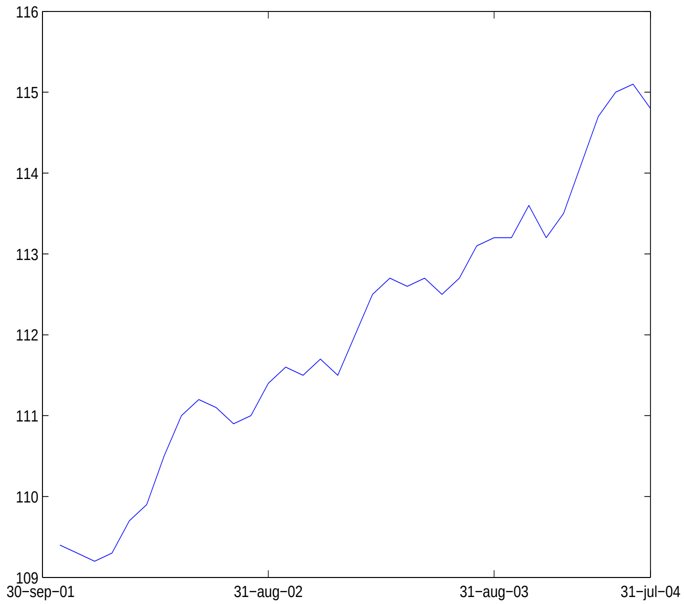
- Denoting by $I(t)$ the CPI's value at time t, the inflation rate over the time interval $[t, T]$ is therefore:

$$i(t, T) := \frac{I(T)}{I(t)} - 1.$$

- In theory, but also in practice, inflation can become negative.

Stylized facts (cont'd)

Historical plots of CPI's



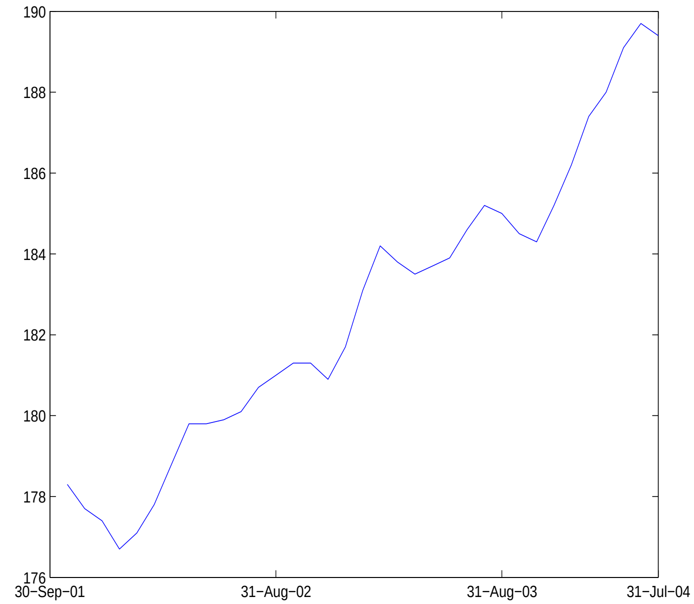


Figure 1: Left: EUR CPI Unrevised Ex-Tobacco. Right: USD CPI Urban Consumers NSA. Monthly closing values from 30-Sep-01 to 21-Jul-04.

Stylized facts (cont'd)

- Banks are used to issue inflation-linked bonds, where a zero-strike floor is offered in conjunction with the "pure" bond.
- To grant positive coupons, the inflation rate is typically floored at zero.
- Accordingly, floors with low strikes are the most actively traded options on inflation rates.
- Other extremely popular derivatives are inflation-indexed swaps.
- Two are the main inflation-indexed swaps traded in the market:
  - the zero coupon (ZC) swap;
  - the year-on-year (YY) swap.

Stylized facts (cont'd)

- Inflation-indexed derivatives require a specific model to be valued.
- Main references: Barone and Castagna (1997), van Bezooyen et al. (1997), Hughston (1998), Kazziha (1999), Cairns (2000), Jamshidian (2002), Jarrow and Yildirim (2003), Belgrade et al. (2004) and Mercurio (2005).
- Inflation derivatives are priced with a foreign-currency analogy (the pricing is equivalent to that of a cross-currency interest-rate derivative).
- In a short rate approach, one models the evolution of the instantaneous nominal and real rates and of the CPI (interpreted as the "exchange rate" between the nominal and real economies).
- Recent approaches are based on market models, where one models forward CPI indices and nominal rates.

Purpose and outline of the talk

- Our purpose is to price analytically, and consistently with no arbitrage, inflation-indexed swaps and options.

- We start by introducing the two main types of inflation swaps and price them by means of a market model.

- We then introduce inflation caps and floors, and derive analytical formulas under the "flat smile" case.

- We finally consider stochastic volatility as in Heston (1993) and derive closed-form formulas for caps and floors.

- Examples of calibration to market data are shown both in the "flat smile" case and in the stochastic volatility case.

Some notations and definitions

We use the subscripts n and r to denote quantities in the nominal and real economies, respectively.

The zero-coupon bond prices at time t for maturity T in the nominal and real economies are denoted, respectively, by $P_n(t,T)$ and $P_r(t,T)$.

The nominal instantaneous rate at time t is denoted by $n(t)$.

The forward LIBOR rates at time t for the future time interval $[T_{i-1}, T_i]$ are

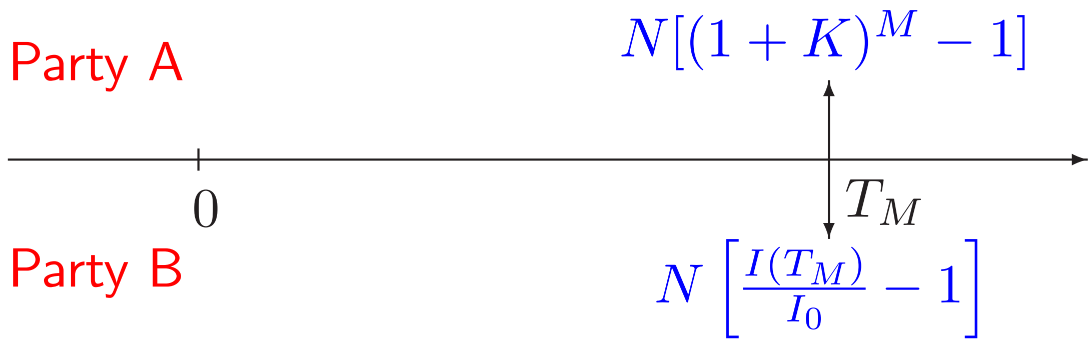
$$F_x(t;T_{i-1},T_i) = \frac{P_x(t,T_{i-1}) - P_x(t,T_i)}{\tau_i P_x(t,T_i)}, \quad x \in \{n, r\},$$

where τ_i is the year fraction for $[T_{i-1}, T_i]$.

We finally define the T_i-forward CPI by

$$\mathcal{I}_i(t) := I(t)\frac{P_r(t,T_i)}{P_n(t,T_i)}$$

Zero-coupon inflation-indexed swaps



In a ZCIIS, at time $T_M = M$ years, Party B pays Party A the fixed amount

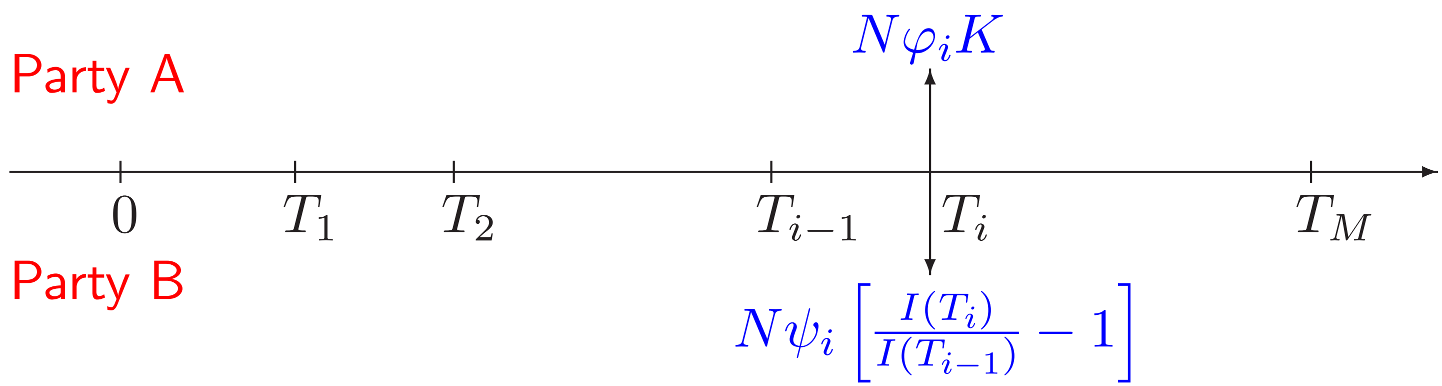
$$N[(1+K)^M - 1],$$

where K and N are, respectively, the contract fixed rate and the contract nominal value.

Party A pays Party B, at the final time T_M, the floating amount

$$N\left[\frac{I(T_M)}{I_0} - 1\right].$$

Year-on-year inflation-indexed swaps



In a YYIIS, at each time T_i, Party B pays Party A the fixed amount

$$N\varphi_i K,$$

while Party A pays Party B the (floating) amount

$$N\psi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1\right],$$

where φ_i and ψ_i are, respectively, the fixed- and floating-leg year fractions for the interval $[T_{i-1}, T_i]$, $T_0 := 0$ and N is again the swap nominal value.

ZCIIS and YYIIS rates

Both ZC and YY swaps are quoted, in the market, in terms of the corresponding fixed rate K.

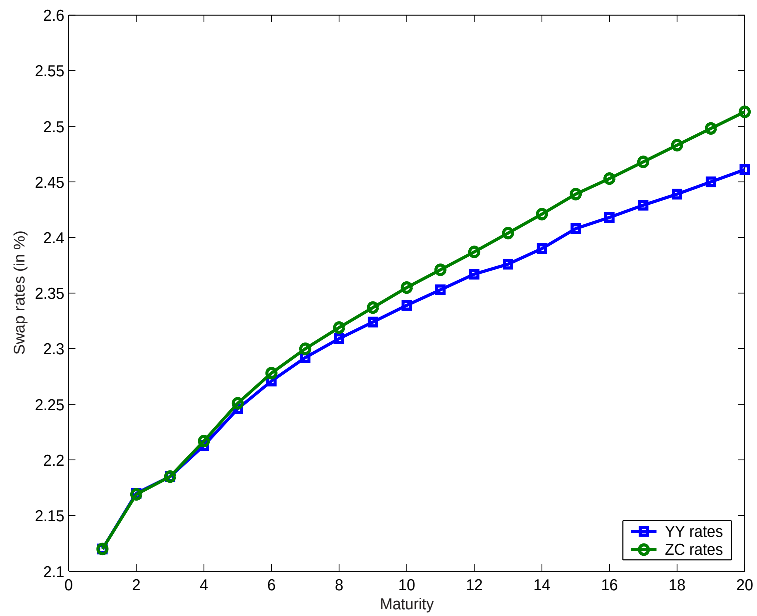


Figure 2: Euro inflation swap rates as of October 7, 2004. The reference CPI is the Euro-zone ex-tobacco index.

Pricing of a ZCIIS

Standard risk-neutral pricing (under the nominal economy) implies that the value at time t, $0 \leq t < T_M$, of the inflation-indexed leg of the ZCIIS is

$$\textbf{ZCIIS}(t, T_M, I_0, N) = N E_n \left\{ e^{-\int_t^{T_M} n(u)\,du} \left[\frac{I(T_M)}{I_0} - 1 \right] \Big| \mathcal{F}_t \right\}.$$

The foreign-currency analogy implies that, for each $t < T$:

	time T		time t
Nominal	$I(T)$	$\cdots$	$E_n\{e^{-\int_t^T n(u)\,du} I(T) \vert \mathcal{F}_t\}$
Real	1		$P_r(t,T)$

$$\Rightarrow \; E_n \left\{ e^{-\int_t^T n(u)\,du} I(T) \Big| \mathcal{F}_t \right\} = I(t) P_r(t,T) \Rightarrow$$

$$\textbf{ZCIIS}(t, T_M, I_0, N) = N \left[\frac{I(t)}{I_0} P_r(t, T_M) - P_n(t, T_M) \right] = N P_n(t, T_M) \left[\frac{\mathcal{I}_M(t)}{I_0} - 1 \right]$$

Pricing of a ZCIIS (cont'd)

The ZCIIS price is therefore model-independent: it is not based on specific assumptions on the interest rates evolution, but simply follows from the absence of arbitrage.

This result is extremely important since it enables us to strip, with no ambiguity, real zero-coupon bond prices (equivalently, forward CPI's) from the quoted prices of zero-coupon inflation-indexed swaps.

The market quotes values of $K = K(T_M)$ for some given maturities T_M.

The ZCIIS corresponding to $(T_M, K(T_M))$ has zero value at time $t = 0$ if and only if

$$N[P_r(0,T_M) - P_n(0,T_M)] = NP_n(0,T_M)[(1+K(T_M))^M - 1]$$

$$\Rightarrow P_r(0,T_M) = P_n(0,T_M)(1+K(T_M))^M \Rightarrow \mathcal{I}_M(0) = I(0)(1+K(T_M))^M$$

Pricing of a YYIIS

The valuation of a YYIIS is less straightforward and, as we shall see, requires the specification of an interest rate model.

The value at time $t < T_i$ of the YYIIS payoff at time T_i is

$$\textbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i E_n \left\{ e^{-\int_t^{T_i} n(u)\,du} \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right] \Big| \mathcal{F}_t \right\},$$

which, assuming $t < T_{i-1}$, can be calculated as

$$N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} E_n \left[e^{-\int_{T_{i-1}}^{T_i} n(u)\,du} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) \Big| \mathcal{F}_{T_{i-1}} \right] \Big| \mathcal{F}_t \right\}$$

$$= N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} [P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)] \Big| \mathcal{F}_t \right\}.$$

This expectation is, in general, model dependent.

Pricing of a YYIIS with a market model

We notice that the forward CPI $\mathcal{I}_i(t) = I(t)P_r(t, T_i)/P_n(t, T_i)$ is a martingale under the nominal T_i-forward measure $Q_n^{T_i}$. Therefore,

$$\begin{aligned}
\textbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i P(t, T_i) E_n^{T_i}\left\{ \frac{I(T_i)}{I(T_{i-1})} - 1 \Big| \mathcal{F}_t \right\} \\
&= N\psi_i P(t, T_i) E_n^{T_i}\left\{ \frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} - 1 \Big| \mathcal{F}_t \right\} \\
&= N\psi_i P(t, T_i) E_n^{T_i}\left\{ \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} - 1 \Big| \mathcal{F}_t \right\}.
\end{aligned}$$

We assume that, under $Q_n^{T_i}$,

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\, dW_i^I(t)$$

and that an analogous evolution holds for $\mathcal{I}_{i-1}$ under $Q_n^{T_{i-1}}$.

Pricing of a YYIIS with a market model (cont'd)

The dynamics of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$ are

$$d\mathcal{I}_{i-1}(t) = -\mathcal{I}_{i-1}(t)\sigma_{I,i-1}\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_iF_n(t;T_{i-1},T_i)}\rho_{I,n,i}\,dt + \sigma_{I,i-1}\mathcal{I}_{i-1}(t)\,dW_{i-1}^I(t),$$

where $\sigma_{I,i-1}$ is a positive constant, W_{i-1}^I is a $Q_n^{T_i}$-Brownian motion with $dW_{i-1}^I(t)\,dW_i^I(t) = \rho_{I,i}\,dt$, and $\rho_{I,n,i}$ is the instantaneous correlation between $\mathcal{I}_{i-1}(\cdot)$ and $F_n(\cdot;T_{i-1},T_i)$.

The evolution of $\mathcal{I}_{i-1}$, under $Q_n^{T_i}$, depends on the nominal rate $F_n(\cdot;T_{i-1},T_i)$.

To avoid unpleasant calculations, we freeze the above drift at its current time-t value, so that $\mathcal{I}_{i-1}(T_{i-1})$ conditional on $\mathcal{F}_t$ is lognormally distributed also under $Q_n^{T_i}$.

Pricing of a YYIIS with a market model (cont'd)

Also the ratio $\mathcal{I}_i(T_{i-1})/\mathcal{I}_{i-1}(T_{i-1})$ conditional on $\mathcal{F}_t$ is lognormally distributed under $Q_n^{T_i}$. This leads to

$$E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})}\Big|\mathcal{F}_t\right\} = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)},$$

where

$$D_i(t) = \sigma_{I,i-1}\left[\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_i F_n(t;T_{i-1},T_i)}\rho_{I,n,i} - \rho_{I,i}\sigma_{I,i} + \sigma_{I,i-1}\right](T_{i-1}-t),$$

so that

$$\textbf{YYIIS}(t,T_{i-1},T_i,\psi_i,N) = N\psi_i P_n(t,T_i)\left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)} - 1\right]$$
$$= N\psi_i P_n(t,T_i)\left[\frac{P_n(t,T_{i-1})P_r(t,T_i)}{P_n(t,T_i)P_r(t,T_{i-1})}\,e^{D_i(t)} - 1\right]$$

Pricing of a YYIIS with a market model (cont'd)

Finally, the value at time t of the inflation-indexed leg of the swap is

$$\textbf{YYIIS}(t, \mathcal{T}, \Psi, N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)})\right]$$

$$+ N \sum_{i=\iota(t)+1}^{M} \psi_i \left[P_n(t, T_{i-1})\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}e^{D_i(t)} - P_n(t, T_i)\right].$$

In particular at $t = 0$,

$$\textbf{YYIIS}(0, \mathcal{T}, \Psi, N) = N \sum_{i=1}^{M} \psi_i P_n(0, T_i) \left[\frac{\mathcal{I}_i(0)}{\mathcal{I}_{i-1}(0)}e^{D_i(0)} - 1\right]$$

$$= N \sum_{i=1}^{M} \psi_i P_n(0, T_i) \left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)}e^{D_i(0)} - 1\right]$$

Inflation-indexed caplets

An Inflation-Indexed Caplet (IIC) is a call option on the inflation rate implied by the CPI index.

Analogously, an Inflation-Indexed Floorlet (IIF) is a put option on the same inflation rate.

In formulas, at time T_i, the IICF payoff is

$$N\psi_i \left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+,$$

where κ is the IICF strike, ψ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, N is the contract nominal value, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

We set $K := 1 + \kappa$.

Inflation-indexed caplets (cont'd)

Standard no-arbitrage pricing theory implies that the value at time $t \leq T_{i-1}$ of the IICF at time T_i is

$$\mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)$$
$$= N\psi_i E_n \left\{ e^{-\int_t^{T_i} n(u)\,du} \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \Big| \mathcal{F}_t \right\}$$
$$= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \Big| \mathcal{F}_t \right\}.$$

The pricing of a IICF is thus similar to that of a forward-start (cliquet) option.

We now derive an analytical formula under the previous market model.

Inflation-indexed caplets (cont'd)

We apply the tower property of conditional expectations to get

$$\begin{aligned}&\mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)\\&= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \frac{E_n^{T_i}\left\{ [\omega(I(T_i) - KI(T_{i-1}))]^+ \,|\mathcal{F}_{T_{i-1}} \right\}}{I(T_{i-1})} \Big| \mathcal{F}_t \right\},\end{aligned}$$

where we assume that $I(T_{i-1}) > 0$.

Sticking to market models, the calculation of the outer expectation depends on whether we model forward rates or the forward CPI's.

We follow, as before, the latter approach, since it allows the derivation of a simpler formula with less input parameters.

Inflation-indexed caplets: a market model (cont'd)

Assuming again that, under $Q_n^{T_i}$,

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\,dW_i^I(t)$$

and remembering that $I(T_i) = \mathcal{I}_i(T_i)$, we have:

$$\begin{aligned}
&E_n^{T_i}\Big\{ \big[\omega(I(T_i) - KI(T_{i-1}))\big]^+ \,|\mathcal{F}_{T_{i-1}}\Big\} \\
&= E_n^{T_i}\Big\{ \big[\omega(\mathcal{I}_i(T_i) - KI(T_{i-1}))\big]^+ \,|\mathcal{F}_{T_{i-1}}\Big\} \\
&= \omega\mathcal{I}_i(T_{i-1})\Phi\left(\omega\frac{\ln\frac{\mathcal{I}_i(T_{i-1})}{KI(T_{i-1})} + \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right) \\
&\quad - \omega KI(T_{i-1})\Phi\left(\omega\frac{\ln\frac{\mathcal{I}_i(T_{i-1})}{KI(T_{i-1})} - \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right).
\end{aligned}$$

Inflation-indexed caplets: a market model (cont'd)

By definition of $\mathcal{I}_{i-1}$, the IICF price thus becomes

$$\omega N \psi_i P_n(t,T_i) E_n^{T_i} \left\{ \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} \Phi\left(\omega \frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{K\mathcal{I}_{i-1}(T_{i-1})} + \frac{1}{2}\sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}} \right) \right.$$

$$\left. -K\Phi\left(\omega \frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{K\mathcal{I}_{i-1}(T_{i-1})} - \frac{1}{2}\sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}} \right) \Big| \mathcal{F}_t \right\}.$$

Remembering the dynamics of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$, and freezing again the drift at its time-t value, we have that under $Q_n^{T_i}$:

$$\ln \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} | \mathcal{F}_t \sim \mathcal{N}\left(\ln \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} + D_i(t) - \frac{1}{2} V_i^2(t), V_i^2(t) \right).$$

where $V_i^2(t) := (\sigma_{I,i-1}^2 + \sigma_{I,i}^2 - 2\rho_{I,i}\sigma_{I,i-1}\sigma_{I,i})(T_{i-1} - t)$. Therefore,

Inflation-indexed caplets: a market model (cont'd)

$$\mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)$$

$$= \omega N \psi_i P_n(t, T_i) \left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} e^{D_i(t)} \Phi\left(\omega \frac{\ln \frac{\mathcal{I}_i(t)}{K\mathcal{I}_{i-1}(t)} + D_i(t) + \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right.$$

$$\left. -K\Phi\left(\omega \frac{\ln \frac{\mathcal{I}_i(t)}{K\mathcal{I}_{i-1}(t)} + D_i(t) - \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right],$$

where $\mathcal{V}_i(t) := \sqrt{V_i^2(t) + \sigma_{I,i}^2(T_i - T_{i-1})}$, and $\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} = \frac{1+\tau_i F_n(t;T_{i-1},T_i)}{1+\tau_i F_r(t;T_{i-1},T_i)}$.

This price depends on the volatilities of the two forward inflation indices and their correlation, the volatility of nominal forward rates, and the correlations between forward inflation indices and nominal forward rates.

Inflation-indexed caps

An inflation-indexed cap is a stream of inflation-indexed caplets.

An analogous definition holds for an inflation-indexed floor.

Given the set of dates $T_0, T_1, \ldots, T_M$, with $T_0 = 0$, a IICapFloor pays off, at each time T_i, $1, \ldots, M$,

$$N\psi_i \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right) \right]^+,$$

where κ is the IICapFloor strike, ψ_i are the contract year fractions for the intervals $[T_{i-1}, T_i]$, $1, \ldots, M$, N is the contract nominal value, $\omega = 1$ for a cap and $\omega = -1$ for a floor.

We again set $K := 1 + \kappa$, $\mathcal{T} := \{T_1, \ldots, T_M\}$ and $\Psi := \{\psi_1, \ldots, \psi_M\}$.

Inflation-indexed caps (cont'd)

From the caplet pricing formula we get:

$$\textbf{IICapFloor}(0, \mathcal{T}, \Psi, K, N, \omega) = \omega N \sum_{i=1}^{M} \psi_i P_n(0, T_i)$$

$$\cdot \left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)} e^{D_i(0)} \Phi \left(\omega \frac{\ln \frac{1+\tau_i F_n(0;T_{i-1},T_i)}{K[1+\tau_i F_r(0;T_{i-1},T_i)]} + D_i(0) + \frac{1}{2}\mathcal{V}_i^2(0)}{\mathcal{V}_i(0)} \right) \right.$$

$$\left. -K\Phi \left(\omega \frac{\ln \frac{1+\tau_i F_n(0;T_{i-1},T_i)}{K[1+\tau_i F_r(0;T_{i-1},T_i)]} + D_i(0) - \frac{1}{2}\mathcal{V}_i^2(0)}{\mathcal{V}_i(0)} \right) \right].$$

This price depends on the volatilities of forward inflation indices and their correlations, the volatilities of nominal forward rates, and the instantaneous correlations between forward inflation indices and nominal forward rates.

Calibration to market data

We consider an example of calibration to Euro market data as of October 7, 2004.

We calibrate the Jarrow and Yildirim (JY) model, the LIBOR market model (MM1) and our market model (MM2) to inflation-indexed swaps.

We use the zero-coupon rates to strip the current real discount factors for the relevant maturities.

Some model parameters are fitted to ATM (nominal) caps volatilities.

The model parameters that best fit the given set of market data are found by minimizing the square absolute difference between model and market YYIIS fixed rates.

To avoid over-parametrization, we introduce some constraints.

Calibration to market data: results

Maturity	Market	JY	MM1	MM2
1	2.120	2.120	2.120	2.120
2	2.170	2.169	2.168	2.168
3	2.185	2.186	2.186	2.184
4	2.213	2.217	2.218	2.215
5	2.246	2.250	2.250	2.247
6	2.271	2.276	2.275	2.272
7	2.292	2.296	2.295	2.293
8	2.309	2.314	2.312	2.310
9	2.324	2.324	2.322	2.320
10	2.339	2.345	2.343	2.341
11	2.353	2.358	2.356	2.355
12	2.367	2.371	2.369	2.369
13	2.383	2.385	2.383	2.383
14	2.390	2.397	2.396	2.396
15	2.408	2.410	2.410	2.410
16	2.418	2.420	2.421	2.421
17	2.429	2.430	2.431	2.432
18	2.439	2.439	2.442	2.443
19	2.450	2.448	2.453	2.454
20	2.461	2.457	2.463	2.465

Calibration to market data: results (cont'd)

The three models are equivalent in terms of calibration to market YYIIS rates. They can however imply quite different prices for zero-strike floors:

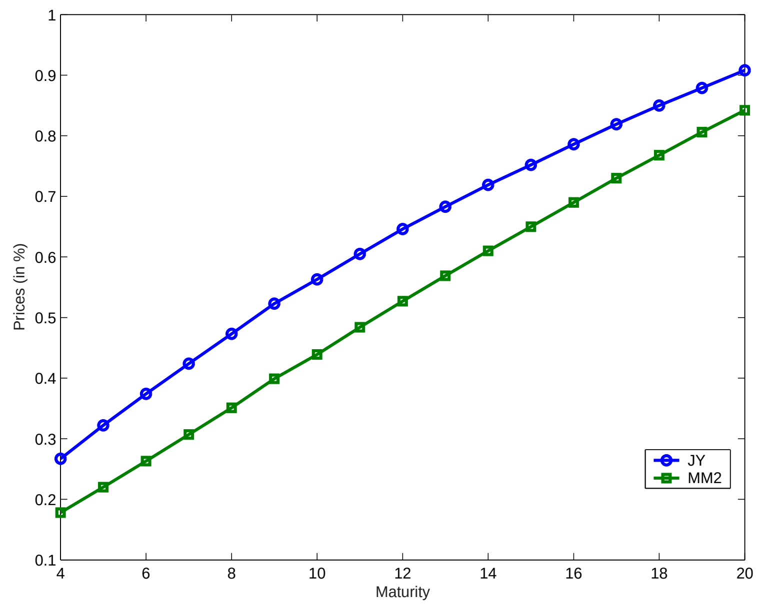


Figure 3: Comparison of zero-strike floors prices implied by the JY model and MM2, for different maturities.

Calibration to swaps and zero-strike floors

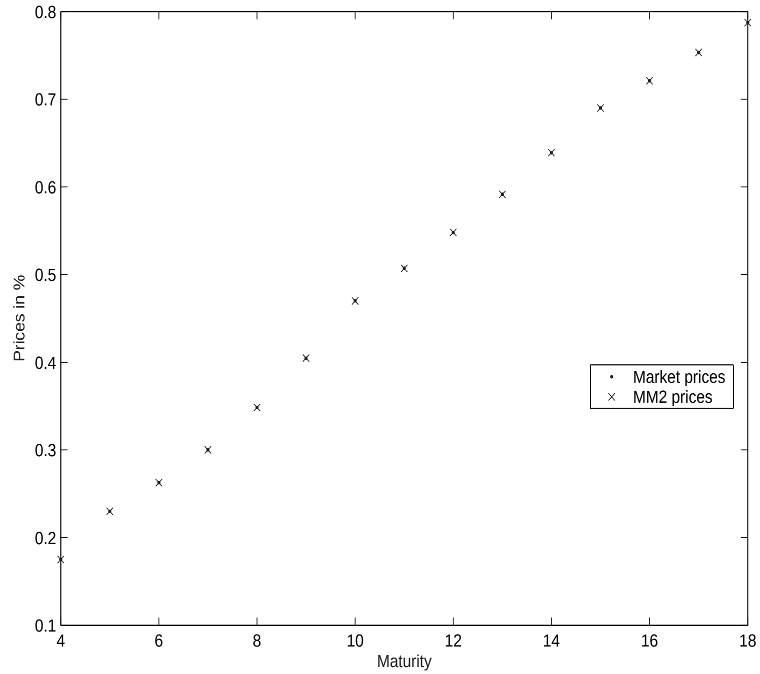


Figure 4: Zero-strike floor prices implied by MM2, after calibration to market quotes (both swaps and floors) as of October 7, 2004.

Including the smile: calibration to one-year caplets

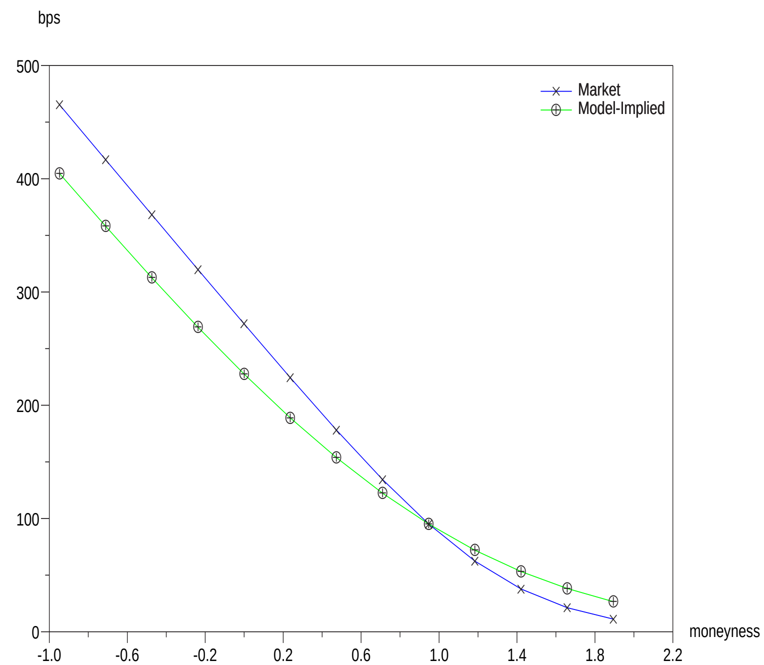


Figure 5: Market prices versus model prices with ATM implied volatility for one-year caplets. Moneyness is defined as: $K/(\mathcal{I}_1(0)/I(0) - 1)$.

A market model with stochastic volatility

We drop the subscript n and assume that, under a reference measure $\mathbb{Q}$:

- Nominal rates F_i are lognormally distributed with constant volatilities;
- Forward CPI's $\mathcal{I}_i$ follow Heston-like dynamics with a common volatility process $V(t)$:

$$\begin{aligned} dF_i(t)/F_i(t) =& (\ldots)\, dt + \sigma_i^F \, dZ_i^{\mathbb{Q},F} \\ d\mathcal{I}_i(t)/\mathcal{I}_i(t) =& (\ldots)\, dt + \sigma_i^I \sqrt{V(t)}\, dZ_i^{\mathbb{Q},I} \\ dV(t) =& \alpha(\theta - V(t))\, dt + \epsilon \sqrt{V(t)}\, dW^{\mathbb{Q}}, \quad V(0) = V_0, \end{aligned}$$

  where σ_i^I, σ_i^F, α, θ, ϵ and V_0 are positive constants, and $2\alpha\theta > \epsilon$ to ensure positiveness of V.

We allow for correlations between Brownian motions $Z_i^{\mathbb{Q},F}, Z_i^{\mathbb{Q},I}, W^{\mathbb{Q}}$.

A market model with stochastic volatility (cont'd)

We take $\mathbb{Q} = \mathbb{Q}^0$, where $\mathbb{Q}^0$ is the spot LIBOR measure corresponding to the numeraire

$$B_d(t) = P(t, \beta(t)) \prod_{l=1}^{\beta(t)} [1 + \tau_l F_l(t)], \quad \beta(t) = T_j \text{ if } T_{j-1} < t \leq T_j.$$

By definition of B_d and the change-of-measure technique, we have, under $\mathbb{Q}^0$,

$$dF_i(t)/F_i(t) = \sigma_i^F \left[- \sum_{l=\beta(t)+1}^{i} \sigma_l^F \rho_{i,l}^F \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} dt + dZ_i^{0,F}(t) \right]$$

$$d\mathcal{I}_i(t)/\mathcal{I}_i(t) = \sqrt{V(t)}\, \sigma_i^I \left[- \sum_{l=\beta(t)+1}^{i} \sigma_l^F \rho_{l,i}^{F,I} \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} dt + dZ_i^{0,I}(t) \right]$$

$$dZ_i^{0,F} dZ_l^{0,F}(t) = \rho_{i,l}^F dt, \quad dZ_i^{0,I} dZ_l^{0,F}(t) = \rho_{l,i}^{F,I} dt$$

The pricing of caplets

The price at time $t \leq T_{j-1}$ of the j-th caplet, is, under the measure Q^{T_j},

$$\textbf{IICplt}_j(t,K) = P(t,T_j)E_t^{T_j}\left(\frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})} - K\right)^+$$
$$= P(t,T_j)\int_{-\infty}^{+\infty}(e^s - e^k)^+ q_t^j(s)\,ds$$

where $k = \ln(K)$ and $q_t^j(s)ds = Q^{T_j}\left\{\ln\left[\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1})\right] \in [s, s+ds] | \mathcal{F}_t\right\}.$

Remark. Instead of having a payoff depending on a single asset $S(t)$, as it is for standard or cliquet options (paying off $[S(T_j)/S(T_{j-1}) - K]^+$ in T_j), here the payoff depends on the ratio between two different assets at two different times.

The pricing of caplets (cont'd)

Following Carr and Madan (1999), we rewrite the caplet price in term of its (renormalized) Fourier transform:

$$\mathbf{IICplt}_j(t, \mathrm{e}^k) = P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-isk}\psi_t^j(\eta, s)ds$$

$$= P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{\pi}\,\mathrm{Re}\int_{0}^{+\infty} \mathrm{e}^{-isk}\psi_t^j(\eta, s)ds$$

$$\psi_t^j(\eta, u) = \frac{\phi_t^j(u - (\eta+1)i)}{(\eta + iu)(\eta + 1 + iu)}$$

where the only unknown is the conditional characteristic function $\phi_t^j(\cdot)$ of $\ln\left(\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1})\right)$, and where $\eta \in \mathbb{R}^+$ is used to ensure L^2-integrability when $k \to -\infty$.

Derivation of the characteristic function

Our objective is now to find an explicit formula for ϕ_t^j.

To this end, we derive the dynamics under the pricing measure Q^{T_j}:

$$d\mathcal{I}_j(t)/\mathcal{I}_j(t) = \sqrt{V(t)}\,\sigma_j^I\,dZ_j^I(t)$$

$$d\mathcal{I}_{j-1}(t)/\mathcal{I}_{j-1}(t) = \sqrt{V(t)}\,\sigma_{j-1}^I\left[-\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\,\sigma_j^F\,\rho_{j,j-1}^{F,I}dt + dZ_{j-1}^I(t)\right]$$

$$dV(t) = \left[\alpha\theta - \epsilon\sqrt{V(t)}\sum_{l=\beta(t)+1}^{j}\frac{\tau_l F_l(t)}{1+\tau_l F_l(t)}\sigma_l^F\,\rho_l^{F,V} - \alpha V(t)\right]dt + \epsilon\sqrt{V(t)}\,dW(t)$$

where $dZ_l^F(t)\,dW(t) = \rho_l^{F,V}\,dt$, for each l.

Derivation of the characteristic function (cont'd)

Setting $Y_j(t) := \ln \mathcal{I}_j(T_j)$, we recall that, by definition of characteristic function and the Markov property:

$$\phi_t^j(u) = E_t^{T_j}\left[\mathrm{e}^{iu \ln \frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})}}\right] = H\left(V(t), Y_j(t), Y_{j-1}(t), F_1(t), \ldots, F_j(t)\right).$$

Applying the Feynman-Kač theorem, H can then be found by solving a related PDE.

Remark. In the general case, due to the unpleasant presence of drift terms of type $\sqrt{V(t)}F_l(t)/(1+\tau_l F_l(t))$, there are no a priori reasons for the PDE to be explicitly solvable. In the following, we thus investigate a particular case allowing for an explicit solution.

Derivation of the characteristic function (cont'd)

We assume that, for each $i, l = 1, \ldots, M$:

$$\rho_{i,l}^{F,I} = \rho_i^{F,V} = 0$$

We allow, however, for non-zero correlations $\rho_{j,l}^I = dZ_j^I dZ_l^I/dt$ (between different forward CPI's) and $\rho_i^{I,V} = dZ_i^I dW/dt$ (between forward CPI's and the volatility).

Setting $X_j(t) := Y_j(t) - Y_{j-1}(t)$, we then have, under Q^{T_j},

$$dY_j(t) = -\frac{1}{2}V(t)(\sigma_j^I)^2\,dt + \sqrt{V(t)}\,\sigma_j^I(t)\,dZ_j^I(t)$$

$$dX_j(t) = \frac{V(t)}{2}((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2)\,dt + \sqrt{V(t)}(\sigma_j^I\,dZ_j^I(t) - \sigma_{j-1}^I\,dZ_{j-1}^I(t))$$

$$dV(t) = [\alpha\theta - \alpha V(t)]\;dt + \epsilon\sqrt{V(t)}\,dW(t)$$

Derivation of the characteristic function (cont'd)

To make ϕ_t^j explicit, we write

$$\phi_t^j(u) = E_t^{T_j}\left[e^{iu\left(Y_j(T_j)-Y_{j-1}(T_{j-1})\right)}\right] = E_t^{T_j}\left[e^{-iuY_{j-1}(T_{j-1})}E_{T_{j-1}}^{T_j}\left(e^{iuY_j(T_j)}\right)\right]$$

Noting that $E_{T_{j-1}}^{T_j}\left(e^{iuY_j(T_j)}\right)$ is the characteristic function of $\ln \mathcal{I}_j(T_j)$ conditional on $\mathcal{F}_{T_{j-1}}$, solving a Heston-like PDE, we have that

$$E_{T_{j-1}}^{T_j}\left[e^{iuY_j(T_j)}\right] = \exp\left\{A_Y(\bar{\tau}_j, u) + B_Y(\bar{\tau}_j, u)V(T_{j-1}) + iuY_j(T_{j-1})\right\}$$

where $\bar{\tau}_j := T_j - T_{j-1}$ and A_Y and B_Y are deterministic complex functions.

Consequently,

$$\phi_t^j(u) = e^{A_Y(\bar{\tau}_j,u)}E_t^{T_j}\left[e^{iuX_j(T_{j-1})+B_Y(\bar{\tau}_j,u)V(T_{j-1})}\right]$$

Derivation of the characteristic function (cont'd)

The last conditional expectation is nothing but the characteristic function of the couple $(X_j(T_{j-1}), V(T_{j-1}))$ evaluated at point $(u, -iB_Y(\bar{\tau}_j, u))$.

By again solving a PDE of Heston's type with suitable boundary conditions, we obtain

$$\phi_t^j(u) = \exp\{A_Y(\bar{\tau}_j, u) + A_X(T_{j-1} - t, u) + B_X(T_{j-1} - t, u)V(t) + iuX_j(t)\}$$

where A_X and B_X are other deterministic complex functions.

The II caplet price is finally calculated by numerical integration:

$$\textbf{IICplt}_j(t, \mathrm{e}^k) = P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{\pi}\,\mathrm{Re}\int_0^{+\infty} \mathrm{e}^{-isk}\frac{\phi_t^j(s-(\eta+1)i)}{(\eta+is)(\eta+1+is)}\,ds$$

Derivation of the characteristic function (cont'd)

The coefficients A_Y and B_Y:

$$B_Y(s,u) = \frac{\gamma - b}{2a}\left[\frac{1 - \mathrm{e}^{\gamma s}}{1 - \frac{b-\gamma}{b+\gamma}\mathrm{e}^{\gamma s}}\right]$$

$$A_Y(s,u) = \frac{\alpha\theta(\gamma - b)}{2a}s - \frac{\alpha\theta}{a}\ln\left[\frac{1 - \frac{b-\gamma}{b+\gamma}\mathrm{e}^{\gamma s}}{1 - \frac{b-\gamma}{b+\gamma}}\right]$$

where

$$a := \epsilon^2/2, \quad c := -iu(\sigma_j^I)^2/2 - (\sigma_j^I)^2 u^2/2,$$

$$b := iu\sigma_j^I \epsilon \rho_j^{I,V} - \alpha, \quad \gamma := \sqrt{b^2 - 4ac}.$$

Derivation of the characteristic function (cont'd)

The coefficients A_X and B_X:

$$B_X(\tau,u) = B_Y(\tau_j,u) + \frac{\bar\gamma - \bar b - 2\bar a B_Y(\tau_j,u)}{2\bar a}\left[\frac{1-\mathrm{e}^{\bar\gamma\tau}}{1-\frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}\mathrm{e}^{\bar\gamma\tau}}\right]$$

$$A_X(\tau,u) = \frac{\alpha\theta(\bar\gamma-\bar b)}{2\bar a}\tau - \frac{\alpha\theta}{\bar a}\ln\left[\frac{1-\frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}\mathrm{e}^{\bar\gamma\tau}}{1-\frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}}\right]$$

where

$$\bar a := \epsilon^2/2, \qquad \bar b := iu\epsilon(\sigma_j^I\rho_j^{I,V} - \sigma_{j-1}^I\rho_{j-1}^{I,V}) - \alpha$$

$$\bar c := iu\big((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2\big)/2 - \big((\sigma_{j-1}^I)^2 + (\sigma_j^I)^2 - 2\sigma_j^I\sigma_{j-1}^I\rho_{j,j-1}^I\big)u^2/2$$

$$\bar\gamma := \sqrt{\bar b^2 - 4\bar a\bar c}$$

Solving the general case with non-zero correlations

As in a LIBOR market model, we can deal with forward-rates-dependent terms in the drifts by resorting to a freezing technique.

The drift terms that involve forward rates are

$$D_l(t) := \sqrt{V(t)}\frac{F_l(t)}{1+\tau_l F_l(t)}$$

and depend on the volatility, too.

A first way to freeze these terms consists in setting

$$D_l(t) \approx D_l(0),$$

thus changing the asymptotic volatility value from θ to

$$\tilde{\theta} := \theta - \frac{\epsilon}{\alpha}\sum_{l=1}^{j} D_l(0)\tau_l \sigma_l^F \rho_l^{F,V}$$

Solving the general case with non-zero correlations (cont'd)

The first freezing leads to the following (approximated) SDEs for X_j and V:

$$\begin{aligned} dX_j(t) \approx & \left[\frac{V(t)}{2}\big((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2\big) + D_j(0)\tau_j\sigma_{j-1}^I\sigma_j^F\rho_{j,j-1}^{F,I}\right] dt \\ & + \sqrt{V(t)}[\sigma_j^I\, dZ_j^I(t) - \sigma_{j-1}^I\, dZ_{j-1}^I(t)] \\ dV(t) \approx & \alpha(\tilde{\theta} - V(t))\, dt + \epsilon\sqrt{V(t)}\, dW(t). \end{aligned}$$

The dynamics of X_j here differs from the previous one for a constant drift term.

Such a term, however, is innocuous and the relevant characteristic functions can still be calculated explicitly.

Solving the general case with non-zero correlations (cont'd)

A second possibility for a tractable approximation is to set

$$D_l(t) \approx \frac{F_l(t)}{1+\tau_l F_l(t)} \frac{V(t)}{\sqrt{V(t)}} \approx \frac{F_l(0)}{1+\tau_l F_l(0)} \frac{V(t)}{\sqrt{V(0)}} = D_l(0) \frac{V(t)}{V(0)},$$

where the freezing is done with the purpose of producing a linear term in $V(t)$. This leads to the following (approximated) SDEs for X_j and V:

$$\begin{aligned} dX_j(t) \approx & V(t) \left[\frac{1}{2} \big((\sigma^I_{j-1})^2 - (\sigma^I_j)^2 \big) + \frac{D_j(0)}{V(0)} \tau_j \sigma^I_{j-1} \sigma^F_j \rho^{F,I}_{j,j-1} \right] dt \\ & + \sqrt{V(t)} [\sigma^I_j \, dZ^I_j(t) - \sigma^I_{j-1} \, dZ^I_{j-1}(t)] \\ dV(t) \approx & \bar{\alpha}(\bar{\theta} - V(t)) \, dt + \epsilon \sqrt{V(t)} \, dW(t), \end{aligned}$$

Solving the general case with non-zero correlations (cont'd)

where

$$\bar{\alpha} := \alpha + \frac{\epsilon}{V(0)} \sum_{l=1}^{j} D_l(0)\tau_l \sigma_l^F \rho_l^{F,V}$$

$$\bar{\theta} := \alpha\theta/\bar{\alpha}$$

Also in this second case, the relevant characteristic functions can be calculated explicitly.

Remark. For the approximated processes $V(t)$ to be meaningful, we must require $\tilde{\theta} > 0$ and $\bar{\alpha} > 0$. Moreover, for the origin to be inaccessible, conditions $2\tilde{\theta}\alpha > \epsilon$ and $2\bar{\theta}\bar{\alpha} > \epsilon$ must be imposed, with the latter that is automatically satisfied since $2\theta\alpha > \epsilon$ by assumption.

Solving the general case with non-zero correlations (cont'd)

To test the goodness of the above approximations, we should perform a Monte Carlo simulation and compare the analytical caplet prices coming from the approximations with the corresponding Monte Carlo price windows calculated numerically.

This procedure, however, is rather cumbersome, since it also requires the joint simulation of all forward rates.

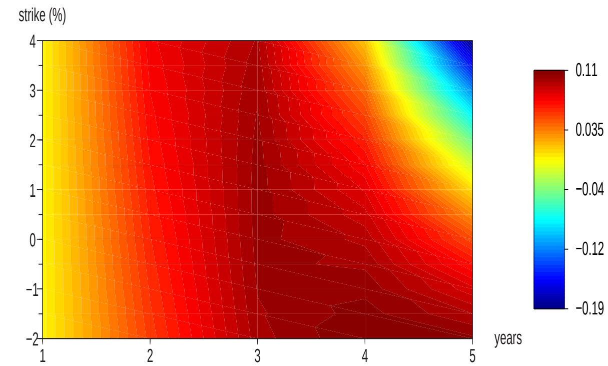
A much quicker test can be conducted by simply comparing the caplet prices implied by the two approximations.

The two approximations, moreover, can be compared with the exact price obtained in the "zero-correlation case".

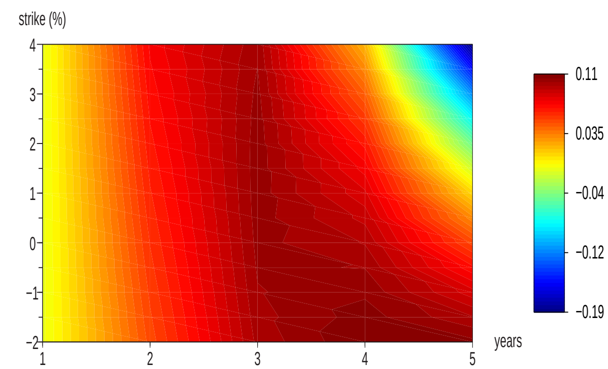
Solving the general case with non-zero correlations (cont'd)

We test the goodness of the above approximations on EUR data as of October 7, 2004.

We set: $\rho_i^{F,V} = \rho_{i,l}^{F,I} = \rho_i^{I,V} = -0.2$,
$\rho_{i,i-1}^{I} = 1 - 1.5\mathrm{e}^{-0.08(i-2)}$,
$\alpha = 0.2$, $\theta = 0.001$, $V(0) = 0.001$,
$\epsilon = 0.01$ and
$\sigma_i^I = 1 - 0.05(i-1)$, for $i, l = 1, \ldots, 5$.



We plot the percentage differences between "exact" and "freezing-based" caplets prices.
The first freezing case is shown on top.



Calibration to a matrix of II caps/floors

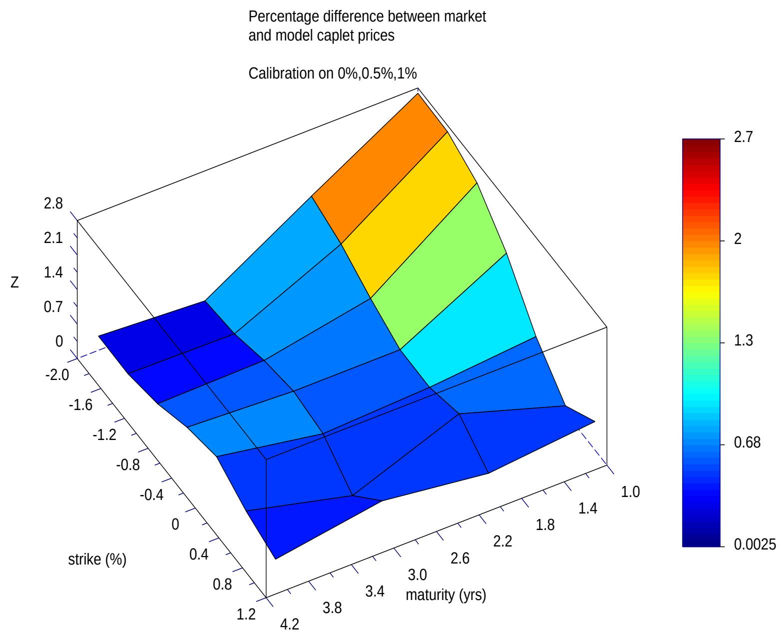


Figure 6: Absolute percentage differences between calibrated prices and market prices (market quotes as of October 7, 2004).

Conclusions

We have reviewed the pricing of II swaps and caps under a (lognormal) market model for forward CPI's.

We have then extended the model by introducing a stochastic volatility as in Heston (1993).

Closed-form formulae for option prices have been obtained when the correlations with forward rates are zero.

In the non-zero correlation case, we have derived efficient price approximations based on classical drift-freezing techniques.

We have also presented an example of calibration to II caplet prices.

Assuming a unique volatility process for all forward CPI's seems too restrictive if we aim at calibrating many maturities simultaneously.

Possible future developments

One may consider a different volatility process for each forward CPI, introducing new volatility and correlation parameters:

$$d\mathcal{I}_j(t)/\mathcal{I}_j(t) = \sqrt{V_j(t)}\,\sigma_j^I\,dZ_j^I(t)$$

$$d\mathcal{I}_{j-1}(t)/\mathcal{I}_{j-1}(t) = \sqrt{V_{j-1}(t)}\,\sigma_{j-1}^I\left[-\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\,\sigma_j^F\,\rho_{j,j-1}^{F,I}dt + dZ_{j-1}^I(t)\right]$$

$$dV_k(t) = \left[\alpha_k\theta_k - \epsilon_k\sqrt{V_k(t)}\sum_{l=\beta(t)+1}^{j}\frac{\tau_l F_l(t)}{1+\tau_l F_l(t)}\sigma_l^F\,\rho_l^{F,V_k} - \alpha_k V_k(t)\right]dt$$
$$+\,\epsilon_k\sqrt{V_k(t)}\,dW_k(t), \quad k = j-1, j$$

However, the calculation of caplet prices is not straightforward, since the pricing of each caplet (but the first one) involves an extra volatility process.