Condition and conquer

Pricing of baskets, Asians and swaptions in general models

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Contents

Problem definition

Consider the following arithmetic average:

$$A(T) = \sum_{i,j=1}^{N,M} w_{ij} S_j(t_i)$$

where $t_1 \leq \ldots \leq t_N = T$ and all weights sum to 1. In this presentation we will consider the problem of pricing European calls on A(T), i.e. options paying the following amount at time T:

$$\left(A(T) - K\right)^+$$

Problem definition (2)

Pure basket:

$$A(T) = \sum_{j=1}^{M} w_j S_j(T)$$

Pure Asian:

$$A(T) = \sum_{i=1}^{N} w_i S(t_i)$$

Problem definition (3)

In the interest rate market the terminology is less straightforward, so we first treat a swap. With a receiver swap we pay floating, and receive fixed:

- Pay $\alpha_i L_i(T_i)$ at T_{i+1}, $i = 1, \ldots, N$
- Receive $\alpha_i K$ at T_{i+1}, $i = 1, \ldots, N$

Note that: $$L_i(T_i) = \frac{1}{\alpha_i}\left(\frac{1}{P(T_i, T_{i+1})} - 1\right)$$

and $P(t,T)$ is the time t price of a zero-coupon bond maturing at time T.

Problem definition (4)

Time T ($\geq T_1$) value of a receiver swap:

$$K\sum_{i=1}^{N}\alpha_i P(T,T_{i+1}) + P(T,T_{N+1}) - P(T,T_1)$$

Usually the swaption maturity (T) coincides with the first reset date of the underlying swap (T_1), so the payoff of a receiver swaption is:

$$\left(\sum_{i=1}^{N} c_i P(T,T_{i+1}) - 1\right)^+$$

where $c_i = \alpha_i K$ for $i < N$ and $c_N = 1+\alpha_N K$. Clearly, a swaption is also an option on an arithmetic average.

Problem definition (5)

Derivatives on arithmetic averages

Baskets
Several underlyings
Several markets
Several processes
Same time

Swaptions
Several underlyings
One market
One/several processes
Same time

Asians
One underlying
One market
One/several processes
Different times

The Black-Scholes case

In the Black-Scholes world:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)\,dt + \sigma_i(t)\,dW_i(t)$$

where $dW_i(t)\,dW_j(t) = \rho_{ij}(t)\,dt$.

- Closed-form solutions not available for options on discretely sampled averages;
- Numerical schemes (PDEs, numerical integration, Laplace/Fourier inversion) can be used, but are too cumbersome when no. of factors is high;

The Black-Scholes case (2)

Conditioning approaches (Curran, Rogers & Shi) use a conditioning variable $\Lambda(T)$ for which we know that:

$$\Lambda(T) \geq K \Rightarrow A(T) \geq K$$

$$\text{e.g. } \Lambda(T) = G(T) = \prod_{i,j=1}^{N,M} S_j(t_i)^{w_{ij}}$$

as the forward price (under the T-forward measure) can then be decomposed as:

$$\mathbb{E}\left[\left(A(T)-K\right)^+\right]$$

$$= \mathbb{E}\left[\left(A(T)-K\right)^+ 1_{[\Lambda(T)<K]}\right] + \mathbb{E}\left[\left(A(T)-K\right)1_{[\Lambda(T)\geq K]}\right]$$

Has to be approximated | Closed-form

The Black-Scholes case (3)

Approximative part: $\mathbb{E}\left[\left(A(T)-K\right)^{+} 1_{[\Lambda(T)<K]}\right]$

One of the most successful approximations is the Curran/Rogers and Shi lower bound, which uses Jensen's inequality:

$$\begin{aligned}
&\mathbb{E}\left[\left(A(T)-K\right)^{+} 1_{[\Lambda(T)<K]}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(A(T)-K\right)^{+} 1_{[\Lambda(T)<K]} \mid \Lambda(T)\right]\right] \\
&\geq \mathbb{E}\left[\left(\mathbb{E}[A(T) 1_{[\Lambda(T)<K]} \mid \Lambda(T)] - K\right)^{+}\right]
\end{aligned}$$

The Black-Scholes case (4)

Lessons from Lord [2005]:

- Closed-form expression for lower bound for any choice of correlation structure, i.e. also for baskets;
- Curran's "naïve" approximation diverges if $K \to \infty$, in the sense that:

$$\lim_{K \to \infty} \mathbb{E}\left[\left(A(T) - K\right)^{+} 1_{[\Lambda(T) < K]}\right] = \infty$$

This is very noticeable for large vols/maturities.

The Black-Scholes case (5)

Lessons from Lord [2005] (cont'd):

- The following approximation:

$$\mathbb{E}\left[\left(A(T)-K\right)^{+}1_{[\Lambda(T)<K]}\right] \approx \mathbb{E}\left[\left(\widetilde{A}(T)-K\right)^{+}1_{[\Lambda(T)<K]}\right]$$

is sharply bounded from above and below, if:

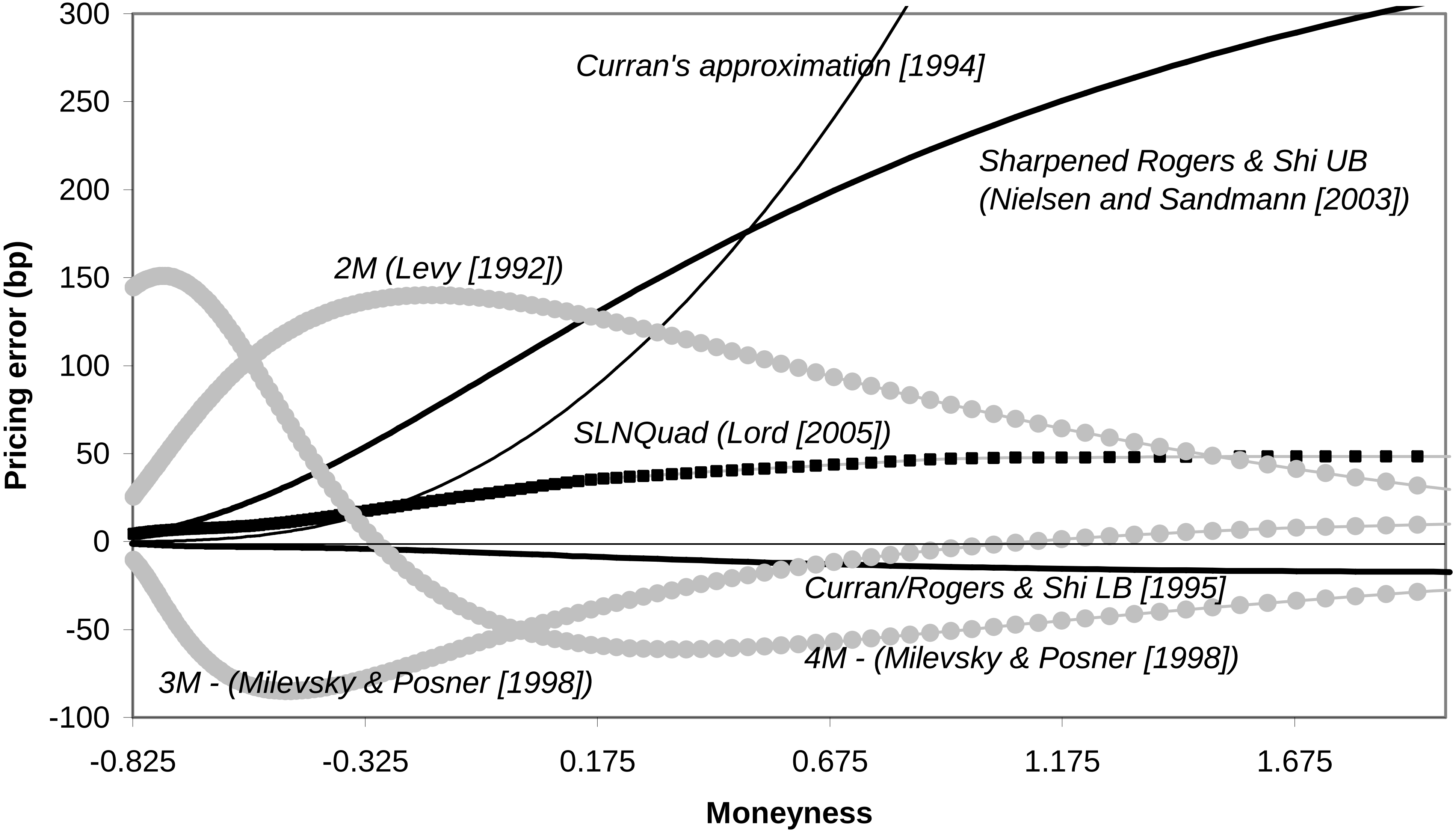
$$\mathbb{E}\left[\widetilde{A}(T) \mid \Lambda(T)=\lambda\right] = \mathbb{E}\left[A(T) \mid \Lambda(T)=\lambda\right]$$

$$\mathrm{Var}\left(\widetilde{A}(T) \mid \Lambda(T)=\lambda\right) \leq \mathrm{Var}\left(A(T) \mid \Lambda(T)=\lambda\right)$$

The resulting approximations are called partially exact and bounded (PEB).

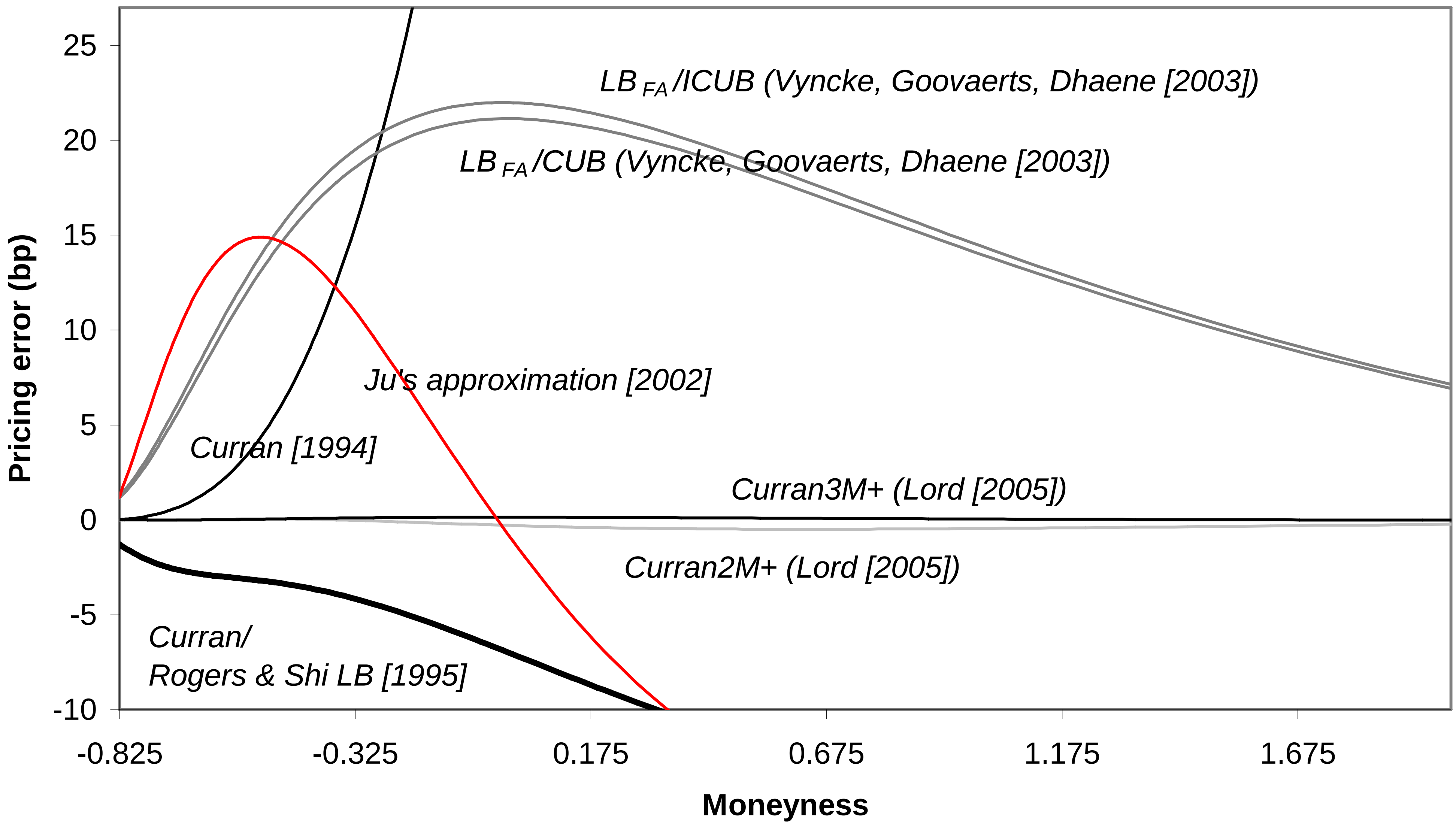
The Black-Scholes case (6)

30y Eurasian call, yearly averaging, r = 5%, σ = 25%



The Black-Scholes case (7)

30y Eurasian call, yearly averaging, r = 5%, σ = 25%



The Black-Scholes case (8)

Lessons from the lognormal/Black-Scholes case:

- Unconditional moment matching is not accurate enough for practical purposes;
- Conditional moment-matching works best;
- ~~Conditional moment-matching is facilitated greatly by analytically known conditional expectations and variances in the multinormal distribution;~~

Will not be the case in general models

Pricing with characteristic functions

For many models the density is not known in closed-form, although the T-forward characteristic function is:

$$\phi(u) = \mathbb{E}\left[\exp\left(iu^{T}\mathbf{X}(T)\right)\right]$$

for $u \in \mathbb{R}^M$, $\mathbf{X}^T = (X_1, \ldots, X_M) = (\ln S_1, \ldots, \ln S_M)$. E.g.:

- *AJD models (Duffie, Pan and Singleton):* Black-Scholes, Merton, Heston, Bates, Hull-White, Cox-Ingersoll-Ross, Dai and Singleton;
- *LQJD models (Gaspar, Cheng and Scaillet):* Stein-Stein, Schöbel-Zhu, Longstaff, Jamshidian, Brown-Schaefer, Beaglehole-Tenney;
- *Exponential Lévy models:* Normal Inverse Gaussian (NIG), Variance Gamma (VG), Carr-Géman-Madan-Yor (CGMY), Barndorff-Nielsen-Shepard (BN-S), time-changed Lévy models, regime-switching Lévy models (Chourdakis [2005]);

Pricing with characteristic functions (2)

Pricing in alternative models has been much facilitated due to the work of Carr and Madan [1999]. For our purposes, consider the following *powerdigital*:

$$\exp\left(ak + \mathbf{b}^T \mathbf{X}(T)\right) 1_{[c + \mathbf{d}^T \mathbf{X}(T) \geq k]}$$

where k = ln K. Its forward price, C(k,t), satisfies:

$$C(k,t) = \tfrac{1}{\pi} \int_0^\infty \mathrm{Re}\left[e^{-k(\alpha + iv)} \psi(v)\right] dv$$

$$\psi(v) = \frac{\exp\left(c(a + \alpha + iv)\right) \cdot \phi\left(\mathbf{d}v - i(\mathbf{b} + (a + \alpha)\mathbf{d})\right)}{a + \alpha + iv}$$

which can be calculated using a numerical integration.

Basket options in general models

Derivatives on arithmetic averages

Baskets
Several underlyings
Several markets
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Same time

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Basket options in general models (2)

Now consider the following arithmetic average:

$$A(T) = \sum_{j=1}^{M} w_j \exp\left(\mathbf{b}_j^T \mathbf{X}(T)\right)$$

where $\mathbf{b_1} = (1,\ldots,0)^T$, …, $\mathbf{b}_M = (0,\ldots,0,1)^T$ if we model the stock prices directly. Conveniently, G(T) is still exponentially affine in the state variables:

$$G(T) = \exp\left(\sum_{j=1}^{M} w_j \mathbf{b}_j^T \mathbf{X}(T)\right)$$

so that $\ln G(T) = \sum_{j=1}^{M} w_j \mathbf{b}_j^T \mathbf{X}(T) \geq k$ implies $A(T) \geq K$.

Basket options in general models (3)

If $\Lambda(T) = c + \mathbf{d}^T\mathbf{X}(T)$ (think of $\Lambda(T)$ as e.g. ln $G(T)$):

$$\mathbb{E}\left[\left(A(T) - K\right)1_{[\Lambda(T)\geq\lambda]}\right]$$

$$= \mathbb{E}\left[\left(\sum_{j=1}^{M} w_j \exp\left(\mathbf{b}_j^T \mathbf{X}(T)\right) - \exp(k)\right)1_{[c+\mathbf{d}^T\mathbf{X}(T)\geq\lambda]}\right]$$

so that we can price such payoffs in closed-form as linear combinations of powerdigitals.

Basket options in general models (4)

Consider again the lower bound of Curran/Rogers and Shi, which can conveniently be rewritten as:

$$\mathbb{E}\left[\left(A(T)-K\right)^{+}\right] \geq \mathbb{E}\left[\left(\mathbb{E}[A(T) \mid \Lambda(T)]-K\right)^{+}\right]$$

This is not a payoff we can price as a linear combination of knock-in forwards. To calculate this lower bound numerically, we have to know the shape of the following set:

$$\mathcal{A}(\Lambda, K) \equiv \left\{\lambda \,\middle|\, \mathbb{E}[A(T) \mid \Lambda(T)=\lambda] \geq K\right\}$$

Basket options in general models (5)

Shape of $\mathcal{A}(\Lambda,K)$:

Consider a derivative paying:

$$\left(A(T)-K\right)1_{[\Lambda(T)\geq\lambda]}$$

Its forward price can be written as:

$$\mathbb{E}\left[\left(\sum_{j=1}^{M} w_j \exp\left(\mathbf{b}_j^T \mathbf{X}(T)\right) - \exp(k)\right) 1_{[c+\mathbf{d}^T\mathbf{X}(T)\geq\lambda]}\right]$$

and is thus a linear combination of powerdigitals, which can be priced in closed-form.

Basket options in general models (6)

Shape of $\mathcal{A}(\Lambda,K)$ (cont'd):

Its first derivative w.r.t. λ equals:

$$-\frac{\partial}{\partial\lambda}\mathbb{E}\left[\left(A(T)-K\right)1_{[\Lambda(T)\geq\lambda]}\right]=\left(\mathbb{E}[A(T)\mid\Lambda(T)=\lambda]-K\right)\cdot f_{\Lambda}(\lambda)$$

where $f_{\Lambda}(\lambda)$ is the density of Λ, evaluated at λ. Clearly, $\mathcal{A}(\Lambda,K)$ consists of those λ for which the above "delta" is positive. Furthermore, by assumption $c+\mathbf{d}^T\mathbf{X}(T) \geq k \Rightarrow A(T) \geq K$, so $[k,\infty) \subset \mathcal{A}(\Lambda,K)$. From Black-Scholes we know that by far the largest remainder comes from an interval of the form $[k^*, k]$.

Basket options in general models (7)

Proposed approximation:

- Determine k^* numerically; important to calculate "delta's" accurately and efficiently;
- Then the lower bound is:

$$\mathbb{E}\left[\left(A(T)-K\right)^+\right] \geq \mathbb{E}\left[\left(A(T)-K\right)1_{[\Lambda(T)\geq k^*]}\right]$$

which can be priced as a linear combination of powerdigitals.

Swaptions in affine Lévy models

Derivatives on arithmetic averages

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Swaptions in affine Lévy models (2)

Unlike basket derivatives, which are “exotic” options, swaptions (along with caps), are the plain vanillas of the interest rate market.

$\Rightarrow$ For pricing purposes it is of the utmost importance to calibrate our preferred model to plain vanillas

Previous approach directly works, provided that:

- The underlyings (zero-coupon bonds) to be exponentially affine in the state variables;
- we know the characteristic function;

$\Rightarrow$ Affine Lévy term-structure models

Swaptions in affine Lévy models (3)

Such models are often formulated as spot rate models, and considered to be superseded. The market standard is BGM/J model with skew and SV. However:

- Andreasen's "Back to the future" article in Risk September 2005 advocates a return to low-dimensional HJM models, for efficiency;
- Gaspar [2004] and Cheng and Scaillet [2005] have shown that, to a certain extent, LQJD models are AJD models, so more realistic dynamics are viable;

Zero-coupon bond options (i.e. also caplets and caps) can be priced analytically, so focus on swaptions.

Swaptions in affine Lévy models (4)

Several methods, other than the traditional Asian moment-matching schemes exist in these models:

- Jamshidian [1989]: closed-form pricing in 1-factor models;
- Munk and Wei [1999] use a stochastic duration to price swaptions as zero bond options;
- Singleton and Umantsev [2002] approximate the exercise region (i.e. $A(T) \geq K$) by an affine function of the state variables. This has to be done for each knock-in forward;
- Collin-Dufresne and Goldstein [2002]: Edgeworth expansion;
- Schrager and Pelsser [2005]: BGM/J-"freezing" approach;

Swaptions in affine Lévy models (5)

Aside from our extension of the Curran/Rogers and Shi lower bound to these models, we also consider a fast alternative to Singleton-Umantsev (FastSU):

- Approximate a coupon bond as a shifted exponentially affine function of the state variables:

$$\mathrm{CB}(\mathrm{T}, \mathbf{X}(\mathrm{T})) \approx \mathrm{C}_{\mathrm{CB}} + \exp\left(\mathrm{A}_{\mathrm{CB}} + \mathbf{B}_{\mathrm{CB}}^{\mathrm{T}} \mathbf{X}(\mathrm{T})\right)$$

- For a “representative” set of values of the state vector, fit the coefficients by NLS;
- Pricing can be done analytically, speed comparable to that of Munk’s stochastic duration approach;

Swaptions in affine Lévy models (6)

Collin-Dufresne and Goldstein [2002], and Schrager and Pelsser [2005], use a 2-factor CIR model:

$$dx_i(t) = -\lambda_i\left(x_i(t) - \overline{x}_i\right)dt + \sigma_i\sqrt{x_i(t)}dW_i(t)$$

$$r(t) = \theta(t) + \sum_{i=1}^{2} x_i(t)$$

to test their approximation. Contrary to their example (Black vols between 4-9.5%), we calibrated the model to the USD vol surface on 21-06-2005, resulting in Black vols between 18-24%.

Swaptions in affine Lévy models (7)

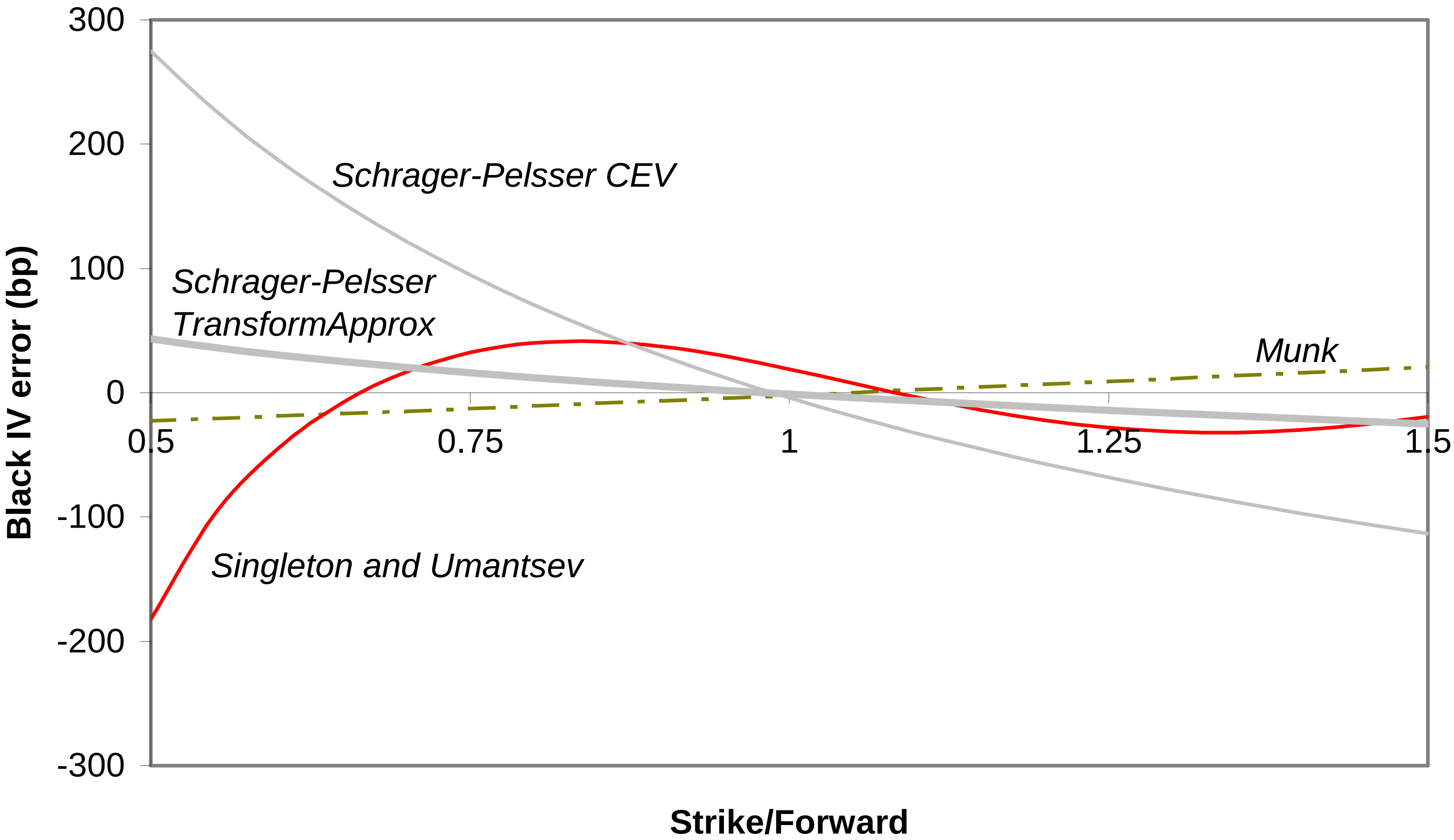
Differences with theoretical price (aside from calibration error) for 12 ATM swaptions with annual payments, swaption maturity equal to 1, 2 or 5 yrs, tenor equal to 1, 2, 5 or 10 yrs:

Method	Absolute Black IV error (bp) Average	Absolute Black IV error (bp) Maximum
Lower bound	2.7E-05	9.1E-05
Singleton-Umantsev	0.02	0.11
FastSU	0.13	0.71
Munk	0.36	2.05
Schrager-Pelsser TransformApprox	1.42	3.41
Collin-Dufresne and Goldstein	7.52	19.02
Schrager-Pelsser CEV	8.61	19.31

Generally desirable to be within 10 bp of mid-quotes.

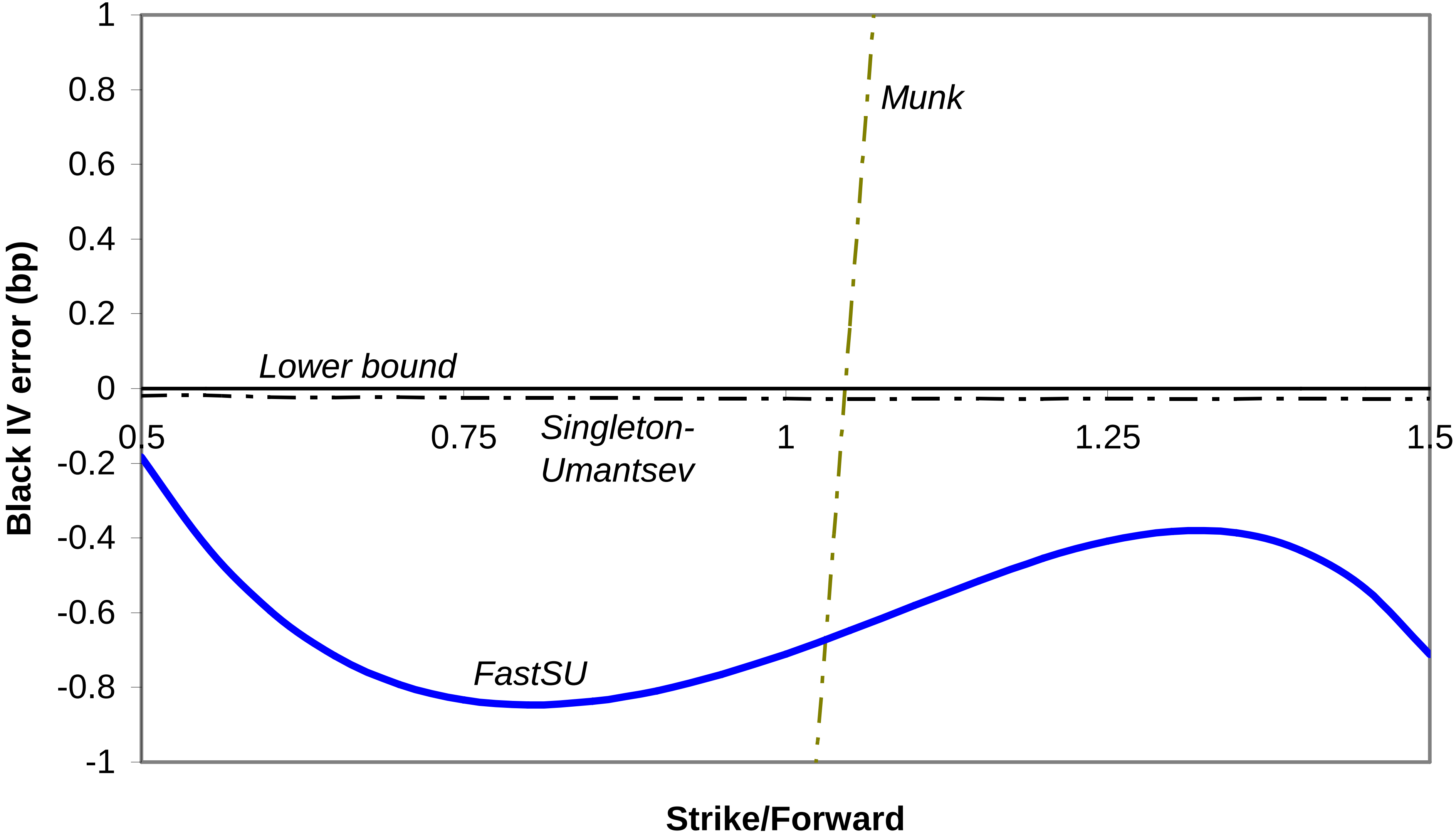
Swaptions in affine Lévy models (8)

5 x 10 swaption:



Swaptions in affine Lévy models (9)

5 x 10 swaption (zoomed in):



Swaptions in affine Lévy models (10)

Question is: what is the computational time, given a certain accuracy? Chosen accuracy here is 1/1000 bp in Black implied vol terms. For the 5 x 10 swaption:

Method	**Time/swaption**	**Swaptions/sec.**
Munk	0.0005	1934
FastSU	0.0031	321
Analytic price	0.0046	219
Lower bound	0.0084	118
Schrager-Pelsser CEV	0.0087	114
Singleton-Umantsev	0.0089	113
Collin-Dufresne and Goldstein	0.2847	4
Schrager-Pelsser TransformApprox	0.6887	1

Asians in affine Lévy models

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Asians in affine Lévy models (2)

Not a lot has been published on Asians in a non-Black-Scholes setting:

- Večeř and Xu [2004]: 1D PIDE for semimartingale models;
- Albrecher et al. [2005] and Albrecher and Schoutens [2005]: Upper bound for Lévy models and SV models;
- Albrecher and Predota [2002, 2004]: Moment-matching approximations and upper bounds for VG and NIG models;
- Zhu [2000]: Tries to apply Vorst's and other approximations in SV models, but has to resort to an approximation to price options on the geometric average;
- Fouque and Han [2003]: use perturbation techniques to approximate price of an Asian option with SV;

Asians in affine Lévy models (3)

If we again focus on affine Lévy models:

- Underlyings are exponentially affine in the state variables;
- … but what about the different timings in $\Lambda(T)$:

$$\Lambda(T) = c + \sum_{i=1}^{N} d(t_i)^T X(t_i)$$

$\Rightarrow$ We need to know the joint characteristic function of $X(t_1)$, …, $X(t_N)$

Asians in affine Lévy models (4)

Using the fact that the characteristic function is exponentially affine, we have in a 1D model:

$$
\begin{aligned}
&\mathbb{E}_t\left[\exp\left(\sum_{i=1}^{N} iu_i X(t_i)\right)\right] \\
&= \mathbb{E}_t\left[\mathbb{E}_{t_{N-1}}\left[\exp\left(\sum_{i=1}^{N} iu_i X(t_i)\right)\right]\right] \\
&= \mathbb{E}_t\left[\exp\left(\sum_{i=1}^{N-1} iu_i X(t_i)\right)\cdot \mathbb{E}_{t_{N-1}}\left[\exp\left(iu_N X(t_N)\right)\right]\right] \\
&= \mathbb{E}_t\left[\exp\left(\sum_{i=1}^{N-1} iu_i X(t_i) + iu_N\left(a_N + b_N X(t_{N-1})\right)\right)\right]
\end{aligned}
$$

Result carries over to models with latent factors, such as SV models, Lévy models with stochastic time, etc.

Asians in affine Lévy models (5)

Note:

This result also allows us to price options on the geometric average in closed-form, just as in the Black-Scholes model. Albrecher and Predota [2002, 2004] and Zhu [2000] had to use approximations to find the value of such an option. Even in Fouque and Han [2004] it is mentioned that closed-form prices only exist for geometric average options in a constant volatility setting.

Asians in affine Lévy models (6)

Example from Albrecher et al. [2005] for a VG model, where the model was calibrated to S&P 500 options. Option maturity of 1y, monthly averaging:

Strike	Moneyness	MC	LB
80	-0.19	20.4940 (1.0E-05)	20.4902 (-0.38)
90	-0.09	11.6938 (7.5E-06)	11.6911 (-0.26)
100	0.01	4.5430 (3.5E-06)	4.5420 (-0.10)
110	0.11	0.9238 (2.4E-06)	0.9233 (-0.05)
120	0.21	0.1999 (3.3E-06)	0.1994 (-0.05)

VG4M	CUB
20.5018 (0.78)	20.7937 (29.97)
11.7075 (1.38)	12.1695 (47.57)
4.5132 (-2.98)	5.0461 (50.31)
0.9336 (0.98)	1.2279 (30.41)
0.2108 (1.09)	0.3382 (13.83)

Conclusions

- Model-independent algorithm for approximating basket options, requiring only the knowledge of the characteristic function;
- Results carry over to swaptions, credit-default swaptions and Asians in affine Lévy models;
- For swaptions and Asians, the approximations are the most accurate to date;
- Room for even better approximations if conditional moments can be calculated efficiently.

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