Investments, wealth and risk tolerance

Lecture V

Topics

- Utility-based measurement of performance
- Variational and stochastic components of optimal utility volume
- Optimal allocations and their stochastic evolution
- Efficient frontier

Utility-based measurement of performance

Stochastic environment Important ingredients

- Time evolution concurrent with the one of the investment universe
- Consistency with up to date information
- Incorporation of available opportunities and constraints
- Meaningful optimal utility volume

Dynamic utility

$$U(x,t)$$
 is an \mathcal{F}_t -adapted process

- ullet As a function of x, U is increasing and concave
- For each self-financing strategy, represented by π , the associated (discounted) wealth X_t satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi}, t) \mid \mathcal{F}_s) \le U(X_s^{\pi}, s) \qquad 0 \le s \le t$$

• There exists a self-financing strategy, represented by π^* , for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t) \mid \mathcal{F}_s) = U(X_s^{\pi^*}, s) \qquad 0 \le s \le t$$

Traditional framework

A deterministic utility datum u(x,T) is assigned at the end of a fixed investment horizon

$$U(x,T) = u(x,T)$$

Backwards in time generation of optimal utility volume

The dynamic utility coincides with the traditional value function.

It remains constant for times beyond T.

Alternative framework

A deterministic utility datum u(x,0) is assigned at the beginning of the trading horizon, t=0

$$U(x,0) = u(x,0)$$

Forward in time generation of optimal utility volume

$$U(X_s^{\pi^*}, x) = E_{\mathbb{P}}(U(X_t^{\pi^*}, t) / \mathcal{F}_s) \qquad 0 \le s \le t$$

- Dynamic utility can be defined for all trading horizons
- Utility and allocations more intuitive
- Difficulties due to the "inverse in time" nature of the problem

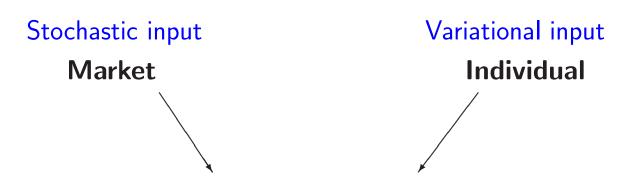
Construction of a class of forward dynamic utilities

Creating the martingale that yields the optimal utility volume

Minimal model assumptions

Stochastic optimization problem "inverse" in time

Key idea



Maximal utility — Optimal allocation

Variational utility input

- Key ingredients: wealth and risk tolerance
- Risk tolerance solves a fast diffusion equation posed inversely in time

$$\begin{cases} r_t + \frac{1}{2}r^2 r_{xx} = 0 \\ r(x,0) = -\frac{u_0'(x)}{u_0''(x)} \end{cases}$$

Utility surface generated by a transport equation

$$\begin{cases} u_t + \frac{1}{2}r(x,t)u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Stochastic market input

Investment universe

Riskless and risky securities

- $\bullet \ (\Omega, \mathcal{F}, \mathbb{P}) \quad ; \quad W = (W^1, \dots, W^d) \quad \text{ standard Brownian Motion}$
- Traded securities

$$1 \le i \le k \qquad \begin{cases} dS_t^i = S_t^i(\mu_t^i dt + \sigma_t^i \cdot dW_t) , & S_0^i > 0 \\ dB_t = r_t B_t dt , & B_0 = 1 \end{cases}$$

 $\mu_t, r_t \in \mathbb{R}$, $\sigma_t^i \in \mathbb{R}^d$ bounded and \mathcal{F}_t -measurable stochastic processes

• Postulate existence of a \mathcal{F}_t -measurable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t \, \mathbb{1} = \sigma_t^T \lambda_t$$

Investment universe

- Self-financing investment strategies $\pi^0_t, \pi^i_t, \quad i=1,\ldots,k$
- Present value of this allocation

$$X_t = \sum_{i=0}^k \pi_t^i$$

$$dX_{t} = \sum_{i=0}^{k} \pi_{t}^{i} (\mu_{t}^{i} - r_{t}) dt + \sum_{i=0}^{k} \pi_{t}^{i} \sigma_{t}^{i} \cdot dW_{t}$$

$$= \sigma_t \pi_t \cdot (\lambda_t \, dt + dW_t)$$

$$\pi_t = (\pi_t^1, \dots, \pi_t^k), \qquad \mu_t - r_t \, \mathbb{1} = \sigma_t^T \lambda_t$$

Market input processes

$$(\sigma_t, \lambda_t)$$
 and (Y_t, Z_t, A_t)

These \mathcal{F}_t -mble processes do **not** depend on the investor's variational utility

They reflect and represent, respectively

 (λ_t, σ_t) : dynamics of traded securites

 Y_t : benchmark numeraire

 Z_t : market view away from market equilibrium feasibility and trading constraints

 A_t : subordination

The processes (Y_t, Z_t, A_t)

Benchmark and/or numeraire

A "replicable" process Y_t satisfying

$$\begin{cases} dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \\ Y_0 = 1 \end{cases}$$

$$\delta_t \in \mathcal{F}_t$$
 , $\sigma_t \sigma_t^+ \delta_t = \delta_t$

 σ_t^+ : Moore-Penrose matrix inverse

Market input processes

• Market views, feasibility and trading constraints

An exponential martingale Z_t satisfying

$$\begin{cases} dZ_t = Z_t \phi_t \cdot dW_t \\ Z_0 = 1, \quad \phi_t \in \mathcal{F}_t \end{cases}$$

Subordination

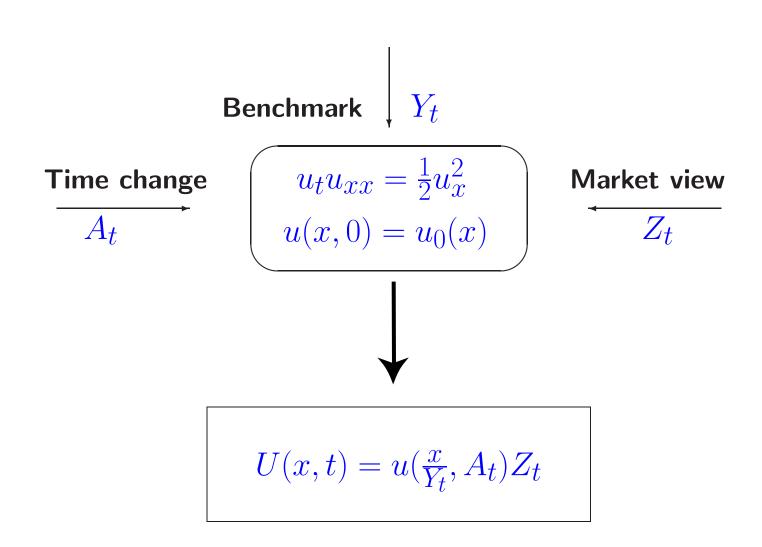
A non-decreasing process A_t solving

$$\begin{cases} dA_t = |\delta_t - \sigma_t \sigma_t^+ (\lambda_t + \phi_t)|^2 dt \\ A_0 = 0 \end{cases}$$

Optimal utility volume Optimal asset allocation

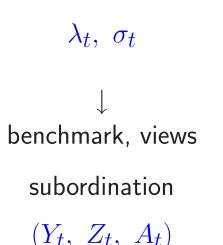
Optimal utility volume

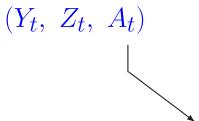
Stochastic input : (Y_t, Z_t, A_t) **Variational input** : u(x, t)



Optimal utility volume

Stochastic market input





Variational input

$$x, r_0(x) = -\frac{u_0'(x)}{u_0''(x)}$$

$$\downarrow$$

$$r_t + \frac{1}{2}r^2r_{xx} = 0 \quad \text{(FDE)}$$

$$u_t + \frac{1}{2}ru_x = 0 \quad \text{(TE)}$$

$$u(x, t)$$

$$U(x,t) = u(\frac{x}{A_t}, Y_t)Z_t$$

Model independent construction!

What is the optimal allocation?

Optimal portfolio processes

$$\pi_t = (\pi_t^0, \pi_t^1, \dots, \pi_t^k)$$

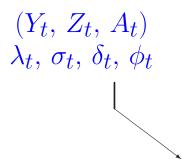
can be directly and explicitly characterized

along with the construction of the forward utility!

The structure of optimal portfolios

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t \, dt + dW_t)$$

Stochastic input Market



Variational input Individual

$$\begin{array}{c} \text{wealth } x \\ \text{risk tolerance } r(x,t) \end{array}$$

$$\frac{1}{Y_t}\pi_t^* \quad \text{is a } \textit{linear} \text{ combination}$$
 of (benchmarked) optimal wealth} and subordinated (benchmarked) risk tolerance

Optimal asset allocation

• Let X_t^* be the optimal wealth, Y_t the benchmark and A_t the subordination processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$
$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$
$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

• Define r_t^* the subordinated (benchmarked) risk tolerance

$$r_t^* = r\left(\frac{X_t^*}{Y_t}, A_t\right)$$

Optimal (benchmarked) portfolios

$$\frac{1}{Y_t} \pi_t^* = \sigma_t^+ \left((\lambda_t + \phi_t) r_t^* + \delta_t \left(\frac{X_t^*}{Y_t} - r_t^* \right) \right)$$

Wealth-risk tolerance stochastic evolution

A system of SDEs at optimum utility volume

$$\widehat{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{ and } \quad \widehat{r}_t^* = r(\widehat{X}_t^*, A_t)$$

$$\begin{cases} d\widehat{X}_t^* = \widehat{r}_t^* (\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t) \\ d\widehat{r}_t^* = r_x (\widehat{X}_t^*, A_t) d\widehat{X}_t^* \end{cases}$$

- Separability of wealth dynamics in terms of risk tolerance and market input
- Sensitivity of risk tolerance in terms of its spatial gradient and changes in optimal wealth
- Utility functional has essentially vanished

Universal representation, no Markovian assumptions

An efficient frontier

Optimal wealth-risk tolerance $(\widehat{X}_t^*, \widehat{r}_t^*)$ system of SDEs in original market configuration

$$\begin{cases} d\widehat{X}_t^* = \widehat{r}_t^* (\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t) \\ d\widehat{r}_t^* = r_x (\widehat{X}_t^*, A_t) d\widehat{X}_t^* \end{cases}$$



change of measure
historical → benchmarked

change of time Levy's theorem

An efficient frontier

Optimal wealth-risk tolerance (x_t^1, x_t^2) system of SDEs in canonical market configuration

$$x_{t}^{1} = \left(\frac{X_{t}^{*}}{Y_{t}}\right)_{A_{t}^{(-1)}} \qquad x_{t}^{2} = r\left(\frac{X_{t}^{*}}{Y_{t}}, A_{t}\right)_{A_{t}^{(-1)}}$$

$$\langle M_{t} \rangle = A_{t} \qquad w_{t} = M_{A^{(-1)}}$$

$$\downarrow$$

$$\begin{cases} dx_{t}^{1} = x_{t}^{2} dw_{t} \\ dx_{t}^{2} = r_{x}(x_{t}^{1}, t)x_{t}^{2} dw_{t} \\ x_{0}^{1} = \frac{x}{u}, \quad x_{0}^{2} = r_{x}(\frac{x}{u}, 0) \end{cases}$$

Analytic solution of the efficient frontier SDE system

$$\begin{cases} dx_t^1 = x_t^2 dw_t \\ dx_t^2 = r_x(x_t^1, t) x_t^2 dw_t \end{cases}$$

• Define the budget capacity function h(x,t) via

$$x = \int_{\underline{x}}^{h(x,t)} \frac{du}{r(u,t)} = \int_{\underline{x}}^{h(x,t)} \gamma(u,t) du$$

 \underline{x} : related to symmetry properties of risk tolerance, reflection point of its spatial derivative and risk aversion front

Analytic solutions

The budget capacity function h solves the (inverse) heat equation

$$\begin{cases} h_t + \frac{1}{2}h_{xx} - \frac{1}{2}r_x(\underline{x}, t)h_x = 0\\ h(x, 0) = h_0(x), \quad x = \int_{\underline{x}}^{h_0(x)} \frac{du}{r(u, 0)} \end{cases}$$

Solution of the efficient frontier SDE system

$$\begin{cases} x_t^1 = h(z_t, t) \\ x_t^2 = h_z(z_t, t) \end{cases}$$
$$z_t = h_0^{-1}(x) - \int_0^t \frac{1}{2} r_x(\underline{x}, s) ds + w_t$$

Using equivalent measure transformations and time change we recover the original pair of optimal (benchmarked) wealth and (benchmark) risk tolerance

Utility-based performance measurement

Market **Investor**

Benchmark, views, constraints

Wealth, risk tolerance

Market input processes

Subordination



$$Y_t, Z_t, A_t$$

 $x, r(x,t), u(x,t)$

Fast diffusion eqn Transport eqn

Optimal utility volume and optimal portfolios

measure change

change

Efficient frontier SDE system

Heat eqn



Fast diffusion eqn

Universal analytic solutions

Dynamic exponential utility

Objective: Find an \mathcal{F}_t -adapted process $U_t(x)$ such that

$$\begin{cases} U_0(x) = -\exp\left(-\frac{x}{y}\right) \\ E_{\mathbb{P}}\left(U_s\left(X_s^{\pi}\right)|\mathcal{F}_t\right) \leq U_t\left(X_t^{\pi}\right) \\ E_{\mathbb{P}}\left(U_s\left(X_s^{\pi^*}\right)|\mathcal{F}_t\right) = U_t\left(X_t^{\pi^*}\right), \quad s \geq t \end{cases}$$

Solution

Variational input

$$u(x, y, z) = -\exp\left(-\frac{x}{y} + z\right)$$

Dynamic exponential utility (continued)

• Stochastic market input consists of a pair of Ito processes, (Y, Z), solving, respectively,

$$\begin{cases} dY_t = Y_t \delta_t \cdot (\kappa_t dt + dW_t) \\ Y_0 = y > 0 \end{cases}$$
$$\begin{cases} dZ_t = \eta_t dt + \xi_t \cdot dW_t \\ Z_0 = 0. \end{cases}$$

and

$$\begin{cases} dZ_t = \eta_t dt + \xi_t \cdot dW_t \\ Z_0 = 0. \end{cases}$$

• Moore-Penrose pseudo-inverse matrix σ^+ of σ $(d \times k)$ is the unique $k \times d$ matrix satisfying

$$\sigma\sigma^+\sigma = \sigma$$
 $\sigma^+\sigma\sigma^+ = \sigma^+$

$$(\sigma\sigma^+)^T = \sigma\sigma^+ \quad (\sigma^+\sigma)^T = \sigma^+\sigma.$$

• The processes $\delta, \kappa, \eta, \xi$ are taken to be bounded and \mathcal{F}_t —progressively measurable. It is, also, assumed that

$$\sigma \sigma^+ \delta = \delta$$
 and $\delta \cdot (\kappa - \lambda) = 0$

• The drift η of the process Z satisfies

$$2\eta = \left|\delta - \sigma\sigma^{+} \left(\lambda + \xi\right)\right|^{2} - \left|\xi\right|^{2}$$

 \bullet Wlog, the dynamics of the benchmark process Y can be written as

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$

Dynamic exponential performance

Solution

$$U_t(x) = -\exp\left(-\frac{x}{Y_t} + Z_t\right)$$

Idea of the proof: applying Ito calculus and using the structural assumptions on the market input yields

$$\begin{split} dU\left(X\right) &= U\left(X\right) \left(-Y^{-1}\beta \cdot dW + XY^{-1}\delta \cdot dW + \xi \cdot dW\right) \\ &+ \frac{1}{2}U\left(X\right) \left(-2Y^{-1}\beta \cdot \lambda + 2XY^{-1}\delta \cdot \kappa + 2\eta + Y^{-2} \left|\beta\right|^2 + 2\left(Y^{-1} - XY^{-2}\right)\delta \cdot \beta \right. \\ &- 2Y^{-1}\xi \cdot \beta + 2XY^{-1}\delta \cdot \xi + \left(X^2Y^{-2} - 2XY^{-1}\right)\left|\delta\right|^2 + \left|\xi\right|^2\right)dt \\ &= U\left(X\right) \left(-Y^{-1}\beta \cdot dW + XY^{-1}\delta \cdot dW + \xi \cdot dW\right) \\ &+ \frac{1}{2}U\left(X\right) \left(\left|Y^{-1}\beta - \left((\lambda + \xi) + \left(XY^{-1} - 1\right)\delta\right)\right|^2 + 2XY^{-1}\delta \cdot (\kappa - \lambda) \right. \\ &+ 2\eta + \left|\xi\right|^2 - \left|\delta - (\lambda + \xi)\right|^2\right)dt \end{split}$$

Idea of the proof (continued)

and, in turn,

$$dU\left(X\right) = U\left(X\right)\left(-Y^{-1}B^{-1}\sigma\pi + XY^{-1}\delta + \xi\right) \cdot dW$$

$$+\frac{1}{2}U\left(X\right)\left(\left|Y^{-1}B^{-1}\sigma\pi - \sigma\sigma^{+}\left((\lambda + \xi) + \left(XY^{-1} - 1\right)\delta\right)\right|^{2} + 2XY^{-1}\delta \cdot (\kappa - \lambda)$$

$$+\left|\left(I - \sigma\sigma^{+}\right)(\lambda + \xi)\right|^{2} + 2\eta + |\xi|^{2} - |\delta - (\lambda + \xi)|^{2}\right)dt$$

..... and, finally,

$$dU\left(X\right) = U\left(X\right)\left(-Y^{-1}B^{-1}\sigma\pi + XY^{-1}\delta + \xi\right) \cdot dW$$

$$\frac{1}{2}U\left(X\right)\left|Y^{-1}B^{-1}\sigma\pi - \sigma\sigma^{+}\left(\lambda + \left(XY^{-1} - 1\right)\delta + \xi\right)\right|^{2}dt$$

At the optimum

Feedback portfolio control process

$$\pi^* = YB\sigma^+ \left(\lambda + \left(X^*Y^{-1} - 1\right)\delta + \xi\right).$$

Optimal wealth process

$$dX^* = B^{-1}\sigma\pi^* \cdot (\lambda dt + dW)$$
$$= \left(Y\left(\sigma\sigma^+ (\lambda + \xi) - \delta\right) + X^*\delta\right) \cdot (\lambda dt + dW).$$

Optimal utility volume

$$dU(X^*) = U(X^*) \left(-Y^{-1}B^{-1}\sigma \pi^* + X^*Y^{-1}\delta + \xi \right) \cdot dW$$
$$= U(X^*) \left(\sigma \sigma^+ (\delta - \lambda) + \left(I - \sigma \sigma^+ \right) \xi \right) \cdot dW.$$

Explicit solutions

Optimal wealth process:

$$X_t^* = \mathcal{E}_t \left(x + \int_0^t \mathcal{E}_s^{-1} Y_s \left(\sigma_s \sigma_s^+ \left(\lambda_s + \xi_s \right) - \delta_s \right) \cdot \left(\left(\lambda_s - \delta_s \right) ds + dW_s \right) \right)$$
$$\mathcal{E}_t = \exp \left(\int_0^t \left(\delta_s \cdot \lambda_s - \frac{1}{2} \left| \delta_s \right|^2 \right) ds + \int_0^t \delta_s \cdot dW_s \right)$$

Optimal portfolio process

$$\pi_t^* = B_t Y_t \sigma_t^+ (\lambda_t + \xi_t - \delta_t) + B_t X_t^* \sigma_t^+ \delta_t$$

$$= x \mathcal{E}_t B_t \sigma_t^+ \delta_t + B_t Y_t \sigma_t^+ (\lambda_t + \xi_t - \delta_t)$$

$$+ B_t \mathcal{E}_t \left(\int_0^t \mathcal{E}_s^{-1} Y_s \left(\sigma_s \sigma_s^+ (\lambda_s + \xi_s) - \delta_s \right) \cdot \left((\lambda_s - \delta_s) \, ds + dW_s \right) \right) \sigma_t^+ \delta_t$$

Optimal utility volume

$$U_t(X_t^*) = \exp\left(-\frac{x}{y} - \int_0^t \frac{1}{2} \left| \sigma_s \sigma_s^+ (\delta_s - \lambda_s) + \left(I - \sigma_s \sigma_s^+\right) \xi_s \right|^2 ds + \int_0^t \left(\sigma_s \sigma_s^+ (\delta_s - \lambda_s) + \left(I - \sigma_s \sigma_s^+\right) \xi_s \right) \cdot dW_s \right)$$

Case I: No benchmark and 'no' views $\delta = \xi = 0$.

Then,
$$Y_t = y$$
, for $t \ge 0$.

The forward performance process takes the form

$$U_t(x) = -\exp\left(-\frac{x}{y} + \int_0^t \frac{1}{2} \left|\sigma_s \sigma_s^{\dagger} \lambda_s\right|^2 ds\right)$$

Note that even in this simple case, the solution is equal to the classical exponential utility only at t=0.

• The optimal discounted wealth and optimal asset allocation are given, respectively, by

$$X_t^* = x + \int_0^t y \left(\sigma_s \sigma_s^+ \lambda_s \right) \cdot (\lambda_s ds + dW_s)$$

and

$$\pi_t^* = y B_t \sigma_t^+ \lambda_t$$

Observe that π^* is **independent** of the initial wealth x.

Case 1: No benchmark and 'no' views $\delta = \xi = 0$ (continued)

Optimal utility volume

$$U_t(X_t^*) = -\exp\left(-\frac{x}{y} - \int_0^t \frac{1}{2} \left|\sigma_s \sigma_s^+ \lambda_s\right|^2 ds - \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot dW_s\right)$$

Observe that π^* is *independent* of the initial wealth x.

Total amount allocated in the risky assets

$$\mathbf{1} \cdot \frac{\pi_t^*}{B_t} = \mathbf{1} \cdot y \sigma_t^+ \lambda_t$$

Amount invested in the riskless asset

$$\pi_t^{0,*} = X_t^* - \mathbf{1} \cdot y \sigma_t^+ \lambda_t$$

Such an allocation is rather **conservative** and is often viewed as an argument **against** the classical exponential utility.

Case 2: Static performance $\sigma \sigma^+ (\delta - \lambda) + (I - \sigma \sigma^+) \xi = 0.$

Then
$$\sigma^+(\delta - \lambda) = 0$$
 and $\sigma\sigma^+\xi = \xi$, and $Z_t = \int_0^t \xi_s \cdot dW_s$

Dynamic exponential utility

$$U_t(x) = -\exp\left(-\frac{x}{Y_t} + \int_0^t \xi_s \cdot dW_s\right)$$

Optimal discounted wealth

$$X_t^* = \mathcal{E}_t \left(x + \int_0^t \mathcal{E}_s^{-1} Y_s \xi_s \cdot ((\lambda_s - \delta_s) \, ds + dW_s) \right)$$

Case 2: Static performance $\sigma \sigma^+ (\delta - \lambda) + (I - \sigma \sigma^+) \xi = 0$. (continued)

Optimal allocation

$$\pi_t^* = x \mathcal{E}_t B_t \sigma_t^+ \delta_t + Y_t B_t \sigma_t^+ \xi_t + B_t \mathcal{E}_t \left(\int_0^t \mathcal{E}_s^{-1} Y_s \xi_s \cdot dW_s \right) \sigma_t^+ \delta_t$$

Optimal utility volume

$$U_t(X_t^*) = U_0(x) = -\exp\left(-\frac{x}{y}\right)$$

Observe that the optimal level of forward performance remains constant across times.

Case 3: No benchmark and risk neutralization $\delta = 0$ and $\lambda + \xi = 0$.

Then,
$$\mathcal{E}_t = 1$$
, $Y_t = y > 0$ and $Z_t = -\int_0^t \frac{1}{2} |\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s$.

Dynamic exponential utility

$$U_t(x) = -\exp\left(-\frac{x}{y} - \frac{1}{2}\int_0^t |\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s\right)$$

Optimal discounted wealth

$$X_t^* = x$$

Case 3: No benchmark and risk neutralization $\delta = 0$ and $\lambda + \xi = 0$. (continued)

Optimal allocations

$$\pi_t^* = 0$$
 and $\pi_t^{0,*} = X_t^* = x$.

Optimal utility volume

$$U_{t}\left(X_{t}^{*}\right)=U_{t}\left(x\right).$$

It is important to notice that, for all trading times, the optimal allocation consists of putting **zero** into the risky assets and, therefore, investing the entire wealth into the riskless asset. Such a solution seems to capture quite accurately the strategy of a **derivatives trader** for whom the underlying **objective is to hedge** as opposed to the asset manager whose objective is to invest.

Case 4: Following the benchmark $\delta = \lambda + \xi$ with $\lambda + \xi \neq 0$.

Then $\delta = \sigma \sigma^+ (\lambda + \xi)$ and, in turn, $Z_t = -\int_0^t \frac{1}{2} |\xi_s|^2 ds + \int_0^t \xi_s \cdot dW_s$.

Dynamic exponential utility

$$U_t(x) = -\exp\left(-\frac{x}{Y_t} - \int_0^t \frac{1}{2} |\xi_s|^2 ds + \int_0^t \xi_s \cdot dW_s\right) .$$

Optimal wealth

$$X_t^* = x\mathcal{E}_t.$$

Returns of wealth and of benchmark

$$\frac{dX_t^*}{X_t^*} = \frac{dY_t}{Y_t}$$

Case 4: Following the benchmark $\delta = \lambda + \xi$ with $\lambda + \xi \neq 0$. (continued)

Optimal allocation

$$\pi_t^* = B_t X_t^* \sigma_t^+ \delta_t$$

Optimal performance level

$$U_t(X_t^*) = -\exp\left(-\frac{x}{Y_t} - \int_0^t \frac{1}{2} |\xi_s|^2 ds + \int_0^t \xi_s \cdot dW_s\right)$$

Observe that, contrary to what we have observed in traditional backward exponential utility problems, the optimal portfolio is a **linear functional of the wealth** and not independent of it.

Case 5: Generating arbitrary portfolio allocations

• Assume that $\mathbf{1} \cdot \sigma_t^+ (\lambda_t + \xi_t) = 1$. Then

$$\mathbf{1} \cdot \frac{\pi_t^*}{B_t} = X_t^* \quad \text{and} \quad \pi_t^{0,*} = 0$$

Hence, the optimal allocation π^* puts zero amount in the riskless asset and invests all wealth in the risky assets, according to the weights specified by the vector $\sigma^+ \left(\lambda + \xi \right)$.

Case 5: Generating arbitrary portfolio allocations (continued)

• Note, also, that for an arbitrary vector ν_t with $\mathbf{1} \cdot \sigma_t^+ \nu_t \neq 0$, the vector

$$\xi_t = \frac{1 - \mathbf{1} \cdot \sigma_t^+ \lambda_t}{\mathbf{1} \cdot \sigma_t^+ \nu_t} \nu_t$$

satisfies the above constraint since
$$\mathbf{1} \cdot \sigma_t^+ \left(\lambda_t + \frac{1 - \mathbf{1} \cdot \sigma_t^+ \lambda_t}{\mathbf{1} \cdot \sigma_t^+ \nu_t} \nu_t \right) = 1$$

Can we generate optimal portfolios that allocate **arbitrary**, but constant, fractions of wealth to the different accounts?

The answer is affirmative. Indeed, for $p \in \mathcal{R}$, set,

$$\mathbf{1} \cdot \sigma_t^+ \left(\lambda_t + \xi_t \right) = p$$

Then, the total investment in the risky assets and the allocation in the riskless bond are

$$\mathbf{1} \cdot \frac{\pi_t^*}{B_t} = pX_t^*$$
 and $\frac{\pi_t^0}{B_t} = (1-p)X_t^*$