MANAGING RISK IN LIFE INSURANCE AND PENSIONS

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Abstract: Stochastic processes in life history analysis, life insurance, and finance (jump processes and their associated random measures and martingales, Levy processes); Traditional paradigms in life insurance (the principle of equivalence, notions of reserves); Management of financial risk and longevity risk (the with profit scheme, unit-linked insurance, securitization of mortality risk); The role of financial instruments in life insurance and pensions - can the markets come to our rescue?

1 Background

Savings contract:
c_t, amount deposited at time $t = 1, 2, ..., m$
b_t, amount withdrawn at time $t = m + 1, ..., T$
S_t, price of unit of asset portfolio at time $t = 0, 1, 2, ...$
Deposit of c_t at time t purchases u_t units of the asset given by $c_t = u_t\, S_t$:

$$u_t = c_t / S_t$$

Balance of account at time $t = 1, 2, ..., m$ is total units purchased times current value of the asset:

$$V_t = S_t\,(u_1 + \cdots + u_t) = S_t \left(\frac{c_1}{S_1} + \cdots + \frac{c_t}{S_t} \right)$$

Likewise, at time $t = m + 1, ..., T$ withdrawal of b_t is financed by selling u_t units of assets given by $b_t = u_t\, S_t$ and balance of account is

$$V_t \;=\; S_t \left(\frac{c_1}{S_1} + \cdots + \frac{c_m}{S_m} - \frac{b_{m+1}}{S_{m+1}} - \cdots \frac{b_t}{S_t} \right)$$

At term T all savings have been withdrawn and account is settled at value $V_T = 0$, giving *Balance equation*:

$$\frac{c_1}{S_1} + \cdots + \frac{c_m}{S_m} = \frac{b_{m+1}}{S_{m+1}} + \cdots + \frac{b_T}{S_T}$$

The role of capital gains is displayed by this relationship. In year t the asset earns interest at rate

$$r_t = \frac{S_t - S_{t-1}}{S_{t-1}}$$

Table 1: The Danish G82 mortality table

Age x	No. of survivors ℓ_x	No of deaths d_x	mortality rate q_x
0	100000	58	0.000579
25	98083	119	0.001206
50	91119	617	0.006774
60	82339	1275	0.015484
70	65024	2345	0.036069
80	37167	3111	0.083711
90	9783	1845	0.188617
100	401	158	0.394000

Pension scheme. Balance equation:

$$\frac{c_1\,\ell_{x+1}}{S_1} + \cdots + \frac{c_m\,\ell_{x+m}}{S_m} = \frac{b_{m+1}\,\ell_{x+m+1}}{S_{m+1}} + \cdots + \frac{b_t\,\ell_{x+T}}{S_T}$$

Numerical illustrations:
Level payments $c_t = c$, $b_t = 1$, $m = 35$, $T = 70$.

No interest, no mortality: $c = 1$
$r = 0.04$, no mortality: $c = 0.2538$
$r = 0.08$, no mortality: $c = 0.0677$

$r = 0.04$, G82 mortality (mean life length 73): $c = 0.1149$
$r = 0.04$, Half of G82 mortality (mean life length 81):
$c = 0.1592$

Managing interest and mortality risk:
1. With profit scheme. Set premium on the safe side. Repay surplus in arrears as *bonus*.

2. Unit-linked scheme:
$c_t \;=\; \underline{c}_t\, S_t/\ell_{x+t},\;\; t = 1, 2, ..., m$
$b_t \;=\; \underline{b}_t\, S_t/\ell_{x+t},\;\; t = m+1, 2, ..., T$
$\underline{c}_t$ and $\underline{b}_t$ *baseline* payments chosen at time 0.
Balance equation:

$$\underline{c}_1 + \ldots + \underline{c}_m \;=\; \underline{c}_{m+1} + \ldots + \underline{b}_T$$

2 The discounted value of a payment stream

S_t asset price at time t
B_t total investments (deposits less withdrawals) in $(0, t]$
U_t balance (investments compounded with interest) at t

Discrete time bank statement with $B_j - B_{j-1}$ invested in S at time $j = 1, 2, \ldots$:

$$U_j - U_{j-1} = U_{j-1} \frac{S_j - S_{j-1}}{S_{j-1}} + B_j - B_{j-1}$$

$$\begin{aligned} U_j &= S_j \sum_{i=1}^{j} S_i^{-1}(B_i - B_{i-1}) \\ &= S_j \left(\sum_{i=1}^{j} S_{i-1}^{-1}(B_i - B_{i-1}) + \sum_{i=1}^{j}(S_i^{-1} - S_{i-1}^{-1})(B_i - B_{i-1}) \right) \\ &= S_j \left(\sum_{i=1}^{j} S_{i-1}^{-1}(B_i - B_{i-1}) - \sum_{i=1}^{j} \frac{S_i - S_{i-1}}{S_{i-1}\, S_i}(B_i - B_{i-1}) \right) \end{aligned}$$

This motivates continuous time analogues:

$$dU_t = U_{t-} \frac{dS_t}{S_{t-}} + dB_t$$

$$U_t = S_t \left(\int_0^t S_{\tau-}^{-1}\, dB_\tau - \int_0^t S_{\tau-}^{-1} S_\tau^{-1}\, d[S, B]_\tau \right)$$

Arbitrage free market with locally risk-free asset $S_t^0 = e^{\int_0^t r}$ and EMM $\tilde{\mathbf{P}}$:

$$\tilde{\mathbf{E}}\left[(S_t^0)^{-1}\, U_t\right] = \tilde{\mathbf{E}}\left[\int_0^t (S_\tau^0)^{-1}\, dB_\tau\right]$$

Reference: Norberg and Steffensen (2005)

3 Life is a process

Policy issued at time 0, terminates at time T.
State of policy at time $t \in [0, T]$ is

$$Z(t) \in \mathcal{Z} = \{0, \ldots, J\}, \quad Z(0) = 0$$

Indicator processes:

$$I_g(t) = 1[Z(t) = g]$$

Counting processes:

$$N_{gh}(t) = \sharp\{\tau;\, Z(\tau-) = g,\, Z(\tau) = h,\ \tau \in (0, t]\}$$

$$dI_g(t) = \sum_{h;h\neq g} dN_{hg}(t) - \sum_{h;h\neq g} dN_{gh}(t)\,, \quad I_g(0) = \delta_{0g}\,.$$

Common assumption: Z is Markov process with transition probabilities

$$p_{jk}(t, u) = \mathbb{P}[Z(u) = k \,|\, Z(t) = j]$$

and intensities

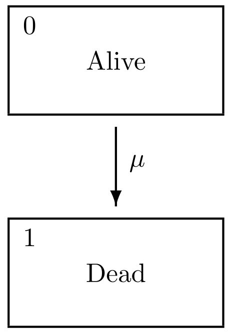
$$\mu_{jk}(t) = \lim_{h \downarrow 0} \frac{p_{jk}(t, t+h)}{h}$$

Compensated counting processes are square integrable orthogonal martingales:

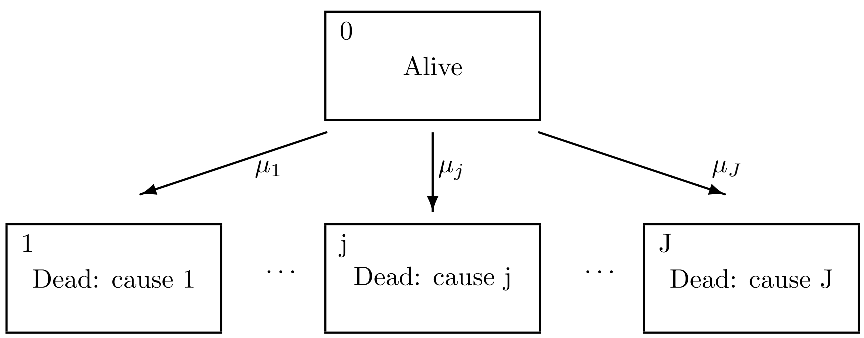
$$dM_{gh}(t) = dN_{gh}(t) - I_g(t)\, \mu_{gh}(t)\, dt$$

$$\mathbf{E}[dM_{gh}(t) \,|\, \mathcal{F}_{t-}] = 0$$

$$\mathbf{E}\,[\, dM_{gh}(t)\, dM_{jk}(t) |\, \mathcal{F}_{t-}] = \delta_{gh,jk}\, I_g(t)\, \mu_{gh}(t)\, dt$$



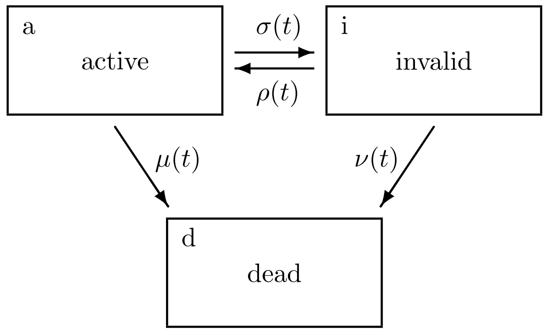
Survival probability: $p_{00}(t,u) = e^{-\int_t^u \mu}$



$$\mu(t) = \sum_{j=1}^{J} \mu_j(t)$$

Probability of death from cause j:

$$p_{0j}(t,u) = \int_t^u e^{-\int_t^\tau \mu}\, \mu_j(\tau)\, d\tau$$



Kolmogorov forward differential equations:

$$\frac{\partial}{\partial t} p_{ij}(s,t) = \sum_{g;g\neq j} p_{ig}(s,t)\,\mu_{gj}(t) \;-\; p_{ij}(s,t) \sum_{g;g\neq j} \mu_{jg}(t)$$

$$p_{ij}(s,s) = \delta_{ij}$$

$$\frac{\partial}{\partial t}\mathbf{P}(s,t) = \mathbf{P}(s,t)\,\mathbf{M}(t)\,, t \in (s,\infty), \quad \mathbf{P}(s,s) = \mathbf{I}\,.$$

Kolmogorov backward differential equations:

$$\frac{\partial}{\partial t} p_{jk}(t,u) = -\sum_{g;g\neq j} \mu_{jg}(t)\,p_{gk}(t,u) \;+\; \sum_{g;g\neq j} \mu_{jg}(t)\,p_{jk}(t,u)$$

$$p_{jk}(u,u) = \delta_{jk}$$

$$\frac{\partial}{\partial t}\mathbf{P}(t,u) = -\mathbf{M}(t)\,\mathbf{P}(t,u)\,, t \in (0,u), \quad \mathbf{P}(u,u) = \mathbf{I}\,.$$

4 Insurance in Life

Individual multi-state policy issued at time 0 and expiring at time n. $B(t)$ total payments of benefits less premiums in $[0,t]$

$$dB(t) = \sum_j I_j(t)\, dB_j(t) + \sum_{j\neq k} b_{jk}(t)\, dN_{jk}(t)$$

$$dB_j(t) \;=\; b_j(t)\, dt \;+\; \Delta B_j(t)$$

Life history:

$$\mathbf{H} = \{\mathcal{H}_t\}_{t\geq 0}; \quad \mathcal{H}_t = \sigma\{Z(\tau);\, 0 \leq \tau \leq t)\}$$

Reserve at time t:

$$V_{\mathbf{H}}(t) = \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r} dB(\tau) \,\middle|\, \mathcal{H}_t \right]$$

In the Markov case this reduces to

$$V_{Z(t)}(t) = \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r} dB(\tau) \,\middle|\, Z(t) \right]$$

Thus, we need only the *state-wise prospective reserves*

$$\begin{aligned} V_j(t) \;&=\; \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r} dB(\tau) \,\middle|\, Z(t) = j \right] \\ &=\; \int_t^T e^{-\int_t^\tau r} \sum_g p_{jg}(t,\tau) \left(dB_g(\tau) + \sum_{h;h\neq g} b_{gh}(\tau)\, \mu_{gh}(\tau)\, d\tau \right) \end{aligned}$$

deterministic functions of t. Backward ODE:

$$\frac{d}{dt} V_j(t) = r(t)\, V_j(t) - b_j(t) - \sum_{k;k\neq j} \mu_{jk}(t)\, R_{jk}(t)$$

$$R_{jk}(t) = b_{jk}(t) + V_k(t) - V_j(t) \quad \text{“sum at risk”}$$

Pasting conditions $\Delta V_j(t) = -\Delta B_j(t)$,
starting from $V_j(T-) = \Delta B_j(T)$.

5 Semi-Markov model and path-dependent payments

State $Z(t)$, *Policy duration* t, *State duration* $S(t)$

$\mu_{jk}(s, S(t))$, intensity of transition
$b_j(s, S(t))$, rate of annuity payment
$b_{jk}(s, S(t-))$ sum assured
$\Delta B_j(S(T))$ terminal endowment

$V_j(s,t)$ reserve in state j at policy duration t and state duration $S(t) = s$.

$$\begin{aligned} V_j(s,t) &= (1 - \mu_{j\,.}(s,t)\,dt)\Big(b_j(s,t)\,dt + e^{-r(t)\,dt}V_j(s+dt, t+dt)\Big) \\ &\quad + \sum_{k;\,k\neq j} \mu_{jk}(s,t)\,dt\ (b_{jk}(s,t) + V_k(0,t)) + o(dt) \end{aligned}$$

First order PDE-s

$$\begin{aligned} \frac{\partial}{\partial t} V_j(s,t) &= r(t)V_j(s,t) - \frac{\partial}{\partial s}V_j(s,t) - b_j(s,t) \\ &\quad - \sum_{k;k\neq j} \mu_{jk}(s,t)\,(b_{jk}(s,t) + V_k(0,t) - V_j(s,t)) \end{aligned}$$

Terminal conditions:

$$V_j(s, n-) = \Delta B_j(s)$$

6 Principle of equivalence

$$\mathbb{E}\left[\int_{0-}^{T} e^{-\int_0^\tau r} dB(\tau)\right] = 0$$

$$\int_{0-}^{T} e^{-\int_0^\tau r} \sum_g p_{0g}(0,\tau) \left(dB_g(\tau) + \sum_{h;h\neq g} b_{gh}(\tau)\, \mu_{gh}(\tau)\, d\tau \right) = 0$$

$$\Delta B_0(0) + V_0(0) = 0$$

7 Martingale techniques

Probability space: $(\Omega, \mathcal{F}, \mathbf{P})$

$\mathbf{E}\, X = \int_\Omega X(\omega)\, d\mathbf{P}(\omega)$

$\mathbf{E}\,[X\,Y] = \mathbf{E}[\,\mathbf{E}\,[\,X\,|\,\mathcal{G}]\,Y]\ \forall\ Y \in \mathcal{G}$

Doob & Co:

Filtration: $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$;
$\mathcal{F}_t \in \mathcal{F}$, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$, $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$

Stopping time: r.v. $T \geq 0$, $[T \leq t] \in \mathcal{F}_t\ \ \forall\ t$

$\mathcal{F}_{_T} = \{A \in \mathcal{F};\ A \cap [T \leq t] \in \mathcal{F}_t\ \ \forall\ t\}$
$(X_t)_{t\geq 0}$ *adapted to* $\mathbf{F}$ if $X_t \in \mathcal{F}_t\ \ \forall\ t$

$\mathbf{F}$-adapted $(M_t)_{t\geq 0}$ is *martingale* (m.g.) $(\mathbf{F}, \mathbf{P})$ if it has RCLL paths, $\mathbf{E}\,|M_t| < \infty$, and

$$\mathbf{E}\,[M_t|\mathcal{F}_s] \;=\; M_s\ \ \forall s \leq t$$

$\mathcal{F}_0 = \{\emptyset, \Omega\}$: $\mathbf{E}[M_t] = M_0$

M.g. *associated with* X, $\mathbf{E}\,|X| < \infty$: $\quad M_t = \mathbf{E}[X\,|\,\mathcal{F}_t]$

Optional sampling: For stopping times $S \leq T$,

$$\mathbf{E}[M_{_T}|\mathcal{F}_{_S}] = M_{_S}$$

Itô & Co:

$$df(X_t) \;=\; f'(X_{t-})\,dX_t + \frac{1}{2} f''(X_{t-})\,d[X,X]^c_t$$

$$+\{f(X_t) \,-\, f(X_{t-}) \,-\, f'(X_{t-})\,\Delta X_t\}$$

Optional variance process:
$[X,X]_t = \lim \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2$, $0 = t_0 < \cdots < t_n = t$

topscript c denotes continuous part
$\Delta X_t = X_t - X_{t-}$ (jump)

X finite variation: $\sup \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}| < \infty$, $0 = t_0 < \cdots < t_n = t$
W Brownian motion: $[W,W]_t = t$. Infinite variation.

M m.g. with paths that are continuous and of bounded variation is constant:

$$M_t^2 = M_0^2 + \int_0^t 2\, M_s \, dM_s$$

$\mathbf{E}\,[M_t^2] = M_0^2 = \mathbf{E}^2[M_t] \Rightarrow \mathbb{V}\mathrm{ar}[M_t] = 0 \Rightarrow M_t = M_0$.

Programme:
To find $\mathbf{E}\,[X]$, analyze the associated m.g. M_t to obtain $M_0 = \mathbf{E}\,[X]$.

8 Martingales in Life

Start from martingale

$$M(t) = \mathbb{E}[X \,|\mathcal{H}_t]$$

$$\begin{aligned}
M(t) &= \mathbb{E}\left[\int_{0-}^{T} e^{-\int_0^\tau r} dB(\tau) \,\middle|\, \mathcal{H}_t\right] \\
&= \int_{0-}^{t} e^{-\int_0^\tau r} dB(\tau) \;+\; e^{-\int_0^t r}\, \mathbb{E}\left[\int_{t}^{T} e^{-\int_t^\tau r} dB(\tau) \,\middle|\, \mathcal{H}_t\right] \\
&= \int_{0-}^{t} e^{-\int_0^\tau r} dB(\tau) \;+\; e^{-\int_0^t r} V_{Z(t)}(t)
\end{aligned}$$

$$\begin{aligned}
& dM(t) \\
&= e^{-\int_0^t r} dB(t) \;+\; e^{-\int_0^t r}(-r(t)\,dt) \sum_j I_j(t)\, V_j(t) \\
&\quad + e^{-\int_0^t r} \sum_j I_j(t)\, dV_j(t) \;+\; e^{-\int_0^t r} \sum_{j \neq k} dN_{jk}(t)\,(V_k(t) - V_j(t-)) \\
&= e^{-\int_0^t r} \sum_j I_j(t) \left(dB_j(t) - r(t) V_j(t)\, dt + dV_j(t) + \sum_{k;\, k \neq j} \mu_{jk}\, dt\, R_{jk}(t) \right) \\
&\quad + e^{-\int_0^t r} \sum_{j \neq k} R_{jk}(t)\, dM_{jk}(t)
\end{aligned}$$

Drift term must be 0, and we obtain the constructive ODE.

9 Relationships in Life

General relationship between present values of annuities, endowments and life assurances:

$$\begin{aligned} e^{-\int_t^T r} I_g(T)\, B_g(T) &= I_g(t)\, B_g(t) \\ &\quad - \int_t^T e^{-\int_t^\tau r} I_g(\tau)\, B_g(\tau)\, r(\tau)\, d\tau \\ &\quad + \sum_{h;\, h\neq g} \int_t^T e^{-\int_t^\tau r} B_g(\tau-)\, dN_{hg}(\tau) \\ &\quad - \sum_{h;\, h\neq g} \int_t^T e^{-\int_t^\tau r} B_g(\tau-)\, dN_{gh}(\tau) \\ &\quad + \int_t^T e^{-\int_t^\tau r} I_g(\tau)\, dB_g(\tau) \end{aligned}$$

10 Those moments in Life

$$V_j^{(q)}(t) = \mathbb{E}\left[V^q(t) \mid Z(t) = j\right], \quad q = 0, 1, 2, \ldots$$

$$\begin{aligned} \frac{d}{dt} V_j^{(q)}(t) &= (q\, r(t) + \mu_{j\cdot}(t)) V_j^{(q)}(t) \,-\, q\, b_j(t)\, V_j^{(q-1)}(t) \\ &\quad - \sum_{k;\, k\neq j} \mu_{jk}(t) \sum_{p=0}^{q} \binom{q}{p} b_{jk}^p(t) V_k^{(q-p)}(t) \end{aligned}$$

$$V_j^{(q)}(t-) = \sum_{p=0}^{q} \binom{q}{p} \Delta B_j(t)^p\, V_j^{(q-p)}(t)$$

$$V_j^{(q)}(n-) = \Delta B_j^q(t)$$

Central moments $m_j^{(q)}(t)$:

$$m_j^{(1)}(t) = V_j^{(1)}(t)$$

$$m_j^{(q)}(t) \;=\; \sum_{p=0}^{q} (-1)^{q-p} \binom{q}{p} V_j^{(p)}(t) \left(V_j^{(1)}(t) \right)^{q-p}$$

Numerical examples in disability model:

$$r = \ln(1.045) = 0.044017$$
$$\mu_x = \nu_x = 0.0005 + 0.000075858 \cdot 10^{0.038x}$$
$$\sigma_x = 0.0004 + 0.0000034674 \cdot 10^{0.06x}$$
$$\rho_x = 0.005$$

Male insured at age 30 for 30 years:
$\mu_{02}(t) = \mu_{12}(t) = \mu_{30+t}$
$\mu_{01}(t) = \sigma_{30+t}$
$\mu_{10}(t) = \rho_{30+t}$

Table 2: Moments for a life assurance with sum 1

Time t	0	6	12	18	24	30
$m_0^{(1)}(t) = m_1^{(1)}(t)$:	0.0683	0.0771	0.0828	0.0801	0.0592	0
$m_0^{(2)}(t) = m_1^{(2)}(t)$:	0.0300	0.0389	0.0484	0.0549	0.0484	0
$m_0^{(3)}(t) = m_1^{(3)}(t)$:	0.0139	0.0191	0.0262	0.0343	0.0369	0

Table 3: Moments for an annuity of 1 per year while active:

Time t	0	6	12	18	24	30
$m_0^{(1)}(t)$:	15.763	13.921	11.606	8.698	4.995	0
$m_1^{(1)}(t)$:	0.863	0.648	0.431	0.230	0.070	0
$m_0^{(2)}(t)$:	5.885	5.665	4.740	2.950	0.833	0
$m_1^{(2)}(t)$:	7.795	5.372	3.104	1.290	0.234	0
$m_0^{(3)}(t)$:	−51.550	−44.570	−32.020	−15.650	−2.737	0
$m_1^{(3)}(t)$:	78.888	49.950	25.099	8.143	0.876	0

Table 4: Moments for combined policy with life assurance of 1 plus disability annuity of 0.5 per year against net premium of 0.013108 per year while active:

Time t	0	6	12	18	24	30
$m_0^{(1)}(t)$:	0.0000	0.0410	0.0751	0.0858	0.0533	0
$m_1^{(1)}(t)$:	7.6451	6.8519	5.8091	4.4312	2.5803	0
$m_0^{(2)}(t)$:	0.4869	0.5046	0.4746	0.3514	0.1430	0
$m_1^{(2)}(t)$:	2.7010	2.0164	1.2764	0.5704	0.0974	0
$m_0^{(3)}(t)$:	2.1047	1.9440	1.5563	0.8686	0.1956	0
$m_1^{(3)}(t)$:	−12.1200	−8.1340	−4.3960	−1.5100	−0.1430	0

11 Stochastic interest

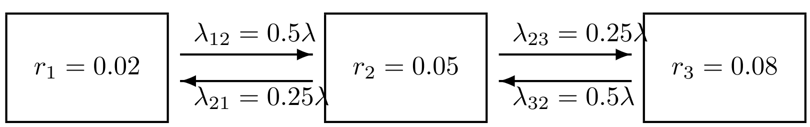


Figure 1: A simple Markov chain interest model.

Economy governed by Markov chain Y on state space $\mathcal{Y} = \{1, \ldots, J^Y\}$ with intensities of transition λ_{ef}, $e, f \in \mathcal{Y}$, $e \neq f$. Force of interest

$$r(t) = \sum_e I_e^Y(t)\, r_e$$

Payment process: Life history is Markov chain Z and payment stream standard type.

The full Markov model:
Y and Z are independent. Then $X = (Y, Z)$ is Markov

chain on $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$ with intensities

$$\kappa_{ej,fk}(t) = \begin{cases} \lambda_{ef}\,, & e \neq f,\ j = k\,, \\ \mu_{jk}(t)\,, & e = f,\ j \neq k\,, \\ 0\,, & e \neq f,\ j \neq k\,. \end{cases}$$

Moments in the combined model:

$$V_{ej}^{(q)}(t) = \mathbb{E}\left[\left(\int_t^n e^{-\int_t^\tau r}\, dB(\tau) \right)^q \,\middle|\, Y(t) = e, Z(t) = j \right] .$$

$$\frac{d}{dt} V_{ej}^{(q)}(t) \;=\; (q r_e + \mu_{j\cdot}(t) + \lambda_{e\cdot}) V_{ej}^{(q)}(t) - q b_j(t) V_{ej}^{(q-1)}(t)$$
$$- \sum_{k;k\neq j} \mu_{jk}(t) \sum_{p=0}^{q} \binom{q}{p} b_{jk}^p(t) V_{ek}^{(q-p)}(t) - \sum_{f;f\neq e} \lambda_{ef} V_{fj}^{(q)}(t)$$

$$V_{ej}^{(q)}(t-) = \sum_{p=0}^{q} \binom{q}{p} (\Delta B_j(t))^p V_{ej}^{(q-p)}(t)$$

Table 5: Central moments $m_{ej}^{(q)}(0)$, $q = 1, 2, 3$ of present value benefits less premiums for *combined policy* in interest state e and policy state j at time 0, for different values of the rate of interest changes, λ. Second column gives net premium π of a policy starting from interest state 2 (medium) and policy state 1 (active).

	e, j :		$1, a$	$1, i$	$2, a$	$2, i$	$3, a$	$3, i$
λ	π	q						
		1	0.15	13.39	0.00	7.65	−0.39	5.03
0	.0131	2	2.55	12.50	0.49	2.70	0.13	0.80
		3	20.45	−99.02	2.11	−12.12	0.37	−2.38
		1	0.06	11.31	0.00	7.90	−0.03	5.78
.05	.0137	2	1.61	12.26	0.62	5.41	0.25	2.43
		3	11.94	−42.87	3.20	−4.33	0.94	−0.08
		1	0.02	8.43	0.00	7.81	−0.02	7.24
.5	.0134	2	0.65	4.90	0.55	4.15	0.46	3.52
		3	3.34	−13.35	2.59	−10.13	2.02	−7.74
		1	0.00	7.77	0.00	7.70	0.00	7.64
5	.0132	2	0.51	2.86	0.50	2.91	0.49	2.86
		3	2.26	−12.51	2.20	−12.19	2.14	−11.88
		1	0.00	7.69	0.00	7.69	0.00	7.69
∞	.0132	2	0.50	2.74	0.50	2.74	0.50	2.74
		3	2.15	−12.37	2.15	−12.37	2.15	−12.37

12 Life in a stochastic Environment

Individual multi-state policy issued at time 0 and expiring at time T. Uncertainty represented by $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$.

Policy history is sub-filtration $\mathbf{H} = \{\mathcal{H}_t\}_{t\in[0,T]} = \mathbf{F}^Z$.

History of economic-demographic environment: $\mathbf{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$. Environmental indices: $\{Y(t)\}_{t\in[0,T]}$, $\mathbf{G}$-adapted process comprising $r(t)$ and mortality factors etc.

(Y, Z) is Markov: Y is Markov, and intensity matrix of Z is $\mathbf{M}(t, Y(t)) = (\mu_{jk}(t, Y(t)))$.
Conditional transition probabilities of Z, given Y,

$$p_{ij}(s,t) = \mathbf{P}[Z(t) = j \,|\, Z(s) = i,\, \mathcal{G}_T] = \mathbf{P}[Z(t) = j \,|\, Z(s) = i,\, \mathcal{G}_t]$$

$$d\mathbf{P}(s,t) = \mathbf{P}(s,t)\,\mathbf{M}(t, Y(t))$$

Problem: Future interest and mortality etc are unknown at time 0.
Equivalence must now mean

$$\mathbb{E}\left[\int_{0-}^{T} e^{-\int_0^\tau r} dB(\tau) \middle| \mathcal{G}_T \right] = 0$$

$$\int_{0-}^{T} e^{-\int_0^\tau r} \sum_g p_{0g}(0,\tau) \left(dB_g(\tau) + \sum_{h;h\neq g} b_{gh}(\tau)\, \mu_{gh}(\tau, Y(\tau))\, d\tau \right) = 0$$

with probability one.
Thus, B must be adapted to $\mathbf{G} \vee \mathbf{H}$.

13 Unit linked

Clear-cut unit linked policy:

$$B_g(t) = e^{\int_0^t r} \frac{p^*_{0g}(0,t)}{p_{0g}(0,t)} \, dB^*_g(t)$$

$$b_{gh}(t) = e^{\int_0^t r} \frac{p^*_{0g}(0,t) \, \mu^*_{gh}(t)}{p_{0g}(0,t) \, \mu_{gh}(t, Y(t))} \, b^*_{gh}(t)$$

Equivalence requirement reduces to

$$\int_{[0,T]} \left(\sum_g p^*_{0g}(0,\tau) \, dB^*_g(\tau) + \sum_{g \neq h} p^*_{0g}(0,\tau) \, \mu^*_{gh}(\tau) \, b^*_{gh}(\tau) \, d\tau \right) = 0$$

Arranged by choice of baseline payments B^*_g and b^*_{gh} at time 0. Environment risk managed perfectly from a solvency point of view.

Predict level of the payments at future time u, e.g.

$$W = \frac{e^{\int_0^u r}}{p_{0g}(0,u) \, \mu_{gh}(u, Y(u))}$$

Start from martingale

$$\begin{aligned} M_t &= \mathbf{E} \left[W \mid \mathcal{G}_t \right] \\ &= \mathbf{E} \left[\left. \frac{e^{\int_0^u r}}{p_{0g}(0,u) \, \mu_{gh}(u, Y(u))} \right| \mathcal{G}_t \right] \\ &= e^{\int_0^t r} \, \mathbf{E} \left[\left. \frac{e^{\int_t^u r}}{\sum_{i \in \mathcal{Z}} p_{0i}(0,t) \, p_{ig}(t,u) \, \mu_{gh}(u, Y(u))} \right| \mathcal{G}_t \right] \\ &= e^{\int_0^t r} \, V\left(t, p_{00}(0,t), \ldots, p_{0J^Z}(0,t), Y(t)\right) , \end{aligned}$$

where

$$V\left(t, p_0, \ldots, p_{J^Z}, y\right) = \mathbf{E} \left[\left. \frac{e^{\int_t^u r}}{\sum_{i \in \mathcal{Z}} p_i \, p_{ig}(t,u) \, \mu_{gh}(u, Y(u))} \right| Y(t) = y \right] .$$

Side condition that applies in all cases is

$$V\left(u, p_0, \ldots, p_{J^Z}, y\right) = \frac{1}{p_g\, \mu_{gh}(u,y)}\,.$$

Unit linked with guarantee:

$$\begin{aligned} dB_g(t) &= e^{\int_0^t r} \left(\frac{p^*_{0g}(0,t)}{p_{0g}(0,t)} \vee m_g \right) dB^*_g(t) \\ b_{gh}(t) &= e^{\int_0^t r} \left(\frac{p^*_{0g}(0,t)\, \mu^*_{gh}(t)}{p_{0g}(0,t)\, \mu_{gh}(t, Y(t))} \vee m_{gh} \right) b^*_{gh}(t) \end{aligned}$$

Take m_g and m_{gh} deterministic.

Guarantees reintroduce environmental risk.

Equivalence would mean that

$$\begin{aligned} W &= \int_{[0,T]} \Bigg(\sum_g \left(p^*_{0g}(0,\tau) \vee p_{0g}(0,\tau)\, m_g \right) dB^*_g(\tau) \\ &\quad + \sum_{g \neq h} \left(p^*_{0g}(0,\tau)\, \mu^*_{gh}(\tau) \vee p_{0g}(0,\tau)\, \mu_{gh}(\tau, Y(\tau))\, m_g \right) b^*_{gh}(\tau)\, d\tau \Bigg) \end{aligned}$$

should be 0 with probability 1. Not possible in general. Now an issue is to measure the risk associated with the guarantees.

One could determine the distribution of W. Easier to make an approximation based on the three first moments and approximate the ε-fractile of W with

$$\mathbf{E}[W] + c_\varepsilon \sqrt{m_W^{(2)}} + \frac{c_\varepsilon^2 - 1}{6} \frac{m_W^{(3)}}{m_W^{(2)}}$$

c_ε is ε-fractile of the standard normal distribution, $m_W^{(q)}$ is q-th central moment of W.

To obtain differential equation from which to determine $\mathbf{E}[W]$, consider

$$\begin{aligned} M_t &= \mathbf{E}\left[W \mid \mathcal{G}_t\right] \\ &= \int_{[0,t]} \left(\sum_g \left(p^*_{0g}(0,\tau) \vee p_{0g}(0,\tau)\, m_g\right) dB^*_g(\tau) \right. \\ &\quad \left. + \sum_{g\neq h} \left(p^*_{0g}(0,\tau)\ \mu^*_{gh}(\tau) \vee p_{0g}(0,\tau)\, \mu_{gh}(t, Y(\tau))\, m_g\right)\ b^*_{gh}(\tau)\, d\tau \right) \\ &\quad + V\left(t, p_{00}(0,t), \ldots, p_{1J^Z}(0,t), Y(t)\right) \end{aligned}$$

where

$$\begin{aligned} &V\left(t, p_0, \ldots, p_{J^Z}, y\right) \\ &= \mathbf{E}\left[\int_{(t,T]} \left(\sum_g \left(p^*_{0g}(0,\tau) \vee \sum_i p_i\, p_{ig}(t,\tau)\, m_g \right) dB^*_g(\tau) \right.\right. \\ &+ \left.\left. \sum_{g\neq h} \left(p^*_{0g}(0,\tau)\ \mu^*_{gh}(\tau) \vee \sum_i p_i\, p_{ig}(t,\tau)\, \mu_{gh}(t, Y(\tau))\, m_g \right)\ b^*_{gh}(\tau)\, d\tau \right) \right| Y(t) \end{aligned}$$

Itô:

$$\begin{aligned} dM_t &= \sum_g \left(p^*_{0g}(0,t) \vee p_{0g}(0,t)\, m_g\right) dB^*_g(t) \\ &\quad + \sum_{g\neq h} \left(p^*_{0g}(0,t)\ \mu^*_{gh}(t) \vee p_{0g}(0,t)\, \mu_{gh}(t, Y(t))\, m_g\right)\ b^*_{gh}(t)\, dt \\ &\quad + \frac{\partial}{\partial t} V\left(t, p_{00}(0,t), \ldots, p_{0J^Z}(0,t), Y(t)\right) dt \\ &\quad + \sum_i \frac{\partial}{\partial p_i} V\left(t, p_{00}(0,t), \ldots, p_{0J^Z}(0,t), Y(t)\right) \sum_h p_{0h}(0,t)\, \mu_{hi}(t, Y(t)) dt \end{aligned}$$

$$+\sum_k \frac{\partial}{\partial y_k} V\left(t, p_{00}(0,t), \ldots, p_{0J^Z}(0,t), Y(t)\right) dY_k^c(t)$$

$$+\frac{1}{2}\sum_{k,\ell} \frac{\partial^2}{\partial y_k \partial y_\ell} V\left(t, p_{00}(0,t), \ldots, p_{0J^Z}(0,t), Y(t)\right) d[Y_k, Y_\ell]^c(t)$$

$$+V\left(t, p_{00}(0,t), \ldots, p_{0J^Z}(0,t), Y(t)\right) - V\left(t, p_{00}(0,t), \ldots, p_{0J^Z}(0,t), Y(t$$

Identify drift part, which must be 0, and obtain a differential equation for the function $V(t, p_0, \ldots, p_{J^Z}, y)$. One side condition that applies in any case is

$$V(T-, p_0, p_2, \ldots, p_{J^Z}, y) = \sum_g \left(p_{0g}^*(0,T) \vee p_g m_g\right) \Delta B_g^*(T)\,.$$

Other side conditions obtained by auxiliary probabilistic reasoning depending on the situation.

Solving the equation numerically, we finally obtain the value we were after, $\mathbf{E}[W] = V(0,1,0,\ldots,0,Y_0)$.

The complexity of the differential equation depends on the nature of the driving process Y. Diffusion processes lead to second order PDE. Jump processes lead to first order PIDE. If Y has finite number of states (e.g. a Markov chain), then much hassle will go away: the function V can be seen as a finite-dimensional vector with elements $V(t, p_0, \ldots, p_{J^Z}, y)$, virtually a reduction of dimension; no second order derivatives; the integral over the jump sizes reduces to just a summation over the finite number of possible directions of transition out of the current state.

14 With Profit

Payments B_g^* and b_{gh}^* are guaranteed at time 0. They are designed in accordance with the equivalence principle using a prudently chosen technical basis with elements r^* and μ_{gh}^*. Surpluses that emerge are paid back as *bonuses* that may be cash dividends or premiums for purchase of additional benefits:
$D(t)$ is the total of dividends paid out cash by time t
$Q(t)$ is the total additional units of additional benefits B^{*+} guaranteed by time t. (At time t the company promises to pay $Q(t)\,(B^{*+}(\tau) - B^{*+}(t))$ for $\tau \in (t, T]$).
D and Q must be non-decreasing, hence of bounded variation, and $D(0) = Q(0) = 0$.
Payments from the company to the insured are

$$dB(t) = dB^*(t) + Q(t-)dB^{*+}(t) + dD(t)\,.$$

B^*, B^{*+} are of the standard form with B_g^*, b_{gh}^*, B^{*+}, b_{gh}^{*+} deterministic.
Q and D are adapted to $\mathbf{H} \vee \mathbf{G}$. They are not stipulated in the contract, but controlled by the company in view of the past experience and with a view to customer needs and solvency.

Company's assessed liability in respect of future payments at time t is

$$V_{Z(t)}^*(t) + Q(t)\,V_{Z(t)}^{*+}(t)$$

Discounted *surplus* at time t is

$$\tilde{W}(t) \;=\; -\int_{0-}^{t} e^{-\int_0^\tau r}\,\left(dB^*(\tau) + Q(\tau-)\,dB^{*+}(\tau) + dD(\tau)\right)$$

$$- e^{-\int_0^t r} \left(V^*_{Z(t)}(t) + Q(t)\, V^{*+}_{Z(t)}(t) \right)$$

$$\tilde{W}(0) = -\Delta B^*(0) - V_1^*(0) = 0$$

$$\tilde{W}(T) = -\int_{0-}^{T} e^{-\int_0^\tau r} \left(dB^*(\tau) + Q(\tau-)\, dB^{*+}(\tau) + dD(\tau) \right)$$

If first order basis can be chosen on the entirely safe side and bonuses are allotted with sufficient prudence, then one can arrange that $\tilde{W}(t) \geq 0$ for all t, and there is no solvency problem. This is possible if the interest rate process is bounded from below and also the transition intensities are suitably bounded as is the case if Y is a Markov chain with finite state space. Otherwise, the guarantees built into this product create environmental risk, and it becomes an issue to calculate (aspects of) the distribution of $\mathbf{E}[\tilde{W}(T) \,|\, \mathcal{G}_T]$.

Solvency requirement:

$$\mathbf{E}\left[\tilde{W}(t) \,\middle|\, \mathcal{G}_t \right] \geq 0\,,\ \ t \in [0,T]\,, \tag{1}$$

Equivalence:

$$\mathbf{E}\left[\tilde{W}(T) \,\middle|\, \mathcal{G}_T \right] = 0\,. \tag{2}$$

Applying Itô to $\tilde{W}(t)$:

$$d\tilde{W}(t) = e^{-\int_0^\tau r} (dC(t) - dD(t) - dQ(t) V^{*+}_{Z(t)}(t) + dM^*(t))$$

$C(t)$ is drift term representing “technical surplus”:

$$dC(t) = (r(t) - r^*) \left(V^*_{Z(t)}(t) + Q(t)\, V^{*+}_{Z(t)}(t) \right)\, dt$$
$$+ \sum_{h;h \neq Z(t)} \left(\mu^*_{Z(t)\,h}(t) - \mu_{Z(t)\,h}(t, Y(t)) \right) \left(R^*_{Z(t)\,h}(t) + Q(t) R^{*+}_{Z(t)\,h}(t) \right)\, dt$$

$$R^*_{gh}(t) = b^*_{gh}(t) + V^*_h(t) - V^*_g(t)\,, \quad R^{*+}_{gh}(t) = b^{*+}_{gh}(t) + V^{*+}_h(t) - V^{*+}_g(t)\,.$$

$M^*(t)$ is martingale (conditional on $\mathcal{G}_T$) representing pure life history randomness:

$$dM^*(t) = \sum_{g \neq h} \left(R^*_{gh}(t) + Q(t-) R^{*+}_{gh}(t) \right) \left(dN^Z_{gh}(t) - I_g(t) \mu^*_{gh}(t, Y(t))\, dt \right) \,,$$

Writing $\tilde{W}(T) = \int_{[0,T]} d\tilde{W}(\tau)$, and forming conditional expectation, equivalence can be recast as

$$\mathbf{E}\left[\int_0^T e^{-\int_0^\tau r} \left(dC(\tau) - dD(\tau) - dQ(t) V^{*+}_{Z(t)}(t) \right) \middle|\, \mathcal{G}_T \right] = 0$$

15 In the market we trust

Equivalence:

$$\mathbb{E}^{\mathbb{Q}}\left[\int_{0-}^{T} e^{-\int_0^\tau r} dB(\tau)\right] = 0$$

$\mathbb{Q}$ is EMM.

Problems:
1. Long term contracts.
2. Gross incompleteness.

Remedy: Financial innovation, e.g. Securitization of mortality risk?

Problems prevail to exist.

16 Modeling stochastic mortality

Vasiček interest model. $(r_t)_{t\geq 0}$ is Ornstein-Uhlenbeck process

$$dr_t = -\alpha\, r_t\, dt + dX_t$$

X is Levy process with LT

$$\ell_{X_t}(\eta) = \mathbf{E}\, e^{\eta\, X_t} = e^{\phi(\eta)\, t}$$

$$r_u = e^{-\alpha\,(u-t)} r_t + \int_t^u e^{-\alpha(u-s)} dX_s$$

Transition probs. of OU process are given by

$$r_u\,|_{r_t} \stackrel{\mathcal{L}}{=} e^{-\alpha(u-t)} r_t + R$$

where $R = \int_0^{u-t} e^{-\alpha s}\, dX_s$ has LT

$$\ell_R(\eta) = e^{\int_0^{u-t} \phi(\eta\, e^{-\alpha s}) ds}$$

First two moments are distribution-free:

$$r_u\,|_{r_t} \sim \left(\rho + e^{-\alpha\,(u-t)}(r_t - \rho)\,,\; \sigma^2 \frac{1 - e^{-2\alpha\,(u-t)}}{2\alpha}\right)$$

$$\int_t^T q_u r_u\, du = \int_t^T q_u e^{-\alpha\,(u-t)}\, du\; r_t + \int_t^T q_u e^{-\alpha u}\, du \int_t^u e^{\alpha s}\, dX_s\,.$$

$$\left.\int_t^T q_u r_u\, du\right|_{r_t} \stackrel{\mathcal{L}}{=} Q_t^T r_t + R$$

where $R = \int_t^T Q_u^T\, dX_u$ has LT

$$\ell_R(\eta) = e^{\int_t^T \phi\left(\eta\, Q_u^T\right) du}$$

The first two moments are distribution-free and are given by

$$\int_t^T q_u r_u \, du \Big|_{r_t} \sim \left(Q_t^T r_t + \mu \int_t^T Q_u^T \, du \, , \, \sigma^2 \int_t^T (Q_u^T)^2 \, du \right)$$

If the random part of X is a subordinator, then r is bounded from below and can be made positive.

If X is compound Poisson process with drift, then r_t and $\int_t^T q_u r_u \, du$ are compound Poisson variates (plus constants).

Stochastic mortality: Mortility rate at age x at calendar time t is

$$\mu_t(x) = \mu^o(x; \, Y(t))$$

where $\mu^o(x;\theta)$ is some parametric mortality intensity function, e.g.

$$\mu^o(x;\theta) = \theta_1 + \theta_2 \, e^{\theta_3 \, x}$$

and $(Y_1(t), Y_2(t), Y_3(t))$ is positive Ornstein-Uhlenbeck process.

Conditional survival probability

$$e^{-\int_0^{y-x} \mu_{t+s}(x+s)}$$

Reference: Norberg (2004)

17 The Markov chain market

The continuous time Markov chain.
$\{Y(t)\}_{t\geq 0}$ Markov chain on $\mathcal{Y} = \{1,\ldots,n\}$. Homogeneous transition probabilities

$$p_{ef}(t) = \mathbb{P}[Y(\tau+t) = f \mid Y(\tau) = e]$$

transition intensities

$$\lambda_{ef} = \lim_{t\searrow 0} \frac{p_{ef}(t)}{t}\,,\ e \neq f$$

$$\lambda_{ee} = -\lambda_{e\cdot} = -\sum_{f;f\in\mathcal{Y}_e} \lambda_{ef}$$

States directly accessible from state e:

$$\mathcal{Y}_e = \{f; \lambda_{ef} > 0\},\ \ n_e = |\mathcal{Y}_e|$$

$\mathbf{P}(t) = (p_{ef}(t))$ and $\mathbf{\Lambda} = (\lambda_{ef})$ are related by

$$\mathbf{\Lambda} = \lim_{t\searrow 0} \frac{1}{t}(\mathbf{P}(t) - \mathbf{I})$$

and by forward and backward Kolmogorov differential equations,

$$\frac{d}{dt}\mathbf{P}(t) = \mathbf{P}(t)\mathbf{\Lambda} = \mathbf{\Lambda}\mathbf{P}(t)$$

Under side condition $\mathbf{P}(0) = \mathbf{I}$ they integrate to

$$\mathbf{P}(t) = \exp(\mathbf{\Lambda} t)$$

Representation:

$$\mathbf{P}(t) = \mathbf{\Phi}\,\mathbf{D}_{e=1,\ldots,n}(e^{\rho_e t})\,\mathbf{\Phi}^{-1} = \sum_{e=1}^{n} e^{\rho_e t} \boldsymbol{\phi}_e \boldsymbol{\psi}_e{}'$$

The first eigenvalue is $\rho_1 = 0$, and corresponding eigenvectors are $\boldsymbol{\phi}_1 = \mathbf{1}$ and $\boldsymbol{\psi}_1{}' = (p_1, \ldots, p_n) = \lim_{t \nearrow \infty}(p_{e1}(t), \ldots, p_{en}(t))$, the stationary distribution of Y. Remaining eigenvalues, $\rho_2, \ldots, \rho_n$, have strictly negative real parts so that the transition probabilities converge exponentially to the stationary distribution as t increases.

Indicator functions

$$I_e(t) = 1[Y(t) = e]$$

Counting processes

$$N_{ef}(t) = |\{\tau;\, 0 < \tau \leq t,\, Y(\tau-) = e\,,\, Y(\tau) = f\}|\,,$$

Information at time t: $\mathcal{F}_t^Y = \sigma\{Y(\tau); 0 \leq \tau \leq t\}$.
Filtration $\mathbf{F}^Y = \{\mathcal{F}_t^Y\}_{t \geq 0}$.
Compensated counting processes

$$dM_{ef}(t) = dN_{ef}(t) - I_e(t)\lambda_{ef}\, dt \tag{3}$$

are zero mean, square integrable, mutually orthogonal martingales w.r.t. $(\mathbf{F}^Y, \mathbb{P})$.

The continuous time Markov chain market.
$Y(t)$ is state of the economy at time t.
In the market there are $m+1$ basic assets, which can be traded freely and frictionlessly.
Asset No. 0 is "locally risk-free" *bank account* with state-dependent interest rate

$$r(t) = r_{Y(t)} = \sum_e I_e(t) r_e$$

state-wise interest rates r_e, $e = 1, \ldots, n$, are constants.
Price process

$$S_0(t) = \exp\left(\int_0^t r(u)\,du\right) = \exp\left(\sum_e r_e \int_0^t I_e(u)\,du\right)$$

$\int_0^t I_e(u)\,du$ is total time spent in state e during $[0,t]$.

$$dS_0(t) = S_0(t)\,r(t)\,dt = S_0(t)\sum_e r_e I_e(t)\,dt$$

Remaining m assets are risky *stocks* with price processes

$$S_i(t) = \exp\left(\sum_e \alpha_{ie}\int_0^t I_e(u)\,du + \sum_e \sum_{f\in\mathcal{Y}_e} \beta_{ief} N_{ef}(t)\right)$$

$i = 1, \ldots, m$: α_{ie}, β_{ief} are constants. Itô:

$$dS_i(t) = S_i(t-)\left(\sum_e \alpha_{ie} I_e(t)\,dt + \sum_e \sum_{f\in\mathcal{Y}_e} \gamma_{ief}\, dN_{ef}(t)\right).$$

$$\gamma_{ief} = \exp(\beta_{ief}) - 1$$

relative price change upon jump $e \to f$.

Take bank account as numeraire. Discounted asset prices are $\tilde{S}_i(t) = S_i(t)/S_0(t)$, $i = 0, \ldots, m$ and $\tilde{S}_0(t) \equiv 1$, (martingale under any measure). Discounted stock prices

$$\tilde{S}_i(t) = \exp\left(\sum_e (\alpha_{ie} - r_e) \int_0^t I_e(u)\, du + \sum_e \sum_{f \in \mathcal{Y}_e} \beta_{ief} N_{ef}(t)\right)$$

$$d\tilde{S}_i(t) = \tilde{S}_i(t-) \left(\sum_e (\alpha_{ie} - r_e) I_e(t)\, dt + \sum_e \sum_{f \in \mathcal{Y}_e} \gamma_{ief} dN_{ef}(t)\right)$$

$i = 1, \ldots, m$.

Portfolios.
A dynamic *portfolio* or *investment strategy* is $m+1$-dimensional stochastic process

$$\boldsymbol{\theta}(t) = (\theta_0(t), \ldots, \theta_m(t)) :$$

$\theta_i(t)$ is number of units of asset No i held at time t. $\boldsymbol{\theta}$ is adapted to $\mathbf{F}^Y$ and the shares of stocks, $(\theta_1(t), \ldots, \theta_m(t))$, must also be $\mathbf{F}^Y$-predictable.

Value of the portfolio $\boldsymbol{\theta}$ at time t is

$$V^\theta(t) = \boldsymbol{\theta}(t)'\mathbf{S}(t) = \sum_{i=0}^m \theta_i(t) S_i(t)\,.$$

Work with discounted prices and values and equip their symbols with a tilde. Discounted value of the portfolio at time t is

$$\tilde{V}^\theta(t) = \boldsymbol{\theta}(t)'\, \tilde{\mathbf{S}}(t)$$

The portfolio $\boldsymbol{\theta}$ is *self-financing* (SF) if $dV^\theta(t) = \boldsymbol{\theta}(t)'\, d\mathbf{S}(t)$ or

$$d\tilde{V}^\theta(t) = \boldsymbol{\theta}(t)'\, d\tilde{\mathbf{S}}(t) = \sum_{i=1}^{m} \theta_i(t)\, d\tilde{S}_i(t)\,. \tag{4}$$

Absence of arbitrage.
Let

$$\tilde{\boldsymbol{\Lambda}} = (\tilde{\lambda}_{ef})$$

be infinitesimal matrix equivalent to $\boldsymbol{\Lambda}$ ($\tilde{\lambda}_{ef} = 0$ if and only if $\lambda_{ef} = 0$).
Girsanov: there exists measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$, under which Y is a Markov chain with infinitesimal matrix $\tilde{\boldsymbol{\Lambda}}$.
The processes $\tilde{M}_{ef}$,

$$d\tilde{M}_{ef}(t) = dN_{ef}(t) - I_e(t)\tilde{\lambda}_{ef}\, dt\,,$$

are zero mean, orthogonal martingales w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$.
Rewrite dynamics as

$$d\tilde{S}_i(t) \;=\; \tilde{S}_i(t-)\left[\sum_e \left(\alpha_{ie} - r_e + \sum_{f\in\mathcal{Y}_e} \gamma_{ief}\tilde{\lambda}_{ef}\right) I_e(t)\, dt + \sum_e \sum_{f\in\mathcal{Y}_e} \gamma_{ief} d\tilde{M}_{ef}(t)\right.$$

$i = 1, \ldots, m$. The discounted stock prices are martingales with respect to $(\mathbf{F}^Y, \tilde{\mathbb{P}})$ if and only if the drift terms on the right vanish:

$$\alpha_{ie} - r_e + \sum_{f\in\mathcal{Y}_e} \gamma_{ief}\tilde{\lambda}_{ef} = 0$$

$e = 1, \ldots, n$, $i = 1, \ldots, m$. In matrix form

$$r_e\mathbf{1} - \boldsymbol{\alpha}_e = \boldsymbol{\Gamma}_e\tilde{\boldsymbol{\lambda}}_e$$

$e = 1, \ldots, n$, where $\mathbf{1}$ is $m \times 1$ and

$$\boldsymbol{\alpha}_e = (\alpha_{ie})_{i=1,\ldots,m} \ , \quad \boldsymbol{\Gamma}_e = (\gamma_{ief})_{i=1,\ldots,m}^{f \in \mathcal{Y}_e} \ , \quad \tilde{\boldsymbol{\lambda}}_e = \left(\tilde{\lambda}_{ef}\right)_{f \in \mathcal{Y}_e}$$

Then

$$d\tilde{S}_i(t) \ = \ \tilde{S}_i(t-) \sum_e \sum_{f \in \mathcal{Y}_e} \gamma_{ief} d\tilde{M}_{ef}(t)$$

Assume martingale measure $\tilde{\mathbb{P}}$ exists. This implies absence of arbitrage:
Cannot have $\tilde{V}^\theta(0) = 0$ and at the same time $\tilde{V}^\theta(T) \geq 0$ almost surely and $\tilde{V}^\theta(T) > 0$ with positive probability.

Insert $\tilde{S}$-dynamics in $\tilde{V}$ dynamics:

$$d\tilde{V}^\theta(t) = \sum_e \sum_{f \in \mathcal{Y}_e} \sum_{i=1}^m \theta_i(t) \tilde{S}_i(t-) \gamma_{ief} d\tilde{M}_{ef}(t)$$

a martingale w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$ and, in particular,

$$\tilde{V}^\theta(t) = \tilde{\mathbb{E}}[\tilde{V}^\theta(T) \,|\, \mathcal{F}_t]$$

for $0 \leq t \leq T$. $\tilde{\mathbb{E}}$ is expectation under $\tilde{\mathbb{P}}$.

Explain the assumptions made about the components of the portfolio $\boldsymbol{\theta}(t)$, adaptedness of portfolio and predictability of shares of stocks.

Attainable claims.
A T-claim is a contractual payment due at time T: an $\mathcal{F}_T^Y$-measurable random variable H with finite expected value. The claim is *attainable* if it can be perfectly duplicated by some SF portfolio $\boldsymbol{\theta}$:

$$\tilde{V}^\theta(T) = \tilde{H}\,.$$

If an attainable claim should be traded in the market, its price must equal the value of the duplicating portfolio in order to avoid arbitrage. Thus, denoting the price process by $\pi(t)$,

$$\tilde{\pi}(t) = \tilde{V}^\theta(t) = \tilde{\mathbb{E}}[\tilde{H}\,|\,\mathcal{F}_t]$$

or

$$\pi(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r} H \,\middle|\, \mathcal{F}_t\right]$$

Dynamics

$$d\tilde{\pi}(t) = \sum_e \sum_{f\in\mathcal{Y}_e} \sum_{i=1}^m \theta_i(t)\tilde{S}_i(t-)\gamma_{ief}\, d\tilde{M}_{ef}(t)$$

Completeness.
An attainable T-claim H can be represented as

$$\tilde{H} = \tilde{\mathbb{E}}[\tilde{H}] + \int_0^T \sum_e \sum_{f\in\mathcal{Y}_e} \eta_{ef}(t)\, d\tilde{M}_{ef}(t)$$

where the $\eta_{ef}(t)$ are $\mathbf{F}^Y$-predictable processes. Conversely, any random variable of this form is a T-claim. Attainability of H means

$$\tilde{H} \;=\; \tilde{V}^\theta(0) + \int_0^T d\tilde{V}^\theta(t)$$

$$= \tilde{V}^{\theta}(0) + \int_0^T \sum_e \sum_{f \in \mathcal{Y}_e} \sum_i \theta_i(t) \tilde{S}_i(t-) \gamma_{ief} d\tilde{M}_{ef}(t)$$

Thus H is attainable iff there exist predictable processes $\theta_1(t), \ldots, \theta_m(t)$ such that

$$\sum_{i=1}^{m} \theta_i(t) \tilde{S}_i(t-) \gamma_{ief} = \eta_{ef}(t)$$

for all e and $f \in \mathcal{Y}_e$. This means that the n_e-vector

$$\boldsymbol{\eta}_e(t) = (\eta_{ef}(t))_{f \in \mathcal{Y}_e}$$

is in $\mathbb{R}(\mathbf{\Gamma}_e{}')$.

The market is *complete* if every T-claim is attainable, that is, if every n_e-vector is in $\mathbb{R}(\mathbf{\Gamma}_e{}')$. This is the case if and only if $\text{rank}(\mathbf{\Gamma}_e) = n_e$, which can be fulfilled for each e only if $m \geq \max_e n_e$, i.e. the number of risky assets is no less than the number of sources of randomness.

Differential equations for the arbitrage-free price. Assume the market is arbitrage-free and complete so that the price of any T-claim is uniquely given as conditional expected value its discounted value under the EMM.

Consider T-claim that depends only on the state of the economy and the price of a given stock at time T. dropping top-script indicating this stock:

$$S(t) = \exp\left(\sum_e \alpha_e \int_0^t I_e(u)\, du + \sum_e \sum_{f \in \mathcal{Y}_e} \beta_{ef} N_{ef}(t) \right)$$

Thus

$$H = h_{Y(T)}(S(T)) = \sum_e I_e(T) h_e(S(T)) \qquad (5)$$

Examples: European call option $H = (S(T) - K)^+$;
Caplet $H = (r(T) - g)^+ = (r_{Y(T)} - g)^+$;
Zero coupon T-bond $H = 1$.

For any claim of the form (5) the relevant state variables involved in the conditional expectation are $(S(t), t, Y(t))$:

$$S(T) = S(t) \exp\left(\sum_e \alpha_e \int_t^T I_e(u)\, du + \sum_e \sum_{f \in \mathcal{Y}_e} \beta_{ef} \left(N_{ef}(T) - N_{ef}(t)\right)\right) \quad (6)$$

due to Markov property. It follows that the price $\pi(t)$ is of the form

$$\pi(t) = \sum_{e=1}^{n} I_e(t) v_e(S(t), t)$$

$$v_e(s,t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r} H \,\middle|\, Y(t) = e,\, S(t) = s \right]$$

are the state-wise prices. By (6) and homogeneity of Y, we obtain

$$v_e(s,t) = \mathbb{E}[h_{Y(T-t)}(s\, S(T-t)) | Y(0) = e] \quad (7)$$

The discounted price (5) is martingale with respect to $(\mathbf{F}^Y, \tilde{\mathbb{P}})$. Assume that the functions $v_e(s,t)$ are continuously differentiable. Applying Itô to

$$\tilde{\pi}(t) = e^{-\int_0^t r} \sum_{e=1}^{n} I_e(t) v_e(S(t), t)$$

we find

$$d\tilde{\pi}(t) \;=\; e^{-\int_0^t r} \sum_e I_e(t) \left(-r_e\, v_e(S(t), t) + \frac{\partial}{\partial t} v_e(S(t), t) + \frac{\partial}{\partial s} v_e(S(t), t) S(t) \alpha_e \right)$$

$$
\begin{aligned}
&+e^{-\int_0^t r}\sum_e\sum_{f\in\mathcal{Y}_e}\left(v_f(S(t-)(1+\gamma_{ef}),t)-v_e(S(t-),t)\right)dN_{ef}(t)\\
= \; & e^{-\int_0^t r}\sum_e I_e(t)\{-r_e\,v_e(S(t),t)+\frac{\partial}{\partial t}v_e(S(t),t)+\frac{\partial}{\partial s}v_e(S(t),t)S(t)\alpha_e\\
&+\sum_{f\in\mathcal{Y}_e}\left(v_f(S(t-)(1+\gamma_{ef}),t)-v_e(S(t-),t)\right)\tilde{\lambda}_{ef}\}dt\\
&+e^{-\int_0^t r}\sum_e\sum_{f\in\mathcal{Y}_e}\left(v_f(S(t-)(1+\gamma_{ef}),t)-v_e(S(t-),t)\right)d\tilde{M}_{ef}(t)\,.
\end{aligned}
$$

We arrive at the non-stochastic PDE

$$
\begin{aligned}
&-r_e\,v_e(s,t)+\frac{\partial}{\partial t}v_e(s,t)+\frac{\partial}{\partial s}v_e(s,t)s\alpha_e\\
&\quad+\sum_{f\in\mathcal{Y}_e}\left(v_f(s(1+\gamma_{ef}),t)-v_e(s,t)\right)\tilde{\lambda}_{ef}=0
\end{aligned}
$$

$$
v_e(s,T)=h_e(s)\,,
$$

$e=1,\ldots,n$.

In matrix form, with

$$
\mathbf{R}=\mathbf{D}_{e=1,\ldots,n}(r_e)\,,\quad \mathbf{A}=\mathbf{D}_{e=1,\ldots,n}(\alpha_e)\,,
$$

$$
-\mathbf{R}\mathbf{v}(s,t)+\frac{\partial}{\partial t}\mathbf{v}(s,t)+s\mathbf{A}\frac{\partial}{\partial s}\mathbf{v}(s,t)+\tilde{\mathbf{\Lambda}}\mathbf{v}(s(1+\gamma),t)=\mathbf{0}
$$

Side conditions

$$
\mathbf{v}(s,T)=\mathbf{h}(s)
$$

Once we have determined $v_e(s,t)$, $e=1,\ldots,n$, the price process is known.

The duplicating SF strategy is obtained as follows. Setting the drift term to 0 we find dynamics of the discounted price;

$$d\tilde{\pi}(t) = e^{-\int_0^t r} \sum_e \sum_{f \in \mathcal{Y}_e} \left(v_f(S(t-)(1+\gamma_{ef}), t) - v_e(S(t-), t)\right) d\tilde{M}_{ef}(t)\,.$$

Identifying coefficients with those in (5), we obtain

$$\sum_{i=1}^{m} \theta_i(t) S_i(t-)\gamma_{ief} = v_f(S(t-)(1+\gamma_{ef}), t) - v_e(S(t-), t)\,, \quad (8)$$

$f \in \mathcal{Y}_e$. The solution $(\theta_{i,e}(t))_{i=1,\ldots,m}$ exists since $\operatorname{rank}(\mathbf{\Gamma}_e) \leq m$, and it is unique iff $\operatorname{rank}(\mathbf{\Gamma}_e) = m$. It is a function of t and $\mathbf{S}(t-)$ and is thus predictable.

Finally, θ^0 is determined upon combining (4), (5), and (8):

$$\theta(t)^0 = e^{-\int_0^t r} \left(\sum_{e=1}^{n} I_e(t) v_e(S(t), t) - \sum_{i=1}^{m} \theta_i(t) S_i(t) \right) .$$

This function is not predictable.

Asian option.

An example of a path-dependent claim is Asian option, $H = \left(\frac{1}{T}\int_0^T S(\tau)\,d\tau - K\right)^+$, where $K \geq 0$. Price process

$$\begin{aligned} \pi(t) &= \tilde{\mathbb{E}}\left[e^{-\int_t^T r} \left(\frac{1}{T}\int_0^T S(\tau)\,d\tau - K \right)^+ \middle| \mathcal{F}_t^Y \right] \\ &= \sum_{e=1}^{n} I_e(t) v_e\left(S(t), t, \int_0^t S(\tau)\,d\tau \right) , \end{aligned}$$

where

$$v_e(s,t,u) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r} \left(\frac{1}{T}\int_t^T S(\tau)\,d\tau + \frac{u}{T} - K \right)^+ \middle| Y(t) = e,\, S(t) = s \right] .$$

The discounted price process is

$$\tilde{\pi}(t) = e^{-\int_0^t r} \sum_{e=1}^{n} I_e(t)\, v_e\left(t, S(t), \int_0^t S(\tau)\, d\tau\right) .$$

We are lead to partial differential equations in three variables.

Interest rate derivatives.
A simple class of claims are those of the form $H = h_{Y(T)}$. Interest rate derivatives of the form $H = h(r(T))$ are included since $r(t) = r_{Y(t)}$.
For such claims the only relevant state variables are t and $Y(t)$, so that the function in (7) depends only on t and e. The PDE (8) reduce to the ODE

$$\frac{d}{dt} v_e(t) = r_e v_e(t) - \sum_{f \in \mathcal{Y}_e} (v_f(t) - v_e(t)) \tilde{\lambda}_{ef}$$

$$v_e(T) = h_e$$

In matrix form:

$$\frac{d}{dt}\mathbf{v}(t) = (\tilde{\mathbf{R}} - \tilde{\mathbf{\Lambda}})\mathbf{v}(t)\,,$$

$$\mathbf{v}(T) = \mathbf{h}\,.$$

Explicit solution

$$\mathbf{v}(t) = \exp\{(\tilde{\mathbf{\Lambda}} - \mathbf{R})\,(T - t)\}\mathbf{h}\,. \tag{9}$$

Depends on t and T only through $T - t$.

In particular, the zero coupon bond with maturity T corresponds to $\mathbf{h} = \mathbf{1}$:

$$\mathbf{p}(t, T) = \exp\{(\tilde{\mathbf{\Lambda}} - \mathbf{R})\,(T - t)\}\mathbf{1}$$

Incompleteness.

The notion of incompleteness pertains to situations where there exist contingent claims that cannot be duplicated by an SF portfolio and, consequently, do not receive unique prices from the no arbitrage postulate alone. Incompleteness arises from scarcity of traded assets, that is, the discounted basic price processes are incapable of spanning the space of all martingales with respect to $(\mathbf{F}^Y, \tilde{\mathbb{P}})$ and, in particular, reproducing the value (5) of every financial derivative.

Risk minimization.

Throughout this section we will mainly be working with discounted prices and values without any other mention than the tilde notation. The reason is that the theory of risk minimization rests on certain martingale representation results that apply to discounted prices under a martingale measure. We will be content to give just a sketchy review of some main concepts and results from the seminal paper of Föllmer and Sondermann (1986) on risk minimization.

Let $\tilde{H}$ be a T-claim that is not attainable. This means that an *admissible* portfolio $\boldsymbol{\theta}$ satisfying

$$\tilde{V}^{\theta}(T) = \tilde{H}$$

cannot be SF. The *cost* by time t of an admissible portfolio $\boldsymbol{\theta}$ is denoted by $\tilde{C}^{\theta}(t)$ and is defined as that part of the portfolio value that has not been gained from trading:

$$\tilde{C}^{\theta}(t) = \tilde{V}^{\theta}(t) - \int_0^t \boldsymbol{\theta}(\tau)' d\tilde{\mathbf{S}}(\tau)\,.$$

The *risk* at time t is defined as the mean squared outstanding cost,

$$\tilde{R}(t)^{\theta} = \tilde{\mathbb{E}}\left[\left(\tilde{C}^{\theta}(T) - \tilde{C}^{\theta}(t)\right)^2 \middle| \mathcal{F}_t\right] . \tag{10}$$

By definition, the risk of an admissible portfolio $\boldsymbol{\theta}$ is

$$\tilde{R}^{\theta}(t) = \tilde{\mathbb{E}}\left[\left(\tilde{H} - \tilde{V}^{\theta}(t) - \int_t^T \boldsymbol{\theta}(\tau)' d\tilde{\mathbf{S}}(\tau)\right)^2 \middle| \mathcal{F}_t\right] ,$$

which is a measure of how well the current value of the portfolio plus future trading gains approximates the claim. The theory of risk minimization takes this entity as its objective function and proves the existence of an optimal admissible portfolio that minimizes the risk (10) for all $t \in [0, T]$.

The proof is constructive and provides a recipe for determining the optimal portfolio. One commences from the *intrinsic value* of $\tilde{H}$ at time t defined as

$$\tilde{V}^H(t) = \tilde{\mathbb{E}}\left[\tilde{H} \mid \mathcal{F}_t\right] . \tag{11}$$

This is the martingale that at any time gives the optimal forecast of the claim with respect to mean squared prediction error under the chosen martingale measure. By the Galchouk-Kunita-Watanabe representation, it decomposes uniquely as

$$\tilde{V}^H(t) \;=\; \tilde{\mathbb{E}}[\tilde{H}] + \int_0^t \boldsymbol{\theta}^H(t)' d\tilde{\mathbf{S}}(t) + L^H(t) \,, \tag{12}$$

where L^H is a martingale with respect to $(\mathbf{F}, \tilde{\mathbb{P}})$ which is orthogonal to the martingale $\tilde{\mathbf{S}}$. The portfolio $\boldsymbol{\theta}^H$ defined by this decomposition minimizes the risk process among all admissible strategies. The minimum risk is

$$\tilde{R}^H(t) = \tilde{\mathbb{E}}\left[\int_t^T d\langle L^H \rangle(\tau) \,\middle|\, \mathcal{F}_t \right] . \tag{13}$$

Unit-linked insurance.

As the name suggests, a life insurance product is said to be *unit-linked* if the benefit is a certain share of an asset (or portfolio of assets). If the contract stipulates a prefixed minimum value of the benefit, then one speaks of *unit-linked insurance with guarantee.*

Let T_x be the remaining life time of an x years old who purchases an insurance at time 0, say. The conditional probability of survival to age $x+u$, given survival to age $x+t$ $(0 \le t < u)$, is

$$\mathbb{P}[T_x > u \,|\, T_x > t] = e^{-\int_t^u \mu_{x+s}\, ds} , \tag{14}$$

where μ_y is the mortality intensity at age y. Introduce the indicator of survival to age $x+t$, $I(t) = 1[T_x > t]$, and the indicator of death before time t, $N(t) = 1[T_x \le t] = 1 - I(t)$. The latter is a (very simple) counting process with intensity $I(t)\,\mu_{x+t}$, and the associated $(\mathbf{F}, \mathbb{P})$ martingale M is given by

$$dM(t) = dN(t) - I(t)\,\mu_{x+t}\, dt . \tag{15}$$

Assume that the life time T_x is independent of the economy Y. We will be working with the martingale measure $\tilde{\mathbb{P}}$ obtained by replacing the intensity matrix $\mathbf{\Lambda}$ of Y with the martingalizing $\tilde{\mathbf{\Lambda}}$ and leaving the rest of the model unaltered.

Consider a unit-linked pure endowment benefit payable at a fixed time T, contingent on survival of the insured, with sum insured equal to the price S_T of the (generic) stock, but guaranteed no less than a fixed amount g. This benefit is a contingent T-claim,

$$H = (S(T) \vee g)\, I(T)\,.$$

The single premium payable as a lump sum at time 0 is to be determined. Let us assume that the financial market is complete so that every purely financial derivative has a unique price process. Then the intrinsic value of H at time t is

$$\tilde{V}^H(t) = \tilde{\pi}(t)\, I(t)\, e^{-\int_t^T \mu}\,,$$

where $\tilde{\pi}(t)$ is the discounted price process of the derivative $S(T) \vee g$, and we have used the somewhat sloppy abbreviation $\int_t^T \mu_{x+u}\, du = \int_t^T \mu$.

Using Itô together with (14) and (15) and the fact that $M(t)$ and $\tilde{\pi}(t)$ almost surely have no common jumps, we find

$$\begin{aligned} d\tilde{V}^H(t) &= d\tilde{\pi}(t)\, I(t-)\, e^{-\int_t^T \mu} + \tilde{\pi}(t-)\, I(t-)\, e^{-\int_t^T \mu}\, \mu_{x+t}\, dt + (0 - \tilde{\pi}(t-)\, e^{-\int_t^T \mu} \\ &= d\tilde{\pi}(t)\, I(t)\, e^{-\int_t^T \mu} - \tilde{\pi}(t)\, e^{-\int_t^T \mu}\, dM(t)\,. \end{aligned}$$

It is seen that the optimal trading strategy is that of the price process of the sum insured multiplied with the conditional probability that the sum will be paid out, and that

$$dL^H(t) = -e^{-\int_t^T \mu}\, \tilde{\pi}(t)\, dM(t)\,.$$

Using $d\langle M\rangle(t) = I(t)\, \mu_{x+t}\, dt$, the minimum risk (13) now assumes the form

$$\begin{aligned}\tilde{R}^H(t) &= \tilde{\mathbb{E}}\left[\int_t^T e^{-2\int_\tau^T \mu}\,\tilde{\pi}(\tau)^2\,I(\tau)\,\mu_{x+\tau}\,d\tau\,\middle|\,\mathcal{F}_t\right]\\ &= I(t)\,e^{-2\int_0^t r}\sum_e I_e(t)\,R_e(S_t,t)\,,\end{aligned}$$

where

$$R_e(s,t) = \tilde{\mathbb{E}}\left[\int_t^T e^{-2\int_\tau^T \mu}\,e^{-2\int_t^\tau r}\,\pi(\tau)^2\,I(\tau)\,\mu_{x+\tau}\,d\tau\,\middle|\,S(t)=s,\,Y(t)=e,\,I(t)=1\right.$$

Working along the lines of the proof of (8), starting from the martingale

$$\begin{aligned}M(t)^R &= \tilde{\mathbb{E}}\left[\int_0^T e^{-2\int_\tau^T \mu}\,\tilde{\pi}(\tau)^2\,I(\tau)\,\mu_{x+\tau}\,d\tau\,\middle|\,\mathcal{F}_t\right]\\ &= \int_0^t e^{-2\int_\tau^T \mu}\,e^{-2\int_0^\tau r}\,\pi(\tau)^2\,I(\tau)\,\mu_{x+\tau}\,d\tau\;+\;I(t)\,e^{-2\int_0^t r}\sum_e I_e(t)R_e(S_t,t)\end{aligned}$$

we obtain the differential equations

$$\begin{aligned}&\big(\pi(t)^2 - R_e(s,t)\big)\,\mu_{x+t} - 2r_eR_e(s,t) + \frac{\partial}{\partial t}R_e(s,t) + \frac{\partial}{\partial s}R_e(s,t)\,s\,\alpha_e\\ &+\sum_{f\in\mathcal{Y}^e}\big(R_f(s(1+\gamma^{ef}),t) - R_e(s,t)\big)\,\tilde{\lambda}^{ef}\,.\end{aligned}$$

These are to be solved in parallel with the differential equations (8) and are subject to the conditions

$$R_e(s,T) = 0\,.$$

18 Quadratic hedging in a Markov chain environment

The Markov chain model for a life insurance policy in a stochastic environment. Markov chain $(Y(t))_{t\in[0,T]}$ with finite state space $\mathcal{Y} = \{0, 1, \ldots, J^Y\}$, starting from $Y(0) = 0$. Natural filtration $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,T]}$ represents the development of the (economic-demographic) *environment*. Assume Y is a Markov chain with intensities $\mathbf{\Lambda}(t) = (\lambda_{ef}(t))_{e\in\mathcal{Y}}^{f\in\mathcal{Y}}$

Assume there exists a market for environmental risk, dictating an EMM $\tilde{\mathbf{P}}$ under which Y is a Markov chain with intensities $\tilde{\lambda}_{ef}(t)$.
Consider life insurance policy issued at time 0 and terminating at time T. State of the policy is a stochastic process $(Z(t))_{t\in[0,T]}$ with finite state space $\mathcal{Z} = \{0, 1, \ldots, J^Z\}$, starting from $Z(0) = 0$. The filtration generated by Z is denoted by $\mathbf{H} = (\mathcal{H}_t)_{t\in[0,T]}$.
Conditional on $\mathcal{G}_T$, Z is a Markov process with intensities $\mu_{Y(t),jk}(t)$, $j, k \in \mathcal{Z}$.
Under $\tilde{\mathbf{P}}$, the process $X = (Y, Z)$ is a Markov chain with state space $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$ and intensities

$$\kappa_{ej,fk}(t) = \begin{cases} \tilde{\lambda}_{ef}(t)\,, & e \neq f,\, j = k\,, \\ \mu_{e,jk}(t)\,, & e = f,\, j \neq k\,, \\ 0\,, & e \neq f,\, j \neq k\,. \end{cases}$$

We assume that there is a money market account with interest rate $r(t) = r_{Y(t)} = \sum_e I_e^Y(t)\, r_e$, that is, r_e is the market interest rate in environment state e.

Martingale analysis of a standard life insurance policy.

Insurance policy is of standard type, with deterministic state-wise annuity payment functions B_j and sums assured b_{jk}.

The analysis goes as in the standard model with deterministic transition intensities and interest rate dependent on the state of the policy. Formally, we have just extended the state space of the individual life history process.

State-wise reserve

$$V_{ej}(t)$$

the conditional expected present value at time t of benefits less premiums in $(t,T]$, given that $(Y(t),Z(t)) = (e,j)$.

$$\begin{aligned} dV_{ej}(t) &= V_{ej}(t)\, r_e\, dt - dB_j(t) \\ &\quad - \sum_{k;\, k\neq j} R_{ej,ek}(t)\, \mu_{e,jk}(t)\, dt \\ &\quad - \sum_{f;\, f\neq e} R_{ej,fj}(t)\, \tilde{\lambda}_{ef}(t)\, dt\,, \end{aligned} \tag{1}$$

subject to terminal conditions

$$V_{ej}(T-) = \Delta B_j(T)\,.$$

$$R_{ej,ek}(t) = b_{jk}(t) + V_{ek}(t) - V_{ej}(t) \tag{2}$$

$$R_{ej,fj}(t) = V_{fj}(t) - V_{ej}(t) \tag{3}$$

Martingale

$$\tilde{M}(t) = \tilde{\mathbf{E}}\left[\int_{0-}^{T} e^{-\int_0^\tau r(s)\, ds}\, dB(\tau) \,\middle|\, \mathcal{G}_t \vee \mathcal{H}_t \right]$$

has dynamics

$$\begin{aligned} d\tilde{M}(t) &= \sum_{e\neq f} d\tilde{M}^Y_{ef}(t) \sum_j I^Z_j(t) \tilde{R}_{ej,fj}(t) \\ &\quad + \sum_{j\neq k} dM^Z_{jk}(t) \sum_e I^Y_e(t) \tilde{R}_{ej,ek}(t)\,, \end{aligned} \tag{4}$$

$\tilde{M}^Y_{ef}$ the compensated counting processes of the environment,

$$d\tilde{M}^Y_{ef}(t) = dN^Y_{ef}(t) \,-\, I^Y_e(t)\, \tilde{\lambda}_{ef}(t)\, dt\,,$$

M^Z_{jk} the compensated counting processes of the policy,

$$dM^Z_{jk}(t) = dN^Z_{jk}(t) \,-\, I^Z_j(t) \sum_e I^Y_e(t)\, \mu_{e,jk}(t)\, dt\,,$$

and the $\tilde{R}_{ej,fk}$ are the discounted sums at risk,

$$\tilde{R}_{ej,fk}(t) = e^{-\int_0^t r(s)\,ds}\, R_{ej,fk}(t)\,. \tag{5}$$

Mortality derivatives; Hedging systematic mortality risk.

Suppose the market has m mortality-related derivatives with discounted price processes $(\tilde{S}_i(t))_{t\in[0,T]}$, $i = 1, \ldots, m$. In an arbitrage-free market these are martingales adapted to $\mathbf{G}$ under $\tilde{\mathbf{P}}$ and, therefore (since $\mathbf{G}$ is the natural filtration of Y), have dynamics of the form

$$d\tilde{S}_i(t) = \sum_{e\neq f} \xi_{i,ef}(t)\, d\tilde{M}^Y_{ef}(t)\,. \tag{6}$$

Consider an SF portfolio consisting of $\theta_i(t)$ units of asset No. $i = 0, \ldots, m$ at time t, asset No. 0 being the money market account.

Discounted value process $\tilde{V}^{\boldsymbol{\theta}}(t)$ is martingale with dynamics

$$d\tilde{V}^{\boldsymbol{\theta}}(t) = \sum_{i=1}^{m} \theta_i(t)\, d\tilde{S}_i(t) = \sum_{e \neq f} \sum_{i=1}^{m} \theta_i(t)\, \xi_{i,ef}(t)\, d\tilde{M}^Y_{ef}(t)\,. \quad (7)$$

Our objective is to minimize the *hedging error* or *risk*, defined as the expected squared difference between the total discounted contractual payments and the discounted terminal value of the portfolio,

$$\rho^\theta = \tilde{\mathbf{E}} \left(\tilde{M}(T) - \tilde{V}^\theta(T) \right)^2 .$$

The risk is nothing but the squared distance between the random variable $\tilde{M}(T)$ and the linear space of T-values of SF portfolios in the Hilbert space of random variables that are square integrable w.r.t. $\tilde{\mathbf{P}}$, the inner product being $\langle X, Y \rangle = \tilde{\mathbf{E}}[XY]$. Inserting

$$\tilde{M}(T) \;=\; \tilde{\mathbf{E}}\tilde{M}_T + \int_0^T d\tilde{M}_t$$

and

$$\tilde{V}^{\boldsymbol{\theta}}(T) \;=\; \tilde{V}^{\boldsymbol{\theta}}(0) \;+\; \int_0^T d\tilde{V}^{\boldsymbol{\theta}}(t)\,,$$

and using the dynamics (4) and (7), we get

$$\begin{aligned} \rho^\theta \;=\; \tilde{\mathbf{E}} \Bigg[\tilde{\mathbf{E}}\tilde{M}(T) - \tilde{V}^{\boldsymbol{\theta}}(0) + \int_0^T \Bigg(& \sum_{e \neq f} d\tilde{M}^Y_{ef}(t) \sum_j I^Z_j(t) \tilde{R}_{ej,fj}(t) \\ & + \sum_{j \neq k} dM^Z_{jk}(t) \sum_e I^Y_e(t) \tilde{R}_{ej,ek}(t) \\ & - \sum_{e \neq f} \sum_{i=1}^{m} \xi_{i,ef}(t)\, \theta_i(t)\, d\tilde{M}^Y_{ef}(t) \Bigg) \Bigg]^2 . \qquad (8) \end{aligned}$$

Recalling some martingale results.
The martingales $\tilde{M}^Y_{ef}$ and M^Z_{jk} are square integrable and mutually orthogonal, i.e the predictable covariance process of any two distinct martingales is zero. Heuristic proof of the orthogonality property:

$$\tilde{\mathbf{E}}\left[N^Y_{ef}(t)\middle|\,\mathcal{G}_{t-}\right] \;=\; I^Y_e(t)\,\tilde{\lambda}_{ef}(t)\,dt + o(dt)\,,$$

$$\tilde{\mathbf{E}}\left[dN^Y_{ef}(t)\,dN^Y_{gh}(t)\middle|\,\mathcal{G}_{t-}\right] \;=\; o(dt)\,,\;\text{ if }\;(e,f)\neq(g,h)\,,$$

and

$$\tilde{\mathbf{E}}\left[\left(dN^Y_{ef}(t)\right)^2\middle|\,\mathcal{G}_{t-}\right] \;=\; \tilde{\mathbf{E}}\left[N^Y_{ef}(t)\middle|\,\mathcal{G}_{t-}\right] + o(dt) \;=\; I^Y_e(t)\,\tilde{\lambda}_{ef}(t)\,dt + o(dt)\,.$$

(The Markov nature of the intensities was not essential to this argument.)

Using these calculations, we can now derive the orthogonality property and also obtain the form of the predictable variance processes. For instance, considering $\tilde{M}^Y_{ef}$ and $\tilde{M}^Y_{gh}$, we find

$$\begin{aligned}
&d\langle\tilde{M}^Y_{ef}\,,\,\tilde{M}^Y_{gh}\rangle(t)\\
&=\tilde{\mathbf{E}}\left[d\tilde{M}^Y_{ef}(t)\,d\tilde{M}^Y_{gh}(t)\middle|\,\mathcal{G}_{t-}\right] + o(dt)\\
&=\tilde{\mathbf{E}}\left[\left(dN^Y_{ef}(t)\;-\;I^Y_e(t)\,\tilde{\lambda}_{ef}(t)\,dt\right)\left(dN^Y_{gh}(t)\;-\;I^Y_g(t)\,\tilde{\lambda}_{gh}(t)\,dt\right)\middle|\,\mathcal{G}_{t-}\right] + o(dt)\\
&=\tilde{\mathbf{E}}\left[dN^Y_{ef}(t)\,dN^Y_{gh}(t)\middle|\,\mathcal{G}_{t-}\right] + o(dt)\\
&=\delta_{ef,gh}\,\tilde{\mathbf{E}}\left[dN^Y_{ef}(t)\middle|\,\mathcal{G}_{t-}\right] + o(dt)\\
&=\delta_{ef,gh}\,I^Y_e(t-)\,\tilde{\lambda}_{ef}(t)\,dt + o(dt)\,.
\end{aligned}\tag{9}$$

Here $\delta_{ef,gh}$ is 1 if $(e,f) = (g,h)$ and 0 otherwise (the Kroenecker delta).

We also need to observe that martingale increments over disjoint time intervals are uncorrelated (conditionally and unconditionally). More precisely, for any two distinct times $t < u$,

$$\tilde{\mathbf{E}}\left[d\tilde{M}^Y_{ef}(t)\, d\tilde{M}^Y_{gh}(u)\middle|\, \mathcal{G}_{t-}\right]$$
$$= \tilde{\mathbf{E}}\left[d\tilde{M}^Y_{ef}(t)\, \tilde{\mathbf{E}}\left[d\tilde{M}^Y_{gh}(u)\middle|\, \mathcal{G}_{u-}\right]\middle|\, \mathcal{G}_{t-}\right] \;=\; 0\,. \quad (10)$$

From all these intermediate results we gather the following expression for the covariance of any two stochastic integrals with respect to the martingales. For instance, for $\int_0^T G(t)\, d\tilde{M}^Y_{ef}(t)$ and $\int_0^T H(t)\, d\tilde{M}^Y_{gh}(t)$, with G and H predictable processes such that the expected values below exist, we have

$$\tilde{\mathbf{E}}\left[\int_0^T G(t)\, d\tilde{M}^Y_{ef}(t)\ \int_0^T H(u)\, d\tilde{M}^Y_{gh}(u)\right]$$
$$= \tilde{\mathbf{E}}\left[\int_0^T \int_0^T G(t)\, H(u)\, d\tilde{M}^Y_{ef}(t)\, d\tilde{M}^Y_{gh}(u)\right]$$
$$= \tilde{\mathbf{E}}\left[\int_0^T G(t)\, H(t)\, \tilde{\mathbf{E}}\left[\, d\tilde{M}^Y_{ef}(t)\, d\tilde{M}^Y_{gh}(t)\middle|\, \mathcal{G}_{t-}\right]\right]$$
$$= \delta_{ef,gh}\, \tilde{\mathbf{E}}\left[\int_0^T G(t)\, H(t)\, I^Y_e(t-)\, \tilde{\lambda}_{ef}(t)\, dt\right]. \quad (11)$$

The off-diagonal terms in the double integral vanished due to (10).

Constructing the optimal hedge.
Now, back to the risk in (8). Since the stochastic integrals have mean 0, they are orthogonal to the constant $\tilde{\mathbf{E}}\tilde{M}(T) - \tilde{V}^{\boldsymbol{\theta}}(0)$. Due to this and to (11), the risk decomposes into

$$\rho^\theta = \rho^\theta_{_0} + \rho^\theta_{_I} + \rho^\theta_{_E} \,,$$

where

$$\rho^\theta_{_0} = \left(\tilde{\mathbf{E}}\tilde{M}(T) - \tilde{V}^{\boldsymbol{\theta}}(0)\right)^2$$

is the *basis risk*,

$$\begin{aligned}\rho^\theta_{_I} &= \tilde{\mathbf{E}}\left[\int_0^T \sum_{j\neq k}\left(\sum_e I^Y_e(t)\tilde{R}_{ej,ek}(t)\right)\, dM^Z_{jk}(t)\right]^2 \\ &= \tilde{\mathbf{E}}\left[\int_0^T \sum_e I^Y_e(t)\sum_{j\neq k} I^Z_j(t)\mu^Z_{e;jk}(t)\tilde{R}^2_{ej,ek}(t)\, dt\right]\end{aligned}$$

is the non-systematic *individual risk*, and

$$\begin{aligned}\rho^\theta_{_E} &= \tilde{\mathbf{E}}\left[\int_0^T \sum_{e\neq f}\left(\sum_j I^Z_j(t)\tilde{R}_{ej,fj}(t) - \sum_{i=1}^m \xi_{i,ef}(t)\,\theta_i(t)\right)\, d\tilde{M}^Y_{ef}(t)\right]^2 \\ &= \tilde{\mathbf{E}}\left[\int_0^T \sum_e I^Y_e(t)\sum_{f\neq e}\tilde{\lambda}_{ef}(t)\left(\sum_j I^Z_j(t)\tilde{R}_{ej,fj}(t) - \sum_{i=1}^m \xi_{i,ef}(t)\,\theta_i(t)\right)^2 dt\right]\end{aligned} \qquad (12)$$

is the systematic *environment risk* or *hedging error*.

The basis risk $\rho^\theta_{_0}$ is minimized by setting

$$\tilde{V}^{\boldsymbol{\theta}}(0) = \tilde{\mathbf{E}}\tilde{M}(T)\,.$$

(Typically, the contract is designed such that $\tilde{\mathbf{E}}\tilde{M}(T) = 0$.) The individual risk $\rho_{_I}^{\theta}$ does not depend on the portfolio strategy. Thus, we are left with the problem of minimizing the environment risk (12). To this end we write the portfolio vector as the sum of its state-wise values,

$$\boldsymbol{\theta}(t) \;=\; \sum_e I_e^Y(t)\,\boldsymbol{\theta}_e(t)\,, \quad \boldsymbol{\theta}_e(t) = (\theta_{e1}(t), \ldots, \theta_{em}(t))'\,,$$

and introduce, for each time t and each state e, the set of states that are directly accessible,

$$\mathcal{Y}_e(t) = \{f;\, \tilde{\lambda}_{ef}(t) > 0\}\,,$$

its dimension

$$n_e(t) \;\text{ the number of elements in }\; \mathcal{Y}^e(t)\,,$$

the $n_e(t)$-vector

$$\boldsymbol{\eta}_e(t) = (\eta_{ef}(t))_{f\in\mathcal{Y}_e(t)}$$

with elements

$$\eta_{ef}(t) = \sum_j I_j^Z(t)\tilde{R}_{ej,fj}(t)\,, \tag{13}$$

the $n_e(t) \times n_e(t)$ diagonal matrix

$$\tilde{\boldsymbol{\Lambda}}_e(t) = \mathbf{D}\mathrm{iag}_{f\in\mathcal{Y}_e(t)}(\tilde{\lambda}_{ef}(t))\,,$$

and the $n_e(t) \times m$ matrix of price coefficients,

$$\boldsymbol{\Xi}_e(t) = (\xi_{i,ef}(t))_{f\in\mathcal{Y}_e(t)}^{i=1,\ldots,m}\,.$$

Rewrite (12) as

$$\rho_{_E}^{\theta}(t) = \tilde{\mathbf{E}}\left[\int_0^T \sum_e I_e^Y(t)\,(\boldsymbol{\eta}_e(t) - \boldsymbol{\Xi}_e(t)\,\boldsymbol{\theta}_e(t))'\;\tilde{\boldsymbol{\Lambda}}_e(t)\;(\boldsymbol{\eta}_e(t) - \boldsymbol{\Xi}_e(t)\,\boldsymbol{\theta}_e(t))\;dt\right]. \tag{1}$$

The best (unrestricted) hedging portfolio,

$$\tilde{\boldsymbol{\theta}}(t) \;=\; \sum_{e} I_e^Y(t)\,\tilde{\boldsymbol{\theta}}_e(t)\,,$$

is obtained by, for each time t and each environment state e, minimizing the quadratic form

$$Q_{e,t}(\boldsymbol{\theta}) = (\boldsymbol{\eta}_e(t) - \Xi_e(t)\,\boldsymbol{\theta})'\;\tilde{\boldsymbol{\Lambda}}_e(t)\;(\boldsymbol{\eta}_e(t) - \Xi_e(t)\,\boldsymbol{\theta})\,, \tag{15}$$

the distance from $\boldsymbol{\eta}_e(t)$ to a point in the linear space spanned by the columns of $\Xi_e(t)$, under the Euclidean weighted inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}'\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\mathbf{y}$. The minimizer is the projection of $\boldsymbol{\eta}_e(t)$ onto the column space of $\Xi_e(t)$, which must be of the form $\Xi_e(t)\,\tilde{\boldsymbol{\theta}}_e(t)$ and satisfies the normal equations

$$\boldsymbol{\theta}'\,\Xi_e'(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\;\Big(\boldsymbol{\eta}_e(t)\;-\;\Xi_e(t)\,\tilde{\boldsymbol{\theta}}_e(t)\Big) = 0\,,\quad \forall\,\boldsymbol{\theta}\,,$$

stating that the difference between $\boldsymbol{\eta}_e(t)$ and its projection is orthogonal to all vectors $\Xi_e(t)\,\boldsymbol{\theta}$ in the column space of $\Xi_e(t)$. This is equivalent to

$$\Xi_e'(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\;\Big(\boldsymbol{\eta}_e(t)\;-\;\Xi_e(t)\,\tilde{\boldsymbol{\theta}}_e(t)\Big)\;=\mathbf{0}\,,$$

(the $m \times 1$ vector with all entries null). Thus, $\tilde{\boldsymbol{\theta}}_e(t)$ is any solution to the equation

$$\Xi_e'(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\Xi_e(t)\,\tilde{\boldsymbol{\theta}}_e(t)\;=\;\Xi_e'(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\boldsymbol{\eta}_e(t)\,. \tag{16}$$

Consider first the case where $m \le n_e(t)$, which means that there are fewer assets than there are martingales at work ("sources of randomness"). Then the dynamics of the assets will typically not span all possible liability dynamics, so that there will be contractual payment streams that cannot be perfectly hedged in state e at time t. We will assume that $\Xi_e(t)$ has full rank m, which is easy to check. (If the rank should be less than m, then some assets would be redundant in state e at time t in the sense that their dynamics are linear combinations of those of a smaller set of assets or maybe null.) Then (16) has the unique solution

$$\tilde{\boldsymbol{\theta}}_e(t) \;=\; \left(\boldsymbol{\Xi}'_e(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\boldsymbol{\Xi}_e(t)\right)^{-1}\,\boldsymbol{\Xi}'_e(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\boldsymbol{\eta}_e(t)\,, \quad (17)$$

and the projection has the explicit form

$$\tilde{\boldsymbol{\eta}}_e(t) = \mathbf{P}_e(t)\,\boldsymbol{\eta}_e(t)\,, \quad (18)$$

where $\mathbf{P}_e(t)$ is the *projection matrix* or just *projector*,

$$\mathbf{P}_e(t) \;=\; \boldsymbol{\Xi}_e(t)\,\left(\boldsymbol{\Xi}'_e(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\boldsymbol{\Xi}_e(t)\right)^{-1}\,\boldsymbol{\Xi}'_e(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,. \quad (19)$$

By Pythagoras, the minimum of $Q_{e,t}(\boldsymbol{\theta})$ in (15) is

$$\begin{aligned}
\min Q_{e,t} \;&=\; \boldsymbol{\eta}'_e(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\boldsymbol{\eta}_e(t) \;-\; \tilde{\boldsymbol{\eta}}'_e(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\tilde{\boldsymbol{\eta}}_e(t) \quad (20)\\
&=\; \boldsymbol{\eta}'_e(t)\,\left(\tilde{\boldsymbol{\Lambda}}_e(t) \;-\; \mathbf{P}'_e(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\mathbf{P}_e(t)\right)\,\boldsymbol{\eta}_e(t)\\
&=\; \boldsymbol{\eta}'_e(t)\,\tilde{\boldsymbol{\Lambda}}_e(t)\,\left(\mathbf{I} - \mathbf{P}_e(t)\right)\,\boldsymbol{\eta}_e(t)\,. \quad (21)
\end{aligned}$$

The case $m > n_e(t)$ is of little interest since it typically means that the all liability dynamics are spanned by the dynamics of the available assets, and there is no hedging error (locally in state e at time t). In this case, with more assets than random sources, there must be linear dependence between the asset dynamics. Thus (at least) one asset is redundant and can be discarded. (In computations it will typically have to be discarded in order to avoid singularity of matrices that need to be inverted.)

Analysis of the hedging error. Inserting (21) into (14) and using (13) and (5), we find that the minimized systematic hedging error $\tilde{\rho} := \rho_{_E}^{\tilde{\theta}}$ is

$$\begin{aligned}\tilde{\rho} \;&=\; \tilde{\mathbf{E}}\,[\int_0^T e^{-2\int_0^\tau r(s)\,ds} \sum_e I_e^Y(\tau) \sum_j I_j^Z(\tau) \\ &\qquad \mathbf{R}'_{ej}(\tau)\,\tilde{\mathbf{\Lambda}}_e(\tau)\,(\mathbf{I}-\mathbf{P}_e(\tau))\,\mathbf{R}_{ej}(\tau)\,d\tau\,]\;, \end{aligned} \tag{22}$$

where $\mathbf{R}_{ej}(t)$ is the $n_e(t)\times 1$ vector with elements $R_{ej,fj}(t)$, $f \in \mathcal{Y}_e(t)$.

For the standard insurance policy we are considering, the state-wise sums at risk $R_{ej,fj}(t)$ are deterministic. Therefore, the issue of computing (22) depends entirely on the properties of the projector $\mathbf{P}_e(t)$.

The simplest situation is when the $\mathbf{P}_e(t)$ are deterministic functions for all e and t. Then, due to the Markov property, we need only consider the state-wise risks at time t,

$$\begin{aligned}\tilde{\rho}_{ej}(t) \;&:=\; \tilde{\mathbf{E}}\,[\int_t^T e^{-2\int_t^\tau r(s)\,ds} \sum_{e'} I_{e'}^Y(\tau) \sum_{j'} I_{j'}^Z(\tau) \\ &\qquad \mathbf{R}'_{e'j'}(\tau)\,\tilde{\mathbf{\Lambda}}_{e'}(\tau)\,(\mathbf{I}-\mathbf{P}_{e'}(\tau))\,\mathbf{R}_{e'j'}(\tau)\,d\tau \Big|\, Y(t)=e,\; Z(t)=j\,]\;. \end{aligned} \tag{23}$$

These are of the same form as state-wise reserves for continuously paid annuity benefits and so are solutions to the Thiele type of backward equations,

$$\begin{aligned}\frac{d}{dt}\tilde{\rho}_{ej}(t) \;&=\; 2\,r_e\,\tilde{\rho}_{ej}(t) \,-\, \mathbf{R}'_{ej}(t)\,\tilde{\mathbf{\Lambda}}_e(t)\,(\mathbf{I}-\mathbf{P}_e(t))\,\mathbf{R}_{ej}(t) \\ &\quad - \sum_{f\in\mathcal{Y}_e} \tilde{\lambda}_{ef}(t)\,(\tilde{\rho}_{fj}(t)-\tilde{\rho}_{ej}(t)) - \sum_{k;k\neq j} \mu_{e;jk}(t)\,(\tilde{\rho}_{ek}(t)-\tilde{\rho}_{ej}(t))\end{aligned}$$

(24)

subject to terminal conditions $\tilde{\rho}_{ej}(T) = 0$. The minimal systematic hedging error at time 0, given $Y(0) = 0$ and $Z(0) = 0$, is $\tilde{\rho}_{00}(0)$.

Question arising: When are the $\mathbf{P}_e(t)$ deterministic? They are if the derivatives are zero-coupon bonds with principals dependent only on the state of the environment upon maturity. More precisely, if derivative No. i pays (only) an amount $h_i(Y(U_i))$ at time U_i, then its price at time $t < U_i$ is

$$S_i(t) = \tilde{\mathbf{E}}\left[e^{-\int_t^{U_i} r(s)\,ds} h_i(Y(U_i)) \middle| \mathcal{G}_t \right] ,$$

a function only of t and $Y(t)$;

$$S_i(t) = F_{i,Y(t)}(t) = \sum_e I_e^Y(t)\, F_{i,e}(t)\,,$$

where the functions $F_{i,e}$ are deterministic. Then, plainly,

$$d\tilde{S}_i(t) = e^{-\int_0^t r(s)\,ds} \sum_{e \neq f} d\tilde{M}_{ef}^Y (F_{i,f}(t) - F_{i,e}(t))\,,$$

hence

$$\xi_{i,ef}(t) = e^{-\int_0^t r(s)\,ds} (F_{i,f}(t) - F_{i,e}(t))\,.$$

The factor $e^{-\int_0^t r(s)\,ds}$ will cancel out in the expression (19) for the projector.

An example with digital mortality rates. Consider a life endowment of b with term T against level premium purchased by an x-year old at time 0. The policy has two states, 0 = 'alive' and 1 = 'dead', and the payment function is

$$dB(t) = I_0^Z(t)\, \left(-c\, 1_{(0,T)}(t)\, dt \,+\, d1_{[T,\infty)}(t)\, b \right)\,.$$

Interest rate is fixed r. Demographic environment is governed by Markov chain Y with state space $\mathcal{Y} = \{0, 1, 2, 3\}$ and with intensity matrix $\tilde{\mathbf{\Lambda}} = (\tilde{\lambda}_{ef})$ of the form

$$\tilde{\mathbf{\Lambda}} = \begin{pmatrix} -(\tilde{\lambda}_1 + \tilde{\lambda}_2) & \tilde{\lambda}_1 & \tilde{\lambda}_2 & 0 \\ 0 & -\tilde{\lambda}_2 & 0 & \tilde{\lambda}_2 \\ 0 & 0 & -\tilde{\lambda}_1 & \tilde{\lambda}_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and that the stochastic mortality rate at age y and calendar time t is of the form

$$\mu_{Y(t)}(y) = \mu^{(0)}(y) + (I_0^Y(t) + I_2^Y(t))\,\mu^{(1)}(y) + (I_0^Y(t) + I_1^Y(t))\,\mu^{(2)}(y)\,. \tag{25}$$

Interpretation: Thee causes of death, e.g. No.1 cardiovascular diseases, No. 2 is cancer, and No. 3 all other causes of death. At time 0 the intensity of dying from cause 1 is $\mu^{(1)}(y)$ at age y, and it remains so until it becomes 0 at all ages at a random time which is exponentially distributed with parameter $\tilde{\lambda}_1$ (medical science eliminates the cardiovascular diseases). Similarly, at time 0 the intensity of dying from cause 2 is $\mu^{(2)}(y)$ at age y, and it remains so until it becomes 0 at all ages at a random time which is exponentially distributed with parameter $\tilde{\lambda}_2$. Mortality by cause No. 0 remains unchanged throughout.

The Thiele differential equations (1) become

$$\begin{aligned} V_{00}'(t) &= V_{00}(t)\,r + c + \left(\mu^{(0)}(x+t) + \mu^{(1)}(x+t) + \mu^{(2)}(x+t)\right) V_{00}(t) \\ &\quad - \tilde{\lambda}_1\,(V_{10}(t) - V_{00}(t)) - \tilde{\lambda}_2\,(V_{20}(t) - V_{00}(t))\,, \\ V_{10}'(t) &= V_{10}(t)\,r + c + \left(\mu^{(0)}(x+t) + \mu^{(2)}(x+t)\right) V_{10}(t) \end{aligned}$$

$$
\begin{aligned}
& \quad - \tilde{\lambda}_2 \left(V_{30}(t) - V_{10}(t) \right) \,, \\
V_{20}'(t) &= V_{20}(t)\, r \,+\, c \,+\, \left(\mu^{(0)}(x+t) + \mu^{(1)}(x+t) \right) V_{20}(t)\,, \\
& \quad - \tilde{\lambda}_1 \left(V_{30}(t) - V_{20}(t) \right) \\
V_{30}'(t) &= V_{30}(t)\, r \,+\, c \,+\, \mu^{(0)}(x+t)\, V_{30}(t)\,. \qquad (26)
\end{aligned}
$$

Terminal conditions:

$$
V_{e0}'(T-) = b\,.
$$

The martingale (4) associated with the discounted payments is now

$$
\begin{aligned}
d\tilde{M}(t) &= I_0^Z(t) \Big[d\tilde{M}_{01}^Y(t)\, \tilde{R}_{00,10}(t) \,+\, d\tilde{M}_{02}^Y(t)\, \tilde{R}_{00,20}(t) \\
& \quad + d\tilde{M}_{13}^Y(t)\, \tilde{R}_{10,30}(t) \,+\, d\tilde{M}_{23}^Y(t)\, \tilde{R}_{20,30}(t) \Big] \\
& \quad + dM_{01}^Z(t) \sum_e I_e^Y(t) \tilde{R}_{e0,e1}(t)\,. \qquad (27)
\end{aligned}
$$

The discounted sums at risk (5) are

$$
\tilde{R}_{e0,f0}(t) \,=\, e^{-rt} \left(V_{f0}(t) - V_{e0}(t) \right) \qquad (28)
$$

and $\tilde{R}_{e0,e1}(t) = -\, e^{-rt}\, V_{e0}(t)$.

Suppose the mortality market consists of only one bond that matures at time $U > T$, with no coupons and with principal at time U given by the survival probability of an y-year old at time 0 (usually called a survivor bond):

$$
S(U) = e^{-\int_0^U \mu_{Y(s)}(y+s)\, ds}\,. \qquad (29)
$$

The discounted price function is

$$
\tilde{S}(t) \;=\; \tilde{E}\left[e^{-rU} e^{\int_0^U \mu_{Y(s)}(y+s)\, ds} \,\middle|\, \mathcal{G}_t \right]
$$

$$= e^{-\int_0^t \left(r+\mu_{Y(s)}(y+s)\right) ds} \, \tilde{E}\left[e^{-\int_t^U \left(r+\mu_{Y(s)}(y+s)\right) ds} \middle| \mathcal{G}_t \right]$$
$$= e^{-\int_0^t \left(r+\mu_{Y(s)}(y+s)\right) ds} \sum_e I_e^Y(t)\, F_e(t)\,,$$

where

$$F_e(t) = \tilde{E}\left[e^{-\int_t^T (r+\mu_{Y(s)}(y+s))\, ds} \middle| Y(t) = e \right] .$$

(The only relevant state variable in this case is $Y(t)$ due to the nice structure of the principal (29). If the pay-offs on the bond would depend on the history of Y in a more complicated manner, then one might have to work with more state variables and would inevitably have to solve partial differential equations in what follows.)

The dynamics of the discounted price function is (recall that $\tilde{\lambda}_{01} = \tilde{\lambda}_1$, $\tilde{\lambda}_{02} = \tilde{\lambda}_2$ etc.)

$$\begin{aligned}
d\tilde{S}(t) &= e^{-\int_0^t \left(r+\mu_{Y(s)}(y+s)\right) ds} \left(-(r+\mu_{Y(t)}(y+t))\, dt\right) \sum_e I_e^Y(t)\, F_e(t) \\
&\quad + e^{-\int_0^t \left(r+\mu_{Y(s)}(y+s)\right) ds} \sum_e I_e^Y(t)\, F_e'(t)\, dt \\
&\quad + e^{-\int_0^t \left(r+\mu_{Y(s)}(y+s)\right) ds} \sum_{e\neq f} dN_{ef}^Y(t)\, (F_f(t) - F_e(t)) \\
&= e^{-\int_0^t \left(r+\mu_{Y(s)}(y+s)\right) ds} \sum_e I_e^Y(t) \\
&\quad \times \left(-(r+\mu_e(y+t))\, F_e(t) + F_e'(t) + \sum_{f;\, f\neq e} \tilde{\lambda}_{ef}(F_f(t) - F_e(t)) \right) dt \\
&\quad + e^{-\int_0^t \left(r+\mu_{Y(s)}(y+s)\right) ds} \sum_{e\neq f} d\tilde{M}_{ef}^Y(t)\, (F_f(t) - F_e(t))\,.
\end{aligned}$$

Here $\mu_0(y) = \mu^{(0)}(y) + \mu^{(1)}(y) + \mu^{(2)}(y)$, $\mu_1(y) = \mu^{(0)}(y) +$

$\mu^{(2)}(y)$, $\mu_2(y) = \mu^{(0)}(y) + \mu^{(1)}(y)$, and $\mu_3(y) = \mu^{(0)}(y)$. Setting the drift term to 0 gives the differential equations

$$F_e'(t) = (r + \mu_e(y+t))F_e(t) - \sum_{f;\, f \neq e} \tilde{\lambda}_{ef}(F_f(t) - F_e(t)) \,,$$

with side conditions

$$F_e(U) = 1 \,,$$

$e = 0, \ldots, 3$. From these we solve the state-wise price functions $F_e(t)$ by the standard numerical technique for ODE. In terms of the basic parameters the differential equations are

$$\begin{aligned}
F_0'(t) &= (r + \mu^{(0)}(y+t) + \mu^{(1)}(y+t) + \mu^{(2)}(y+t))F_0(t) \\
&\quad - \tilde{\lambda}_1(F_1(t) - F_0(t)) - \tilde{\lambda}_2(F_2(t) - F_0(t)) \,, \\
F_1'(t) &= (r + \mu^{(0)}(y+t) + \mu^{(2)}(y+t))F_1(t) - \tilde{\lambda}_2(F_3(t) - F_1(t)) \,, \\
F_2'(t) &= (r + \mu^{(0)}(y+t) + \mu^{(1)}(y+t))F_2(t) - \tilde{\lambda}_1(F_3(t) - F_2(t)) \,, \\
F_3'(t) &= (r + \mu^{(0)}(y+t))F_3(t) \,.
\end{aligned}$$

The price dynamics reduces to

$$\begin{aligned}
d\tilde{S}(t) &= e^{-\int_0^t \left(r + \mu_{Y(s)}(y+s)\right) ds} \times \\
&\quad \Big[d\tilde{M}_{01}^Y(t)(F_1(t) - F_0(t)) + d\tilde{M}_{02}^Y(t)(F_2(t) - F_0(t)) \\
&\quad + d\tilde{M}_{13}^Y(t)(F_3(t) - F_1(t)) + d\tilde{M}_{23}^Y(t)(F_3(t) - F_2(t)) \Big] \,.
\end{aligned}$$

The coefficients $\xi_{ef}(t)$ are (subscript i is not needed as there is only one derivative)

$$\begin{aligned}
\xi_{01}(t) &= e^{-\int_0^t \left(r + \mu_{Y(s)}(y+s)\right) ds}(F_1(t) - F_0(t)) \,, \\
\xi_{02}(t) &= e^{-\int_0^t \left(r + \mu_{Y(s)}(y+s)\right) ds}(F_2(t) - F_0(t)) \,, \\
\xi_{13}(t) &= e^{-\int_0^t \left(r + \mu_{Y(s)}(y+s)\right) ds}(F_3(t) - F_1(t)) \,, \\
\xi_{23}(t) &= e^{-\int_0^t \left(r + \mu_{Y(s)}(y+s)\right) ds}(F_3(t) - F_2(t)) \,. \quad (30)
\end{aligned}$$

The coefficients $\eta_{ef}(t)$ in (13) are

$$\eta_{ef}(t) \;=\; I_0^Z(t)\, \tilde{R}_{e0,f0}(t)\,. \tag{31}$$

We obtain the optimal trading strategy by minimizing the quadratic form (15) for each t and e.

First consider $e = 0$. The quadratic form is

$$\tilde{\lambda}_1 \left(\eta_{01}(t) - \theta\, \xi_{01}(t)\right)^2 \,+\, \tilde{\lambda}_2 \left(\eta_{02}(t) - \theta\, \xi_{02}(t)\right)^2 \,.$$

One easily calculates the minimizing θ,

$$\tilde{\theta}(t) = \frac{\tilde{\lambda}_1\, \eta_{01}(t)\, \xi_{01}(t) \,+\, \tilde{\lambda}_2\, \eta_{02}(t)\, \xi_{02}(t)}{\tilde{\lambda}_1\, \xi_{01}^2(t) \,+\, \tilde{\lambda}_2\, \xi_{02}^2(t)}\,,$$

and the minimum

$$\tilde{\lambda}_1\, \eta_{01}^2(t) \,+\, \tilde{\lambda}_2\, \eta_{02}^2(t) \,-\, \frac{\Big(\tilde{\lambda}_1\, \eta_{01}(t)\, \xi_{01}(t) \,+\, \tilde{\lambda}_2\, \eta_{02}(t)\, \xi_{02}(t)\Big)^2}{\tilde{\lambda}_1\, \xi_{01}^2(t) \,+\, \tilde{\lambda}_2\, \xi_{02}^2(t)}$$

$$= I_0^Z(t)\, e^{-2rt} \Big[\tilde{\lambda}_1 \left(V_{10}(t) - V_{00}(t)\right)^2 \,+\, \tilde{\lambda}_2 \left(V_{20}(t) - V_{00}(t)\right)^2$$

$$- \frac{\Big(\tilde{\lambda}_1 \left(V_{10}(t) - V_{00}(t)\right) \left(F_1(t) - F_0(t)\right) \,+\, \tilde{\lambda}_2 \left(V_{20}(t) - V_{00}(t)\right) \left(F_2(t) - F_0(t)\right)}{\tilde{\lambda}_1 \left(V_{10}(t) - V_{00}(t)\right)^2 \,+\, \tilde{\lambda}_2 \left(V_{20}(t) - V_{00}(t)\right)^2}$$

Terms dependent on the past have factored out, which is very convenient.

Next, consider $e = 1$. The quadratic form is now just

$$\tilde{\lambda}_2 \left(\eta_{13}(t) - \theta\, \xi_{13}(t)\right)^2 \,.$$

The minimizing θ is $\tilde{\theta}(t) \,=\, \eta_{13}(t)/\xi_{13}(t)$ and the minimum is 0; there is only one martingale (source of randomness) in state 1, and the single asset spans its dynamics. The analysis for state $e = 2$ is similar. In state

$e = 3$ there is no environment risk left. Calculation of the state-wise environment risk is easy in this case. One can now discuss how the hedging efficiency depends on y and U.

One can invent other derivatives, e.g. a bond with principal

$$S(U) = e^{-\int_0^U \left(I_0^Y(s)+I_2^Y(s)\right)\mu^{(1)}(y+s)\,ds},$$

which is related to cause 1 only. The discounted price function of this bond is

$$\begin{aligned}\tilde{S}(t) &= \tilde{E}\left[e^{-rU}e^{-\int_0^U \left(I_0^Y(s)+I_2^Y(s)\right)\mu^{(1)}(y+s)\,ds}\middle|\,\mathcal{G}_t\right]\\ &= e^{-\int_0^t\left(r+\left(I_0^Y(s)+I_2^Y(s)\right)\mu^{(1)}(y+s)\right)ds}\sum_e I_e^Y(t)\,F_e(t)\,,\end{aligned}$$

where

$$F_e(t) = \tilde{E}\left[e^{-\int_t^U\left(r+\left(I_0^Y(s)+I_2^Y(s)\right)\mu^{(1)}(y+s)\right)ds}\middle|\,Y(t)=e\right].$$

The dynamics of the discounted price function is

$$\begin{aligned}d\tilde{S}(t) &= e^{-\int_0^t\left(r+\left(I_0^Y(s)+I_2^Y(s)\right)\mu^{(1)}(y+s)\right)ds}\left(-r-\left(I_0^Y(t)+I_2^Y(t)\right)\mu^{(1)}(y+t)\right)dt\sum\\ &\quad + e^{-\int_0^t\left(r+\left(I_0^Y(s)+I_2^Y(s)\right)\mu^{(1)}(y+s)\right)ds}\sum_e I_e^Y(t)\,F_e'(t)\,dt\\ &\quad + e^{-\int_0^t\left(r+\left(I_0^Y(s)+I_2^Y(s)\right)\mu^{(1)}(y+s)\right)ds}\sum_{e\neq f} dN_{ef}^Y(t)\,(F_f(t)-F_e(t))\\ &= e^{-\int_0^t\left(r+\left(I_0^Y(s)+I_2^Y(s)\right)\mu^{(1)}(y+s)\right)ds}\sum_e I_e^Y(t)\\ &\quad\times\Bigg(-\left(r+\left(I_0^Y(t)+I_2^Y(t)\right)\mu^{(1)}(y+t)\right)F_e(t)+F_e'(t)+\sum_{f;\,f\neq e}\tilde{\lambda}_{ef}(F_f(\end{aligned}$$

$$+ \, e^{-\int_0^t \left(r + \left(I_0^Y(s) + I_2^Y(s)\right)\mu^{(1)}(y+s)\right) ds} \sum_{e \neq f} d\tilde{M}_{ef}^Y(t) \left(F_f(t) - F_e(t)\right).$$

Setting the drift term to 0, and using the obvious relationship $I_e^Y(t)\, I_f^Y(t) = \delta_{ef}$, gives the differential equations

$$F_e'(t) = \left(r + (\delta_{e0} + \delta_{e2})\mu^{(1)}(y+t)\right) F_e(t) - \sum_{f;\, f \neq e} \tilde{\lambda}_{ef}(F_f(t) - F_e(t)),$$

with side conditions

$$F_e(U) = 1,$$

$e = 0, \ldots, 3$. In states 1 and 3 this bond (essentially) reduces to the bank account and becomes redundant. This we can realize by direct reasoning, but let us derive it by brute force using the differential equations and their side conditions. Firstly, the simplest one is

$$F_3'(t) \;=\; r\, F_3(t),$$

subject to $F_3(U) = 1$, which trivially has the solution $F_3(t) = e^{-r\,(U-t)}$. Secondly,

$$F_1'(t) \;=\; r\, F_1(t) - \tilde{\lambda}_{13}(F_3(t) - F_1(t)) = (r + \tilde{\lambda}_{13})\, F_1(t) - \tilde{\lambda}_{13}\, e^{-r\,(U-t)},$$

subject to $F_1(U) = 1$, has the solution (can be taken direct from Theorem 3.1 in the life insurance lecture notes)

$$\begin{aligned}
F_1(t) \;&=\; \int_t^U e^{-(r+\tilde{\lambda}_{13})\,(\tau - t)}\, \tilde{\lambda}_{13}\, e^{-r\,(U-\tau)}\, d\tau \;+\; e^{-(r+\tilde{\lambda}_{13})\,(U-t)} \\
&=\; \int_t^U e^{-\tilde{\lambda}_{13}\,(\tau - t)}\, \tilde{\lambda}_{13}\, d\tau\, e^{-r\,(U-t)} \;+\; e^{-(r+\tilde{\lambda}_{13})\,(U-t)} \\
&=\; \left(1 - e^{-\tilde{\lambda}_{13}\,(U-t)}\right) e^{-r\,(U-t)} \;+\; e^{-(r+\tilde{\lambda}_{13})\,(U-t)} \\
&=\; e^{-r\,(U-t)}.
\end{aligned}$$

This bond can be compared with a similar bond related to cause 2.

One could consider different designs of the derivatives, e.g. $S(U) = I_1^Y(U) + I_3^Y(U)$ which has value 1 at term U if cause No. 1 has been eliminated by that time and else has value 0. Also this derivative has nice coefficients in the martingale representation, and the optimal hedging risk can be easily calculated. You should invent some reasonable and mathematically tractable derivatives and carry through the calculations along the lines above. You may consider models with more causes of deaths or more general state spaces, but your choice should be clearly motivated.

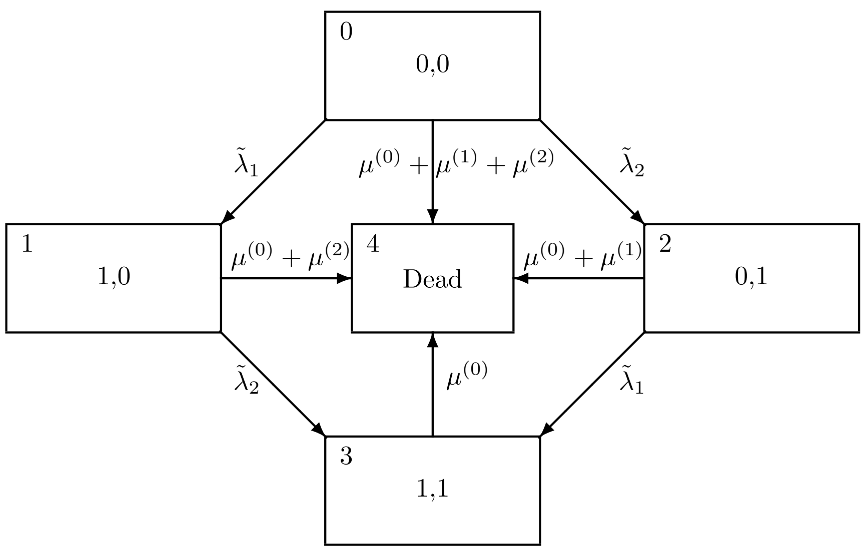


Figure 2: A Markov model for three causes of death, two of which are digital.

Figure 2 shows a flow-chart for the Markov model.

The best buy and hold strategy. There are reasons why one could be interested in restricted trading strategies, in particular constant portfolios also known as *buy-and-hold* strategies. With $\boldsymbol{\theta}$ constant (14) reduces to

$$\begin{aligned}
\rho^{\theta}_{E}(t) &= \tilde{\mathbf{E}}\left[\int_0^T \sum_e I^Y_e(t)\, \boldsymbol{\eta}'_e(t)\, \tilde{\boldsymbol{\Lambda}}_e(t)\, \boldsymbol{\eta}_e(t)\, dt\right] \\
&\quad - 2\,\boldsymbol{\theta}'\, \tilde{\mathbf{E}}\left[\int_0^T \sum_e I^Y_e(t)\, \boldsymbol{\Xi}'_e(t)\, \tilde{\boldsymbol{\Lambda}}_e(t)\, \boldsymbol{\eta}_e(t)\, dt\right] \\
&\quad + \boldsymbol{\theta}'\, \tilde{\mathbf{E}}\left[\int_0^T \sum_e I^Y_e(t)\, \boldsymbol{\Xi}'_e(t)\, \tilde{\boldsymbol{\Lambda}}_e(t)\, \boldsymbol{\Xi}_e(t)\, dt\right] \boldsymbol{\theta}.
\end{aligned}$$

The problem boils down to minimizing a quadratic form.

One easily derives a system of ODE for the determination of the optimal solution.

Some further remarks on essential part of the price processes. Suppose that, for each e and t, the coefficient matrix $\boldsymbol{\Xi}_e(t)$ is of the form

$$\boldsymbol{\Xi}_e(t) \;=\; \boldsymbol{\Xi}_{0,e}(t)\, \Psi_e(t)\,, \tag{32}$$

where $\boldsymbol{\Xi}_{0,e}(t)$ is an $n_e(t) \times m$ matrix that is deterministic and $\Psi_e(t)$ is some predictable $m \times m$ matrix that has full rank. Then the projector (19) reduces to

$$\begin{aligned}
\mathbf{P}_e(t) &= \boldsymbol{\Xi}_{0,e}(t)\, \Psi_e(t)\, \left(\Psi'_e(t)\, \boldsymbol{\Xi}'_{0,e}(t)\, \tilde{\boldsymbol{\Lambda}}_e(t)\, \boldsymbol{\Xi}_{0,e}(t)\, \Psi_e(t)\right)^{-1} \Psi'_e(t)\, \boldsymbol{\Xi}'_{0,e}(t)\, \tilde{\boldsymbol{\Lambda}}_e(t) \\
&= \boldsymbol{\Xi}_{0,e}(t)\, \left(\boldsymbol{\Xi}'_{0,e}(t)\, \tilde{\boldsymbol{\Lambda}}_e(t)\, \boldsymbol{\Xi}_{0,e}(t)\right)^{-1} \boldsymbol{\Xi}'_{0,e}(t)\, \tilde{\boldsymbol{\Lambda}}_e(t)\,,
\end{aligned} \tag{33}$$

which is deterministic. Points in case are survivor bonds of the form (29) and bonds with principal dependent only on the environment state at maturity (discussed in Paragraph I). For such bonds the discounted price processes are of the form $\tilde{S}_i(t) = c_i(t)\, F_{i,Y(t)}(t)$, where the c_i involves a discount factor (could well be stochastic) and maybe a stochastic survival probability, and the $F_{i,e}$ are functions of t only. The coefficients in the martingale dynamics are $\xi_{i,ef}(t) = c_i(t)\,(F_{i,f}(t) - F_{i,e}(t))$, and so $\mathbf{\Xi}_e(t)$ is of the form (32) with $\Psi_e(t) = \mathbf{D}\mathrm{iag}(c_1(t), \ldots, c_m(t))$.

The transform (32) means that the derivatives are obtained by forming m linearly independent SF portfolios in m basic risky derivatives. The two sets of derivatives will span the same space of martingale dynamics and are therefore equivalent for the purpose of hedging.

Another worked example. Two digital causes of death: cause 0 with intensity μ_0 which becomes inactive with intensity λ_1, and cause 1 with intensity μ_1 which becomes inactive with intensity λ_2. Other causes of death are switching from high mortality μ_2 to low mortality μ_3 with intensity λ_3 and from low mortality to high mortality with intensity λ_4. There are thus 8 Y-states:
0 = (cause 0 active, cause 1 active, other causes high),
1 = (cause 0 active, cause 1 active, other causes low),
2 = (cause 0 inactive, cause 1 active, other causes high),
3 = (cause 0 inactive, cause 1 active, other causes low),
4 = (cause 0 active, cause 1 inactive, other causes high),
5 = (cause 0 active, cause 1 inactive, other causes low),
6 = (cause 0 inactive, cause 1 inactive, other causes high),
7 = (cause 0 inactive, cause 1 inactive, other causes high),

Two derivatives: Derivative No 1 is a survivor bond with

principal $exp(-\int_0^U mu_{Y(s)}(s)ds)$ at maturity $U = T + 10$, the other derivative is a digital bond with principal $I_2^Y(T) + I_3^Y(T) + I_6^Y(T) + I_7^Y(T)$ (1 if cause 0 is inactive) at maturity T. Derivative No 1 is effective in all states (its dynamics involves all martingales generated by the Y-process). Derivative No 2 reduces to the bank account in states 2,3,6,7 where Cause 0 has become inactive, and can thus be disregarded in these states. Calculation of the state-wise goes as follows: In states 0 and 1 there are three sources of randomness (directly accessible states) and two derivatives, and we use proj2. In states 2 and 3 there are two sources of randomness and effectively just 1 derivative, and we use proj1. In states 3 and 4 there are two sources of randomness and two effective derivatives which means that there is no hedging error. In states 6 and 7 there is one source of randomness and one effective derivative, which means there is no hedging error.
Results:

Table 6: Portfolio at time 0 for derivative i in state j $\theta[j, i]$

$\theta[0,0] =$	0.0604500	$\theta[0,1] =$	1.2539787	$\theta[0,2] =$	−0.0016435
$\theta[1,0] =$	0.0607967	$\theta[1,1] =$	1.2540398	$\theta[1,2] =$	−0.0016512
$\theta[2,0] =$	0.0601127	$\theta[2,1] =$	1.2521867	$\theta[2,2] =$	0.0000000
$\theta[3,0] =$	0.0604579	$\theta[3,1] =$	1.2522477	$\theta[3,2] =$	0.0000000
$\theta[4,0] =$	−0.0173425	$\theta[4,1] =$	1.4088099	$\theta[4,2] =$	−0.0023568
$\theta[5,0] =$	−0.0173393	$\theta[5,1] =$	1.4088099	$\theta[5,2] =$	−0.0023677
$\theta[6,0] =$	−0.0180721	$\theta[6,1] =$	1.4067967	$\theta[6,2] =$	0.0000000
$\theta[7,0] =$	−0.0180721	$\theta[7,1] =$	1.4067967	$\theta[7,2] =$	0.0000000

e	$V[e]$	$F[1,e]$	$F[2,e]$	ρ
0	0.6735644	0.4893173	0.2914893	$3.6182138986E-07$
1	0.6769010	0.4916795	0.2914893	$3.6163455095E-07$
2	0.6751562	0.4911755	0.7408182	$3.6239222220E-07$
3	0.6785006	0.4935467	0.7408182	$3.6220491125E-07$
4	0.6905159	0.5029390	0.2914893	$4.1700518684E-11$
5	0.6939364	0.5053669	0.2914893	$4.1700536355E-11$
6	0.6921477	0.5048489	0.7408182	$-8.3572715992E-19$
7	0.6955763	0.5072861	0.7408182	$-8.3572715992E-19$

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