Comonotonicity Applied in Finance

Michèle Vanmaele[1]

[1]Department of Applied Mathematics and Computer Science
Ghent University, Belgium

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Outline

Applications in finance: References

1. pricing problem of European type exotic options

- Chen, Deelstra, Dhaene & Vanmaele (2007). Static Super-replicating strategy for a class of exotic options. (submitted)
- Vyncke & Albrecher (2007). Comonotonic control variates for multi-asset option pricing. *Third Brazilian Conference on Statistical Modelling in Insurance and Finance*, 260-265

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2. Minimizing risk of a financial product using a put option

- Deelstra, Ezzine, Heyman & Vanmaele (2007). Managing Value-at-Risk for a bond using put options. *Computational Economics*. 29(2), 139-149.
- Annaert, Deelstra, Heyman & Vanmaele (2007). Risk management of a bond portfolio using options. *Insurance: Mathematics and Economics*. (in press)
- Deelstra, Vanmaele & Vyncke (2008). Minimizing the risk of a financial product using a put option. (in preparation)

European type exotic options

option with pay-off at maturity T

$$(\mathbb{S} - K)_+ \;\; \text{(call)} \quad \text{or} \quad (K - \mathbb{S})_+ \;\; \text{(put)}$$

- discrete case: weighted sum of asset prices at T_i, $0 \le T_i \le T$

$$\mathbb{S} = \sum_{i=1}^{n} w_i X_i, \quad w_i \text{ positive weights}$$

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examples: Asian, basket, pure unit-linked contract

$$X_i = S(T - i + 1) \qquad S_i(T) \qquad P\frac{S(T)}{S(T-i)}$$

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- continuous case: continuous averaging of asset prices

$$\mathbb{S} = \int_0^T w(s)X(s)ds \quad \text{(Asian)}$$

European type exotic options: call option price

model-based approach

$$C[K] = e^{-rT} E[(\mathbb{S} - K)_+]$$

under probability measure Q (all discounted gain processes are martingales, with a gain process being the sum of processes of discounted prices and accumulated discounted dividends)

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$$C[K] = e^{-rT} E[(\mathbb{S} - K)_+] = e^{-rT} \int_K^{+\infty} (1 - F_{\mathbb{S}}(x)) dx$$

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- Cumulative distribution function (cdf) of $\mathbb{S}$: $F_{\mathbb{S}}(x) = \Pr(\mathbb{S} > x)$ explicitly known?

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- via comonotonicity: comonotonic approximations for cdf, lower and upper bounds, comonotonic MC simulation

European type exotic options: call option price

model-free approach

- price $C[K]$ of option with pay-off $(\mathbb{S} - K)_+$ at T not observable in the market
- market of plain vanilla option prices

$$C_i[K] = e^{-rT_i} E[(X_i - K)_+], \quad i = 1, \ldots, n$$

for (finite or infinite) number of strikes K

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- via comonotonicity:
  - largest possible fair price for this option, given the available information from the market
  - price of cheapest super-replicating strategy consisting of buying a linear combination of available plain vanilla options

Minimizing risk of a financial product using a put option

- Classical hedging example: hedging exposure to price risk of an asset
  - minimize VaR of position in share by using put options
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$\Rightarrow$ *comonotonic* and *non-comonotonic*

Stochastic order and comonotonicity: References

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Stochastic order

Definition

A random variable X is said to precede another random variable Y in the stop-loss order sense, notation $X \leq_{sl} Y$, in case

$$E\left[(X-d)_+\right] \leq E\left[(Y-d)_+\right], \quad \text{for all } d.$$

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interpretation:

- X has uniformly smaller upper tails than Y
- any risk-averse decision maker would prefer to pay X instead of Y
- also called increasing convex order and denoted $\leq_{icx}$

  $$X \leq_{icx} Y \quad \Leftrightarrow \quad E[v(X)] \leq E[v(Y)]$$

  for all non-decreasing convex functions v
- if $X \leq_{sl} Y$ then $E[X] \leq E[Y]$

Definition

A random variable X is said to precede another random variable Y in the convex order sense, notation $X \leq_{cx} Y$, if and only if

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- extreme values are more likely to occur for Y than for X
- equivalent formulation:

$$X \leq_{cx} Y \quad \Leftrightarrow \quad E[v(X)] \leq E[v(Y)]$$

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- if $X \leq_{cx} Y$ then $\text{var}[X] \leq \text{var}[Y]$, inverse implication does not hold

$$\frac{1}{2}(\text{var}[Y] - \text{var}[X]) = \int_{-\infty}^{+\infty} |E[(Y-k)_+] - E[(X-k)_+]| dk$$

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  if in addition $\text{var}[X] = \text{var}[Y]$ then X and Y are equal in distribution

General inverse distribution function

Definition

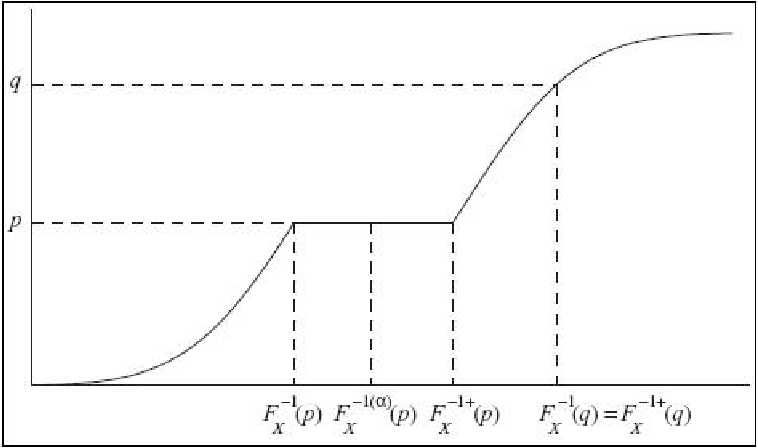
The α-inverse of the cumulative distribution function F_X of a random variable X is defined as a convex combination of the inverses F_X^{-1} and F_X^{-1+} of F_X:

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p)$$

$$p \in (0,1)\,,\ \alpha \in [0,1],$$

with $$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}\,, \quad p \in [0,1]$$

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}\,, \quad p \in [0,1]\,.$$


q
p
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Comonotonicity

Definitions

- A set $A \subseteq \mathbb{R}^n$ is comonotonic if for any $\underline{x}$ and $\underline{y}$ in A, $x_i < y_i$ for some i implies that $x_j \leq y_j$ for all j
- A random vector $(X_1, \ldots, X_n)$ is called comonotonic if it has a comonotonic support

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Equivalent Characterizations

A random vector $(X_1, \ldots, X_n)$ with marginal cdf's $F_{X_i}(x) = \Pr[X_i \leq x]$ is said to be comonotonic if

- for $U \sim Uniform(0,1)$, we have $(X_1, \ldots, X_n) \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right).$
- $\exists$ a r.v. Z and non-decreasing functions f_i, $(i = 1, \ldots, n)$, s.t. $(X_1, \ldots, X_n) \stackrel{d}{=} (f_1(Z), \ldots, f_n(Z)).$

1. Interpretation
   - very strong positive dependence structure
   - if $\underline{x}$ and $\underline{y}$ are possible outcomes of $\underline{X}$, then they must be ordered componentwise
   - common monotonic
   - the higher the value of one component X_i, the higher the value of any other component X_j
   - all components driven by one and the same random variable $\Rightarrow$ one-dimensional

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2. Comonotonicity has some interesting properties that can be used to facilitate various complicated problems
   - Several functions are additive for comonotonic variables

   $\Rightarrow$ multivariate problem is reduced to univariate ones for which quite often analytical expressions are available
   - Comonotonicity leaves the marginals F_{X_i} intact

   $\Rightarrow$ for MC simulation: simulated samples needed in univariate cases are readily available from the main simulation routine

Comonotonic counterpart

The comonotonic counterpart $(Y_1^c, \ldots, Y_n^c)$ of a random vector $(Y_1, \ldots, Y_n)$ with marginal distribution functions F_{Y_i}, $i = 1, \ldots, n$ is given by $\left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right)$, for $U \sim Uniform(0, 1)$.

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Comonotonic sum

$$S^c = Y_1^c + \cdots + Y_n^c$$

with cdf: $F_{S^c}(x) = \sup\left\{p \in [0,1] \mid \sum_{i=1}^n F_{Y_i}^{-1}(p) \le x\right\}$ and

$$F_{S^c}^{-1+}(0) = \sum_{i=1}^n F_{Y_i}^{-1+}(0) \quad \text{and} \quad F_{S^c}^{-1}(1) = \sum_{i=1}^n F_{Y_i}^{-1}(1)$$

Properties

- Additivity: general inverse cdf is additive for comonotonic variables

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- Convex order: For any random vector $(Y_1, \ldots, Y_n)$ with given marginals, the sum $S = \sum_{i=1}^{n} Y_i$ satisfies $S \leq_{cx} S^c$, i.e.

$$E[S] = E[S^c] \quad \text{and} \quad E\left[(S-K)_+\right] \leq E\left[(S^c-K)_+\right]$$

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- always: for $K = \sum_{i=1}^{n} K_i$

$$(S-K)_+ \; = \; (\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} K_i)_+ \leq \sum_{i=1}^{n} \; (Y_i - K_i)_+$$

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- equality for $S = S^c$ and $K_i = F_{Y_i}^{-1(\alpha)}(F_{S^c}(K))$

Properties (continued)

- **Decomposition**: for $K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right)$

$$E\left[(S^c - K)_+\right] = \sum_{i=1}^{n} E\left[\left(Y_i - F_{Y_i}^{-1(\alpha)}\left(F_{S^c}(K)\right)\right)_+\right]$$

with $\alpha \in [0,1]$ such that

$$F_{S^c}^{-1(\alpha)}\left(F_{S^c}(K)\right) = \sum_{i=1}^{n} F_{Y_i}^{-1(\alpha)}\left(F_{S^c}(K)\right) = K$$

$$\Longleftrightarrow \qquad \alpha = \frac{F_{S^c}^{-1+}\left(F_{S^c}(K)\right) - K}{F_{S^c}^{-1+}\left(F_{S^c}(K)\right) - F_{S^c}^{-1}\left(F_{S^c}(K)\right)}$$

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Note: second term is zero when all marginal cdf's F_{X_i} are strictly increasing and at least one is continuous

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known for all strikes K

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- $C[K]$: fair price rational decision maker is willing to pay for option with pay-off $(\mathbb{S} - K)_+$

Theorem

- For any $K \in \left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$, any fair price $C[K]$ of the option with pay-off $(\mathbb{S} - K)_+$ at time T satisfies

$$C[K] \leq e^{-rT} E\left[(\mathbb{S}^c - K)_+\right]$$
$$= \sum_{i=1}^{n} w_i e^{-r(T-T_i)} C_i \left[F_{X_i}^{-1(\alpha)}(F_{\mathbb{S}^c}(K))\right]$$

with α given by

$$\alpha = \frac{F_{\mathbb{S}^c}^{-1+}(F_{\mathbb{S}^c}(K)) - K}{F_{\mathbb{S}^c}^{-1+}(F_{\mathbb{S}^c}(K)) - F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(K))}$$

in case $F_{\mathbb{S}^c}^{-1+}(F_{\mathbb{S}^c}(K)) \neq F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(K))$ and $\alpha = 1$ otherwise.

Theorem (continued)

- For $K \notin \left(F_{\mathbb{S}^c}^{-1+}(0), F_{\mathbb{S}^c}^{-1}(1)\right)$, the exact exotic option price $C[K]$ is given by

$$C[K] = \begin{cases} \sum_{i=1}^{n} w_i e^{-r(T-T_i)} C_i[0] - e^{-rT} K & \text{if } K \leq F_{\mathbb{S}^c}^{-1+}(0) \\ 0 & \text{if } K \geq F_{\mathbb{S}^c}^{-1}(1). \end{cases}$$

Sketch of Proof

- first step

$$E\left[(\mathbb{S}^c - K)_+\right] = \sum_{i=1}^{n} w_i E\left[\left(X_i - F_{X_i}^{-1(\alpha)}\left(F_{\mathbb{S}^c}(K)\right)\right)_+\right]$$

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- Nielsen & Sandmann (2003). *JFQA*, **38**, 449-473: Lagrange optimization + B&S setting

Optimality of super-replicating strategy

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- UB optimal static super-replicating strategy

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- optimal in much broader class of admissible strategies that super-replicate pay-off $(\mathbb{S} - K)_+$:

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subclass:

$$\nu_i(k) = \begin{cases} w_i e^{-r(T-T_i)} & \text{if } k \geq F_{X_i}^{-1(\alpha)}(F_{\mathbb{S}^c}(K)) \\ 0 & \text{if } k < F_{X_i}^{-1(\alpha)}(F_{\mathbb{S}^c}(K)) \end{cases}$$

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- in setting of primal and dual problems

Laurence & Wang (2004). What's a basket worth? *Risk Magazine*, **17**, 73-77.

Hobson, Laurence & Wang (2005). Static-arbitrage upper bounds for the price of basket options. *Quantitative Finance*, **5**, 329-342.

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- UB is largest possible expectation given the marginal pricing distributions of underlying asset prices
- worst possible case is comonotonic case

Application 1: Finite market case

Derivation of the upper bound

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- finite dataset of option prices
- for each i: strikes $0 = K_{i,0} < K_{i,1} < K_{i,2} < \cdots < K_{i,m_i} < \infty$
- pay-offs $(X_i - K_{i,j})_+$ at $T_i \leq T$ and option price

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- model-free UB for $C[K]$ in terms of observed $C_i[K_{i,j}]$ via comonotonicity

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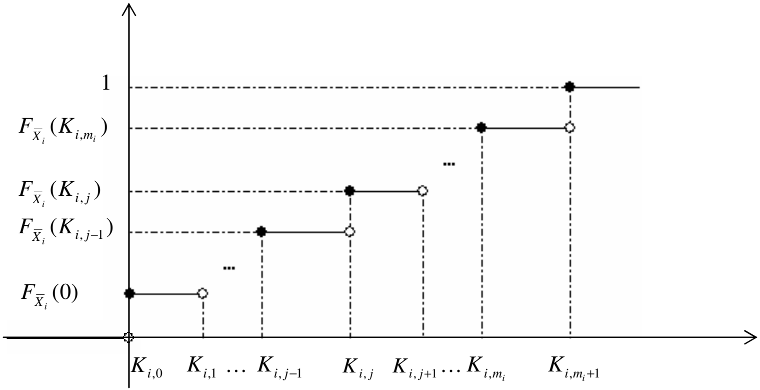
- unifying approach of Chen, Deelstra, Dhaene & Vanmaele (2007)
  (1) construct r.v. $\overline{X}_i$ with discrete distribution $F_{\overline{X}_i}$:

$$F_{\overline{X}_i}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 + e^{rT_i} \dfrac{C_i\left[K_{i,j+1}\right] - C_i\left[K_{i,j}\right]}{K_{i,j+1} - K_{i,j}} & \text{if } K_{i,j} \leq x < K_{i,j+1},\ j = 0, 1, \ldots, m_i \\ 1 & \text{if } x \geq K_{i,m_i+1} \end{cases}$$


1
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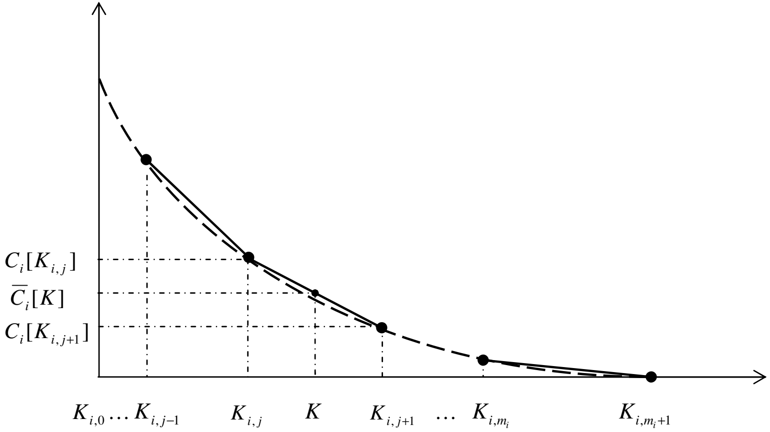

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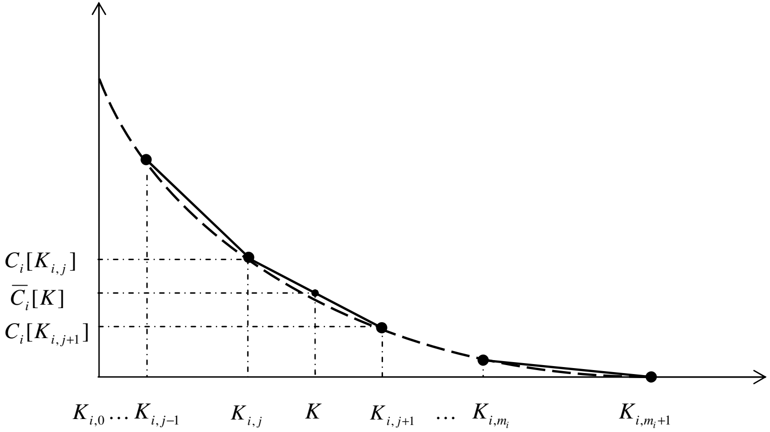

Theorem

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$$C[K] \leq e^{-rT} E\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$$
$$= \sum_{i \in \overline{N}_K} w_i e^{-r(T-T_i)} \left(\alpha C_i [K_{i,j_i}] + (1-\alpha) C_i [K_{i,j_i+1}]\right)$$
$$+ \sum_{i \in N_K} w_i e^{-r(T-T_i)} C_i [K_{i,j_i}]$$

with α given by

$$\alpha = \frac{\sum_{i \in N_K} w_i K_{i,j_i} + \sum_{i \in \overline{N}_K} w_i K_{i,j_i+1} - K}{\sum_{i \in \overline{N}_K} w_i \left(K_{i,j_i+1} - K_{i,j_i}\right)}$$

in case $N_K \neq \{1, 2, \ldots, n\}$ and $\alpha = 1$ otherwise.

Theorem

- For any $K \in (0, \sum_{i=1}^{n} w_i K_{i,m_i+1})$, any fair price $C[K]$ of the option with pay-off $(\mathbb{S} - K)_+$ at time T is constrained from above as follows:

$$C[K] \le e^{-rT} E\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] = \sum_{i \in \overline{N}_K} w_i e^{-r(T-T_i)} \left(\alpha C_i[K_{i,j_i}] + (1-\alpha) C_i[K_{i,j_i+1}]\right) + \sum_{i \in N_K} w_i e^{-r(T-T_i)} C_i[K_{i,j_i}]$$

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$$\alpha = \frac{\sum_{i \in N_K} w_i K_{i,j_i} + \sum_{i \in \overline{N}_K} w_i K_{i,j_i+1} - K}{\sum_{i \in \overline{N}_K} w_i \left(K_{i,j_i+1} - K_{i,j_i}\right)}$$

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Theorem(continued)

- For any $K \notin (0, \sum_{i=1}^n w_i K_{i,m_i+1})$, the option price $C[K]$ is given by:

$$C[K] = \begin{cases} \sum_{i=1}^n w_i e^{-r(T-T_i)} C_i[0] - e^{-rT} K & \text{if } K \leq 0 \\ 0 & \text{if } K \geq \sum_{i=1}^n w_i K_{i,m_i+1}. \end{cases}$$

Sketch of Proof

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- first step: decomposition & comonotonicity

$$E\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right] = \sum_{i=1}^{n} w_i E\left[\left(\overline{X}_i - F^{-1(\alpha)}_{\overline{X}_i}\left(F_{\bar{\mathbb{S}}^c}(K)\right)\right)_+\right]$$

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$$= \begin{cases} C_i\left[K_{i,j_i}\right] & \text{if } i \in A_K \\ \alpha C_i\left[K_{i,j_i}\right] + (1-\alpha)C_i\left[K_{i,j_i+1}\right] & \text{if } i \notin A_K \end{cases}$$

Sketch of Proof (continued)

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- second step

$$(\mathbb{S} - K)_+ \leq \sum_{i=1}^{n} w_i \left(X_i - F_{\overline{X}_i}^{-1(\alpha)} \left(F_{\bar{\mathbb{S}}^c}(K) \right) \right)_+$$

Sketch of Proof (continued)

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\begin{aligned}
(\mathbb{S}-K)_+ &\leq \sum_{i=1}^{n} w_i \left(X_i - F^{-1(\alpha)}_{\overline{X}_i}\left(F_{\overline{\mathbb{S}}^c}(K)\right)\right)_+ \\
&\leq \sum_{i\in\overline{N}_K} w_i \left(\alpha\left(X_i - K_{i,j_i}\right)_+ + (1-\alpha)\left(X_i - K_{i,j_i+1}\right)_+\right) \\
&\quad + \sum_{i\in N_K} w_i \left(X_i - K_{i,j_i}\right)_+
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Sketch of Proof (continued)

- second step: RHS: pay-off of strategy

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\Rightarrow \quad C[K] &\le \sum_{i\in\overline{N}_K} w_i e^{-r(T-T_i)} \left(\alpha C_i\left[K_{i,j_i}\right] + (1-\alpha) C_i\left[K_{i,j_i+1}\right]\right) \\
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&\quad + \sum_{i\in N_K} w_i e^{-r(T-T_i)} C_i\,[K_{i,j_i}]
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■

Remark 1

relation between UB infinite and finite market case

$$\mathbb{S}^c \leq_{sl} \bar{\mathbb{S}}^c \quad \Rightarrow \quad e^{-rT} E\left[(\mathbb{S}^c - K)_+\right] \leq e^{-rT} E\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$$

moreover

$$E\left[\mathbb{S}^c\right] = E\left[\bar{\mathbb{S}}^c\right] \quad \Rightarrow \quad \mathbb{S}^c \leq_{cx} \bar{\mathbb{S}}^c$$

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assumption: $C[K] = e^{-rT} E[(\mathbb{S} - K)_+]$ then from $\mathbb{S} \leq_{cx} \mathbb{S}^c \leq_{sl} \bar{\mathbb{S}}^c$ immediately

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Theorem (convergence result)

The upper bound $e^{-rT} E[(\bar{\mathbb{S}}^c - K)_+]$ in the finite market case converges to the upper bound $e^{-rT} E[(\mathbb{S}^c - K)_+]$ in the infinite market case when $m \to +\infty$ and $h \to 0$.

Optimality of super-replicating strategy

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Definition

$$\bar{\mathcal{A}}_K = \left\{ \underline{\nu} \mid \left(\sum_{i=1}^{n} w_i X_i - K \right)_+ \leq \sum_{i=1}^{n} \sum_{j=0}^{m_i} e^{r(T-T_i)} \nu_{i,j} (X_i - K_{i,j})_+ \right\}$$

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cheapest super-replicating strategy $\underline{\nu} \in \bar{\mathcal{A}}_K$

Theorem

Consider the finite market case. For any $K \in (0, \sum_{i=1}^{n} w_i K_{i,m_i+1})$ we have that

$$e^{-rT} E\left[\left(\bar{\mathbb{S}}^c - K \right)_+ \right] = \min_{\underline{\nu} \in \bar{\mathcal{A}}_K} \sum_{i=1}^{n} \sum_{j=0}^{m_i} \nu_{i,j} C_i \left[K_{i,j} \right].$$

Sketch of Proof

analogous to infinite market case by noting infimum is reached for subclass

$$\nu_{i,j} = \begin{cases} w_i e^{-r(T-T_i)} & \text{if } i \in N_K \text{ and } j = j_i \\ w_i e^{-r(T-T_i)}\alpha & \text{if } i \in \overline{N}_K \text{ and } j = j_i \\ w_i e^{-r(T-T_i)}(1-\alpha) & \text{if } i \in \overline{N}_K \text{ and } j = j_i + 1 \end{cases}$$

and equals UB $e^{-rT} E\left[\left(\bar{\mathbb{S}}^c - K\right)_+\right]$

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Largest possible fair price

- worst case expectation

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Theorem

In the finite market case it holds that for any $K \in (0, \sum_{i=1}^{n} w_i K_{i,m_i+1})$

$$e^{-rT} E\left[(\bar{\mathbb{S}}^c - K)_+\right] = \max_{\underline{Y} \in \overline{\mathcal{R}}_n} e^{-rT} E\left[(\sum_{i=1}^{n} w_i Y_i - K)_+\right]$$

with

$$\overline{\mathcal{R}}_n = \{\underline{Y} \mid Y_i \geq 0 \wedge e^{-rT_i} E[(Y_i - K_{i,j})_+] = C_i[K_{i,j}]\ j = 0, \ldots, m_i+1,\ i = 1, \ldots$$

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- UB is largest possible expectation given the finite number of observable plain vanilla call prices
- worst possible case is comonotonic case

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For more details see Vyncke & Albrecher (2007).

(Comonotonic) lower bound by conditioning

Theorem

For any random vector $(X_1, \ldots, X_n)$ and any random variable Λ, we have

$$E[S \mid \Lambda] = \sum_{i=1}^{n} E[X_i \mid \Lambda] \leq_{cx} S = \sum_{i=1}^{n} X_i$$

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Remarks

- conditional expectation $\Rightarrow$ eliminates randomness that cannot be explained by $\Lambda \Rightarrow S^\ell$ less risky than S
- Λ and S mutually independent $\Rightarrow$ trivial result $E[S] \leq_{cx} S$
- Λ completely determines $S \Rightarrow S^\ell$ coincides with S
- $(E[X_1 \mid \Lambda], \ldots, E[X_n \mid \Lambda])$ in general not same marginals as $(X_1, \ldots, X_n)$
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The random variable Λ is such that

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- additivity of inverse cdf

  $F_{S^\ell}^{-1}(p) = \sum_{i=1}^{n} F_{E[X_i|\Lambda]}^{-1}(p)$
- cdf of S^ℓ: $F_{S^\ell}(x) = \sup\{p \in (0,1) \mid \sum_{i=1}^{n} F_{E[X_i|\Lambda]}^{-1}(p) \leq x\}$

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Properties

- additivity of inverse cdf and some property

$$F_{S^\ell}^{-1}(p) = \sum_{i=1}^{n} F_{E[X_i|\Lambda]}^{-1}(p) = \sum_{i=1}^{n} E[X_i \mid \Lambda = F_\Lambda^{-1+}(1-p)]$$

- cdf of S^ℓ: $F_{S^\ell}(x) = \sup\{p \in (0,1) \mid \sum_{i=1}^{n} F_{E[X_i|\Lambda]}^{-1}(p) \leq x\}$
- cdf of S^ℓ also strictly increasing and continuous and uniquely determined by

$$\sum_{i=1}^{n} F_{E[X_i|\Lambda]}^{-1}(F_{S^\ell}(x)) = x$$

Properties (continued)

Decomposition: for $K \in (F_{S^\ell}^{-1+}(0), F_{S^\ell}^{-1}(1))$

$$E[(S^\ell - K)_+] = \sum_{i=1}^{n} E\left[\left(E[X_i \mid \Lambda] - F_{E[X_i|\Lambda]}^{-1(\alpha)}(F_{S^\ell}(K))\right)_+\right]$$

with $\alpha \in [0,1]$ such that

$$F_{S^\ell}^{-1(\alpha)}(F_{S^\ell}(K)) = \sum_{i=1}^{n} F_{E[X_i|\Lambda]}^{-1(\alpha)}(F_{S^\ell}(K)) = K$$

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Note that under assumptions 1 and 2 the second term is zero.

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- $$E[(S^\ell - K)_+] = \int_{-\infty}^{+\infty} (\sum_{i=1}^{n} E[X_i \mid \Lambda] - K)_+ dF_\Lambda(\lambda)$$
- analytical closed-form expression when all X_i lognormal cdf and Λ normal r.v., see

Deelstra, Diallo & Vanmaele (2007). Bounds for Asian basket options. *JCAM*, in press.

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- From convex ordering: $\text{var}[S^\ell] \le \text{var}[S]$ and

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$$\text{var}[\mathbb{S}] = \sum_{i=1}^n \sum_{j=1}^n w_i w_j E[e^{Z_i}] E[e^{Z_j}] (e^{\text{cov}(Z_i, Z_j)} - 1)$$

$$\text{var}[\mathbb{S}^\ell] = \sum_{i=1}^n \sum_{j=1}^n w_i w_j E[e^{Z_i}] E[e^{Z_j}] (e^{r_i r_j \sigma_{Z_i} \sigma_{Z_j}} - 1)$$

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r_i all same sign $\Rightarrow \mathbb{S}^\ell$ comonotonic sum

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$$\mathrm{CTE}_p[X] = E[X \mid X > F_X^{-1}(p)], \quad p^{\in}(0,1)$$

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   - maximal variance approach: maximize 1st order approx of var$[\mathbb{S}^\ell]$, cfr. Vanduffel, Dhaene & Goovaerts (2005)

$$\text{var}[\mathbb{S}^\ell] \approx \left(\text{corr}(\sum_{j=1}^{n} w_j E[e^{Z_j}], \Lambda) \right)^2 \text{var}[\sum_{j=1}^{n} w_j E[e^{Z_j}] Z_j]$$

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locally optimal choice cfr. Vanduffel et al. (2007)

$$\mathrm{CTE}_p[\mathbb{S}^\ell] = \frac{1}{1-p}\sum_{i=1}^{n} w_i E[e^{Z_i}]\Phi(r_i\sigma_{Z_i} - \Phi^{-1}(p))$$

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maximize 1st order approximation of $\text{CTE}_p[\mathbb{S}^\ell]$

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\end{aligned}$$

$$\Rightarrow \quad \Lambda^{(p)} = \sum_{i=1}^n w_i E[e^{Z_i}]\Phi'[r_i^{MV}\sigma_{Z_i} - \Phi^{-1}(p)]Z_i$$

- Asian options

Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002). The concept of comonotonicity in actuarial science and finance: Applications. *IME*, **31**(2), 133-161.

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Application 2: Minimizing risk by using put option
Risk measures

- consider a set of risks Γ and probability space $(\Omega, \mathcal{F}, P)$
- elements $Y \in \Gamma$ are random variables, representing losses
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Definition

A risk measure ρ is a functional

$$\rho : \Gamma \mapsto \mathbb{R}.$$

Properties risk measures

Properties

- Monotonicity: $Y_1 \leq Y_2$ implies $\rho[Y_1] \leq \rho[Y_2]$, for any $Y_1, Y_2 \in \Gamma$
- Positive homogeneity: $\rho[aY] = a\rho[Y]$, for any $Y \in \Gamma$ and $a > 0$
- Translation invariance: $\rho[Y + b] = \rho[Y] + b$, for any $Y \in \Gamma$ and $b \in \mathbb{R}$
- Subadditivity: $\rho[Y_1 + Y_2] \leq \rho[Y_1] + \rho[Y_2]$, for any $Y_1, Y_2 \in \Gamma$
- Additivity of comonotonic risks: for any $Y_1, Y_2 \in \Gamma$ which are comonotonic: $\rho[Y_1 + Y_2] = \rho[Y_1] + \rho[Y_2]$

Artzner, Delbaen, Eber & Heath (1999). Coherent measures of risk. *Mathematical Finance*, **9**, 203-229.

coherent risk measure: monotonic, positive homogeneous, translation invariant and subadditive

Some well-known risk measures

- Value-at-Risk at level p: p-quantile risk measure

$$\mathrm{VaR}_p[Y] = F_Y^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_Y(x) \geq p\}$$

  related risk measure:
  $\mathrm{VaR}_p^+[Y] = F_Y^{-1+}(p) = \sup\{x \in \mathbb{R} \mid F_Y(x) \leq p\}$
  monotonic, positive homogeneous, translation invariant, additive for comonotonic risks but **not subadditive** $\Rightarrow$ **not coherent**
- Tail Value-at-Risk at level p or Conditional VaR

$$\mathrm{TVaR}_p[Y] = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q[Y]dq$$

  coherent risk measure and additive for comonotonic risks
- Conditional Tail Expectation at level p:

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- for translation invariant and positive homogeneous risk measure

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- put option price: $P(0,T,K) = \text{disc} \cdot \mathrm{E}[(K - X(T))_+]$ and $F_{X(T)}$ continuous

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  e.g. basket of asset prices or coupon-bearing bond
- for some real constants a_i, $i = 1, \ldots, n$:

$$X = \sum_{i=1}^{n} a_i X_i$$

- optimal strike for constrained risk minimization problem again obtained from

$$P(0, T, K) - (K + \rho[-X(T)])\frac{\partial P}{\partial K}(0, T, K) = 0$$

- formula further elaborated under additional assumptions
- distinguish two cases: comonotonic and non-comonotonic sum

Comonotonic sum

- additional assumptions:

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    1. sum $X(T)$ is comonotonic

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- decomposition of risk:

$$\rho[-X(T)] = \rho[-\sum_{i=1}^{n} a_i X_i(T)] = \sum_{i=1}^{n} a_i \rho[-X_i(T)]$$

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- decomposition of put option price:

$$P(0, T, K) = \sum_{i=1}^{n} a_i P_i(0, T, K_i) \quad \text{with} \quad \sum_{i=1}^{n} a_i K_i = K,$$

put option $P_i(0, T, K_i)$ with X_i as underlying, maturity T, strike K_i

- decomposition of put option price:

  characterisation of the components K_i:

$$K_i = F^{-1(\alpha)}_{X_i(T)}(F_{X(T)}(K)) \text{ with } \sum_{i=1}^{n} a_i F^{-1(\alpha)}_{X_i(T)}(F_{X(T)}(K)) = K$$

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  from where

$$\alpha = \frac{K - \sum_{i=1}^{n} a_i F^{-1+}_{X_i(T)}(F_{X(T)}(K))}{\sum_{i=1}^{n} a_i (F^{-1}_{X_i(T)}(F_{X(T)}(K)) - F^{-1+}_{X_i(T)}(F_{X(T)}(K))}$$

  when $F^{-1}_{X_i(T)}(F_{X(T)}(K)) \neq F^{-1+}_{X_i(T)}(F_{X(T)}(K))$ and without loss of generality $\alpha = 1$ otherwise

- decomposition of derivative of put option price

$$\frac{\partial P}{\partial K}(0,T,K)=\sum_{i=1}^{n}a_i\frac{\partial P_i(0,T,K_i)}{\partial K_i}\frac{\partial K_i}{\partial K}$$

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$$\frac{\partial P}{\partial K}(0,T,K) = \sum_{i=1}^{n} a_i \frac{\partial P_i(0,T,K_i)}{\partial K_i} \frac{\partial K_i}{\partial K}$$

assume marginals F_{X_i} are continuous
by Breeden and Litzenberger (1978) and characterisation of K_i

$$\frac{\partial P_i(0,T,K_i)}{\partial K_i} = \text{disc} \cdot F_{X_i(T)}(K_i) = disc \cdot F_{X(T)}(K)$$

thus independent of i

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Algorithm

Step 1 Denote $A_K := F_{X(T)}(K)$ and solve following equation for A_K:

$$\sum_{i=1}^{n} a_i P_i(0, T, F_{X_i(T)}^{-1(\alpha)}(A_K)) - \text{disc} \cdot A_K \sum_{i=1}^{n} a_i (F_{X_i(T)}^{-1(\alpha)}(A_K) + \rho[-X_i(T)]) = 0$$

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$$K^* = \sum_{i=1}^{n} a_i F_{X_i(T)}^{-1(\alpha)}(A_K)$$

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practical application in

Annaert, Deelstra, Heyman & Vanmaele (2007). Risk management of a bond portfolio using options. *Insurance: Mathematics and Economics*. (in press)

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- appromixations of $X(T)$

$$X^{\nu}(T) := \sum_{i=1}^{n} a_i X_i^{\nu}(T), \qquad \nu = \ell, c$$

with

$$X_i^{\ell}(T) := \mathrm{E}[X_i(T)|\Lambda] \quad \text{and} \quad X_i^{c}(T) := F_{X_i(T)}^{-1}(U)$$

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with $X^c(T)$ comonotonic and $X^{\ell}(T)$ also when Λ carefully chosen

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$$\rho[-X^{\nu}(T)] = \sum_{i=1}^{n} a_i \rho[-X_i^{\nu}(T)]$$

original constrained minimization problem:

$$\min_{K,h} X(0) + C - hK + (1-h)\rho[-X(T)]$$

$$\text{s.t. } C = hP(0,T,K) \text{ and } h \in (0,1)$$

approximate constrained minimization problem:

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$$C = h_\nu P^\nu(0, T, K_\nu^*)$$

Step 4 Minimized approximate risk equals

$$X(0) + C - h_\nu^* K_\nu^* + (1 - h_\nu^*) \sum_{i=1}^{n} a_i \rho[-X_i^\nu(T)]$$

Quality of approximations?

- ordering of risk measures based on stochastic dominance, stop-loss order, convex order

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- study applications
  1. coupon-bearing bond and two-additive-factor Gaussian model
  2. basket of shares

see

Deelstra, Vanmaele & Vyncke (2008). Minimizing the risk of a financial product using a put option. (in preparation)

Thanks for your attention!