Noname manuscript No.
(will be inserted by the editor)

Equivalence of the minimax martingale measure

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Received: date / Accepted: date

Abstract This paper states that for financial markets with continuous filtrations, the minimax local martingale measure defined by Frittelli is equivalent to the objective measure for nondecreasing but not strictly increasing utility functions if it exists, provided the dual utility function satisfies some boundedness assumptions for the relative risk aversion, and there exists an equivalent local martingale measure which has enough integrability property. The result in this paper essentially extends an earlier result of Delbaen/Schachermayer, who proved this for the case where the dual utility function is quadratic. Examples for this situation are specifically q-optimal measures for $q > 1$. The generalization is done using Young functions on Orlicz spaces, and proving a conditional version of the Hölder inequality in this general setup.

Keywords Incomplete Markets · Martingale Measure · Duality methods · Utility maximization · Risk aversion

Mathematics Subject Classification (2000) 64N10 · 60G44 · 91B16 · 91B28

JEL Classification G11 · C61

1 Introduction

In an incomplete market model, option prices cannot be determined from arbitrage considerations alone. A well known technique to deal with this situation is the utility indifference argument stated for instance in [10] and [4], namely that the option price should be such that an investor investing optimally with respect to his utility function u should be indifferent of first order between whether or not to invest a small amount in the option. Following [4], this means

$$\sup_{X \in C(x)} E[u(X)] = \sup_{X \in L^\infty : p(X) \leq x} E[u(X)]$$

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where $C(x)$ is the set of superreplicable claims at initial portfolio value x, whereas on the right-hand side, the optimization is done over all claims with price less than or equal to x.

The minimax martingale measure [4] for a given initial wealth x is the probability measure $\hat{Q}_x$ which minimizes the maximal attainable utility at $U(Q,x)$ at a price x,

$$U(\hat{Q}_x, x) \leq U(Q,x) \; \forall Q \in M_1$$

where M_1 is the set of absolutely continuous separating measures, where a separating measure is, as in [4], an absolutely continuous probability measure under which all claims that are replicable at initial portfolio value 0 have a nonpositive expectation, and

$$U(Q,x) := \sup\{E[u(X)] : X \in L^\infty, E^Q[X] \leq x\}$$

Existence of such a minimax measure has been proved in a very general setup in [4]. As already mentioned there, this minimax measure is exactly the pricing rule required in order to guarantee that the supremum attainable expected utility with respect to all (not necessarily replicable) claims with price smaller than or equal to the initial wealth is not larger than the maximal expected utility when considering only replicable claims.

But this minimax measure has another interesting property: by the duality theory, following for example [10] or earlier papers such as [6], it turns out that a good candidate for the optimal terminal value of a portfolio at initial value x in an incomplete market is

$$\hat{X}_T(x) = I\left(\lambda \frac{d\hat{Q}_x}{dP}\right)$$

where I is the inverse of the derivative of u, and λ minimizes

$$\lambda x + E\left[u^*\left(\lambda \frac{d\hat{Q}_x}{dP}\right)\right]$$

where u^* is the convex conjugated function to u.

In general, even if the minimax measure exists, is it not necessarily equivalent to the objective probability measure P. Using the quadratic utility function, already in [2] has been stated that even for a model with only three states, this measure is not necessarily equivalent.

Why is the question of equivalence an interesting one? Firstly, when assuming a market which is free of arbitrage, and thinking about a representative investor with utility function u and initial wealth x, then, when completing the market by the minimax martingale measure, one would like to have that the completed market is also free of arbitrage. This is not the case when the minimax measure is only absolutely continuous with respect to P. Indeed, let Q_x be the minimax measure. If it is not equivalent, there is a measurable set A with $P(A) > 0$ and $Q_x(A) = 0$. It follows that the L^∞-claim 1_A is nonnegative and positive with strictly positive probability. However, this claim has, due to the pricing rule $p(1_A) = E^{\hat{Q}_x}[1_A] = 0$ zero price, and is therefore an arbitrage opportunity. A price system like that seems therefore not to be very reasonable. In this sense, one can see the question of equivalence as a test of the model on its reasonability.

A second reason why the question of equivalence might be interesting has been stated in [10]: if the utility function satisfies the suitable regularity conditions as well as the property of a reasonable asymptotic elasticity (see [10] for details), it follows that the optimal claim, defined by

$$\hat{X}_T(x) = I\left(\lambda \frac{d\hat{Q}_x}{dP}\right)$$

is replicable at price x using only the traded assets in the incomplete market.

Having motivated why the question about equivalence may be interesting, we turn now to the answer of it. In specific situations, equivalence has already been proved in literature. For example, if the price processes are continuous, and using a quadratic utility function (or equivalently a mean-variance optimization), equivalence of the minimal martingale measure has already been proved in [2]. Even if the definition of this measure is different from the definition in [4], it is stated in the latter paper that for continuous processes, the two definitions agree.

Another situation where continuity is not needed has been proved in [3] for the case of exponential utility functions, or for more general utility functions in [5]. In the case of utility functions that are unbounded from above, the equivalence of the minimax martingale measure has been proved in [4].

The present paper generalizes the results of [2]. In the case of a continuous filtration, equivalence will be proved for a very broad class of utility functions which have a satiation point, using only a slight assumption about boundedness of the relative risk aversion. It will turn out as a consequence that all q-optimal measures with $q > 1$ are equivalent, provided the filtration is continuous, and there exists an equivalent local martingale measure which is in L^q. This includes obviously the case of the variance-optimal martingale measure. The technique for this proof follows the idea of [2], but uses a generalized version of the Hölder inequalities for Young functions in Orlicz spaces.

For the case of unsatiated investors with bounded utility functions, there is already a quite general result in [5]. We will show that this result, combined with the one for unbounded utility functions stated in [4], holds for all strictly increasing utility functions.

The outline of this paper is as follows. In section 2, Young functions and Orlicz spaces are introduced. Section 3 introduces a generalized definition of the relative risk aversion which can also be applied if the Young function is not differentiable. Furthermore, some important consequences of the boundedness of the relative risk aversion are proved. In section 4, a conditional version of the generalized Hölder inequality for Young functions is proved, a topic that may be interesting on its own. The main proof concerning the equivalence of the minimax martingale measure for the situation of satiated investors in is presented in section 5. Section 6 then proves the equivalence for all bounded concave utility functions of unsatiated investors. Applications to q-optimal measures are given in section 7. Section 8 concludes.

2 Young functions and Orlicz spaces

If the utility function u is nondecreasing but not strictly increasing, that is if there exists a satiation point $c \in \mathbb{R}$ with $u(x) \le u(c)$ for all $x \in \mathbb{R}$ and $u(x) < u(c)$ for $x < c$, then the function

$$\Phi^*(x) := u(c) - u(c - |x|)$$

is a Young function, with a slight generalization which we will state next. It makes sense to apply Young functions for studying the equivalence problem for utility functions that have a satiation point. We will therefore firstly prove the theorems for Young functions, and subsequently we will show how this is connected to utility functions.

2.1 Generalized Young functions

Definition 2.1 (Young function) A Young function $\Phi(x)$ is an even function $\mathbb{R} \to \mathbb{R}_+$ with

- $\Phi(x)$ is convex
- $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$
- $\lim_{x\to \infty} \frac{\Phi(x)}{x} = \infty$

The general theory of Young functions and Orlicz spaces can be found for example in [1]. For our purposes, we need a slightly more general class of functions which we call generalized Young functions.

Definition 2.2 (Generalized Young function) A function $\Phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a generalized Young function if it satisfies the following properties:

1. $\Phi(0) = 0$
2. $\Phi(x)$ is convex and lower semicontinuous for $x \in \mathbb{R}$
3. There exists a constant $c > 0$ such that $\Phi(x) < \infty$ for all $|x| \leq c$
4. $\Phi(x) \to \infty$ for $x \to \infty$
5. $\Phi(x) = \Phi(-x)$

Definition 2.3 (Conjugate function) A function $\Phi^*(y)$ is conjugate to a generalized Young function $\Phi(x)$ if it satisfies

$$\Phi^*(y) = \sup_{x\in\mathbb{R}} (xy - \Phi(x)) \tag{2.1}$$

Lemma 2.1 *Let Φ be a generalized Young function. Then the following statements hold:*

- *$\Phi(x)$ is continuous on the interior of $\{\Phi(x) < \infty\}$.*
- *$\Phi(x)$ is nonnegative.*
- *If Φ^* is defined by equation (2.1), then Φ^* is also a generalized Young function.*
- *$(\Phi^*)^* = \Phi$.*

Proof The continuity on the interior of $\{\Phi(x) < \infty\}$ follows from the convexity.

For the nonnegativity, let there be a point x_0 where $\Phi(x_0) < 0$. Then $\Phi(-x_0) < 0$ by property (5) in Definition 2.2. By the convexity and property (1), we have that

$$0 = \Phi(0) = \Phi\left(-\tfrac{1}{2}x_0 + \tfrac{1}{2}x_0\right) \leq \tfrac{1}{2}\left(\Phi(-x_0) + \Phi(x_0)\right) = \Phi(x_0) < 0$$

which is an obvious contradiction. If follows that $\Phi(x) \geq 0$.

Let now $\Phi^*(y)$ be the conjugate function of $\Phi(x)$. We have to check the properties (1)-(5).

We have that $\Phi^*(0) = \sup_x(-\Phi(x)) \leq 0$, because $\Phi(x)$ is nonnegative. But by property (1), $\Phi(0) = 0$, and therefore $\Phi^*(0) = 0$.

Property (5) is also easy to check:

$$\Phi(-y) = \sup_x (x(-y) - \Phi(x)) = \sup_x ((-x)y - \Phi(-x)) = \sup_x (xy - \Phi(x)) = \Phi(y)$$

The convexity can be checked as follows:

$$\begin{aligned}\Phi^*(ty_1 + (1-t)y_2) &= \sup_x (txy_1 + (1-t)xy_2 - t\Phi(x) - (1-t)\Phi(x)) \\ &\leq t\sup_x (xy_1 - \Phi(x)) + (1-t)\sup_x (xy_2 - \Phi(x)) \\ &= t\Phi^*(y_1) + (1-t)\Phi^*(y_2)\end{aligned}$$

For showing the lower semicontinuity, we have to prove that the set

$$U := \{y \in \mathbb{R} : \Phi^*(y) > \alpha\}$$

is open for all $\alpha \in \mathbb{R}$. Because this is trivial for $\alpha < 0$, we only show it for $\alpha \geq 0$. Let $y \in U$. Then there exists an $x^* \in \mathbb{R}$ with $x^*y - \Phi(x^*) > \alpha$. It follows obviously $\Phi(x^*) < \infty$ and $x^* > 0$. Because $\{y > \frac{1}{x^*}(\alpha + \Phi(x^*))\}$ is obviously an open set, we have an open ball $U_\epsilon(y)$ around y with

$$\Phi^*(\eta) = \sup_x (\eta x - \Phi(x)) \geq \eta x^* - \Phi(x^*) > \alpha \quad \forall \quad \eta \in U_\epsilon(y)$$

It follows that $U_\epsilon(y) \subset U$ and therefore the lower semicontinuity.

For property (3), there should exist an $\epsilon > 0$ such that

$$\sup_x (x\epsilon - \Phi(x)) < \infty$$

If there exists an $x_0 > 0$ such that $\Phi(x_0) = \infty$, this is obviously the case, because then

$$\sup_x (x\epsilon - \Phi(x)) = \sup_{|x| \leq x_0} (x\epsilon - \Phi(x)) \leq x_0\epsilon < \infty$$

Because of property (4), if $\Phi(x) < \infty \; \forall x \in \mathbb{R}$, we must have that there exists an $x_0 > 0$ with $\Phi(x_0) > 0$. By the convexity of $\Phi(x)$, it follows for $x \neq x_0$ that

$$\frac{\Phi(x) - \Phi(x_0)}{x - x_0} \geq \frac{\Phi(x_0)}{x_0} =: \gamma > 0$$

and therefore

$$\Phi^*(y) = \sup_x (xy - \Phi(x)) \leq \sup_x (xy - \Phi(x_0) - \gamma(x - x_0))$$

Because $\gamma > 0$, we are done, because we can choose $0 < \epsilon < \gamma$, and the right-hand side of the above equation is finite for all $y \leq \epsilon$, which proves property (3).

For property (4), we see that, because there exists a $c > 0$ such that $\Phi(x) < \infty$ on $|x| \leq c$ and the fact that this implies continuity on $[0, c]$, that $\Phi(x)$ is Lipschitz-continuous on $[0, c]$ because of the convexity. Let $L > 0$ be the Lipschitz constant, then, for $y > L$,

$$\begin{aligned}\sup_x (xy - \Phi(x)) &\geq \sup_{x\in[0,c]} (xy - \Phi(x)) \geq \sup_{x\in[0,c]} (xy - Lx) \\ &\geq c(y - L) \to \infty \quad \text{as} \quad y \to \infty\end{aligned}$$

Recognizing that from the properties (1)-(5) it follows that $\Phi(x) < \infty$ on an open set around 0, the last statement of the lemma follows from the Fenchel-Moreau theorem ([9]). Therefore the lemma is completely proved. $\square$

Definition 2.4 (Orlicz space) Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\Phi(x)$ be a generalized Young function. The Orlicz space L^{Φ} defined by Φ is the space of all $\mathcal{F}$-measurable random variables X, for which there exists a $\lambda > 0$ such that

$$E\left[\Phi\left(\frac{X}{\lambda}\right)\right] < \infty$$

The Orlicz space can then be endowed with the Luxemburg norm, which is defined by

$$||X||_{\Phi} := \inf\{\lambda > 0 : E\left[\Phi\left(\frac{X}{\lambda}\right)\right] \leq 1\} \tag{2.2}$$

Example 2.1 For a $c > 0$ set

$$\Phi(x) := \begin{pmatrix} 0 & \text{for } |x| \leq c \\ \infty & \text{for } |x| > c \end{pmatrix}$$

It is easily checked that this is an example of a generalized Young function. Looking at the Luxemburg norm for a random variable X, one finds that

$$||X||_{\Phi} = \frac{1}{c}||X||_{\infty} = \frac{1}{c}\operatorname*{ess\,sup}_{\omega}|X(\omega)|$$

Example 2.2 $\Phi(x) := |x|$. It is again easily checked that this is a Young function, and the corresponding Luxemburg norm is

$$||X||_{\Phi} = ||X||_{L^1}$$

One can see furthermore that example 2 is the conjugate function of example 1, if $c = 1$ in example 1.

We will see later that we want sometimes to exclude 'singular' cases. Therefore we state here an additional assumption that we will need sometimes, namely that a generalized Young function Φ is differentiable at 0. We will say sometimes also that Φ is smooth at 0.

As can be seen, example 1 is smooth at 0, whereas example 2 is not.

Proposition 2.1 *If Φ and Φ^* are conjugate, and if $X \in L^{\Phi}$ and $Y \in L^{\Phi^*}$, then XY is in L^1 and we have the Hölder inequality*

$$E[XY] \leq 2||X||_{\Phi}||Y||_{\Phi^*} \tag{2.3}$$

For Young functions this statement is shown in [1]. The proof for generalized Young functions follows from the one of the conditional Luxemburg norm, which we will prove later in the paper.

It is easily checked that this inequality holds also for the pair (L^1, L^{∞}).

2.3 A Property of generalized Young functions

In this section we prove an auxiliary property that requires only tools of real analysis.

Proposition 2.2 *Let the generalized Young function $\Phi(x)$ be smooth at 0. Then there exists an $y_0 > 0$ such that the conjugate function $\Phi^*(y)$ is invertible for $0 \leq y \leq y_0$, and there exists a constant $x_0 \leq (\Phi^*)^{-1}(y_0)$ such that*

$$\frac{1}{2} \leq \frac{\Phi(x)}{x(\Phi^*)^{-1}(\Phi(x))} \leq 1 \quad \forall \quad 0 \leq x \leq x_0 : \Phi(x) > 0 \tag{2.4}$$

Proof By the Young inequality, we have for any x, y that

$$xy \leq \Phi(x) + \Phi^*(y)$$

In particular, this inequality holds for $y := (\Phi^*)^{-1}(\Phi(x))$, where by the continuity of $\Phi(x)$ for small values of x, we have that the inverse exists for x sufficiently small. It follows that

$$x(\Phi^*)^{-1}(\Phi(x)) \leq \Phi(x) + \Phi^*((\Phi^*)^{-1}(\Phi(x))) = 2\Phi(x)$$

Division by $2x(\Phi^*)^{-1}(\Phi(x))$ gives the lefthand inequality.

On the other hand, we have for every $y \geq 0$ an $\infty \geq x^{opt} \geq 0$ such that

$$x^{opt} y = \Phi(x^{opt}) + \Phi^*(y) \tag{2.5}$$

This follows from the fact that

$$\Phi^*(y) = \sup_{x \geq 0} (xy - \Phi(x))$$

If there is an $x_{max} > 0$ with $x_{max} y \leq \Phi(x_{max})$ and $\Phi(x_{max}) < \infty$, then $xy - \Phi(x)$ is a continuous function on a compact interval $[0, x_{max}]$ and attains therefore its maximum at x^{opt}. Because of the convexity of Φ, $\Phi^*(y)$ cannot be larger than $x^{opt} y - \Phi(x^{opt})$.

If $x_{max} y \leq \Phi(x)$ holds only when $\Phi(x_{max}) = \infty$, but still $x_{max} < \infty$, then there is, by the convexity and the lower semicontinuity of Φ, an $\tilde{x}$ such that $\Phi(\tilde{x}) < \infty$ and $\Phi(x) = \infty$ for $x > \tilde{x}$. Again, $xy - \Phi(x)$ is a continuous function on the compact interval $[0, \tilde{x}]$, and attains therefore its maximum at x^{opt}. Because for any larger x, $xy - \Phi(x) = -\infty$, the maximum at x^{opt} is also the value of $\Phi^*(y)$. If there exists no $x_{max} < \infty$ with $x_{max} y \leq \Phi(x_{max})$, then, by the convexity of $\Phi(x)$, the function $xy - \Phi(x)$ is increasing for all $x \geq 0$. This means $x^{opt} = \infty$.

Now let $y = (\Phi^*)^{-1}(\Phi(x))$. Again, this number is well defined and small if x is sufficiently small. If $x < x^{opt}$, then $\tilde{x}y - \Phi(\tilde{x})$ is monotonically increasing on $[0, x^{opt}[$, and therefore $\tilde{x}y \geq \Phi(\tilde{x})$ for all $\tilde{x} < x^{opt}$. This holds in particular for $\tilde{x} = x$, and therefore

$$x(\Phi^*)^{-1}(\Phi(x)) = xy \geq \Phi(x)$$

and equation (2.4) is proved. If $x \geq x^{opt}$, then $x^{opt} < \infty$ and

$$x(\Phi^*)^{-1}(\Phi(x)) \geq x^{opt}(\Phi^*)^{-1}(\Phi(x)) = \Phi(x^{opt}) + \Phi^*((\Phi^*)^{-1}(\Phi(x))) \geq \Phi(x)$$

because by the nonnegativity of Φ we have $\Phi(x^{opt}) \geq 0$, and for the equality equation (2.5) has been applied. □

3 Generalized Young functions and relative risk aversion

It will turn out that relative risk aversion of a generalized Young function ϕ is an important issue in order to be able to replace any absolutely continuous local martingale measure by an equivalent one which has a smaller L^{ϕ}-expectation, as has been done in [2] for the case $\phi(x) = x^2$. The relative risk aversion is connected to a power function property, that is that if $\phi(x)$ has in a region a constant relative risk aversion of γ, it 'behaves' in this region like a power function with power $\gamma+1$. We will extend this fact to generalized Young functions with bounded relative risk aversion. Furthermore, we will see how the relative risk aversion of the primal and its dual function is connected.

In the standard literature, relative risk aversion is only defined if the utility function is twice differentiable. We will extend this definition to all generalized Young functions. Furthermore, a strict requirement of bounded relative risk aversion may be easily violated, even if this is only so at a few 'irrelevant' points. We will therefore introduce the definition of essential relative risk aversion, in order to be able to exclude irrelevant points.

The usual definition of relative risk aversion from the literature is the following:

Definition 3.1 (Relative risk aversion) Let $\phi(x)$ be a generalized Young function with $\phi(x) > 0$ for all $x > 0$ which is at least twice differentiable. Then one may define the relative risk aversion of ϕ through

$$rra(x) := \frac{x\phi''(x)}{\phi'(x)} \tag{3.1}$$

Now let us go to a more general definition. We can restrict to the interval $]0, \infty[$, as a generalized Young function is symmetric. By the fact that ϕ is convex, we have that the subdifferential $\delta\phi(x)$ always exists and is nonempty in the region where $\phi(x)$ is finite, and by the fact that ϕ is a generalized Young function on $]0, \infty[$, it is nondecreasing in the sense that for $x < y$ we have $z_x \leq z_y$ for every $z_x \in \delta\phi(x)$, $z_y \in \delta\phi(y)$. We may therefore uniquely define

$$\phi_r'(x) := \sup_y \{y \in \delta\phi(x)\} \tag{3.2}$$

It is easy to show that $\phi_r'(x) \in \delta\phi(x)$ and the function $\phi_r'(x)$ is right-continuous and monotonic and therefore of finite variation, and we may define the Lebesgue-Stieltjes measure $d\phi_r'(x)$ without ambiguity if $\phi(x) < \infty$. It is clear that if $x = \inf\{y : \phi(y) = \infty\}$, we must have $\phi_r'(x) = \infty$. For consistency and for preserving the monotonicity of ϕ_r', we define therefore $\phi_r' = \infty$ for all x with $\phi(x) = \infty$. Rearranging the terms in equation (3.1), we have, for a twice continuously differentiable function with $\phi(x) > 0$ for all $x > 0$, that $d\ln\phi'$ is absolutely continuous with respect to $d\ln x$, and there exists a unique Radon-Nikodym derivative $\gamma(x)$. We may therefore say that $\phi(x)$ has a relative risk aversion $\gamma(x)$ on an interval $I \subset]0, \infty[$ if

$$\int_B d\ln\phi' = \int_B \gamma(x) d\ln x \tag{3.3}$$

or in differential notation

$$d\ln\phi' = \gamma(x) d\ln x$$

for every Borel set $B \subset I$.

We assumed here that the measure $d\ln\phi'$ is defined on the whole inteval $]0,\infty[$. This is not necessarily the case for generalized Young functions. Indeed, if $\phi(x)=0$ or if $\phi(x)=\infty$, the measure $d\ln\phi'_r$ is not defined. We define therefore the domain

$$D := \{x \in]0,\infty[: \exists\epsilon > 0 : 0 < \phi(y) < \infty \ \forall y \in B_\epsilon(x)\}$$

which is something like an effective domain for the measure $d\ln\phi'_r$. It follows that this measure is sigma-finite on D. On D^c, we define the measure

$$d\ln\phi'_r(\{x\}) := \infty$$

for every point $x \in D^c$, and it follows that on this subset the measure $d\ln\phi'_r$ is obviously not sigma-finite.

Remark 3.1 If ϕ is a (not generalized) Young function, then $D =]0,\infty[$.

We turn now to the general definition of the integrated relative risk aversion.

Definition 3.2 (Integrated relative risk aversion) Let $\phi(x)$ be a generalized Young function. Then the integrated relative risk aversion of ϕ is defined as the measure $d\ln\phi'_r$.

If this measure is absolutely continuous with respect to $d\ln x$, then ϕ has a relative risk aversion $\gamma(x)$ on an interval $I \subset]0,\infty[$ if the measure $d\ln\phi'_r$ satisfies equation (3.3) on every Borel subset $B \subset I$.

Remark 3.2 If $\phi'(x)$ exists, the measure $d\phi'(x)$ is equivalent to $d\phi'_r(x)$. But replacing $d\phi'(x)$ by $d\phi'_r(x)$, we may apply the notion of integrated relative risk aversion for all generalized Young functions.

Remark 3.3 If the measure $d\ln\phi'_r$ is absolutely continuous with respect to the measure $d\ln x$, then the relative risk aversion $\gamma(x)$ is nothing else than the Radon-Nikodym derivative. However, in general, this measure is not absolutely continuous, and therefore the expression $\gamma(x)$ makes no sense by its own, but only in the integrated version stated in equation (3.3), where the righthand side is only a notion for the defined expression on the lefthand side.

Before stating the main inequalities, we would like to define a partial ordering of the relative risk aversion.

Definition 3.3 (Comparison of relative risk aversions) Let $I \subset]0,\infty[$ be an interval, and $\gamma_1(x), \gamma_2(x)$ two relative risk aversion functions corresponding to the generalized Young functions ϕ_1 and ϕ_2. Then we say that $\gamma_1 \le \gamma_2$ on I if for all Borel sets $B \subset I$ we have for their integrated relative risk aversion that

$$d\ln\phi'_{1,r}(B) \le d\ln\phi'_{2,r}(B)$$

Remark 3.4 It follows that if $\gamma_1 \le \gamma_2$, then $d\ln\phi'_{1,r}$ is absolutely continuous with respect to $d\ln\phi'_{2,r}$. Furthermore, if $d\ln\phi'_{2,r}$ is absolutely continuous with respect to $d\ln x$, it follows that $\gamma_1(x) \le \gamma_2(x)$ $d\ln x$- almost surely on I. On the other hand, if $\gamma(x) \le \gamma_{max}$ is a bounded function, then the integrated relative risk aversion defined in equation (3.3) is absolutely continuous with respect to $d\ln x$.

We will now prove the important relationship which connects relative risk aversion to the power property of a function.

Proposition 3.1 *The relative risk aversion of a generalized Young function ϕ is uniformly bounded from below (above) by a constant $\gamma > 0$ in a region $R :=]0, b]$ if and only if for every $0 < x < y < b$ we have the inequality*

$$\phi_r'(y) \geq \phi_r'(x) \left(\frac{y}{x}\right)^\gamma \tag{3.4}$$

or inequality in the other direction if bounded from above.

Proof Let firstly x be in D, and $x < y < b$. If $y \in D^c$, then $\phi_r'(y) = \infty$, and nothing is to prove, because the left-hand side is already ∞.

Let therefore $y \in D$. By assumption we have

$$d \ln \phi_r' \geq \gamma d \ln \xi$$

on $]x, y]$. Therefore by integration

$$\ln \phi_r'(y) - \ln \phi_r'(x) \geq \gamma (\ln y - \ln x)$$

and by the rules of the logarithm and the monotonicity of the exponential function

$$\frac{\phi_r'(y)}{\phi_r'(x)} \geq \left(\frac{y}{x}\right)^\gamma$$

which is equation (3.4).

If $x \in D$, then either $\phi_r'(x) = 0$ from which equation (3.4) follows directly, or $\phi(\xi) > 0$ for all $\xi > x$. In the latter case, inequality (3.4) holds for all $\xi > x$, and by the right-continuity of ϕ_r' then also for x.

Let now equation (3.4) hold and consider the interval $]x, y]$. We consider firstly the case where $x, b \in D$. Then the measures $d \ln \phi_r'$ and $d \ln x$ are sigma-finite. By taking the logarithm which is monotonic we have

$$\ln \phi_r'(y) - \ln \phi_r'(x) \geq \gamma (\ln y - \ln x)$$

on every half-open interval in $D \cap]0, b]$. The result follows by the following Lemma 3.1 for sigma-finite measures.

If $x \in D^c$, then $d \ln \phi_r'(\{x\}) = \infty$ but $\gamma d \ln y(\{x\}) = 0$, for every $\gamma > 0$, and therefore the relative risk aversion must be bounded from below by any constant $\gamma > 0$. $\square$

Lemma 3.1 *Let μ_1 and μ_2 two sigma-finite measures on the Borel set with $\mu_1(I) \leq \mu_2(I)$ for every half-open interval I. Then $\mu_1(B) \leq \mu_2(B)$ for every Borel set B.*

Proof Let μ_1 and μ_2 be two sigma-finite measures on the Borel set with $\mu_1(I) \leq \mu_2(I)$ for all half-open intervals I. If there would be a Borel set B on which $\mu_1(B) > \mu_2(B)$, this inequality would also have to hold on one of the countably many sets on which μ_1 and μ_2 are finite. We may therefore assume that both measures are finite. We define then a signed measure $\lambda := \mu_2 - \mu_1$. This measure is obviously countably additive, positive on all half-open intervals, and negative on B. Because of the additivity of λ, we have that $\lambda \geq 0$ for all finite unions of half-open intervals, which form an algebra. Because this algebra is a subset of the Borel sets, λ is also countably additive on this algebra. By Carathéodory's extension theorem [12], one can therefore extend λ (defined on this algebra, where it is positive) to a positive measure $\tilde{\lambda}$, defined on the whole Borel set. It follows that $\tilde{\lambda}(B) \geq 0$. The two measures λ and $\tilde{\lambda}$ coincide on a π-system generating the Borel set, and by the uniqueness lemma which holds also for signed measures it follows that $\lambda = \tilde{\lambda}$ on the Borel set, therefore $\lambda(B) \geq 0$, a contradiction. $\square$

The following proposition connects the relative risk aversion of a generalized Young function to the one of its dual function. This issue will be needed later when proving the equivalence of the minimax martingale measure.

Proposition 3.2 *Let ϕ be a generalized Young function. Then its dual function $\phi^*(y)$ has a relative risk aversion bounded from above (below) by $\frac{1}{\gamma}$ on $]0, b]$ if $\phi(x)$ has a relative risk aversion bounded from below (above) by γ on $]0, \phi^*(b)]$.*

Proof Let $0 < x < y \leq b$ and assume firstly that $x, y \in D_{\phi^*}$. We have that

$$\int_{]x,y]} d\ln(\phi^*)'_r = \ln(\phi^*)'_r(y) - \ln(\phi^*)'_r(x)$$

and

$$\int_{](\phi^*)'_r(x),(\phi^*)'_r(y)]} d\ln x = \ln(\phi^*)'_r(y) - \ln(\phi^*)'_r(x)$$

and therefore the integrals are the same. Because for $x > 0$ the measure $d\ln x$ is absolutely continuous with respect to the Lebesgue measure, we may exclude the point $(\phi^*)'_r(y)$ from the integration without changing the value. By assumption,

$$\int_{]x,y]} d\ln(\phi^*)'_r = \int_{](\phi^*)'_r(x),(\phi^*)'_r(y)[} d\ln x \leq \frac{1}{\gamma} \int_{](\phi^*)'_r(x),(\phi^*)'_r(y)[} d\ln\phi'_r$$

and the right-hand side is then equal to

$$\frac{1}{\gamma}\left(\ln\phi'_r((\phi^*)'_r(y)_-) - \ln\phi'_r((\phi^*)'_r(x))\right) \leq \frac{1}{\gamma}(\ln y - \ln x) = \frac{1}{\gamma}\int_{]x,y]} d\ln\xi$$

which is the required result. The first inequality follows from the fact that

$$(\phi^*)'_r(y) \in \delta\phi^*(y) \Rightarrow y \in \delta\phi((\phi^*)'_r(y))$$

by the general duality rules of subdifferentials, and $\phi'_r((\phi^*)'_r(y)_-)$ is the infimum of those subdifferentials, and therefore smaller. This holds by the general rule that $\phi'_r(z_-) = \inf\{\delta\phi(z)\}$. On the other hand, by the same argument, we have that $x \in \delta\phi((\phi^*)'_r(x))$ and therefore smaller than or equal to the supremum of the subdifferential, which is $\phi'_r((\phi^*)'_r(x))$.

We will show now that $x, y \in D_{\phi^*}$ is always true if ϕ has a relative risk aversion bounded from below. We have in general $D =]d_{min}, d_{max}[$. If $d_{min} > 0$, we have that $0 \in \delta\phi^*(d_{min})$ and thus $d_{min} \in \delta\phi(0)$. It follows that $\phi(x) \geq d_{min}x$ and because ϕ'_r is monotonically increasing we have $\phi'_r \geq d_{min}$. For $\epsilon > 0$, it follows that

$$\int_{]0,\epsilon]} d\ln\phi'_r \leq \frac{1}{d_{min}}\int_{]0,\epsilon]} d\phi'_r = \frac{1}{d_{min}}\left(\phi'_r(\epsilon) - \phi'_r(0)\right) \to 0$$

as $\epsilon \to 0$. On the other hand, $d\ln x(]0,\epsilon]) = \infty$ for every $\epsilon > 0$. It follows that the relative risk aversion of ϕ cannot be bounded from below.

Let on the other side $d_{max} \leq b$. It follows that

$$\phi(x) = \sup_y \left(xy - \phi^*(y)\right) \leq xb$$

because $\phi^*(y) = \infty$ for $y > b$. By the fact that ϕ'_r is nondecreasing, it follows that $\phi'_r(x) \leq b$ for all $x \in]0, \infty[$. Furthermore we have $(\phi^*)'_r(b) = \infty$. If we choose a $K > 0$ so large that $\phi'_r(K) > 0$, we get

$$d \ln \phi'_r(]K, \infty[) < \infty$$

but $d \ln x(]K, \infty[= \infty$. Again, the relative risk aversion of ϕ cannot be bounded from below.

We have therefore that the boundedness from above holds on any half-open interval, and because $]0, b] \subset D$, we have that the measures $d \ln x$ and $d \ln(\phi^*)'_r$ are sigma-finite. The result follows now by Lemma 3.1. □

Equation (3.3) and Definition 3.3 would imply that there cannot be jumps in ϕ'_r if the relative risk aversion is bounded from above. It is clear that the relative risk aversion is not bounded from above at the jumps. On the other hand, Definitions 3.2 and 3.3 are too strict: for having that the right-hand side of inequality (3.3) can be estimated by the left-hand side with a constant that may be different from 1, it is only necessary that the relative risk aversion is bounded from below (or bounded from above) "mostly". There may be some x where this statement is not satisfied.

A first idea of a generalization of risk aversion would be if we would say that a generalized Young function ϕ has essentially relative risk aversion of $\gamma(x)$ on $]0, b]$ if

$$\sup_{B' \in \mathcal{B}]0,b]} \int_{B'} (d \ln \phi'_r - \gamma d \ln x) < \infty$$

and

$$\inf_{B' \in \mathcal{B}]0,b]} \int_{B'} (d \ln \phi'_r - \gamma d \ln x) > -\infty$$

For obtaining again an if and only if statement analoguous to Proposition 3.1, we define it again slightly more generally.

Definition 3.4 (Essential bounds for relative risk aversion) A generalized Young function $\phi(x)$ has essentially a relative risk aversion bounded from above by $\gamma(x)$, $0 < \gamma \leq \gamma_{max} < \infty$, on a Borel set $]0, b]$ if the supremum

$$\sup_{I =]x,y] : 0 < x < y \leq b} \left(\int_I d \ln \phi'_r - \int_I \gamma(x) d \ln x \right) < \infty \tag{3.5}$$

It has a relative risk aversion essentially bounded from below by $\gamma(x)$ if the infimum

$$\inf_{I =]x,y] : 0 < x < y \leq b} \left(\int_I d \ln \phi'_r - \int_I \gamma(x) d \ln x \right) > -\infty \tag{3.6}$$

It has essentially a risk aversion of $\gamma(x)$ if both (3.5) and (3.6) are valid.

Remark 3.5 We have to restrict here to the case where $\gamma(x)$ is bounded, in order to get that at least the second integral is finite for all fixed intervals that we have considered.

Example 3.1 For $n \geq 0$ define

$$d \ln \phi_r'(x) = \begin{matrix} \ln 2 \text{ if } & x = \frac{3}{4} 2^{-n} \\ 0 & \text{otherwise} \end{matrix}$$

Then we have that $\int_B (d \ln \phi_r' - \gamma d \ln x)$ is unbounded from above as well as from below with respect to all $B \in \mathcal{B}]0, b]$. But looking only at intervals, one can see that the function $\phi(x)$ has essentially a relative risk aversion of 1 due to Definition 3.4. Looking only at intervals, the positive and negative parts of the 'measure' $d \ln \phi_r' - \gamma d \ln x$ cancel out to a uniformly bounded number, even if this (signed) 'measure' is infinite from below as well as from above and does therefore not define a true signed measure.

Remark 3.6 A sufficient condition for ϕ having a relative risk aversion essentially bounded from above is that there exists a Borel-measurable set $C \subset]0, b]$ such that

$$\int_{B'} d \ln \phi_r' \leq \int_{B'} \gamma d \ln x \tag{3.7}$$

for each Borel set $B' \subset (]0, b] \setminus C)$, where

$$\int_C d \ln \phi_r'(x) < \infty \tag{3.8}$$

Example 3.2 If ϕ_r' has a finite amount of jumps in $0 < x_1 < ... < x_n < \infty$ it satisfies assumption (3.8) if $\phi_r'(x_{1-}) > 0$. By the transformation of variable formula for finite variation processes equation (3.8) then gives for $C = \{x_1, ..., x_n\}$

$$\int_C d \ln \phi_r'(x) = \sum_i \ln \left(\frac{\phi_r'(x_i)}{\phi_r'(x_i^-)} \right) < \infty$$

Example 3.3 The function $\phi(x) := x^{\frac{1}{x}}$ is convex if $x > 0$ is sufficiently small, and $\phi(0) = 0$. But the relative risk aversion according to to Definition 3.4 is not essentially bounded from above on any interval $]0, b]$.

Remark 3.7 A sufficient condition for ϕ having a relative risk aversion essentially bounded from below is that there exists a Borel set $C \subset]0, b]$ such that

$$\int_{B'} d \ln \phi_r' \geq \gamma \int_{B'} d \ln x \tag{3.9}$$

for each Borel set $B' \subset (]0, b] \setminus C)$, where

$$\int_C d \ln(x) < \infty \tag{3.10}$$

Example 3.4 Assumption (3.10) is satisfied for a finite amount of closed intervals away from 0: $C = [x_1, y_1] \cup ... \cup [x_n, y_n]$ with $0 < x_1 < y_1 < x_2 < y_2 < ... < y_n < \infty$.

Example 3.5 The function $\phi(x) := -\frac{x}{\ln x}$ is convex if $x > 0$ is sufficiently small, and $\phi(0) = 0$. The relative risk aversion is not essentially bounded from below by a constant $\gamma > 0$ on any interval $]0, b]$. The function is asymptotically linear.

Remark 3.8 Equations (3.5) and (3.6) indeed hold for all intervals $I \subset]0,b]$ which are bounded away from 0. Let (3.5) be satisfied for half-open intervals $I =]a,c]$ with $a > 0$ and $a < c \leq b$. By the fact that $d\ln x$ is absolutely continuous and $d\ln\phi'_r$ is a positive measure,

$$\int_{]a,c[} d\ln\phi'_r - \int_{]a,c[} \gamma d\ln x \leq \int_{]a,c]} d\ln\phi'_r - \int_{]a,c]} \gamma d\ln x < \infty$$

Now consider the interval $[a,c]$. Then, with $0 < a_1 < a$, we have

$$\int_{[a,c]} d\ln\phi'_r - \int_{[a,c]} \gamma d\ln x \leq \int_{]a_1,c]} d\ln\phi'_r - \int_{]a_1,c]} \gamma d\ln x + \gamma_{max} \int_{]a_1,a]} d\ln x < \infty$$

because $d\ln x$ is absolutely continuous with respect to the Lebesgue measure, and $\gamma(x)$ is bounded.

On the other hand, let equation (3.6) be satisfied for all half-open intervals $]a,c] \subset]0,b]$. By the fact that $d\ln x$ is absolutely continuous and $d\ln\phi'_r$ is a positive measure, we have that

$$\int_{[a,c]} d\ln\phi'_r - \int_{[a,c]} \gamma d\ln x \geq \int_{]a,c]} d\ln\phi'_r - \int_{]a,c]} \gamma d\ln x \geq K > -\infty$$

Now consider the interval $]a,c[$. By the absolute continuity of $d\ln x$ and the positivity of $d\ln\phi'_r$ we have

$$\int_{]a,c[} d\ln\phi'_r - \int_{]a,c[} \gamma d\ln x \geq \sup_{\tilde{c}<c} \left(\int_{]a,\tilde{c}]} d\ln\phi'_r - \int_{]a,\tilde{c}]} \gamma d\ln x \right) \geq K > -\infty$$

□

We now reformulate Propositions 3.1 and 3.2 for the case of essentially bounded relative risk aversion:

Proposition 3.3 *Let ϕ be a generalized Young function. Then the relative risk aversion of ϕ is essentially bounded from above (below) by a constant $0 < \gamma < \infty$ for $0 < x \leq b$, with $b > 0$ if and only if $\phi(b) > 0$ and there exists a constant $K > 0$ such that for all $0 < x < y < b$, we have the inequality*

$$\phi'_r(y) \leq K\phi'_r(x) \left(\frac{y}{x}\right)^\gamma \tag{3.11}$$

Proof Let $\phi(x)$ satisfy equation (3.5) of Definition 3.4 on $]0,b]$. Let $0 < x < y < b$. Because $]x,y] \subset]0,b]$ is an interval, we have

$$\int_{]x,y]} d\ln\phi'_r \leq \gamma \int_{]x,y]} d\ln x + K_1$$

where $K_1 > 0$ is the supremum from equation (3.5). It follows by the rules of the logarithm that if $\phi(x) > 0$

$$\ln\phi'_r(y) \leq \ln\phi'_r(x) + \gamma \ln\left(\frac{y}{x}\right) + K_1$$

By the monotonicity of the exponential function, equation (3.11) follows with the constant $K = \exp(K_1)$. If there would be an $x > 0$ with $\phi(x) = 0$, then also $\phi(x_1) = 0$ with $x_1 < x$, and by definition, $d\ln\phi'_r(]x_1,x]) = \infty$. On the other hand, $d\ln x(]x_1,x]) <$

∞, and therefore equation (3.5) could not be valid. It follows that $\phi_r' > 0$ for all $x > 0$, and therefore also $\phi(b) > 0$.

On the other hand, let there be a constant $K > 0$ such that for all $0 < x < y < b$, equation (3.11) is satisfied. If $\phi_r'(x) = 0$ for an $x > 0$, then equation (3.11) could not be valid for $y = b$, because if $\phi(b) > 0$ we must also have $\phi_r' > 0$. It follows that we can always take the logarithm. Then, by doing this, we have

$$\ln \phi_r'(y) - \ln \phi_r'(x) \leq \gamma(\ln y - \ln x) + \ln K$$

Therefore, the supremum in equation (3.5) must be bounded by $\ln K$.

The statement for the case with boundedness from below is proved in the same way. $\square$

Corollary 3.1 *Let ϕ be a generalized Young function with relative risk aversion essentially bounded from above by a constant γ on $]0, b]$. Then for all $b > y > x > 0$ the following inequality holds:*

$$\phi(y) \leq K\phi(x) \left(\frac{y}{x}\right)^{\gamma+1} \tag{3.12}$$

Proof Because $\phi_r'(x)$ is nondecreasing, it is continuous with exception of at most countably many points, which are a Lebesgue-Nullset. Therefore ϕ_r' is Riemann integrable. By the fact that $\phi(0) = 0$, we have for all partitions $0 = x_0 < .. < x_n = x$

$$\phi(x) = \sum_{j=0}^{n-1} \phi(x_{j+1}) - \phi(x_j)$$

and by the convexity of ϕ and the fact that $\phi_r'(\xi) \in \delta\phi(\xi)$, it follows

$$\phi(x_{j+1}) - \phi(x_j) \geq \phi_r'(x_j)\Delta x_j$$

where $\Delta x_j := x_{j+1} - x_j$, and

$$\phi(x_{j+1}) - \phi(x_j) \leq \phi_r'(x_{j+1})\Delta x_j$$

It follows that

$$\sum_{j=0}^{n-1} \phi_r'(x_j)\Delta x_j \leq \phi(x) \leq \sum_{j=0}^{n-1} \phi_r'(x_{j+1})\Delta x_j$$

Because of the Riemann integrability, the lefthand side as well as the righthand side converge to the integral of ϕ_r'.

The same thing holds obviously also for $\phi(y)$. The result follows now by performing an integration with respect to x as well as another with respect to y, by the monotonicity of the integral. $\square$

Proposition 3.4 *The relative risk aversion of a generalized Young function ϕ is essentially bounded from below by a constant γ on a the set $]0, b]$ if its conjugate function $\phi^*(y)$ has a relative risk aversion essentially bounded from above by the constant $\frac{1}{\gamma}$ on the set $]0, \phi_r'(b)]$. Furthermore, the statement holds also if we exchange the words 'above' and 'below'.*

Proof Let $0 < x < y < b$ be given and let firstly $x, y \in D$. We have

$$\int_{]x,y]} d\ln\phi_r' = \ln\phi_r'(y) - \ln\phi_r'(x) = \int_{]\phi_r'(x),\phi_r'(y)]} d\ln x$$

and on the right-hand side we can take the closed interval instead, because the difference is only a Lebesgue-Nullset.

By the fact that ϕ^* has a relative risk aversion essentially bounded from above by $\gamma > 0$, we have

$$\int_{[\phi_r'(x),\phi_r'(y)]} d\ln x + \gamma K \geq \gamma\left(\int_{[\phi_r'(x),\phi_r'(y)]} d\ln(\phi^*)'\right)$$

where K is the supremum in equation (3.5), and therefore independent of the choice of x and y. A similar argument as in the proof of Proposition 3.2 gives that

$$\gamma K + \int_{]x,y]} d\ln\phi_r' \geq \gamma\int_{[\phi_r'(x),\phi_r'(y)]} d\ln(\phi^*)' \geq \gamma(\ln y - \ln x) = \gamma\int_{]x,y]} d\ln\xi$$

from which equation (3.6) follows, because this holds uniformly for all intervals $]x, y]$.

For intervals which contain elements in D^c, equation (3.6) holds trivially, because by definition the measure $d\ln\phi_r'$ would be infinite on this set, whereas the measure $d\ln x$ would be finite. $\square$

Proposition 3.5 *Let $\phi(x) > 0$ for all $x > 0$, and let on a set $]0, b]$ with $\phi(b) < \infty$, $\phi(x)$ satisfy for all $0 < x < y < b$ the inequality (3.12) for a constant $\gamma > 0$ (respectively the reverse inequality). Then $\phi(x)$ is invertible on $[0, b]$, and its inverse $\phi^{-1}(\xi)$ satisfies for all $0 < \xi < \eta < \phi(b)$*

$$\frac{\phi^{-1}(\eta)}{\phi^{-1}(\xi)} \geq \frac{1}{K}\left(\frac{\eta}{\xi}\right)^{\frac{1}{\gamma+1}} \tag{3.13}$$

Proof Because of the fact that $\phi(x) > 0$ for $x > 0$ and $\phi(0) = 0$, it follows that $\phi(x)$ is strictly increasing on $\{x > 0\}$, and by the convexity and the fact that $\phi(b) < \infty$ it is also continuous on $[0, b]$, and therefore invertible. Equation (3.13) then follows from equation (3.12) by setting $x = \phi^{-1}(\xi)$ and $y = \phi^{-1}(\eta)$. $\square$

The final corollary of this section will give the important technical condition which makes it possible to prove the equivalence of the minimax martingale measure.

Corollary 3.2 *Let the generalized Young function $\phi(x)$ have a relative risk aversion which is essentially bounded from below by $\gamma > 0$ in a region around 0, and let $\phi^*(y) > 0$ for all $y > 0$. Then we have for all sequences $x_n \to 0$ and $p_n \to 0$ that*

$$\frac{(\phi^*)^{-1}(p_n x_n)}{(\phi^*)^{-1}(x_n)} \to 0$$

Proof By Proposition 3.4, we have that $\phi^*(y)$ has a relative risk aversion which is essentially bounded from above by $\frac{1}{\gamma}$ if $y > 0$ is sufficiently small. By Proposition 3.5, it follows that $(\phi^*)^{-1}(x)$ satisfies the inequality (3.13) with γ replaced by $\frac{1}{\gamma}$, that is

$$\frac{y}{x} \leq \left(K\frac{(\phi^*)^{-1}(y)}{(\phi^*)^{-1}(x)}\right)^{\frac{\gamma+1}{\gamma}}$$

Applying this for x_n as y and $p_n x_n$ as x we get

$$\frac{(\phi^*)^{-1}(p_n x_n)}{(\phi^*)^{-1}(x_n)} \leq K p_n^{\frac{\gamma}{\gamma+1}}$$

Because $\gamma > 0$, the right-hand side tends to 0 as $p_n \to 0$. □

4 Conditional Luxemburg norm and Hölder inequality

The aim of this section is to get a conditional version of the Hölder inequality for general Luxemburg norms. We also have to make a further generalization with respect to the definition (2.2) of the Luxemburg norm. Indeed, the norm defined there gives in principle a comparison of random variables at the point where $E[\Phi(\lambda X)] = 1$. In L^p spaces, this does not matter, because $E[\Phi(\lambda X)] = \lambda^p E[\Phi(X)]$, such that if we would replace the constant 1 in (2.2) by another constant c, we would get an equivalent norm. However, this is not true any more for a general function $\Phi(x)$. The fact that a random variable X has a larger Luxemburg norm than a random variable Y does not imply that $E[\Phi(X)] \geq E[\Phi(Y)]$, if we are not in the L^p case. For finding later an equivalent martingale measure $\tilde{Z}$ which contradicts that an absolutely continuous measure Z^{opt} is a minimax measure, we have to compare the measures at the right point. The point 1 is completely arbitrary. In general, for the conditional version of the Luxemburg norm, we take only a strictly positive random variable which is measurable with respect to the sub-sigma-algebra.

A final issue is that a power function $\Phi(x) = x^p$ always satisfies the so-called Δ_2 condition stated for example in [1], which in particular implies for a specific $t_0 > 0$ that

$$E[\Phi(t_0 X)] < \infty \Rightarrow E[\Phi(tX)] < \infty \ \forall t \geq 0$$

This is in general not satisfied by a generalized Young function. Therefore, on some subsets of Ω, equation (2.2) still may be satisfied for a specific λ, even if the expectation is infinity on the total Ω for this λ. Therefore, for a definition of a conditional Luxemburg norm, one would like to have a conditional expectation even if the random variable is not integrable. On the other hand, by the definition of the generalized Young functions, the random variables are always nonnegative. We therefore have to use the nonnegative version of the conditional expectation, which has been discussed in [11].

4.1 Conditional expectation for nonnegative random variables

The aim of this section is to prove a statement about existence of a left-continuous as well as monotonically increasing version of the conditional expectation for nonnegative random variables, which will be needed in order to prove the existence of the conditional Luxemburg norm. The statement would be a consequence of the standard result for supermartingales, as soon as we assume integrability. One may expect that this holds also for the case of nonnegative random variables. For completeness, we give here the proofs.

We repeat here the definition of the conditional expectation for nonnegative random variables, as in [11].

Definition 4.1 (Conditional expectation) Let $X \geq 0$ be a random variable which may attain the value ∞, and let $\mathcal{G}$ be a sub-sigma-algebra. Then a random variable $Y : \Omega \to [0, \infty]$ is said to be a version of the conditional expectation of X conditionally upon $\mathcal{G}$, denoted by $E[X|\mathcal{G}]$, if

1. Y is nonnegative
2. Y is $\mathcal{G}$-measurable
3. For every subset $G \in \mathcal{G}$ we have

$$\int_G X dP = \int_G Y dP \tag{4.1}$$

   where equality in $[0, \infty]$ means that either both are finite and equal or both are infinite.

It has been stated in [11] that all rules about the conditional expectation with integrability condition can also been used for the nonnegative version of it.

With the existence of the conditional expectation for nonnegative random variables, we can define, for a nondecreasing, nonnegative and left-continuous process X_t, a process Y_t by

$$Y_t := E[X_t|\mathcal{G}] \tag{4.2}$$

We will now show that Y_t has a nonnegative, nondecreasing, left-continuous modification. For this, we would like to proceed as in [7], but we cannot directly apply the results, because there integrability has been assumed. For doing so, we need firstly a new upcrossing lemma:

Lemma 4.1 *Let X_t be a nonnegative and nondecreasing process, $\mathcal{G}$ a sub-sigma-algebra, and Y_t as in (4.2). Then the following is true:*

1. *For any increasing sequence t_n, the process Y_{t_n} is almost surely nondecreasing*
2. *The process Y_t is almost surely nondecreasing for $t \in \mathbb{Q}$*
3. *For any $a < b$, the amount of upcrossings $U_N[a, b]$ is almost surely bounded by 1*

Proof Let $t_1 < t_2$. Then, by the monotonicity of the conditional expectation, $E[X_{t_1}|\mathcal{G}] \leq \tilde{E}[X_{t_2}|\mathcal{G}]$ almost surely, and therefore $P[Y_{t_1} > Y_{t_2}] = 0$. It follows that

$$P[\exists t_n < t_m : Y_{t_n} > Y_{t_m}] = P\left[\cup_{n<m}\{Y_{t_n} > Y_{t_m}\right] = 0$$

and therefore almost every process Y_{t_n} is nondecreasing, which shows item 1.

For item 2, define the set

$$A := \{\omega \in \Omega : \exists t_1 < t_2 \in \mathbb{Q} : Y_{t_1} > Y_{t_2}\}$$

Then A is a countable union of sets $A_{t_1,t_2} := \{Y_{t_1} > Y_{t_2}\}$ for fixed $t_1 < t_2$, which have probability 0. Therefore A has probability 0.

For proving 3, we have that Y_{t_n} is a nonnegative nondecreasing process, and therefore for any $a < b$, we have

$$U_N[a, b] \leq 1$$

□

From Lemma 4.1, it follows also that $U_\infty[a,b] \leq 1$ almost surely, and therefore, for any $a < b$,

$$E[U_\infty[a,b]] \leq 1 < \infty$$

We proceed now in a similar way as in [7]. Firstly, we define, from Y_t, a process $\tilde{Y}_t$ which is nonnegative, nondecreasing and left-continuous, and subsequently we prove that this process is a modification of Y_t.

Proposition 4.1 *For any version of the proces Y_t defined above, define the process*

$$\tilde{Y}_t := \sup_{s<t; s\in\mathbb{Q}} Y_s \tag{4.3}$$

Then there exists a subset $\Omega^* \subset \Omega$ with $P[\Omega^*] = 1$, such that for all $\omega \in \Omega^*$ the following is true:

1. *$\tilde{Y}_t(\omega)$ is nonnegative for all t*
2. *$\tilde{Y}_t(\omega)$ is nondecreasing*
3. *$\tilde{Y}_t(\omega)$ is left-continuous*
4. *$\tilde{Y}_t$ is a modification of Y_t*

Proof We define $\Omega^* := \Omega \setminus A$, with the set A from the proof of the previous Lemma. It follows that $P[\Omega^*] = 1$. Nonnegativity is due to the definition of the conditional expectation for nonnegative random variables. That $\tilde{Y}_t$ is nondecreasing follows from the definition of the supremum.

We now show the left-continuity. Let $t_n < t$ be any sequence converging monotonically to t. Because $\tilde{Y}_t$ is nondecreasing it follows that $\tilde{Y}_{t_n}$ is a nondecreasing sequence, bounded by $\tilde{Y}_t$, or $\tilde{Y}_t = \infty$. Let the limit for $n \to \infty$ be strictly smaller than $\tilde{Y}_t$. Then there exists a sequence $s_m < t, s_m \in \mathbb{Q}$ such that $Y_{s_m} > \lim_{n\to\infty} \tilde{Y}_{t_n}$ for all m and n. Because $t_n \to t$, we may choose a subsequence $t_{n_m} =: t_m$ such that $t_m > s_m$, and a sequence $q_m \in \mathbb{Q}$ with $t_m > q_m > s_m$. By the definition of the supremum, it follows that

$$Y_{q_m} \leq \tilde{Y}_{t_m} < Y_{s_m}$$

Because $q_m > s_m$, this realization is not nondecreasing on $\mathbb{Q}$, and can therefore not be in Ω^*. It follows that $\tilde{Y}_t$ is left-continuous.

That $\tilde{Y}_t$ is a modification of Y_t is shown as follows. By definition, there exists a sequence $t_n \in \mathbb{Q}$ such that $\tilde{Y}_t = \lim_{n\to\infty} Y_{t_n}$, where $t_n < t$. By the fact that Y_t is nondecreasing on $\mathbb{Q}$ (Lemma 4.1), we may choose this sequence in such a way that $t_n \to t$. Furthermore, we may choose a subsequence which is increasing. Then Y_{t_n} is nondecreasing. Therefore, by the monotone convergence theorem, for $G \in \mathcal{G}$

$$\int_G \tilde{Y}_t dP = \lim_{n\to\infty} \int_G Y_{t_n} dP = \lim_{n\to\infty} \int_G X_{t_n} dP = \int_G X_t dP$$

where the second equality follows from the fact that Y_{t_n} is a version of the conditional expectation of X_{t_n}, and the last by the fact that X_t is left-continuous and again the monotone convergence theorem. $\square$

Proposition 4.2 *Let the process X_t be continuous at a point t_0, and integrable for a $t_1 > t_0$. Then $\tilde{Y}_{t_0} = \hat{Y}_{t_0}$ almost surely, and $\tilde{Y}_t$ is continuous at t_0 too.*

Proof Define the process

$$\hat{Y}_t := \inf_{s>t; s\in\mathbb{Q}} Y_s$$

Then with the same arguments as in Proposition 4.1, we have that $\hat{Y}_t$ is nonnegative, nondecreasing and right-continuous on t_0, for all $\omega \in \Omega^*$. Furthermore, for every $s, q \in \mathbb{Q}$ with $s < t < q$, we have $Y_s \leq Y_q$ on Ω^*. Taking on the left-hand side the supremum and on the right-hand side the infimum yields

$$\tilde{Y}_t \leq \hat{Y}_t \ \forall \omega \in \Omega^*, \ \forall \ t$$

Because of the right-continuity of $\hat{Y}_t$ at t_0 and the fact that $\tilde{Y}_t$ is nondecreasing, we have that for any sequence $t_n \to t_0$, $t_n > t_0$ that

$$\tilde{Y}_{t_0} \leq \tilde{Y}_{t_n} \leq \hat{Y}_{t_n} \to \hat{Y}_{t_0}$$

for all $\omega \in \Omega^*$. If there is a $t > t_0$ for which X_t is integrable, then, by the dominated convergence theorem, $\hat{Y}_{t_0}$ is a version of the conditional expectation too, and therefore $\hat{Y}_{t_0} = \tilde{Y}_{t_0}$ almost surely. The result follows. □

Corollary 4.1 *Let the process X_t be left-continuous, integrable at $t = t_0 > 0$ and continuous at 0 with $X_0 = 0$. Then the process $\tilde{Y}_t$ converges to 0 almost surely as $t \to 0$.*

Proof By the dominated convergence theorem, setting $Y_0 = 0$ gives a modification of $\hat{Y}_0$. The result follows by the right-continuity of $\hat{Y}_t$ at $t = 0$. □

4.2 Conditional Luxemburg norm

Definition 4.2 (Conditional Luxemburg norm) For a generalized Young function Φ, a subsigma-algebra $\mathcal{G} \subset \mathcal{F}$, a nonnegative random variable X in the Orlicz space L^Φ and for a $\mathcal{G}$-measurable, nonnegative and integrable random variable ξ, the conditional Luxemburg norm is the $\mathcal{G}$-measurable random variable $\Omega \to [0, \infty]$ given by

$$lux[\Phi(X)|\mathcal{G}]_\xi(\omega) := \begin{array}{ll} \inf \Lambda(\omega) & \text{if } \Lambda(\omega) \neq \emptyset \\ \infty & \text{if } \Lambda(\omega) = \emptyset \end{array} \tag{4.4}$$

where the set $\Lambda(\omega)$ is defined as

$$\Lambda(\omega) := \{\lambda > 0, E^{rc}\left[\Phi\left(\frac{X}{\lambda}\right)|\mathcal{G}\right] \leq \xi\} \tag{4.5}$$

and the notion of E^{rc} means a version of the conditional expectation for nonnegative random variables which is right-continuous in λ.

Theorem 4.1 *The conditional Luxemburg norm as defined above exists and is unique in the sense that if λ and $\tilde{\lambda}$ are two versions of the conditional Luxemburg norm, then*

$$P[\lambda = \tilde{\lambda}] = 1$$

Furthermore, if λ is the conditional Luxemburg norm defined in (4.4) and (4.5), we have on $\{\lambda > 0\}$

$$E\left[\Phi\left(\frac{X}{\lambda}\right)|\mathcal{G}\right] \leq \xi \ a.s. \tag{4.6}$$

whereas on $\{\lambda = 0\}$*, we have* $E[\Phi(X)|\mathcal{G}] = 0$*.*

Moreover, if ξ *is strictly positive, then the conditional Luxemburg norm is almost surely finite.*

Proof We firstly have to show that such a right-continuous version as stated in (4.5) exists. This follows from the fact that with $t := \frac{1}{\lambda}$

$$Z_t := E\left[\Phi\left(tX\right)|\mathcal{G}\right] = E\left[\Phi\left(\frac{X}{\lambda}\right)|\mathcal{G}\right] \tag{4.7}$$

satisfies the assumptions of Proposition 4.1, and therefore has a left-continuous version, which is furthermore monotonically increasing. Therefore $Z_\lambda = Z_{\frac{1}{t}}$ has a right-continuous version which is monotonically decreasing in λ.

Consider now for the moment a specific right-continuous version of the conditional expectation. Then for each $\omega \in \Omega^*$, $P[\Omega^*] = 1$, the set $\Lambda(\omega)$ is uniquely defined as a subset in $\mathbb{R}$ which is bounded by 0 from below. Therefore the infimum exists for all $\omega \in \Omega^*$ for which $\Lambda(\omega)$ is nonempty. We may therefore define the conditional Luxemburg norm by (4.4) without ambiguity on Ω^* .

We show now that this definition is independent of the choice of the version of the right-continuous conditional expectation. Let Y_1 and Y_2 be two right-continuous versions of the conditional expectation. It follows that they are indistinguishable, that is there exists a set $\Omega^{**} \subset \Omega$ where $Y_1 = Y_2$. The set $\Omega^* \cap \Omega^{**}$ has probability 1, and the sets $\Lambda(\omega)$ from (4.5) are the same for every $\omega \in \Omega^* \cap \Omega^{**}$. It follows that the definition is unique on a set with probability 1.

Because $X \in L^\Phi$, we have by definition that X is almost surely finite, and $\frac{X}{\lambda} \to 0$ as $\lambda \to \infty$, and the same property holds for $\Phi(\frac{X}{\lambda})$, by the continuity of generalized Young functions at 0. It follows that the left-continuous modification of the process in (4.7) satisfies the assumptions of corollary 4.1, and converges therefore to 0 almost surely. If $\xi > 0$ strictly, it follows that $\Lambda(\omega)$ is almost surely nonempty, and therefore the infimum in (4.4) exists and is finite.

Let now the random variable defined in (4.4) and (4.5) be denoted by $\lambda^*(\omega)$. We have seen that this random variable exists and is unique. We need to show that $\lambda^*(\omega)$ is $\mathcal{G}$-measurable. Let $\gamma \in \mathbb{R}$. We need to show that $\{\omega : \lambda^*(\omega) \leq \gamma\} \in \mathcal{G}$. But for $\gamma < \infty$, we have

$$\{\lambda^* \leq \gamma\} = \{Y_\gamma := E^{rc}\left[\Phi\left(\frac{X}{\gamma}\right)|\mathcal{G}\right] \leq \xi\}$$

Let namely ω be in the set on the left. Then the set $\Lambda(\omega)$ is nonempty, and there exists a sequence λ_n converging to λ^* from above. Because $\lambda_n \in \Lambda(\omega)$, it satisfies equation (4.5), and $Y_{\lambda_n} \leq \xi$. Because Y_λ is right-continuous, also $Y_\gamma \leq \xi$. If on the other hand $Y_\gamma \leq \xi$, then by definition $\gamma \in \Lambda(\omega)$, and its infimum λ^* is always smaller than or equal to γ. Because Y_γ is obviously $\mathcal{G}$-measurable, the result follows.

It remains to prove equation (4.6). On the set $\{\lambda^* = \infty\}$, we have $\Phi(\frac{X}{\lambda^*}) = \Phi(0) = 0$, and therefore equation (4.6) is satisfied, because ξ is nonnegative. Now consider the set $\{\lambda^* < \infty\}$. It is enough to show the inequality for the sets $S := \{\lambda^* \leq \lambda_{max}\}$ for all $\lambda_{max} > 0$. Because $\lambda^* \in \mathcal{G}$ and bounded from above on S, we may approximate it from above by step functions $\lambda_n := \sum_k \lambda_{nk} 1_{A_{nk}}$ with $A_{nk} \in \mathcal{G}$. On the set A_{nk}, we have therefore that $\lambda(\omega) \leq \lambda_{nk}$, and by the monotonicity and right-continuity of Z_λ in (4.7) it follows that $\lambda_{nk} \in \Lambda(\omega)$ on A_{nk}. This implies by definition (4.5)

$$E\left[\Phi\left(\frac{X}{\lambda_{nk}}\right)|\mathcal{G}\right] \leq \xi$$

on the set A_{nk}, and therefore

$$E\left[\Phi\left(\frac{X}{\lambda_n}\right)|\mathcal{G}\right] = \sum_k E\left[\Phi\left(\frac{X}{\lambda_{nk}}\right)|\mathcal{G}\right] 1_{A_{nk}} \leq \xi$$

on the whole set S. Because this holds for every step function on S, we conclude inequality (4.6) by the monotone convergence theorem. Finally, on $\{\lambda^* = 0\}$, we may take the sequence $\lambda_{nk} = \frac{1}{n}$, and it follows that

$$E[\Phi(nX)|\mathcal{G}] \leq \xi$$

for all $n \in \mathbb{N}$. But because $\Phi(nX)$ tends to infinity on $\{X > 0\}$ and ξ is almost surely finite, this can only happen if $X = 0$ and therefore $E[\Phi(X)|\mathcal{G}] = 0$ on the set $\{\lambda^* = 0\}$. Theorem 4.1 is now proved. □

4.3 Hölder inequality

We now prove a generalization of the Hölder inequality in conditional form, which will be important for the development below:

Theorem 4.2 *Let Φ and Φ^* be complementary Young functions, that vanish only at 0, and let ξ be a strictly positive random variable. Then*

$$E[|XY||\mathcal{G}] \leq 2\xi lux[\Phi(X)|\mathcal{G}]_\xi lux[\Phi^*(Y)|\mathcal{G}]_\xi \tag{4.8}$$

Furthermore, if $X \in L^\Phi$, $Y \in L^{\Phi^*}$, then XY is integrable.

Proof It is enough to prove the statement for X and Y nonnegative. Let us firstly assume that the conditional Luxemburg norms are almost surely strictly positive, and let λ and μ be two strictly positive $\mathcal{G}$-measurable random variables such that

$$\begin{aligned} E\left[\Phi\left(\tfrac{X}{\lambda}\right)|\mathcal{G}\right] &\leq \xi \text{ a.s.} \\ E\left[\Phi^*\left(\tfrac{Y}{\mu}\right)|\mathcal{G}\right] &\leq \xi \text{ a.s.} \end{aligned} \tag{4.9}$$

By the Young inequality, we have almost surely that

$$\frac{XY}{\lambda\mu} \leq \Phi\left(\frac{X}{\lambda}\right) + \Phi^*\left(\frac{Y}{\mu}\right)$$

By the monotonicity of the conditional expectation, this yields

$$E\left[\frac{XY}{\lambda\mu}|\mathcal{G}\right] \leq E\left[\Phi\left(\frac{X}{\lambda}\right) + \Phi^*\left(\frac{Y}{\mu}\right)|\mathcal{G}\right]$$

almost surely. But by (4.9) the right-hand side is almost surely smaller than or equal to 2ξ. Because λ and μ are $\mathcal{G}$-measurable, we get

$$E[XY|\mathcal{G}] \leq 2\xi\lambda\mu \text{ a.s.}$$

By (4.6), inequality (4.9) holds in particular for $\lambda = lux[\Phi(X)|\mathcal{G}]_\xi$ and $\mu = lux[\Phi^*(Y)|\mathcal{G}]_\xi$, which yields (4.8).

If on a set with nonzero probability at least one of the conditional Luxemburg norms is 0, then, by Theorem 4.1, either $E[\Phi(X)|\mathcal{G}] = 0$ or $E[\Phi^*(Y)|\mathcal{G}] = 0$. Becase Φ and Φ^* vanish only at 0, we have $E[XY|\mathcal{G}] = 0$ on this set, and (4.8) is still satisfied.

□

5 Minimax measures for satiated utility functions

5.1 Definitions and assumptions

Throughout this section, we work in an environment with continuous filtration. It is known that without this assumption, equivalence of the minimax martingale measure can even be violated in a three-state model (see [2]).

Assumption 5.1 The probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ has a continous filtration $\mathcal{F}_t$.

It follows that every price process, as well as every density process, has to be continuous. We work here in the environment of [4], and use therefore the same assumptions about the utility functions, with the additional requirement that there is a satiation point.

Assumption 5.2 The utility function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous and concave on $\mathbb{R}$, and nondecreasing in its effective domain, which is assumed to have a nonempty interior. Furthermore u has a satiation point in the interior of its effective domain, that is a point c with $u(x) < u(c)$ for $x < c$ and $u(x) = u(c)$ for $x \geq c$.

Remark 5.1 Because c is in the interior of the effective domain of u, it follows that there is a point $x < c$ with $u(x) > -\infty$.

We state here again the definition of the minimax martingale measure as in [4].

Definition 5.1 (Minimax martingale measure) An absolutely continuous separating measure $\hat{Q}_x$ is a minimax martingale measure if it satisfies

$$\sup_{w \in L^\infty : E^{\hat{Q}_x}[w \leq 0]} \{E^P[u(x+w)]\} = \min_{Q \in M_1} \sup_{w \in L^\infty : E^Q[w] \leq 0} \{E^P[u(x+w)]\}$$

where M_1 is the set of all absolutely continuous separating measures, that is

$$M_1 := \{z \in L^1_+(P) : E^P[zw] \leq 0 \ \forall w \in C, E^P[z] = 1\}$$

where C is the convex cone of superreplicable claims at zero initial portfolio value.

Remark 5.2 Because we are working in an environment where the filtration is continuous, it follows by Lemma 1.1 of [4] that the set of absolutely continuous separating measures corresponds to the set of absolutely continuous local martingale measures. We will therefore not make a distinction between those two notions, as long as we work in an environment with continuous filtration.

Remark 5.3 By Corollary 2.1 of [4], it follows that if $\hat{Q}_x$ is a minimax martingale measure it must satisfy

$$\min_{\lambda \geq 0} \left[\lambda x - E^P \left[u^* \left(\lambda \frac{d\hat{Q}_x}{dP} \right) \right] \right] \leq \min_{\lambda \geq 0} \left[\lambda x - E^P \left[u^* \left(\lambda \frac{dQ}{dP} \right) \right] \right] \tag{5.1}$$

where Q is any absolutely continuous local martingale measure, and $u^*(y)$ is the concave conjugated function of u, that is

$$u^*(y) := \inf_x (xy - u(x)) \tag{5.2}$$

Assumption 5.3 There exists at least one equivalent separating measure $Z_\infty^{(0)}$, which is in L^1.

Assumption 5.4 The conditions stated in [4] are satisfied in order to guarantee that the minimax measure Z_∞^{opt} exists, defined in Definition 5.1.

Because Z_∞^{opt} is integrable, it follows that the martingales

$$\begin{aligned} Z_t^{(0)} &:= E[Z_\infty^{(0)}|\mathcal{F}_t] \\ Z_t^{opt} &:= E[Z_\infty^{opt}|\mathcal{F}_t] \end{aligned} \tag{5.3}$$

are uniformly integrable, and with the assumption that the filtration $\mathcal{F}_t$ is continuous, they are also continuous. The stopping times T_n and T are defined through

$$\begin{aligned} T\ &:= \inf\{t \geq 0 : Z_t^{opt} = 0\} \\ T_n &:= \inf\{t \geq 0 : Z_t^{opt} = \tfrac{1}{n}\} \wedge n \end{aligned} \tag{5.4}$$

It follows that $T_n < T_{n+1} < T$ and $T_n \to T$ almost surely.

5.2 Utility functions and generalized Young functions

With $u(x)$ a utility function satisfying Assumption 5.2, it is easy to see that

$$\Phi^*(x) := u(c) - u(c - |x|) \tag{5.5}$$

is a generalized Young function (Definition 2.2).

The next lemma shows why we can restrict our considerations to generalized Young functions:

Lemma 5.1 *Let the utility function satisfy Assumption 5.2. Then*

1. *For all $y \geq 0$, the concave conjugate function as defined in (5.2) satisfies*
$$u^*(y) = yc - u(c) - \Phi(y) \tag{5.6}$$
where $\Phi(y) = \Phi^{**}(y)$ is the conjugate function to $\Phi^*(x)$, defined in (5.5).
2. *If $\hat{Q}_x$ is a minimax martingale measure (Definition 5.1), it satisfies*
$$\hat{\lambda} x_0 + E[\Phi(\hat{\lambda} Z_\infty^{opt})] \leq \lambda x_0 + E[\Phi(\lambda Z_\infty)] \quad \forall Z_\infty \in \mathcal{M},\ \forall \lambda > 0 \tag{5.7}$$
where $\mathcal{M}$ is the set of all absolutely continuous local martingale measures, and $\hat{\lambda}$ is the minimum of the left-hand side of (5.1).

Proof For $y \geq 0$ we have

$$\begin{aligned} \Phi(y) &= \sup_x (xy - \Phi^*(x)) = \sup_x (xy - u(c) + u(c - |x|)) \\ &= \sup_{x \geq 0} (xy - u(c) + u(c - x)) \\ &= \sup_{x' \leq c} \big((c - x')y - u(c) + u(x')\big) \\ &= cy - u(c) + \sup_{x' \leq c} \big(u(x') - x'y\big) \\ &= cy - u(c) - \inf_{x' \leq c} \big(x'y - u(x')\big) \\ &= cy - u(c) - \inf_{x'} \big(x'y - u(x')\big) \\ &= cy - u(c) - u^*(y) \end{aligned}$$

From this statement 1 follows.

For any $\lambda \geq 0$, for any local martingale measure Q, it follows that

$$\lambda x - E^P\left[u^*\left(\lambda \frac{dQ}{dP}\right)\right] = \lambda x - \lambda c + u(c) + E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right]$$

Hence, if $\hat{Q}_x$ is a minimax martingale measure, we have from Remark 5.3 that

$$\min_{\lambda \geq 0}\left[\lambda x + E^P\left[\Phi\left(\lambda \frac{d\hat{Q}_x}{dP}\right)\right]\right] \leq \min_{\lambda \geq 0}\left[\lambda x + E^P\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right]\right]$$

for all absolutely continuous local martingale measures Q. Performing the minimization over λ on the left-hand side yields statement 2. □

5.3 Boundedness of the relative risk process

We come now to the key reasons of the above considerations. We would like to prove an extension of the argument stated in [2]. Indeed we would like to show that if the relative risk aversion near 0 is essentially bounded from below away from 0, then the relative risk, as will be defined next, converges to infinity for a continuous martingale on the set where it converges to 0. On the other hand, if the equivalent martingale measure has enough integrability property, the relative risk of it will remain bounded.

Definition 5.2 (Relative risk) Let X be a L^{Φ}-integrable random variable, and c a constant. Then the relative risk of X at the point c is defined as

$$RR(c) := \frac{E[\Phi(cX)]}{\Phi(c)} \leq \infty \tag{5.8}$$

Proposition 5.1 *Let Φ be a generalized Young function with $\Phi(x) > 0$ for all $x > 0$ and let the relative risk aversion of $\Phi(x)$ be essentially bounded from above by $0 \leq \gamma \leq \infty$ for all $x \geq 0$. Furthermore, let X be a random variable which is $L^{\gamma+1}$-integrable, and let $\mathcal{F}_n$ be a filtration. If $\gamma < \infty$, then for every sequence $a_n \in \mathcal{F}_n$ with $a_n > 0$ which is uniformly bounded from below away from 0 for all ω, and for every sequence $c_n \in \mathcal{F}_n$ such that $c_n \to 0$ almost surely, the relative risk remains almost surely bounded, that is*

$$\frac{E\left[\Phi(c_n X)\mathcal{F}_n\right]}{\Phi(a_n c_n)} \leq K(\omega) < \infty \ a.s. \tag{5.9}$$

If $\gamma = \infty$, then there exists for every ω a lower bound $\beta(\omega) > 0$ such that for every sequence $a_n > 0$, $a_n \in \mathcal{F}_{T_n}$, uniformly bounded from below by $\beta(\omega)$, the statement still holds for n large enough.

Proof We firstly assume that $\gamma < \infty$. Then, by corollary 3.1, we have

$$\Phi(c_n X) \leq K\Phi(a_n c_n)\left(\frac{X}{a_n}\right)^{\gamma+1} 1_{\frac{X}{a_n} \geq 1} + \Phi(a_n c_n) 1_{\frac{X}{a_n} \leq 1}$$

and taking the expectations

$$E\left[\Phi(c_n X)\right] \leq \Phi(a_n c_n)\left(KE\left[\left(\frac{X}{a_n}\right)^{\gamma+1}\right] + 1\right)$$

By the assumption that X in $L^{\gamma+1}$ with $\gamma \geq 0$ and that a_n is almost surely bounded from below away from 0, the expression within the brackets at the right-hand side is a bounded, and the result follows.

If $\gamma = \infty$ and $\Phi(x) > 0$ for all $x > 0$, then, by the monotonicity of Φ, we have

$$\Phi(c_n X) \leq \Phi(c_n ||X||_\infty)$$

and therefore

$$E\left[\Phi(c_n X)|\mathcal{F}_{T_n}\right] \leq \Phi(c_n ||X||_\infty)$$

Because Φ is a generalized Young function, there is a constant $1 > b > 0$ with $\Phi(b) < \infty$. Because $c_n \to 0$, we must have an $N \in \mathbb{N}$ such that $c_n ||X||_\infty \leq b < 1$ for all $n \geq N$. Because Φ is convex, we have

$$\Phi(c_n ||X||_\infty) \leq c_N ||X||_\infty \Phi\left(\frac{c_n}{c_N}\right)$$

for $n \geq N$. Hence, with a lower bound of $a_n \geq \frac{1}{c_N}$ for $n \geq N$, we get, for $n \geq N$,

$$\frac{E[\Phi(c_n X)|\mathcal{F}_{T_n}]}{\Phi(a_n c_n)} \leq c_N ||X||_\infty$$

□

We will prove a generalization of Lemma 3.4 in [2]. The idea of the proof is the same, with the Hölder inequality for Orlicz spaces instead of the Cauchy-Schwarz inequality, and some additional arguments.

Proposition 5.2 *Let X_t be a continuous uniformly integrable martingale with stopping times as in (31) for the process X_t instead of Z_t^{opt}. If the relative risk aversion of $\Phi(x)$ is essentially bounded from below away from 0 in a region around 0, and $\Phi(y) > 0$ as well as $\Phi^*(y) > 0$ for all $y > 0$, then for all $\mathcal{F}_{T_n}$-measurable sequences $a_n > 0$ which are bounded from above in n for every ω, we have that the relative risk*

$$\frac{E[\phi(X_\infty)|\mathcal{F}_{T_n}]}{\phi(a_n X_{T_n})} \to \infty$$

on the set $\{X_T = 0\}$.

Proof We take $\xi_n = \Phi(a_n X_{T_n})$ and apply the Hölder inequality, which is possible because X_{T_n} and therefore ξ is strictly positive. Then

$$E[X_\infty 1_{X_T \neq 0}|\mathcal{F}_{T_n}] \leq 2\Phi(a_n X_{T_n}) lux[\Phi(X_\infty|\mathcal{F}_{T_n})]_{\Phi(a_n X_{T_n})} lux[\Phi^*(1_{T\neq 0})|\mathcal{F}_{T_n}]_{\Phi(a_n X_{T_n})}$$

We have that Φ^* is invertible in a region around 0 and

$$\begin{aligned} lux[\Phi^*(1_{T\neq 0})|\mathcal{F}_{T_n}]_{\phi(a_n X_{T_n})} &= \inf\{\lambda > 0 : \Phi^*\left(\tfrac{1}{\lambda}\right) p_n \leq \Phi(a_n X_{T_n})\} \\ &= \left((\Phi^*)^{-1}\left(\tfrac{\phi(a_n X_{T_n})}{p_n}\right)\right)^{-1} \end{aligned}$$

with $p_n = P[Z_T \neq 0|\mathcal{F}_{T_n}] \to 0$ on the set $\{Z_\infty = 0\}$, where the second equality follows if we define

$$(\Phi^*)^{-1}(x) := \inf\{y \geq 0 : \Phi^*(y) \geq x\}$$

so that by the fact that $\Phi^*(y)$ is strictly increasing it coincides with the usual inverse as long as $\Phi^*(y)$ is finite, and it remains constant after the point where $\Phi^*(y)$ jumps to ∞.

It follows that

$$\frac{1}{2} \leq lux[\Phi(X_\infty|\mathcal{F}_{T_n})]_{\Phi(a_n X_{T_n})} \frac{\Phi(a_n X_{T_n})}{X_{T_n}(\Phi^*)^{-1}(\Phi(a_n X_{T_n}))} \frac{(\Phi^*)^{-1}(\Phi(a_n X_{T_n}))}{(\Phi^*)^{-1}(\frac{\Phi(a_n X_{T_n})}{p_n})} \quad (5.10)$$

By Proposition 2.4, we have that the expression

$$\frac{\Phi(a_n X_{T_n})}{a_n X_{T_n}(\Phi^*)^{-1}(\Phi(a_n X_{T_n}))}$$

converges to a finite constant away from 0 if $a_n X_{T_n} \to 0$. Because a_n is bounded in n, this is always satisfied if $X_{T_n} \to 0$. By the fact that $(\Phi^*)^{-1}(x) \to 0$ as $x \to 0$ and $\Phi(a_n X_{T_n}) \to 0$ as well, the last fraction of equation (5.10) can only converge to a value different from 0 (or not converge) if $\frac{\Phi(a_n X_{T_n})}{p_n} \to 0$ as well. But in this case, the Corollary 3.2 guarantees that this last fraction of equation (5.10) still converges to 0 if the relative risk aversion is essentially bounded from below away from 0, in a region of 0. It follows, again by the boundedness of the sequence a_n from above, that

$$lux[\Phi(X_\infty|\mathcal{F}_{T_n})]_{\Phi(a_n X_{T_n})} \to \infty$$

on $\{Z_{T_n} \neq 0\}$. This means that, almost surely, for every $\lambda > 0$, we may find an $n \in \mathbb{N}$ with

$$E[\Phi(\frac{X_\infty}{\lambda})|\mathcal{F}_{T_n}] \geq \Phi(a_n X_{T_n})$$

By the convexity of Φ and $\Phi(0) = 0$, we have that $\Phi(\frac{X_\infty}{\lambda}) \leq \frac{1}{\lambda}\Phi(a_n X_\infty)$ for $\lambda \geq 1$, and thus

$$\frac{E[\Phi(X_\infty)|\mathcal{F}_{T_n}]}{\Phi(a_n X_{T_n})} \geq \lambda$$

Because λ can be made arbitrarily large, the relative risk converges to infinity. □

5.4 Equivalence of minimax martingale measures

Theorem 5.1 *Let Φ be a generalized Young function, and let Assumptions (5.1) to (5.4) be satisfied, where $\mathcal{F}_t$ is a continuous filtration. Assume that*

- *$\Phi(x) > 0$ for $x > 0$, and $\Phi(x)$ is smooth at 0.*
- *The relative risk aversion of $\Phi(x)$ is essentially bounded from below away from 0 in a neighborhood of 0.*
- *The relative risk aversion of $\Phi(x)$ is essentially bounded from above by a constant $\gamma \leq \infty$.*
- *There exists an equivalent martingale measure $Z^{(0)}_\infty$ which is in $L^{\gamma+1}$.*

Then any minimal martingale measure satisfying equation (5.7) is equivalent to the original measure P.

Proof We follow here the arguments in [2], which we can extend to this general situation using Propositions 5.1 and 5.2. Assume that Z^{opt}_∞ is not equivalent, but satisfies equation (5.7). We have therefore with T_n and T as in section 5.1 that $\{Z^{opt}_T = 0\}$ has nonzero probability. Define, as in [2], the process

$$Z_t := \begin{array}{ll} Z^{opt}_t & \text{on } A^c_n \cup \{T_n > t\} \\ Z^{(0)}_t \frac{Z^{opt}_{T_n}}{Z^{(0)}_{T_n}} & \text{on } A_n \cap \{T_n \le t\} \end{array} \tag{5.11}$$

for any $\mathcal{F}_{T_n}$-measurable set A_n. Because the martingale $Z^{(0)}_\infty$ is strictly positive, it follows that the sequence $Z^{(0)}_{T_n}$ is uniformly bounded from below away from 0, almost surely. Therefore the $\mathcal{F}_{T_n}$-measurable random variables

$$c_n := \hat{\lambda} \frac{Z^{opt}_{T_n}}{Z^{(0)}_{T_n}} \to 0 \text{ a.s.}$$

Because $Z^{(0)}_\infty \in L^{\gamma+1}$, it follows by Proposition 5.1 that the relative risk

$$\frac{E\left[\Phi(c_n Z^{(0)}_\infty)|\mathcal{F}_{T_n}\right]}{\Phi(a_n c_n)}$$

remains bounded as $n \to \infty$, for every sequence $a_n \in \mathcal{F}_{T_n}$ with a sufficiently large lower bound. We may therefore choose a sequence which is also bounded from above.

On the other hand, the assumption that $\Phi(x)$ is smooth at 0 implies that $\Phi^*(y) > 0$ for all $y > 0$, and therefore the assumptions for Proposition 5.2 are satisfied, and the relative risk converges to ∞ as $n \to \infty$ on the set $\{Z^{opt}_T = 0\}$, that is

$$\frac{E\left[\Phi(X^{opt}_\infty)|\mathcal{F}_{T_n}\right]}{\Phi(a_n c_n)} \to \infty$$

with the same choice of a_n and c_n as above, because $\frac{a_n}{Z^{(0)}_{T_n}}$ is almost surely bounded from above. For almost every ω on $\{Z^{opt}_T = 0\}$ we may find therefore an $N \in \mathbb{N}$ with

$$E[\Phi(\hat{\lambda} Z^{opt}_\infty)|\mathcal{F}_{T_n}] > E[\Phi(\hat{\lambda} Z^{(0)}_\infty)|\mathcal{F}_{T_n}]$$

for all $n \ge N$. But this means that for n large enough, we have a set A_n which is $\mathcal{F}_{T_n}$-measurable and has strictly positive probability, on which

$$E[\Phi(\hat{\lambda} Z^{opt}_\infty)|\mathcal{F}_{T_n}] > E[\Phi(\hat{\lambda} Z^{(0)}_\infty)|\mathcal{F}_{T_n}]$$

Taking this set for equation (5.11), we have that the martingale measure Z_t defined in (5.11) satisfies

$$\hat{\lambda} x_0 + E[\Phi(\hat{\lambda} Z^{opt}_\infty)] > \hat{\lambda} x_0 + E[\Phi(\hat{\lambda} Z_\infty)]$$

But this means that Z^{opt}_∞ cannot satisfy equation (5.7). □

Corollary 5.1 *Let the Assumptions 5.1 to 5.4 be satisfied. Let furthermore the utility function $u(x)$ satisfy the following properties:*

- *u satisfies Assumption 5.2.*
- *u is smooth at c.*

- *there is an $\epsilon > 0$ such that, $u(x)$ has a relative risk aversion essentially bounded from above for $x \in (c - \epsilon, c)$.*
- *for $x < c$, $u(x)$ has a relative risk aversion essentially bounded from below by $\frac{1}{\gamma}$ with $\gamma > 0$.*
- *there exists an equivalent local martingale measure $Z^{(0)}_\infty$ which is in $L^{\gamma+1}$.*

Then the minimax martingale measure, as defined in Definition 5.1, is equivalent.

Proof Consider the generalized Young function $\Phi^*(x)$ defined in equation (5.5). By Assumption 5.2 it follows that $\Phi^*(x)$ is smooth at 0 and $\Phi^*(x) > 0$ for all $x > 0$. It is obvious that its conjugate function $\Phi(y)$ satisfies also those properties.

Furthermore, by the assumptions of the corollary, $\Phi^*(x)$ has a relative risk aversion which is essentially bounded from above in a region of 0, as well as essentially bounded from below by $\frac{1}{\gamma}$. By Proposition 3.4, it follows that $\Phi(y)$ has a relative risk aversion bounded from below away from 0 around 0, as well as a relative risk aversion bounded from above by γ. It follows that the assumptions for Theorem 5.1 are satisfied.

By Lemma 5.1, the minimax martingale measure satisfies the property (5.7). It follows from Theorem 5.1 that this measure must be equivalent. □

6 Second case: Nonsatiated investors

We begin with an easy lemma.

Lemma 6.1 *Let the utility function $u(x)$ be increasing, concave, unsatiated and bounded. Then the dual function $\Phi(y)$ satisfies*

$$\frac{\Phi(y) - \Phi(0)}{y} \to -\infty \text{ as } y \to 0 \tag{6.1}$$

Proof Without loss of generality we assume that $\sup_x u(x) = 0$. We have

$$\Phi(y) = \sup_x (u(x) - xy)$$

For every $n \in \mathbb{N}$, we can find an y_n such that for all $x \leq n$, the subdifferential of u at x contains only values larger than y_n. We have therefore that

$$\sup_x \left(\frac{u(x)}{y_n} - x \right) \leq \sup_x \frac{u(x)}{y_n} - n \leq -n \to -\infty$$

□

We have therefore, for the class of nonsatiated investors with bounded utility function, the important property (6.1). An important example of such a utility function is the exponential one treated in [3]. An extension of the arguments there, using mainly the property (6.1) of the dual utility function, leads to a general proof of equivalence for this class of utility functions.

The case of unbounded utility functions has already been treated in [4] and therefore does not need to be discussed any more.

Theorem 6.1 *Let Z_0 be the minimax measure for a strictly increasing, concave and bounded utility function, such that the dual function $\Phi(y)$ satisfies property (6.1). Furthermore let there be an equivalent separating measure Z_1 and a constant $\lambda > 0$ such that*

$$E[\Phi(\lambda Z_1)] < \infty$$

Then the minimax measure Z_0 is equivalent.

Remark 6.1 Notice that in this situation we do not need that the filtration is continuous.

Remark 6.2 Theorem 6.1 is similar to the Proposition 3.1 in [5]. The difference is that here we do not assume differentiability nor boundedness from below for the dual utility function. We will therefore give a proof under our assumptions.

Proof Firstly we would like to mention a fact that has already been proved in [8] for the case when the wealth cannot become negative, namely that $\Phi^-(Z)$ are integrable random variables for separating measures Z (Lemma 3.2, integrability follows from uniform integrability). This holds for any convex function Φ if Z is integrable, because by the convexity,

$$\Phi(y) \geq \Phi(y_0) + (y - y_0)\delta\Phi(y_0)$$

where y_0 is a point where the subdifferential is finite, and therefore

$$\Phi^-(y) \leq C + Dy^+$$

where C and D are positive constants. The result follows by the integrability of Z.

We again assume without loss of generality that $\Phi(0) = 0$. With the assumptions of Theorem 6.1 it follows that $\Phi(\hat{\lambda} Z_0)$ and $\Phi(\lambda Z_1)$ are integrable, where $\hat{\lambda}$ is the minimal λ from the minimax measure.

Because Z_0 and Z_1 are separating measures, for $0 \leq x \leq 1$, also Z_x is a separating measure, where

$$\lambda_x Z_x := x\lambda Z_1 + (1 - x)\hat{\lambda} Z_0$$

and

$$\lambda_x := x\lambda + (1 - x)\hat{\lambda} \tag{6.2}$$

It follows that also $\Phi^-(\lambda_x Z_x)$ is integrable. From the convexity of Φ, the function

$$\frac{1}{x}\left(\Phi(\lambda_x Z_x) - \Phi(\hat{\lambda} Z_0)\right) \tag{6.3}$$

is nondecreasing almost surely in x, and as $x \to 1$, the function converges almost surely to the integrable random variable

$$\Phi(\lambda Z_1) - \Phi(\hat{\lambda} Z_0)$$

We may therefore apply the monotone convergence theorem to conclude that

$$\lim_{x\to 0} \frac{1}{x} E\left[\Phi(\lambda_x Z_x) - \Phi(\hat{\lambda} Z_0)\right] = E\left[\lim_{x\to 0} \frac{1}{x}\left(\Phi(\lambda_x Z_x) - \Phi(\hat{\lambda} Z_0)\right)\right] \tag{6.4}$$

Let now Z_1 be equivalent, and $A := \{Z_0 = 0\}$ be a set with $P(A) > 0$. Then we have on A that

$$\frac{1}{x}\left(\Phi(\lambda_x Z_x) - \Phi(\hat{\lambda} Z_0)\right) = \frac{1}{x}\Phi(x\lambda Z_1) \to -\infty$$

because $P(\{Z_1 = 0\}) = 0$ as Z_1 is equivalent. Because the sequence in (6.3) is decreasing as $x \to 0$, the positive part is bounded from above by an integrable function, and therefore the positive part of the integrand on the right-hand side in (6.4) is integrable. From the fact that the integrand is $-\infty$ on a set with nonzero probability, we conclude that the expectation on the right-hand side is $-\infty$, and so is the left-hand side.

This means that for every constant $C > 0$ we find an $\hat{x} > 0$ such that

$$E\left[\Phi(\lambda_x Z_x)\right] + Cx \leq E\left[\Phi(\hat{\lambda} Z_0)\right] \ \ \forall x \leq \hat{x} \tag{6.5}$$

On the other hand, because Z_0 is the minimax measure, we must have for all $\lambda > 0$ and all $0 \leq x \leq 1$ that

$$E\left[\Phi(\lambda_x Z_x)\right] + \lambda_x x_0 \geq E\left[\Phi(\hat{\lambda} Z_0)\right] + \lambda_0 x_0$$

where x_0 means the initial wealth. With (6.2), we get

$$E\left[\Phi(\lambda_x Z_x)\right] + x\left(\lambda_1 - \hat{\lambda}\right) x_0 \geq E\left[\Phi(\hat{\lambda} Z_0)\right]$$

But this is a contradiction to (6.5), if we choose the constant $C > 0$ large enough. $\square$

7 Counterexamples

The aim of this section is to provide examples for what may happen if the assumptions are not satisfied.

7.1 An L^Φ integrable random variable for which the relative risk does not remain bounded

We consider the following generalized Young function:

$$\Phi(x) := \begin{matrix} x^2 & \text{if } |x| \leq 1 \\ 2|x| - 1 & \text{if } |x| > 1 \end{matrix} \tag{7.1}$$

This is a Young function which even has a continuous derivative. The relative risk aversion as $x \to 0$ is 1. As $x \to \infty$, the function behaves as a linear function, and therefore every integrable random variable is in L^Φ. The relative risk aversion is therefore uniformly bounded from above by 1, and from below by 1 as $x \to 0$. Consider now the following random variable:

$$X := \frac{1}{U^{\frac{2}{3}}} - 1 \tag{7.2}$$

where U is a uniformly distributed random variable. Using a Brownian filtration, this random variable may be generated for example by

$$U = \tilde{\Phi}(W_n - W_{n-1})$$

where $\tilde{\Phi}$ is the cumulative standard normal distribution, n is an integer (indeed it may be any real number) and W_t a Wiener process.

We have that $X > 0$ with probability 1 and

$$E[X] = \int_0^1 \frac{du}{u^{-\frac{2}{3}}} - 1 = 3u^{\frac{1}{3}}|_{u=1} - 1 = 2$$

and therefore X is integrable. But looking at the second moment, we have

$$E[X^2] = \int_0^1 \left(u^{-\frac{2}{3}} - 1\right)^2 du = \int_0^1 u^{-\frac{4}{3}} du - 2\int_0^1 u^{-\frac{2}{3}} du + 1 = -3u^{-\frac{1}{3}}|_0^1 - 5 = \infty$$

It follows that X is not square-integrable and therefore does not satisfy the assumption of Proposition 5.1.

Let now $c_n := \frac{1}{n}$ be a sequence converging to 0. Then we have

$$E[\Phi(c_n X)] = E[\frac{1}{n^2} X^2 1_{X \le n}] + E[(\frac{2X}{n} - 1)1_{X>n}]$$

To evaluate this expression, we recognize that $\{X \le n\} = \{U \ge \frac{1}{(n+1)^{\frac{3}{2}}}\}$ and thus with $u_0 := \frac{1}{(n+1)^{\frac{3}{2}}}$

$$\begin{aligned} E[\Phi(\tfrac{1}{n}X)] &= \tfrac{1}{n^2}\int_{u_0}^1 \left(u^{-\frac{2}{3}} - 1\right)^2 du + \int_0^{u_0} \left(\tfrac{2}{n}(u^{-\frac{2}{3}} - 1) - 1\right) du \\ &= \tfrac{1}{n^2}\left(3(u_0^{-\frac{1}{3}} - 1) + 6(1 - u_0^{\frac{1}{3}}) + 1 - u_0\right) \\ &\quad + \tfrac{1}{n}\left(6u_0^{\frac{1}{3}} - 2u_0\right) - u_0 \\ &= \tfrac{1}{n^2}\left(3\sqrt{n+1} - \tfrac{6}{\sqrt{n+1}} - \tfrac{1}{(n+1)^{\frac{3}{2}}} + 4\right) \\ &\quad + \tfrac{1}{n}\left(\tfrac{6}{\sqrt{n+1}} - \tfrac{2}{(n+1)^{\frac{3}{2}}}\right) - \tfrac{1}{(n+1)^{\frac{3}{2}}} \end{aligned}$$

The dominating terms behave as $\frac{8}{(n+1)^{\frac{3}{2}}}$, and therefore, as $n \to \infty$, the relative risk satisfies

$$\frac{E[\Phi(\frac{X}{n})]}{\Phi(\frac{1}{n})} \sim 8\frac{n^2}{(n+1)^{\frac{3}{2}}} \to \infty$$

This example shows that, if X is not in $L^{\gamma+1}$ where γ is the essential upper bound of the relative risk aversion of Φ, the statement of Proposition 5.1 does not need to be true.

7.2 A uniformly integrable martingale for which the relative risk does not converge to infinity

We start with the following discrete-time martingale with $Z_1 = 1$:

$$Z_n := \begin{array}{lll} Z_{n-1} & \text{if} & Z_{n-1} \ge Z_{n-2} \text{ and } n > 2 \\ C_n & \text{otherwise with probability } p_n = \frac{1}{n^2} & \\ \frac{1}{n^4} & \text{otherwise with probability } (1 - p_n) = \frac{n^2-1}{n^2} & \end{array} \tag{7.3}$$

and C_n chosen in such a way that the process is indeed a martingale. Obviously this discrete-time martingale is bounded, nonnegative, and converges to 0 as $n \to \infty$ with nonzero probability, which can be seen by

$$\ln\left(\prod_n \frac{n^2-1}{n^2}\right) = \sum_n \ln\left(1 - \frac{1}{n^2}\right) \sim \sum_n -\frac{1}{n^2} > -\infty$$

which shows that the product of the probabilities that this martingale goes down converges to a number strictly larger than 0.

With an underlying Brownian motion, we define

$$p_n := P\left[W_{n^4} - W_{(n-1)^4} \leq a_n\right] \tag{7.4}$$

where obviously a_n is chosen in a way that the probabilities fit. With

$$Z_\infty := \lim_{n\to\infty} Z_n$$

the process

$$X_t := E[Z_\infty|\mathcal{F}_t] \tag{7.5}$$

with the filtration generated by the Brownian motion W_t defines therefore a bounded continuous nonnegative martingale which converges to 0 on a set with nonzero probability.

For $t = n^4$, we have furthermore that $X_t = Z_n$. If T_n is the announcing sequence of stopping times, that is $X_{T_n} = \frac{1}{n}$, we would like to show that the supremum

$$y_n := \sup_\omega\{X_\infty(\omega) : X_{T_n}(\omega) = \frac{1}{n}\}$$

converges to 0 with $\sqrt{X_{T_n}}$. Firstly, this is true if $n = k^4$ for a $k \in \mathbb{N}$, because then, by the definition (7.3) of the martingale $X_{T_n} = Z_k$,

$$C_k = \frac{k^2}{(k-1)^4} - \frac{k^2-1}{k^4} \leq \frac{K}{k^2}$$

for a constant $K > 0$. But also if $n \in \left](k-1)^4, k^4\right]$, by the construction, the maximum that the random variable X_∞ can achieve must be bounded by C_{k-1}, and therefore, for $n \in \left((k-1)^4, k^4\right]$,

$$C_n \leq C_{k-1} \leq \frac{K}{(k-1)^2} \leq \frac{K_2}{k^2} \leq \frac{K_2}{\sqrt{n}} = K_2\sqrt{Z_{T_n}}$$

where K_2 is another constant. This was to prove.

Now we take our dual utility function ϕ which has a relative risk aversion which is not essentially bounded from below, that is

$$\phi(x) := \frac{x}{\ln(\frac{1}{x})}$$

when x is small enough. We want to calculate the fraction

$$\frac{E[\phi(Z_\infty)|\mathcal{F}_{T_n}]}{\phi(Z_{T_n})}$$

On the set where $Z_T = 0$, conditional on $\mathcal{F}_{T_n}$, we have $Z_{T_n} = \frac{1}{n}$, and the supremum the random variable can reach if $Z_{T_n} = \frac{1}{n}$ is $K_2 \frac{1}{\sqrt{n}}$. We have therefore

$$E\left[\frac{Z_\infty}{\ln(\frac{1}{Z_\infty})}|\mathcal{F}_{T_n}\right] \leq E\left[\frac{Z_\infty}{\ln(\frac{\sqrt{n}}{K_2})}|\mathcal{F}_{T_n}\right] = \frac{1}{n\ln(\frac{\sqrt{n}}{K_2})} \tag{7.6}$$

On the other hand,

$$\phi(Z_{T_n}) = \frac{1}{n\ln(\frac{1}{n})}$$

and the relative risk becomes

$$\frac{E[\phi(Z_\infty)|\mathcal{F}_{T_n}]}{\phi(Z_{T_n})} \leq \frac{\ln n}{\ln(\sqrt{n}K_2)} = \frac{\ln n}{0.5\ln n - \ln K_2}$$

which is obviously bounded as $n \to \infty$. We have therefore an example which shows that if the relative risk aversion of ϕ is not essentially bounded from below away from 0, the conclusion of Proposition 5.2 does not need to hold, and the relative risk does not need to converge to ∞.

7.3 A continuous market with a non-equivalent q-optimal martingale measure

Let W^1 and W^2 be two independent Brownian motions, and let the stock price process until time $t = 1$ satisfy

$$dS = dW^1 \tag{7.7}$$

The filtration is generated by W^1 and W^2 until time $t = 1$. It is clear that $Z_t = 1$ is the density process of an equivalent martingale measure for $t \leq 1$ and that the market admits an absolutely continuous martingale measure which is not equivalent, that is there exists a density process Z_t^{abs} of an absolutely continuous martingale measure as well as a set A with strictly positive probability and with $Z_1^{abs}(\omega) = 0$ for $\omega \in A$. We may even choose Z_1^{abs} in such a way that it is bounded.

For $1 \leq t \leq 2$, the price process is constructed in the following way. Let $F(x)$ be the cumulative distribution function of a strictly positive random variable which is integrable but has bad integrability property in the sense that it is not, say, p-integrable. Let ψ be its inverse, and

$$X := \psi(\Phi(W_2^2 - W_1^2)) \tag{7.8}$$

Furthermore, the filtration for $t \geq 1$ is now only the one generated by W_1^1 and W_t^2, that is for $t > 1$, the first Brownian motion does not play a role any more. It is clear that X is independent of $\mathcal{F}_1$, and that X is integrable but not p- integrable. Because $A \in \mathcal{F}_1$, we have also that $X1_A$ is integrable but not p-integrable:

$$E[(X1_A)^p] = E[E[X^p|\mathcal{F}_1]1_A] = E[X^p]E[1_A] = \infty$$

Because $X1_A$ is integrable, we may chose it in a way that $E[X1_A] = P[A]$. Now we define for $1 \leq t \leq 2$ the martingale

$$X_t := E[X1_A|\mathcal{F}_t] + 1_{A^c} \tag{7.9}$$

It is clear that X_t is a strictly positive uniformly integrable martingale for all $1 \leq t \leq 2$. We may therefore take the stochastic logarithm of X_t and define the process $\lambda(t)$ in such a way that

$$X_t = \mathcal{E}\left(-\int_1^t \lambda(s)dW_s^2\right)$$

where we applied here also the martingale representation theorem.

Now we define the stock price process for $1 \leq t \leq 2$ in the following way:

$$dS := \lambda(t)dt + dW^2 \tag{7.10}$$

With this construction, we have the following properties:

1. For any choice of the martingale measure Z_1 for $t \leq 1$, we have that the measure $Z := Z_1 X_2$ is a uniformly integrable martingale measure
2. The martingale measure $\hat{Z} := Z_1^{abs} X_2$ is not equivalent but bounded and therefore in L^q for any $q \geq 1$.
3. All equivalent local martingale measures in this market are of the form $Z = Z_1 X_2$
4. The market does not admit an equivalent martingale measure which is p-integrable

Proof Because $Z_1 \in \mathcal{F}_1$ and X is independent of $\mathcal{F}_1$, we have for $t \geq 1$

$$E[Z|\mathcal{F}_t] = Z_1 \left(E[X 1_A + 1_{A^c}|\mathcal{F}_t]\right) = Z_1 X_t$$

and for $t \leq 1$ we have

$$E[Z|\mathcal{F}_t] = E[Z_1 E[X_2|\mathcal{F}_1]|\mathcal{F}_t] = E[X_2]E[Z_1|\mathcal{F}_t] = Z_t$$

and therefore Z_t is a martingale. For $t \leq 1$, the construction of Z_t already guarantees that Z_t is a martingale measure. For $1 \leq t \leq 2$, we have

$$dZ_t = Z_1 dX_t = -\lambda(t) Z_1 X_t dW_t^2 = -\lambda(t) Z_t dW_t^2$$

and thus

$$d(SZ) = \lambda(t) Z_t dt - \lambda(t) Z_t d\langle W^2, W^2\rangle_t + \text{loc. martingale}$$

which is obviously a local martingale.

From the proof before, it is clear that $\hat{Z}$ is a martingale measure. Because $\hat{Z} = 0$ on A and $P[A] > 0$, it is also clear that this martingale measure is not equivalent. By construction, we have that Z_1^{abs} is bounded. But on A^c, we have that $X_2 = 1$, and therefore $\hat{Z}$ remains bounded on this subset. But on A, it is 0 and therefore bounded too. It follows that $\hat{Z}$ is bounded.

Let now be Z any absolutely continuous local martingale measure. Because the density process is a martingale, we must have $E[Z|\mathcal{F}_1] = Z_1$. For the economy up to time $t = 1$, we have therefore $E[SZ|\mathcal{F}_t] = E[SZ_1|\mathcal{F}_t]$, and therefore Z_1 must be one of the martingale measures chosen in the economy for $t \leq 1$.

For $t \geq 1$, we will follow a market completeness argument. Firstly we have that X_2 is a strictly positive integrable random variable with expectation 1. Therefore, the measure Q defined by $\frac{dQ}{dP} = X_2$ is equivalent. By construction, the density process Z_t follows the stochastic differential equation

$$dZ = -\lambda(t) Z_t dW_t^2$$

with $\lambda(t) = 0$ for $t < 1$ and $\lambda(t)$ as above for $t \geq 1$. By the Girsanov theorem, the process $\tilde{W}_t^2$ defined by

$$d\tilde{W}^2 := dW^2 + \lambda dt$$

is a Brownian motion under Q. We have therefore that for $t \geq 1$, S follows under Q the dynamics

$$dS = \lambda(t)dt + dW_t^2 = \lambda(t)dt + (d\tilde{W}_t^2 - \lambda(2)dt) = d\tilde{W}_t^2$$

Let now $\tilde{Q}$ be any equivalent local martingale measure, and let $\tilde{Z} := \frac{d\tilde{Q}}{dQ}$ its density. Then the process $\tilde{Z}_t$ is a martingale under Q, and by the martingale representation theorem it can be represented for $t \geq 1$ by

$$\tilde{Z}_t = \tilde{Z}_1 + \int_0^t H_s d\tilde{W}_s^2 = \tilde{Z}_1 + \int_0^t H_s dS$$

for a predictable process H_s. Because $\tilde{Q}$ is also a local martingale measure, we have that $\tilde{Z}S$ must be a local martingale under Q. This implies that the quadratic variation of $\tilde{Z}S$ must vanish, because both $\tilde{Z}$ and S are martingales under Q. It follows for $t \geq 2$ that

$$0 = d\langle \tilde{Z}, S\rangle = d\langle \int H dS, S\rangle = Hd\langle S\rangle = Hd\langle W^2\rangle = Hdt$$

and therefore $H = 0$. It follows that any density process of an equivalent local martingale measure is $\frac{d\tilde{Q}}{dQ} = Z_1$, and from this

$$\frac{d\tilde{Q}}{dP} = \frac{d\tilde{Q}}{dQ}\frac{dQ}{dP} = Z_1 X_2$$

Let Z now be an equivalent martingale measure. It follows that $Z_1 > 0$, and therefore there exists a set $A_n \subset A$, $A_n \in \mathcal{F}_t$, with $P[A_n] > 0$ and on which $Z_1 \geq \frac{1}{n}$. Because $A_n \in \mathcal{F}_1$, it is independent of X. Because $Z = Z_1 X_2$, we have

$$E[Z^p] = E\left[Z_1^p E\left[(1_A X + 1_{A^c})^p | \mathcal{F}_1\right]\right] \geq \frac{1}{n^p} P[A_n] E[X^p] = \infty$$

□

The financial market defined before shows that, if the assumption of the existence of a square integrable local martingale measure in Theorem 1.3 in [2] is dropped, the variance-optimal martingale measure does not need to be equivalent.

Proof With the example before, it is clear that there is an absolutely continuous local martingale measure which is bounded and therefore square-integrable, but for which there does not exist any equivalent local martingale measure which is p-integrable. With the choice $p = 2$ (and therefore the appropriate choice of the distribution function $F(x)$), we have that no equivalent local martingale measure is square-integrable. It follows that an equivalent local martingale measure cannot be variance-optimal. □

8 Application: $q-$optimal measures

From section 5, it follows that q-optimal local martingale measures, that is martingale measures $\hat{Q}$ which minimize the expression

$$E\left[\left(\frac{dQ}{dP}\right)^q\right] \tag{8.1}$$

for $q > 1$ over all absolutely continuous local martingale measures Q, are always equivalent, provided there exists an equivalent local martingale measure which is bounded in L^q. By [4], this q-optimal measure always exists if $q > 1$. Because the function $\Phi(x) = x^q$ has constant relative risk-aversion of $q-1$ which is obviously bounded from below away from 0, the assumptions of Theorem 5.1 are obviously satisfied.

On the other hand, section 7.3 shows how to construct a financial market for which there does not exist an equivalent local martingale measure which is in L^q, and for which the q-optimal measure is only absolutely continuous and not equivalent.

From section 6, it follows that if there exists an equivalent local martingale measure for which the expectation (8.1) remains bounded, we still have that the q-optimal local martingale measure is equivalent provided it exists. But from [4], this existence is not guaranteed any more.

9 Conclusion

In this paper, we have shown that the minimax martingale measure in the sense of [4] is equivalent to the objective probability measure under some conditions on the utility function as well as on the existence of an equivalent local martingale measure which is sufficiently integrable, and, for the case of satiated investors, the continuity of the filtration. Whereas the case with strictly increasing utility functions has essentially already been treated in [5], the situation with a utility function that has a maximum has only been treated in the specific case of the variance-optimal martingale measure in [2]. In our paper, we use essentially the same method as there for proving a substantial generalization of this result. Furthermore, we provide an example which shows that the condition of the existence of an equivalent local martingale measure which is square-integrable cannot be dropped without possibly additional assumptions on the financial market.

For further research, one could try to find a sharper distinction whether or not the minimax martingale measure is equivalent for situations where the relative risk either remains bounded or converges to infinity for the absolutely continuous as well as for the equivalent local martingale measure. Furthermore, for finding counterexamples, we had to assume quite specific market situations, which are different from the models that are normally used. It may therefore be advantageous to find conditions on the market rather than on the utility function which guarantee that the minimax martingale measure is equivalent.

One main question remains, namely if we really need the stronger condition of the existence of an equivalent local martingale measure which is in $L^{\gamma+1}$, where γ is the upper bound of the essential relative risk aversion, rather than only the weaker one, namely the existence of an equivalent local martingale measure which is in L^{Φ}. From the counterexamples, it becomes clear that we will not be able to prove the stronger result using this method of proof. Combining the counterexamples, it would even be possible to construct a situation where the relative risk of every equivalent local martingale measure tends faster to infinity than the one for the absolutely continuous one on the set where the absolutely continuous local martingale measure tends to 0. But the question is then still what happens on the set where the absolutely continuous local martingale measure does not converge to 0.

Finally, we had, as in [4], always the assumption of the existence of a risk-free asset, or equivalently, that the investor optimizes his terminal wealth by discounting everything by a numéraire. If this assumption is dropped, the optimal portfolio may more easily hit the maximum point of the utility function, which mostly implies by the duality results that the dual minimizer is 0 with nonzero probability.

Acknowledgements I would like to thank Professors J.M. Schumacher, M. Frittelli, F. Delbaen, P.J.C. Spreij, F.C. Drost, B.J.M. Werker and P. Cizek for their useful comments.

References

1. S. Bloom and R. Kerman. Weighted Orlicz space integral inequalities for the Hardy-Littlewood maximal operator. *Studia Mathematica*, 110(2):149–167, 1994.
2. F. Delbaen and W. Schachermayer. The variance-optimal martingale measure for continuous processes. *Bernoulli*, 2(1):81–105, 1996.
3. M. Frittelli. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance*, 10(1):39–52, 2000.
4. M. Frittelli. On the existence of minimax martingale measures. *Mathematical Finance*, 12(1):1–21, 2002.
5. Y. Kabanov and Chr. Stricker. On the optimal portfolio for the exponential utility maximization: Remarks to the six-author paper. *Mathematical Finance*, 12(2):125–134, 2002.
6. I. Karatzas, J. Lehoczky, S. Shreve, and G. Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29(3):702–730, 1991.
7. I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, New York, 2nd edition, 1998.
8. D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999.
9. J. P-Aubin. *Mathematical methods of game and economic theory*. North-Holland, Amsterdam, 2nd edition, 1982.
10. W. Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability*, 11(2):694–734, 2001.
11. A. N. Shiryaev. *Probability*. Springer, New York, 2nd edition, 1996.
12. D. Williams. *Probability with Martingales*. Cambridge University Press, Cambridge, 1991.