Equivalence of the minimax martingale measure

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January 14, 2008

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Contents

Minimax measure and problem formulation

Minimax measure

- Incomplete market
- Definition (Frittelli): $\hat{Q}_x$ such that

$$\sup_{w\in L^\infty : E^{\hat{Q}_x}[w]\leq 0}\{E^P[u(x+w)]\} = \sup_{w\in L^\infty : E^{Q}[w]\leq 0\ \forall Q\in M_1}\{E^P[u(x+w)]\} \qquad (1)$$

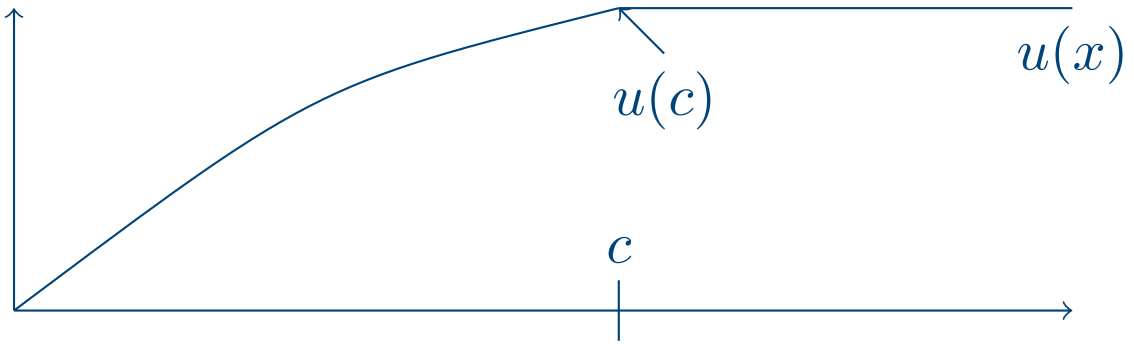
where

$$M_1 := \left\{z \in L^1_+(P) : E^P[zw] \leq 0 \quad \forall w \in C, E^P[z] = 1\right\}$$

C : Cone of superreplicable claims at 0

- Existence: Frittelli (2002)
- Equivalence: Partially Frittelli, Kabanov-Stricker for strictly increasing utility functions

Satiated utility functions



Definition: $\exists c \in \mathbb{R} : u(x) = u(c) \quad \forall \quad x \geq c$

- One-sided risk functions
- Expected shortfall risk (Föllmer/Leukert)
- Power utility functions with $p > 1$

$\longrightarrow$ Research question: Is minimax measure equivalent?
$\longrightarrow$ This paper: Focus to satiated utility functions

Dual formulation of minimax MM

- Lemma 5.1: If $Z_\infty^{opt} := \frac{d\hat{Q}_x}{dP}$ density of minimax martingale measure, then there exists $\hat{\lambda} > 0$ with

$$\hat{\lambda}x + E[\phi(\hat{\lambda}Z_\infty^{opt})] \leq \lambda x + E[\phi(\lambda Z_\infty)] \quad \forall Z_\infty \in M_1, \quad \forall \lambda > 0$$

  with ϕ: Conjugate function of u

- In particular $\phi(x) = x^q$

$$E[(Z_\infty^{opt})^q] \leq E[Z_\infty^q] \quad \forall Z_\infty \in M_1$$

- Restriction to Young functions: If $u(x)$ satiated utility function $\rightarrow$ With $\phi^*(x) := -u(c - |x|) + u(c)$, ϕ^* is Young function

- For conjugate function $\phi(x) = \phi^{**}(x)$, Lemma 5.1 still valid

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Earlier results

Example: Variance-optimal measure

- Continuous processes $\longrightarrow$ Variance-optimal measure nonnegative and coincides with minimax measure
- Variance-optimal measure:

$$\min_{Q \in M_1} E^P \left[\left(\frac{dQ}{dP} \right)^2 \right]$$

- Delbaen/Schachermayer: If price processes continuous $\longrightarrow$ measure equivalent, provided $\exists Z_\infty^{eq} \in L^2$ which is equivalent
- Price processes discontinuous $\longrightarrow$ simple counterexamples exist

Method of proof (DS)

- Z_t: density process of absolutely continuous MM
- $T_n < T_{n+1} < T, T_n \to T$ announcing sequence of stopping times

$$T := \inf\{t > 0 : Z_t = 0\}$$

Then

$$\begin{aligned} \frac{E\left[\left(Z_{T_n}\frac{Z_\infty^{eq}}{Z_{T_n}^{eq}}\right)^2 |\mathcal{F}_{T_n}\right]}{(Z_{T_n})^2} &< \infty \quad \text{a.s.} \\ \frac{E\left[(Z_\infty)^2|\mathcal{F}_{T_n}\right]}{(Z_{T_n})^2} &\to \infty \quad \text{on} \quad \{Z_T = 0\} \end{aligned} \tag{2}$$

Relative risk aversion

Can this be extended

- Replace $x \to x^2$ by Young function $\phi(x)$
- $$\frac{E\left[\phi\left(Z_{T_n}\frac{Z_\infty^{eq}}{Z_{T_n}^{eq}}\right)|\mathcal{F}_{T_n}\right]}{\phi\left(Z_{T_n}\right)} < \infty \quad ?$$

  $\longrightarrow$ Not necessarily if $Z_\infty^{eq} \notin L^{\gamma+1}$

  γ: Upper bound of relative risk aversion of ϕ
- $$\frac{E\left[\phi\left(Z_\infty\right)|\mathcal{F}_{T_n}\right]}{\phi\left(Z_{T_n}\right)} \longrightarrow \infty \quad ?$$

  $\longrightarrow$ Not necessarily if $\inf_x\{RRA(\phi)\} = 0$

Relative risk aversion

- Classical definition of RRA assumes $\phi \in C^2$
- Definition 3.2: Integrated relative risk aversion: Measure $d \ln \phi_r'$, where ϕ_r': Right-differential of ϕ
- Connection to classical definition

$$RRA(x) = \gamma(x) \Leftrightarrow d \ln \phi'(x) = \gamma(x) d \ln x$$

- Definition 3.3: Comparison of (integrated) RRA

$$IRRA(\phi_1) \leq IRRA(\phi_2)$$

$$\Leftrightarrow d \ln \phi_{1,r}'(B) \leq d \ln \phi_{2,r}'(B) \quad \forall \quad B \in \mathcal{B}$$

Essential bounds for RRA

- Definition 3.4: $RRA(\phi)$ essentially bounded by γ on $]0, b]$ if

$$\sup_{I=]x,y]:0<x<y\leq b} \left(\int_I d\ln\phi_r' - \int_I \gamma(x) d\ln x \right) < \infty$$

$$\inf_{I=]x,y]:0<x<y\leq b} \left(\int_I d\ln\phi_r' - \int_I \gamma(x) d\ln x \right) > -\infty$$

- Proposition 3.3: $RRA(\phi)$ essentially bounded from above (below) iff

$$\phi_r'(y) \leq K\phi_r'(x) \left(\frac{y}{x}\right)^\gamma \quad \forall \quad 0 < x < y \leq b$$

$$\left(K\phi_r'(y) \geq \phi_r'(x) \left(\frac{y}{x}\right)^\gamma \quad \forall \quad 0 < x < y \leq b \right)$$

Main results

Assumptions

- A1: $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ has continuous filtration $\mathcal{F}_t$
- A2: Utilty function has satiation point c, and is upper semicontinuous, concave, and nondecreasing
- A3: There exists an equivalent separating measure (filtration continuous $\longrightarrow$ equivalent local martingale measure)
- A4: The minimax martingale measure exists

Proposition 5.1

- $\phi(x) > 0 \quad \forall \quad x > 0$
- Essentially $RRA(\phi) \leq \gamma \quad \forall \quad x > 0$
- $\exists Z_\infty^{eq} \in L^{\gamma+1}$, with $Z_\infty^{eq} > 0$

Then

$$\frac{E\left[\phi\left(Z_{T_n}\frac{Z_\infty^{eq}}{Z_{T_n}^{eq}}\right)|\mathcal{F}_{T_n}\right]}{\phi\left(Z_{T_n}\right)} < \infty \quad a.s.$$

Proof: Application of proposition 3.3

Proposition 5.2

- $\phi(x) > 0 \quad \forall \quad x > 0$ and $\phi(x)$ differentiable at 0
- Z_t continuous uniformely integrable martingale, $P[Z_\infty = 0] > 0$
- $RRA(\phi(x))$ essentially bounded from below away from 0, in region around 0
- T_n: Announcing sequence of stopping times
- $T = \inf\{t > 0 : Z_t = 0\}$ as above

Then

$$\frac{E\left[\phi\left(Z_\infty\right)|\mathcal{F}_{T_n}\right]}{\phi\left(Z_{T_n}\right)} \to \infty \quad \text{on} \quad \{Z_\infty = 0\}$$

Idea of proof

- Bases of an idea applied in Delbaen/Schachermayer

- Cauchy-Schwartz $\longrightarrow$ Hölder for Luxemburg norms

$$Z_{T_n} = E[Z_\infty 1_{Z_T \neq 0} | \mathcal{F}_{T_n}] \leq$$
$$2\phi(Z_{T_n}) lux[\phi(Z_\infty | \mathcal{F}_{T_n})]_{\phi(Z_{T_n})} lux[\phi^*(1_{T\neq 0}) | \mathcal{F}_{T_n}]_{\phi(Z_{T_n})}$$

- Difficulty: Define conditional version of Luxemburg norm (4.2)

$$lux[\phi(X)|\mathcal{G}]_\xi(\omega) := \begin{matrix} \inf \Lambda(\omega) & \text{if} & \Lambda(\omega) \neq \emptyset \\ \infty & \text{if} & \Lambda(\omega) = \emptyset \end{matrix}$$

where

$$\Lambda(\omega) := \left\{ \lambda > 0 : E^{rc}\left[\phi\left(\frac{X}{\lambda}\right) |\mathcal{G}\right] \leq \xi \right\}$$

Theorem 5.1

- Assumptions (A1)-(A4) satisfied
- Assumptions of Propositions 5.1 and 5.2 about RRA satisfied

Then the minimax martingale measure is equivalent

Proof: Similar to Delbaen/Schachermayer
Proposition 5.1 and 5.2

Conclusion

Conclusion and further research

- Minimax martingale measure equivalent for most satiated utility functions
- Generalization of Delbaen/Schachermayer if filtration continuous
- In particular applicable for q-optimal measures
- Generalization of RRA
- Conditinonal version of Luxemburg norm
- Further research:
  - Situation when $Z_\infty^{eq} \in L^\phi$, not in $L^{\gamma+1}$
  - Conditions for market instead utility function
  - Only asset prices continuous (not filtration)