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The critical price for the American put in an exponential Lévy model

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Abstract This paper considers the behavior of the critical price for the American put in the exponential Lévy model when the underlying stock pays dividends at a continuous rate. We prove the continuity of the free boundary and give a characterization of the critical price at maturity, generalizing a recent result of S.Z. Levendorskiĭ (Int. J. Theor. Appl. Finance 7:303–336, 2004).

Keywords American options · Optimal stopping · Exponential Lévy model

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JEL Classification G10 · G12 · G13

1 Introduction

The introduction of stochastic processes with discontinuous paths in financial modeling goes back to Merton (see [9]). Merton's jump-diffusion model can be derived from the classical Black–Scholes model by adding to the logarithm of the stock price a compound Poisson process, which is the simplest example of a Lévy process with jumps. More recently, general Lévy processes were introduced in financial modeling, and we refer to the monographs [3, 5] for an account of the literature in this direction. A large number of papers have been devoted to the pricing of European options in this setting. In this paper, we focus on American options and especially on the study of the exercise boundary.

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The first results on American option pricing within jump-diffusion models are due to Zhang (see [14–16]) and Pham (see [10]). Zhang developed an approach based on variational inequalities in the spirit of Bensoussan and Lions [1]. Pham characterized the option price as the solution of a free-boundary problem and derived some properties of the exercise boundary. Zhang and Pham's results rely in a crucial way on the diffusion part of the underlying Lévy process.

For general Lévy processes, Levendorskiĭ [8] studied the behavior of the exercise boundary of the American put near maturity. In particular, he observed that if the Lévy measure satisfies some conditions, especially if the intensity of positive jumps is not too small, the limit of the critical price at maturity is smaller than the strike price, which is in contrast to the Black–Scholes setting. Levendorskiĭ also showed that, for some Lévy models, the behavior of the limit of the critical price as the interest rate goes to 0 is more natural than in the standard Black–Scholes case.

The purpose of the present paper is to clarify the basic properties of the early exercise boundary of the American put on a dividend paying stock in general exponential Lévy models. To this end, we characterize the price of the American option as the unique solution of a variational inequality in the sense of distributions. This enables us to recover and extend Levendorskiĭ's results and to prove the continuity of the early exercise boundary (a result proved by Pham [11] for jump-diffusions). Note that our approach is completely different from that of Levendorskiĭ and that we have less stringent assumptions on the Lévy measure.

The paper is organized as follows. In Sect. 2, we recall some basic facts about multidimensional Lévy processes and characterize the value function of an optimal stopping problem as the unique solution of a variational inequality in the sense of distributions. In Sect. 3, we describe the exponential Lévy model with dividends and set up the basic properties of the American put price in this model. The fourth section is devoted to properties of the exercise boundary. We first establish the continuity of the free boundary; then we study the limit of the critical price at maturity and discuss some particular cases.

2 Multidimensional Lévy processes and optimal stopping

2.1 Infinitesimal generator

A d-dimensional Lévy process X is a càdlàg[1] stochastic process with values in $\mathbb{R}^d$, starting from 0, with stationary and independent increments. The random process X can be interpreted as the independent superposition of a Brownian motion with drift and an infinite superposition of independent (compensated) Poisson processes. More precisely the Lévy–Itô decomposition (see [13]) gives the following representation of X:

$$\begin{aligned} X_t &= \gamma t + B_t + \tilde{X}_t + \lim_{\varepsilon \to 0} \tilde{X}_t^\varepsilon, \\ \tilde{X}_t &= \int_0^t \int_{|x|>1} x J_X(ds, dx), \qquad \tilde{X}_t^\varepsilon = \int_0^t \int_{\varepsilon \le |x| \le 1} x \tilde{J}_X(ds, dx), \end{aligned} \tag{2.1}$$

[1] The sample paths of X are right-continuous with left limits.

where γ is a vector in $\mathbb{R}^d$, $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion with covariance matrix A, J_X is a Poisson measure on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$, and $\tilde{J}_X$ is the compensated Poisson measure $\tilde{J}_X(dt, dx) = J_X(dt, dx) - dt\,\nu(dx)$. The measure ν is a positive Radon measure on $\mathbb{R}^d\setminus\{0\}$, called the Lévy measure of X, and it satisfies

$$\int_{\mathbb{R}^d} \big(1 \wedge |x|^2\big)\nu(dx) < \infty,$$

where $|x|$ denotes the Euclidean norm of x. Note that the terms on the right-hand side of (2.1) are independent, and the convergence of the last term is almost sure and uniform with respect to t on $[0, T]$. The Lévy–Itô decomposition entails that the distribution of X is uniquely determined by (A, γ, ν), called the characteristic triplet of the process X. The characteristic function of X has the following Lévy–Khinchin representation (see [13]). Denote by $u.v$ the scalar product of two vectors u, v in $\mathbb{R}^d$. We have

$$\mathbb{E}\big[e^{iz.X_t}\big] = \exp\big[t\Psi(z)\big], \quad z \in \mathbb{R}^d,$$

with

$$\Psi(z) = -\frac{1}{2}z.Az + i\gamma.z + \int \big(e^{iz.x} - 1 - iz.x\mathbf{1}_{|x|\leq 1}\big)\nu(dx).$$

The Lévy process X is a Markov process, and its infinitesimal generator $\mathcal{A}_X$ is defined as follows (see [13]). Let $\mathcal{C}_b^2(\mathbb{R}^d)$ denote the set of all bounded $\mathcal{C}^2$ functions with bounded derivatives. For $g \in \mathcal{C}_b^2(\mathbb{R}^d)$, we have

$$\mathcal{A}_X(g)(x) = \mathcal{A}_X^0(g)(x) + \mathcal{B}_X(g)(x),$$

where

$$\mathcal{A}_X^0(g)(x) = \frac{1}{2}\sum_{i,j=1}^{d} A_{i,j}\frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d}\gamma_i \frac{\partial g}{\partial x_i}(x)$$

and

$$\begin{aligned}\mathcal{B}_X(g)(x) &= \int \nu(dy)\left(g(x+y) - g(x) - \sum_{i=1}^{d} y_i \frac{\partial g}{\partial x_i}(x)\mathbf{1}_{|y|\leq 1}\right)\\ &= \int \nu(dy)\big(g(x+y) - g(x) - y.\nabla g(x)\mathbf{1}_{|y|\leq 1}\big).\end{aligned}$$

Here, ∇g denotes the gradient of g. Note that if g is a locally integrable function on $\mathbb{R}^d$, $\mathcal{A}_X^0(g)$ can be defined in the sense of distributions. We now show how $\mathcal{B}_X(g)$ can be defined as a distribution if g is a bounded and continuous function. We need to recall some facts and notation from the theory of distributions. If $\mathcal{O}$ is an open subset of $\mathbb{R}^d$, we denote by $\mathcal{D}(\mathcal{O})$ the set of all $\mathcal{C}^\infty$ functions with compact support in $\mathcal{O}$ and by $\mathcal{D}'(\mathcal{O})$ the space of distributions on $\mathcal{O}$. If $u \in \mathcal{D}'(\mathcal{O})$ and $\varphi \in \mathcal{D}(\mathcal{O})$, $\langle u, \varphi\rangle$

denotes the evaluation on the test function φ of the distribution u. Note that if u is a locally integrable function on $\mathcal{O}$,

$$\langle u, \varphi \rangle = \int_{\mathcal{O}} u(x)\varphi(x)\,dx.$$

We now introduce the adjoint operator $\mathcal{B}_X^*$ of $\mathcal{B}_X$. For $\varphi \in \mathcal{D}(\mathbb{R}^d)$, define

$$\mathcal{B}_X^*(\varphi)(x) = \int \big(\varphi(x-y) - \varphi(x) + y.\nabla\varphi(x)\mathbf{1}_{|y|\le 1}\big)\nu(dy), \quad x \in \mathbb{R}^d.$$

For the next proposition, we use the notation

$$\big\|D^2\varphi\big\|_\infty = \sup_{x\in\mathbb{R}^d}\sup_{|y|\le 1}\left|\sum_{i=1}^d\sum_{j=1}^d y_i y_j \frac{\partial^2\varphi}{\partial x_i \partial x_j}(x)\right|, \qquad \|\varphi\|_{L^1} = \int_{\mathbb{R}^d}\big|\varphi(y)\big|\,dy,$$

$$B_1 = \big\{y \in \mathbb{R}^d \mid |y| \le 1\big\}.$$

We also denote by $\lambda_d(A)$ the Lebesgue measure of a Borel set $A \subset \mathbb{R}^d$ and by A^c the complement of A.

Proposition 2.1 *If $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the function $\mathcal{B}_X^*(\varphi)$ is continuous and integrable on $\mathbb{R}^d$, and we have*

$$\big\|\mathcal{B}_X^*(\varphi)\big\|_{L^1} \le \frac{1}{2}\big\|D^2\varphi\big\|_\infty \lambda_d(K+B_1)\int_{B_1}|y|^2\nu(dy) + 2\|\varphi\|_{L^1}\nu\big(B_1^c\big),$$

where K is the support of φ. Moreover, if $g \in \mathcal{C}_b^2(\mathbb{R}^d)$, we have

$$\big\langle \mathcal{B}_X(g), \varphi\big\rangle = \int_{\mathbb{R}^d} g(x)\mathcal{B}_X^*(\varphi)(x)\,dx.$$

Proof We have

$$\mathcal{B}_X^*(\varphi)(x) = \int_{B_1}\big(\varphi(x-y) - \varphi(x) + y.\nabla\varphi(x)\big)\nu(dy) + \int_{B_1^c}\big(\varphi(x-y) - \varphi(x)\big)\nu(dy).$$

It follows from Taylor's formula that

$$\big|\varphi(x-y) - \varphi(x) + y.\nabla\varphi(x)\big| \le \frac{1}{2}\big\|D^2\varphi\big\|_\infty |y|^2.$$

Note that if $x \notin K + B_1$, then $x \notin K$ and $x - y \notin K$ for $y \in B_1$, so that we have $\varphi(x-y) - \varphi(x) + y.\nabla\varphi(x) = 0$. Hence,

$$\int_{\mathbb{R}^d}\big|\mathcal{B}_X^*(\varphi)(x)\big|\,dx \le \frac{1}{2}\big\|D^2\varphi\big\|_\infty \int_{K+B_1}dx\int_{B_1}\nu(dy)|y|^2 + \int_{\mathbb{R}^d}dx\int_{B_1^c}\nu(dy)\big|\varphi(x-y) - \varphi(x)\big|$$

$$
\begin{aligned}
&= \frac{1}{2}\|D^2\varphi\|_\infty \lambda_d(K+B_1)\int_{B_1}|y|^2\nu(dy) \\
&\quad + \int_{B_1^c}\nu(dy)\int_{\mathbb{R}^d}\big(|\varphi(x-y)|+|\varphi(x)|\big)\,dx \\
&= \frac{1}{2}\|D^2\varphi\|_\infty \lambda_d(K+B_1)\int_{B_1}|y|^2\nu(dy) + 2\|\varphi\|_{L^1}\nu\big(B_1^c\big).
\end{aligned}
$$

Now, if $g \in \mathcal{C}_b^2(\mathbb{R}^d)$, we have, using Fubini's theorem and integration by parts,

$$
\begin{aligned}
\langle \mathcal{B}_X(g), \varphi\rangle &= \int_{\mathbb{R}^d} \mathcal{B}_X(g)(x)\varphi(x)\,dx \\
&= \int\bigg(\int\big(g(x+y)-g(x)-y.\nabla g(x)\mathbf{1}_{|y|\le 1}\big)\nu(dy)\bigg)\varphi(x)\,dx \\
&= \int\bigg(\int_{B_1}\big(g(x+y)-g(x)-y.\nabla g(x)\big)\nu(dy)\bigg)\varphi(x)\,dx \\
&\quad + \int dx\int_{B_1^c}\nu(dy)\big(g(x+y)-g(x)\big)\varphi(x) \\
&= \int_{B_1}\nu(dy)\bigg(\int\big(g(x+y)-g(x)-y.\nabla g(x)\big)\varphi(x)\,dx\bigg) \\
&\quad + \int_{B_1^c}\nu(dy)\bigg(\int\big(g(x+y)\varphi(x)-g(x)\varphi(x)\big)\,dx\bigg) \\
&= \int_{B_1}\nu(dy)\bigg(\int g(x)\big(\varphi(x-y)-\varphi(x)+y.\nabla\varphi(x)\big)\,dx\bigg) \\
&\quad + \int_{B_1^c}\nu(dy)\bigg(\int g(x)\big(\varphi(x-y)-\varphi(x)\big)\,dx\bigg) \\
&= \int g(x)\mathcal{B}_X^*(\varphi)(x)\,dx.
\end{aligned}
$$

□

Remark 2.2 It follows from Proposition 2.1 that, if g is a bounded Borel measurable function on $\mathbb{R}^d$, we can define the distribution $\mathcal{B}_X(g)$ by setting

$$
\langle \mathcal{B}_X(g), \varphi\rangle = \int g(x)\mathcal{B}_X^*(\varphi)(x)\,dx. \tag{2.2}
$$

We also define $\mathcal{A}_X(g) = \mathcal{A}_X^0(g) + \mathcal{B}_X(g)$. Note that from (2.2) it follows that if $(g_n)_{n\in\mathbb{N}}$ is a sequence of bounded measurable functions on $\mathbb{R}^d$ such that

$$
\forall x \in \mathbb{R}^d, \quad \lim_{n\to\infty} g_n(x) = g(x),
$$

with, for some positive constant C,

$$
|g_n| \le C, \quad n \in \mathbb{N},
$$

then the sequence $(\mathcal{B}_X(g_n))_{n\in\mathbb{N}}$ converges to $\mathcal{B}_X(g)$ in $\mathcal{D}'(\mathbb{R}^d)$.

The following proposition will be useful in regularization arguments.

Proposition 2.3 *If $g \in L^\infty(\mathbb{R}^d)$, we have, for every $\theta \in \mathcal{D}(\mathbb{R}^d)$,*

$$\mathcal{A}_X(g * \theta) = \mathcal{A}_X(g) * \theta.$$

Proof Recall that partial derivatives commute with the convolution. Therefore, it suffices to show that $\mathcal{B}_X(g*\theta) = \mathcal{B}_X(g) * \theta$. We have, for $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\begin{aligned}\langle \mathcal{B}_X(g*\theta), \varphi\rangle &= \int (g*\theta)(x)\mathcal{B}_X^*(\varphi)(x)\,dx \\ &= \iint g(x-y)\theta(y)\mathcal{B}_X^*(\varphi)(x)\,dx\,dy \\ &= \iint g(x)\theta(y)\mathcal{B}_X^*(\varphi)(x+y)\,dx\,dy.\end{aligned}$$

Now,

$$\begin{aligned}&\int \theta(y)\mathcal{B}_X^*(\varphi)(x+y)\,dy \\ &\quad= \int \theta(y)\,dy \int \nu(dz)\big(\varphi(x+y-z) - \varphi(x+y) + z.\nabla\varphi(x+y)\mathbf{1}_{|z|\le 1}\big) \\ &\quad= \int \nu(dz) \int \theta(y)\big(\varphi(x+y-z) - \varphi(x+y) + z.\nabla\varphi(x+y)\mathbf{1}_{|z|\le 1}\big)\,dy \\ &\quad= \int \nu(dz)\big(\varphi*\check{\theta}(x-z) - \varphi*\check{\theta}(x) + z.\nabla(\varphi*\check{\theta})(x)\mathbf{1}_{|z|\le 1}\big) \\ &\quad= \mathcal{B}_X^*(\varphi*\check{\theta})(x),\end{aligned}$$

where $\check{\theta}(x) = \theta(-x)$. Hence,

$$\langle \mathcal{B}_X(g*\theta), \varphi\rangle = \int g(x)\mathcal{B}_X^*(\varphi*\check{\theta})(x)\,dx = \langle \mathcal{B}_X(g), \varphi*\check{\theta}\rangle = \langle \mathcal{B}_X(g)*\theta, \varphi\rangle. \qquad \square$$

2.2 Analytic characterization of the supermartingale property

Let X be a d-dimensional Lévy process and fix g in $\mathcal{C}_b^2(\mathbb{R}^d)$. By the Itô formula, we can show the following semimartingale decomposition for X (see [5], Proposition 8.18):

$$g(X_t) = M_t + V_t, \tag{2.3}$$

where M is a martingale given by

$$M_t = g(0) + \int_0^t \nabla g(X_{s-}).dB_s + \int_{[0,t]\times\mathbb{R}^d} \tilde{J}_X(ds,dy)\big[g(X_{s-}+y) - g(X_{s-})\big],$$

and

$$V_t = \int_0^t \mathcal{A}_X(g)(X_s)\,ds.$$

Given an open set $U \subset \mathbb{R}^d$ and $x \in U$, we introduce the following stopping time, which is the exit time of the process $x + X$ from U:

$$\tau_U^x = \inf\{t \geq 0 \mid x + X_t \notin U\}.$$

The following proposition provides a characterization of the supermartingale property for a smooth function of the stopped process.

Proposition 2.4 *Let f be a function in $\mathcal{C}_b^2(\mathbb{R}^d)$ and U an open set in $\mathbb{R}^d$. Then the following two conditions are equivalent*:

1. *The process $(f(X_{t\wedge\tau_U^x} + x))_{t\geq 0}$ is a supermartingale for every $x \in U$,*
2. *For all $x \in U$, $\mathcal{A}_X(f)(x) \leq 0$.*

Proof 1 $\Longrightarrow$ 2. Fix x in U. Since $(f(X_{t\wedge\tau_U^x} + x))_{t\geq 0}$ is a supermartingale, we have, for every $t \in (0, T)$,

$$\mathbb{E}\left[\frac{1}{t}\big(f(X_{t\wedge\tau_U^x} + x) - f(x)\big)\right] \leq 0.$$

So, by the decomposition (2.3) we get

$$\mathbb{E}\left[\frac{1}{t}\int_0^{t\wedge\tau_U^x} \mathcal{A}_X(f)(X_s + x)\,ds\right] \leq 0, \quad \forall t \in (0, T). \tag{2.4}$$

Note that, due to the right-continuity of X, we have $\tau_U^x > 0$ almost surely. Therefore, letting t go to zero in (2.4), we have, by dominated convergence, $\mathcal{A}_X(f)(x) \leq 0$.

1 $\Longleftarrow$ 2. By the martingale/drift decomposition (2.3), we have, for $x \in U$ and $0 \leq s \leq t$,

$$\begin{aligned}
\mathbb{E}\big[f(X_{t\wedge\tau_U^x} + x)\big|\mathcal{F}_s\big] &= \mathbb{E}\left[f(x) + M_{t\wedge\tau_U^x} + \int_0^{t\wedge\tau_U^x} \mathcal{A}_X(f)(x + X_\ell)\,d\ell\,\middle|\,\mathcal{F}_s\right] \\
&= f(x) + M_{s\wedge\tau_U^x} + \int_0^{s\wedge\tau_U^x} \mathcal{A}_X(f)(x + X_\ell)\,d\ell \\
&\quad + \mathbb{E}\left[\int_{s\wedge\tau_U^x}^{t\wedge\tau_U^x} \mathcal{A}_X(f)(x + X_\ell)\,d\ell\,\middle|\,\mathcal{F}_s\right] \\
&= f(X_{s\wedge\tau_U^x} + x) + \mathbb{E}\left[\int_{s\wedge\tau_U^x}^{t\wedge\tau_U^x} \mathcal{A}_X(f)(x + X_\ell)\,d\ell\,\middle|\,\mathcal{F}_s\right] \\
&\leq f(X_{s\wedge\tau_U^x} + x),
\end{aligned}$$

since $\mathcal{A}_X(f)(x + X_\ell) \leq 0$ a.s. for $\ell \in (s \wedge \tau_U^x, t \wedge \tau_U^x)$. □

The following result shows that the previous proposition can be extended to bounded continuous functions, provided that the inequality $\mathcal{A}_X(f) \le 0$ is taken in the sense of distributions.

Proposition 2.5 *Let f be a bounded continuous function on $\mathbb{R}^d$ and U an open set in $\mathbb{R}^d$. Then the following two conditions are equivalent*:

1. *The process $(f(X_{t\wedge\tau_U^x} + x))_{t\ge 0}$ is a supermartingale for every $x \in U$.*
2. *The distribution $\mathcal{A}_X(f)$ is a nonpositive measure on U.*

Proof 1 $\Longrightarrow$ 2. Fix x_0 in U and define the stopping time τ_U by

$$\tau_U = \inf\{t \ge 0 \,|\, \exists y \in B(x_0, a) \text{ such that } y + X_t \notin U\},$$

where $B(x_0, a)$ is the open ball in $\mathbb{R}^d$ with center x_0 and radius a. We choose $a > 0$ so that $B(x_0, 2a) \subset U$. Note that $\tau_U \le \tau_U^{x-y}$ for every $(x, y) \in B(x_0, a/2)B(0, a/2)$, and, since $(f(X_{t\wedge\tau_U^{x-y}} + x - y))$ is a supermartingale,

$$\mathbb{E}\big[f(X_{t\wedge\tau_U} + x - y)\big] \le f(x - y). \tag{2.5}$$

Now, consider a regularizing sequence ρ_n such that for every $n \in \mathbb{N}$, ρ_n is an even nonnegative $\mathcal{C}^\infty$ function, $\text{supp}(\rho_n) \subset B(0, a/2n)$, and $\int \rho_n(x)dx = 1$. We deduce from (2.5) that

$$\int_{\mathbb{R}^d} \mathbb{E}\big[f(X_{t\wedge\tau_U} + x - y)\big]\rho_n(y)\,dy \le \int_{\mathbb{R}^d} f(x - y)\rho_n(y)\,dy,$$

which is equivalent via the Fubini theorem to

$$\mathbb{E}\big[f * \rho_n(X_{t\wedge\tau_U} + x)\big] \le f * \rho_n(x).$$

So, for every $t > 0$,

$$\frac{1}{t}\big[\mathbb{E}\big(f * \rho_n(X_{t\wedge\tau_U} + x)\big) - \rho_n * f(x)\big] \le 0.$$

Note that $f * \rho_n \in \mathcal{C}_b^2(\mathbb{R}^d)$, so that we get

$$\mathbb{E}\left[\frac{1}{t}\int_0^{t\wedge\tau_U} \mathcal{A}_X(f * \rho_n)(X_s + x)\,ds\right] \le 0.$$

But $\mathcal{A}_X(f * \rho_n)(X_s + x)$ is a right-continuous bounded function with respect to s and $t \wedge \tau_U = t$ for t close to 0 (because $\tau_U > 0$ a.s.). So, letting t go to 0, we obtain

$$\mathcal{A}_X(f * \rho_n)(x) \le 0, \quad \forall x \in B(x_0, a/2).$$

Using Proposition 2.4, we deduce $\mathcal{A}_X(f) * \rho_n \le 0$ on $B(x_0, a/2)$ and, by letting n go to infinity, $\mathcal{A}_X(f) \le 0$ in the sense of distributions in $B(x_0, a/2)$. Since x_0 is arbitrary in U, we conclude, using a standard partition of unity argument, that $\mathcal{A}_X(f) \le 0$ on U.

$1 \Longleftarrow 2$. Suppose that $\mathcal{A}_X(f)$ is a nonpositive measure on U. For a positive integer n, consider the open set U_n defined by

$$U_n = \{x \in U \,|\, d(x, U^c) > 1/n\},$$

where $d(x, U^c) = \inf_{y \in U^c} |x - y|$. Note that the sequence $(U_n)_{n\geq 1}$ is increasing and that $\bigcup_{n\geq 1} U_n = U$. Consider a sequence $(\rho_n)_{n\geq 1}$ of even nonnegative $\mathcal{C}^\infty$ functions such that

$$\mathrm{supp}(\rho_n) \subset B(0, 1/n), \qquad \int \rho_n(x)\, dx = 1, \quad n \geq 1.$$

If φ is a nonnegative function in $\mathcal{D}(U_n)$, the inclusion

$$\mathrm{supp}(\varphi * \rho_n) \subset \mathrm{supp}(\varphi) + \mathrm{supp}(\rho_n)$$

gives $\varphi * \rho_n \in \mathcal{D}_+(U)$, where $\mathcal{D}_+(U)$ is the set of all nonnegative functions in $\mathcal{D}(U)$. Since $\mathcal{A}_X(f) \leq 0$ on U, we deduce $\langle \mathcal{A}_X(f), \varphi * \rho_n \rangle \leq 0$. On the other hand, we have $\langle \mathcal{A}_X(f), \varphi * \rho_n \rangle = \langle \mathcal{A}_X(f) * \rho_n, \varphi \rangle$, because ρ_n is even, and $\mathcal{A}_X(f) * \rho_n = \mathcal{A}_X(f * \rho_n)$, by Proposition 2.3. Hence, the function $\mathcal{A}_X(f * \rho_n)$ is nonpositive on the open set U_n. Note that we also have $\mathcal{A}_X(f * \rho_n) \leq 0$ on U_k for $k \leq n$, because $U_k \subset U_n$. Since $f * \rho_n \in \mathcal{C}_b^2(\mathbb{R}^d)$, we can apply Proposition 2.4 and deduce that, for $k \leq n$ and $x \in U_k$, the process $(f * \rho_n(X_{t\wedge \tau^x_{U_k}} + x))_{t\geq 0}$ is a supermartingale. Recall that $\tau^x_{U_k}$ is the exit time of the process $x + X$ from the open set U_k. We now have, for $0 \leq s \leq t$ and $n \geq k$,

$$\mathbb{E}\big[f * \rho_n(X_{t\wedge \tau^x_{U_k}} + x) \big| \mathcal{F}_s\big] \leq f * \rho_n(X_{s\wedge \tau^x_{U_k}} + x).$$

We now fix k and let n go to infinity. The sequence $(f * \rho_n)_{n\geq 1}$ converges to f and is uniformly bounded. So, by the dominated convergence theorem we have

$$\mathbb{E}\big[f(X_{t\wedge \tau^x_{U_k}} + x) \big| \mathcal{F}_s\big] \leq f(X_{s\wedge \tau^x_{U_k}} + x). \tag{2.6}$$

Now, the sequence of stopping times $(\tau^x_{U_k})_{k\geq 1}$ is clearly increasing. Let $\bar{\tau} = \lim_{k\to\infty} \tau^x_{U_k}$. We prove that $\bar{\tau} = \tau^x_U$ a.s. We obviously have $\bar{\tau} \leq \tau^x_U$. On the other hand, by the right-continuity of X, we have, on $\{\bar{\tau} < \infty\}$,

$$d(x + X_{\tau_{U_k}}, U_k^c) \leq \frac{1}{k},$$

and by the quasi-left-continuity of X (see Chap. 8, p. 279 in [13]), we also have

$$d(x + X_{\bar{\tau}}, U^c) = 0 \quad \text{a.s.}$$

on $\{\bar{\tau} < \infty\}$. Therefore, $\tau^x_U \leq \bar{\tau}$ a.s. To complete the proof, we let k go to infinity in (2.6). By the quasi-left-continuity of X and the dominated convergence theorem, we have

$$\mathbb{E}\big[f(X_{t\wedge \tau_U} + x) \big| \mathcal{F}_s\big] \leq f(X_{s\wedge \tau_U} + x),$$

which proves that the process $(f(X_{t\wedge \tau^x_U} + x))$ is a supermartingale. □

Remark 2.6 It follows from Proposition 2.5 that a continuous bounded function f satisfies $\mathcal{A}_X(f) \geq 0$ (resp. $\mathcal{A}_X(f) = 0$) on U (in the sense of distributions) if and only if for all $x \in U$, the process $(f(X_{t\wedge\tau_U^x} + x))_{t\geq 0}$ is a submartingale (resp. a martingale).

2.3 Optimal stopping and variational inequality

Let $X = (X_t)_{t\geq 0}$ be a d-dimensional Lévy process. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural complete filtration of X. Recall that $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions (cf. [2], Chap. 1). For $0 \leq t \leq T$, we denote by $\mathcal{T}_{t,T}$ the set of all stopping times with values in $[t, T]$.

For a continuous and bounded function f on $\mathbb{R}^d$, define

$$u_f(t,x) = \sup_{\tau\in\mathcal{T}_{0,t}} \mathbb{E}\big[f(x+X_\tau)\big], \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d.$$

Theorem 2.7 *The function u_f is continuous on $[0,+\infty)\times\mathbb{R}^d$ and, for all $T > 0$ and $x \in \mathbb{R}^d$, the process $(u_f(T-t, x+X_t))_{0\leq t\leq T}$ is the Snell envelope with horizon T of the process $(f(x+X_t))_{0\leq t\leq T}$, which means that*

$$u_f(T-t, x+X_t) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\big[f(x+X_\tau)\big|\mathcal{F}_t\big], \quad 0 \leq t \leq T.$$

Proof This is essentially a classical result; see [6, 7] for related general results. In our setting, a proof can be given by approximation. Indeed, if $f \in \mathcal{C}_b^2(\mathbb{R}^d)$, the continuity of u_f is easy to deduce from the equality

$$\mathbb{E}\big[f(x+X_{\tau_2})\big] - \mathbb{E}\big[f(x+X_{\tau_1})\big] = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \mathcal{A}_X f(x+X_s)\,ds\right],$$

valid for bounded stopping times τ_1 and τ_2. The connection with the Snell envelope can be derived by discretizing the stopping times. For an arbitrary bounded and continuous function f, we first approximate f by a sequence of continuous functions with compact supports as follows. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$, with values in $[0,1]$, such that $\varphi = 1$ on the unit ball. For a positive integer n, let $f_n(x) = f(x)\varphi_n(x)$ with $\varphi_n(x) = \varphi(x/n)$. We have, for any bounded stopping time τ,

$$\mathbb{E}\big[f(x+X_\tau)\big] - \mathbb{E}\big[f_n(x+X_\tau)\big] = \mathbb{E}\big[f(x+X_\tau)\big(1-\varphi_n(x+X_\tau)\big)\big].$$

Therefore,

$$\big|u_f(t,x) - u_{f_n}(t,x)\big| \leq \|f\|_\infty \sup_{\tau\in\mathcal{T}_{0,t}} \mathbb{E}\big[1-\varphi_n(x+X_\tau)\big].$$

Note that if $|x| \leq n$, we have $\mathbb{E}[\varphi_n(x+X_\tau)] = 1 + \mathbb{E}[\int_0^\tau \mathcal{A}_X\varphi_n(x+X_s)\,ds]$, so that $|u_f(t,x) - u_{f_n}(t,x)| \leq \|f\|_\infty \mathbb{E}[\int_0^t |\mathcal{A}_X\varphi_n(x+X_s)|\,ds]$. It can be proved that the sequence $(\mathcal{A}_X\varphi_n)$ converges uniformly to 0, so that (u_{f_n}) converges uniformly on compact sets to u_f. Note that we also have the convergence of the Snell envelopes.

Now, if f is continuous and compactly supported, it can be approximated uniformly by a sequence of functions in $\mathcal{C}_b^2(\mathbb{R}^d)$. □

We can now characterize the value function u_f of an optimal stopping problem with reward function f as the unique solution of a variational inequality. Note that in the following statement, $\partial_t v + \mathcal{A}_X v$ is to be understood as a distribution (cf. Remark 2.2).

Theorem 2.8 *Fix $T > 0$ and let f be a continuous and bounded function on $\mathbb{R}^d$. The function v defined by $v(t,x) = u_f(T-t,x)$ is the only continuous and bounded function on $[0,T] \times \mathbb{R}^d$ satisfying the following conditions*:

1. $v(T,.) = f$.
2. $v \geq f$.
3. *On* $(0,T) \times \mathbb{R}^d$, $\partial_t v + \mathcal{A}_X v \leq 0$.
4. *On the open set* $\{(t,x) \in (0,T) \times \mathbb{R}^d \mid v(t,x) > f(x)\}$, $\partial_t v + \mathcal{A}_X v = 0$.

Proof Clearly, if $v(t,x) = u_f(T-t,x)$, the first two conditions are satisfied. The other two conditions follow from properties of the Snell envelope. Indeed, if $U = (U_t)_{0\leq t\leq T}$ is the Snell envelope of a quasi-left-continuous and right-continuous process $(Z_t)_{0\leq t\leq T}$, the process U is a supermartingale, and the stopped process $(U_{\tau^*\wedge t})_{0\leq t\leq T}$ is a martingale, where $\tau^* = \inf\{t \geq 0 \mid U_t = Z_t\}$ (see [6] for the theory of the Snell envelope in continuous time). Using Theorem 2.7, we deduce that $(v(t,x+X_t))_{0\leq t\leq T}$ is a supermartingale. We then apply Proposition 2.5 to the $(d+1)$-dimensional process (t,X_t) and the function v. This yields $\partial_t v + \mathcal{A}_X v \leq 0$ on $(0,T) \times \mathbb{R}^d$. Similarly, the condition $\partial_t v + \mathcal{A}_X v = 0$ on $\{v > f\}$ follows from the martingale property of the Snell envelope stopped at the optimal stopping time.

Conversely, if a bounded and continuous function v satisfies the four conditions of the theorem, we see that the process $(v(t,x+X_t))_{0\leq t\leq T}$ is a supermartingale which dominates the process $Z = (f(x+X_t))_{0\leq t\leq T}$. Therefore, $v(t,x+X_t) \geq U_t$, where U is the Snell envelope of Z. Moreover, if $\tau = \inf\{t \geq 0 \mid v(t,x+X_t) = f(x+X_t)\}$, then $(v(t\wedge\tau, x+X_{t\wedge\tau}))_{0\leq t\leq T}$ is a martingale, so that $v(0,x) = \mathbb{E}[f(x+X_\tau)]$. We easily deduce therefrom that $v(0,x) = u_f(T,x)$. By changing the time origin we also have $v(t,x) = u_f(T-t,x)$. □

3 The American put price in the exponential Lévy model

3.1 The exponential Lévy model

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modeled as a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_0)$. We suppose that there exists an equivalent (risk neutral) probability $\mathbb{P}$ under which the discounted underlying is a martingale, and therefore the absence of arbitrage is satisfied. In the exponential Lévy model, the risk neutral dynamics of S_t is given by

$$S_t = S_0 e^{(r-\delta)t+X_t}, \tag{3.1}$$

where the interest rate r and the dividend rate δ are nonnegative constants, and $(X_t)_{t\in[0,T]}$ is a real Lévy process with characteristic triplet (σ^2,γ,ν). We include r and δ in (3.1) for ease of notation. The infinitesimal generator of the process X is given by

$$Lf(x) = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + \gamma\frac{\partial f}{\partial x}(x) + \int \left(f(x+y) - f(x) - y\frac{\partial f}{\partial x}(x)\mathbf{1}_{|y|\le 1}\right)\nu(dy). \tag{3.2}$$

Under $\mathbb{P}$, the discounted dividend-adjusted stock price $(e^{-(r-\delta)t}S_t)_{t\in[0,T]}$ is a martingale, which is equivalent to the following two conditions on the characteristic triplet (see [5] Proposition 3.17):

$$\int_{|x|\ge 1} e^x \nu(dx) < \infty, \tag{3.3}$$

and

$$\frac{\sigma^2}{2} + \gamma + \int \left(e^x - 1 - x\mathbf{1}_{|x|\le 1}\right)\nu(dx) = 0. \tag{3.4}$$

We suppose that conditions (3.3) and (3.4) are satisfied in the sequel. We deduce from (3.4) that the infinitesimal generator defined in (3.2) can be written as

$$Lf(x) = \frac{\sigma^2}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\right)(x) + \int \left(f(x+y) - f(x) - \left(e^y - 1\right)\frac{\partial f}{\partial x}(x)\right)\nu(dy). \tag{3.5}$$

The stock price $(S_t)_{t\in[0,T]}$ is also a Markov process and $S_t = S_0 e^{\tilde{X}_t}$, where $\tilde{X}$ is a Lévy process with characteristic triplet $(\sigma^2, r-\delta+\gamma, \nu)$. We denote by $\tilde{L}$ the infinitesimal generator of $\tilde{X}$. So, from (3.5) we have

$$\tilde{L}f(x) = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial x}(x) + \tilde{\mathcal{B}}f(x), \tag{3.6}$$

where

$$\tilde{\mathcal{B}}f(x) = \int \nu(dy)\left(f(x+y) - f(x) - \left(e^y - 1\right)\frac{\partial f}{\partial x}(x)\right).$$

To finish this section, we recall the following proposition (see [5], Proposition 3.10).

Proposition 3.1 *Let X be a Lévy process with characteristic triplet (σ^2,γ,ν). Then the following two conditions are equivalent*:

(i) $X_t \ge 0$ *a.s. for some* $t > 0$,
(ii) $\sigma = 0$, $\nu((-\infty,0]) = 0$, $\int_0^\infty (x\wedge 1)\nu(dx) < \infty$, *and* $\gamma - \int_{(0,1]} x\nu(dx) \ge 0$.

3.2 The American put price

In this model, the value at time t of an American put with maturity T and strike price K is given by

$$P_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\big[e^{-r\tau}\psi(S_\tau)\big|\mathcal{F}_t\big],$$

where $\psi(x) = (K - x)_+$, and $\mathcal{T}_{t,T}$ denotes as above the set of stopping times τ satisfying $t \le \tau \le T$. Due to the Markov property, we have

$$P_t = P(t, S_t),$$

where

$$P(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}\big[e^{-r\tau}\psi\big(S_\tau^x\big)\big] \tag{3.7}$$

with $S_t^x = xe^{\tilde{X}_t} = xe^{(r-\delta)t + X_t}$. The following proposition follows easily from (3.7).

Proposition 3.2 *For $t \in [0,T]$, the function $x \mapsto P(t,x)$ is nonincreasing and convex on $[0,+\infty)$. For $x \in [0,+\infty)$, the function $t \mapsto P(t,x)$ is continuous and nonincreasing on $[0,T]$.*

We now state the variational inequality related to the American put in the exponential Lévy model. Note that this variational inequality was already established by X. Zhang (see [15]) in the jump-diffusion model and by H. Pham (see [10] and [12]) for more general models, in the sense of viscosity solutions. It is more convenient to state the variational inequality after a logarithmic change of variable. Define

$$\tilde{P}(t,x) = P\big(t, e^x\big), \quad (t,x) \in [0,T] \times \mathbb{R}.$$

We have

$$\tilde{P}(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}\big[e^{-r\tau}\tilde{\psi}(x + \tilde{X}_\tau)\big],$$

where $\tilde{\psi}(x) = \psi(e^x) = (K - e^x)_+$.

Theorem 3.3 *The distribution $(\partial_t + \tilde{L} - r)\tilde{P}$ is a nonpositive measure on $(0,T) \times \mathbb{R}$, and on the open set $\{(t,x) \in (0,T) \times \mathbb{R} \mid \tilde{P}(t,x) > \tilde{\psi}(x)\}$, we have $(\partial_t + \tilde{L} - r)\tilde{P} = 0$.*

Proof It suffices to apply Theorem 2.8 to the two-dimensional Lévy process (t, X_t) and to the function $f(t,x)$ defined by $f(t,x) = e^{-rt}\tilde{\psi}(x)$. □

4 Properties of the free boundary

Throughout this section, we assume that at least one of the following conditions is satisfied:

$$\sigma \neq 0, \quad \nu\big((-\infty, 0)\big) > 0, \quad \text{or} \quad \int_{(0,+\infty)} (x \wedge 1)\nu(dx) = +\infty.$$

It follows that, for any $A > 0$ and any $t > 0$, we have $\mathbb{P}[X_t + At < 0] > 0$ (see Proposition 3.1) and, consequently,

$$\forall t > 0,\ \forall M > 0, \quad \mathbb{P}[X_t < -M] > 0.$$

From this property we easily derive that

$$\forall t \in [0, T),\ \forall x \in [0, +\infty), \quad P(t, x) > 0. \tag{4.1}$$

We also assume that $r > 0$.

We now define the *critical price* at time $t \in [0, T)$ by

$$b(t) = \inf\big\{x \geq 0 \mid P(t, x) > \psi(x)\big\}.$$

Note that, since $t \mapsto P(t, x)$ is nonincreasing, the function $t \mapsto b(t)$ is nondecreasing. It follows from (4.1) that $b(t) \in [0, K)$. We clearly have $P(t, x) = \psi(x)$ for $x \in [0, b(t))$ and also for $x = b(t)$, due to the continuity of P and ψ. We also deduce from the convexity of $x \mapsto P(t, x)$ and (4.1) that

$$\forall t \in [0, T),\ \forall x > b(t), \quad P(t, x) > \psi(x).$$

In other words, the continuation region C can be written as

$$C = \big\{(t, x) \in [0, T) \times [0, +\infty) \mid x > b(t)\big\}.$$

The graph of b is called the *exercise boundary* or *free boundary*.

Our first observation is that $b(t)$ is positive.

Proposition 4.1 *For $t \in [0, T)$, we have $b(t) > 0$.*

Proof Since T is arbitrary and the put price is a function of $T - t$, we may assume without loss of generality that $0 < t < T$. Suppose that $b(t^*) = 0$ for some $t^* \in (0, T)$. We then have $b(t) = 0$ for $t \leq t^*$, so that

$$\forall (t, x) \in (0, t^*) \times (0, +\infty), \quad P(t, x) > \psi(x).$$

Therefore, on the set $(0, t^*) \times \mathbb{R}$, we have $\tilde{P} > \tilde{\psi}$ and $(\partial_t + \tilde{L} - r)\tilde{P} = 0$. Since $t \mapsto P(t, x)$ is nonincreasing, we deduce that, for $t \in (0, t^*)$, $(\tilde{L} - r)\tilde{P}(t, .) \geq 0$. In fact, the inequality $(\tilde{L} - r)\tilde{P} \geq 0$ in the sense of distributions implies that for any nonnegative test functions θ and φ in $\mathcal{D}(\mathbb{R})$, which have support respectively in $(0,t^*)$ and $(-\infty,+\infty)$, we have

$$\begin{aligned}\int_{(0,t^*)}\theta(t)\,dt\int_{\mathbb{R}}\tilde{P}(t,x)\Big(-\tilde{\gamma}\varphi'(x)+\frac{\sigma^2}{2}\varphi''(x)+\mathcal{B}^*(\varphi)(x)\Big)\,dx\\ \geq r\int_{(0,t^*)}\theta(t)\,dt\int_{\mathbb{R}}\big(K-e^x\big)\varphi(x)\,dx,\end{aligned}$$

where $\tilde{\gamma}=r-\delta+\gamma$ and

$$\mathcal{B}^*(\varphi)(x)=\int\big(\varphi(x-y)-\varphi(x)+y\varphi'(x)\mathbf{1}_{|y|\leq 1}\big)\nu(dy).$$

So, by the continuity of $t\mapsto P(t,.)$, we must have

$$\begin{aligned}\int_{\mathbb{R}}\tilde{P}(t,x)\Big(-\tilde{\gamma}\varphi'(x)+\frac{\sigma^2}{2}\varphi''(x)+\mathcal{B}^*(\varphi)(x)\Big)\,dx\\ \geq r\int_{\mathbb{R}}\big(K-e^x\big)\varphi(x)\,dx.\end{aligned}\tag{4.2}$$

Let χ be a nonnegative $\mathcal{C}^\infty$ function with support in the interval $[-1,0]$ and $\int\chi(x)dx=1$. Apply (4.2) with $\varphi(x)=\lambda\chi(\lambda x)$, where $\lambda>0$. We have

$$r\int_{\mathbb{R}}\big(K-e^x\big)\varphi(x)\,dx=rK-\int e^{x/\lambda}\chi(x)\,dx.$$

Note that, since $\operatorname{supp}\chi\subset[-1,0]$, $\lim_{\lambda\to 0}\int e^{x/\lambda}\chi(x)\,dx=0$. On the other hand, we have

$$\begin{aligned}&\int_{\mathbb{R}}\tilde{P}(t,x)\Big(-\tilde{\gamma}\varphi'(x)+\frac{\sigma^2}{2}\varphi''(x)+\mathcal{B}^*(\varphi)(x)\Big)\,dx\\ &\quad=\int_{\mathbb{R}}\tilde{P}(t,x/\lambda)\Big(-\tilde{\gamma}\lambda\chi'(x)+\frac{\sigma^2}{2}\lambda^2\chi''(x)\Big)\,dx+\int_{\mathbb{R}}\tilde{P}(t,x)\big(\mathcal{B}^*(\varphi)(x)\big)\,dx.\end{aligned}$$

Since $\tilde{P}$ is bounded, we have

$$\lim_{\lambda\to 0}\int_{\mathbb{R}}\tilde{P}(t,x/\lambda)\Big(-\tilde{\gamma}\lambda\chi'(x)+\frac{\sigma^2}{2}\lambda^2\chi''(x)\Big)\,dx=0.$$

We also have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\tilde{P}(t,x)\big(\mathcal{B}^*(\varphi)(x)\big)\,dx\right|\\ &\quad\leq\int dx\,\tilde{P}(t,x/\lambda)\int\nu(dy)\big|\chi(x-\lambda y)-\chi(x)+\lambda y\chi'(x)\mathbf{1}_{|y|\leq 1}\big|\\ &\quad\leq\|\tilde{P}\|_\infty\big(j_1(\lambda)+j_2(\lambda)\big),\end{aligned}$$

where

$$j_1(\lambda) = \int_{\{|y|>1\}} \nu(dy) \int dx \left|\chi(x-\lambda y) - \chi(x)\right|$$

and

$$j_2(\lambda) = \int_{\{|y|\le 1\}} \nu(dy) \int dx \left|\chi(x-\lambda y) - \chi(x) + \lambda y \chi'(x)\right|.$$

It is easy to prove, by dominated convergence, that $\lim_{\lambda\to 0} j_1(\lambda) = 0$ and $\lim_{\lambda\to 0} j_2(\lambda) = 0$. Therefore, since $r > 0$, we get a contradiction by letting $\lambda \to 0$ in (4.2). □

4.1 Continuity of the free boundary

Theorem 4.2 *The function $t \mapsto b(t)$ is continuous on $[0, T)$.*

For the proof of the left-continuity, we need the following lemma.

Lemma 4.3 *For $t \in [0, T)$ and $x \in [0, b(t))$, let*

$$\varphi_t(x) = \int \left(P\left(t, xe^y\right) + xe^y - K\right)\nu(dy).$$

1. *For each $t \in [0, T)$, the function φ_t is nonnegative, convex, and continuous on the interval $[0, b(t))$.*
2. *Let $\tilde{E} = \{(t, x) \in (0, T) \times \mathbb{R} \mid x < \tilde{b}(t)\}$ with $\tilde{b}(t) = \ln b(t)$. On the open set $\tilde{E}$, we have*

$$(\tilde{L} - r)\tilde{P}(t, x) = \tilde{\varphi}_t(x) + \delta e^x - rK,$$

where $\tilde{\varphi}_t(x) = \varphi_t(e^x)$.

Proof Note that, if $x < b(t)$, we have

$$P\left(t, xe^y\right) + xe^y - K = 0 \quad \text{for } y < \ln\frac{b(t)}{x},$$

so that, due to (3.3), $\varphi_t(x)$ is finite. The continuity of φ_t follows from the continuity of $P(t, .)$ by dominated convergence. The convexity of φ_t is a consequence of the convexity of $P(t, .)$.

For the second part of the lemma, note that on $\tilde{E}$, we have $\tilde{P} = \tilde{\psi}$, so that, using (3.6),

$$(\tilde{L} - r)\tilde{P} = \left(r - \delta - \frac{\sigma^2}{2}\right)\tilde{\psi}' + \frac{\sigma^2}{2}\tilde{\psi}'' + \tilde{\mathcal{B}}\tilde{P} - r\tilde{\psi}.$$

For $(t, x) \in \tilde{E}$, we have $\tilde{\psi}(x) = K - e^x$, so that

$$\begin{aligned}
&\left(r-\delta-\frac{\sigma^2}{2}\right)\tilde{\psi}'(x)+\frac{\sigma^2}{2}\tilde{\psi}''(x)-r\tilde{\psi}(x)\\
&\quad=\left(r-\delta-\frac{\sigma^2}{2}\right)\left(-e^x\right)+\frac{\sigma^2}{2}\left(-e^x\right)-r\left(K-e^x\right)\\
&\quad=\delta e^x-rK.
\end{aligned}$$

On the other hand, it is easy to check that on $\tilde{E}$, the distribution $\tilde{\mathcal{B}}\tilde{P}$ coincides with the function defined by

$$\begin{aligned}
\tilde{\mathcal{B}}\tilde{P}(t,x)&=\int \nu(dy)\left(\tilde{P}(t,x+y)-\tilde{\psi}(x)-\left(e^y-1\right)\tilde{\psi}'(x)\right)\\
&=\int \nu(dy)\left(\tilde{P}(t,x+y)-\left(K-e^x\right)-\left(e^y-1\right)\left(-e^x\right)\right)\\
&=\int \nu(dy)\left(\tilde{P}(t,x+y)+e^{x+y}-K\right)=\varphi_t\left(e^x\right).
\end{aligned}$$

□

Proof of Theorem 4.2 We first show the right-continuity. Fix $t\in[0,T)$ and let $(t_n)_{n\geq 1}$ be a decreasing sequence such that $\lim_{n\to\infty} t_n=t$. Since the function b is nondecreasing, the sequence $(b(t_n))$ is nonincreasing and $\lim_{n\to\infty} b(t_n)\geq b(t)$. On the other hand, we have

$$P\left(t_n,b(t_n)\right)=\psi\left(b(t_n)\right),\quad n\geq 1,$$

and by the continuity of P and ψ,

$$P\left(t,\lim_{n\to\infty} b(t_n)\right)=\psi\left(\lim_{n\to\infty} b(t_n)\right).$$

Hence, $\lim_{n\to\infty} b(t_n)\leq b(t)$. Therefore, $\lim_{n\to\infty} b(t_n)=b(t)$, and right-continuity is proved.

We now prove that b is left-continuous. Equivalently, we prove that the mapping $t\mapsto\tilde{b}(t)=\ln b(t)$ is left-continuous. Fix $t\in(0,T)$ and denote by $\tilde{b}(t^-)$ the left limit of $\tilde{b}$ at t. Recall that $\tilde{b}$ is nondecreasing, so that the limit exists and $\tilde{b}(t^-)\leq\tilde{b}(t)$.

Suppose that $\tilde{b}(t^-)<\tilde{b}(t)$ and let $(s,x)\in(0,t)\times(\tilde{b}(t^-),\tilde{b}(t))$. We then have $x>\tilde{b}(t^-)\geq\tilde{b}(s)$, so that $\tilde{P}(s,x)>\tilde{\psi}(x)$. Therefore, on the open set $(0,t)\times(\tilde{b}(t^-),\tilde{b}(t))$, we have, using Theorem 3.3, $(\partial_t+\tilde{L}-r)\tilde{P}=0$. Hence,

$$(\tilde{L}-r)\tilde{P}=-\partial_t\tilde{P}\geq 0\quad\text{on }(0,t)\times\left(\tilde{b}(t^-),\tilde{b}(t)\right),$$

where the last inequality follows from the fact that $t\mapsto\tilde{P}(t,x)$ is nonincreasing. Using the continuity of $\tilde{P}$, we deduce that for every $s\in(0,t)$, we have $(\tilde{L}-r)\tilde{P}(s,.)\geq 0$ on the open interval $(\tilde{b}(t^-),\tilde{b}(t))$. Using Lemma 4.3, we get $\tilde{\varphi}_t(x)+\delta e^x-rK\geq 0$ for $x\in(\tilde{b}(t^-),\tilde{b}(t))$. Equivalently, we have

$$\varphi_t(x)+\delta x-rK\geq 0,\quad x\in\left(b\left(t^-\right),b(t)\right).$$

On the other hand, on the set $(t,T) \times (-\infty, \tilde{b}(t))$, we have $\tilde{P} = \tilde{\psi}$, and it follows from $(\partial_t + \tilde{L} - r)\tilde{P} \le 0$ that $(\tilde{L} - r)\tilde{P} \le 0$. Therefore, using Lemma 4.3 again,

$$\tilde{\varphi}_s(x) + \delta e^x - rK \le 0, \quad (s,x) \in (t,T) \times \big(-\infty, \tilde{b}(t)\big).$$

Hence, by continuity, $\tilde{\varphi}_t(x) + \delta e^x - rK \le 0$ for $x \in (-\infty, \tilde{b}(t))$. We then have

$$\varphi_t(x) + \delta x = rK \quad \text{for } x \in \big(b\big(t^-\big), b(t)\big). \tag{4.3}$$

Now, let $\hat{\varphi}_t(x) = \varphi_t(x) + \delta x$. Note that $\hat{\varphi}_t$ is continuous, convex on $[0, b(t))$, non-negative, and that $\hat{\varphi}_t(0) = 0$. Therefore, if $\hat{\varphi}_t(x) > 0$ for some $x \in [0, b(t))$, $\hat{\varphi}_t$ must be strictly increasing on $[x, b(t))$. This contradicts (4.3). □

4.2 Critical price near maturity

The following result characterizes the limit of the critical price $b(t)$ as t approaches T.

Theorem 4.4 *If* $\int (e^x - 1)_+ \nu(dx) \le r - \delta$, *we have* $\lim_{t\to T} b(t) = K$. *On the other hand, if* $\int (e^x - 1)_+ \nu(dx) > r - \delta$, *we have* $\lim_{t\to T} b(t) = \xi$, *where* ξ *is the unique real number in the interval* $(0, K)$ *such that*

$$\varphi(\xi) = rK, \tag{4.4}$$

where φ *is the function defined by*

$$\varphi(x) = \varphi_T(x) + \delta x, \quad \textit{and} \quad \varphi_T(x) = \int \big(xe^y - K\big)_+ \nu(dy), \quad x \in (0, K).$$

Proof Define $b(T) = \lim_{t\to T} b(t)$ and $\tilde{b}(T) = \ln b(T)$. We clearly have $b(T) \le K$. Recall that due to Lemma 4.3, on the set $\{(t,x) \in (0,T) \times \mathbb{R} \mid x < \tilde{b}(t)\}$, the inequality $(\partial_t + \tilde{L} - r)\tilde{P} \le 0$ reads $\tilde{\varphi}_t(x) + \delta e^x - rK \le 0$. Equivalently, we have, for $t \in (0,T)$ and $x \in (0, b(t))$,

$$\varphi_t(x) + \delta x - rK \le 0.$$

Observe that for $0 < x < K$, we have

$$\lim_{t\to T} \varphi_t(x) = \int \big(xe^y - K\big)_+ \nu(dy) = \varphi_T(x).$$

Hence,

$$\forall x \in \big(0, b(T)\big), \quad \varphi_T(x) + \delta x - rK \le 0. \tag{4.5}$$

On the other hand, on the set $\{(t,x) \in (0,T) \times \mathbb{R} \mid x > \tilde{b}(t)\}$, we have

$$(\tilde{L} - r)\tilde{P} = -\partial_t \tilde{P} \ge 0.$$

Therefore, for $t \in (0, T)$, we have $(\tilde{L} - r)\tilde{P}(t, .) \geq 0$ on the interval $(\tilde{b}(t), +\infty)$ (see Remark 2.2). Note that $\lim_{t\to T}(\tilde{L} - r)\tilde{P}(t, .) = (\tilde{L} - r)\tilde{\psi}$ in the sense of distributions. Therefore, we have $(\tilde{L} - r)\tilde{\psi} \geq 0$ on the interval $(\tilde{b}(T), +\infty)$. It is easy to check that on the interval $(-\infty, \ln K)$, we have $(\tilde{L} - r)\tilde{\psi} = \bar{\varphi}_T - rK$, where $\bar{\varphi}_T(x) = \tilde{\varphi}_T(x) + \delta e^x$. Hence,

$$\forall x \in (b(T), +\infty) \cap (0, K), \quad \varphi_T(x) + \delta x \geq rK. \tag{4.6}$$

Note that the function φ defined by

$$\varphi(x) = \varphi_T(x) + \delta x$$

is nondecreasing, convex on $[0, K)$ and satisfies $\varphi(0) = 0$. Therefore, if $\varphi(x) > 0$ for some $x \in [0, K)$, then φ must be strictly increasing on $(0, K)$.

Now, if $\int (e^x - 1)_+ \nu(dx) \leq r - \delta$, then $\int (Ke^y - K)_+ \nu(dy) \leq rK - \delta K$, so that $\lim_{x\to K} \varphi(x) \leq rK$. It follows that $\varphi \leq rK$ on $[0, K)$, and since φ is strictly increasing on any interval where it is positive, we have $\varphi(x) < rK$ for all $x \in [0, K)$. We then deduce from (4.6) that $b(T) = K$.

Finally, suppose that $\int (e^x - 1)_+ \nu(dx) > r - \delta$. We then have $\lim_{x\to K} \varphi(x) > rK$, and since $\varphi(0) = 0$, the equation $\varphi(\xi) = rK$ has a solution in $(0, K)$, and this solution is unique since φ is strictly increasing on any interval where it is positive. We have $\varphi(x) < rK$ for $x < \xi$ and $\varphi(x) > rK$ for $x > \xi$. We deduce from (4.5) and (4.6) that $b(T) = \xi$. □

Remark 4.5 If $\nu([0, +\infty)) = 0$, we deduce from Theorem 4.4 that $b(T) = K \wedge \frac{rK}{\delta}$ as in the standard Black–Scholes model.

Remark 4.6 In the *CGMY* model, we remark from Theorem 4.4 that the critical price at maturity $b(T)$ does not depend on the parameter G. Furthermore, $b(T)$ can be smaller than the strike K in some cases. This is illustrated in Table 1. In fact, we observe that for typical values of the parameters taken from [4], the limit $b(T)$ is

Table 1 Critical price at maturity in *CGMY* model

C	M	Y	$b(T)$
65.65	46.98	−0.0719	49.9964
21.34	48.40	0.0037	49.9973
25.72	31.72	0.0931	49.9909
1.50	27.12	0.7836	49.9999
4.94	45.66	−0.7904	50.0000
10.52	108.06	0.7515	49.9976
280.11	102.53	0.1191	49.9995
25.72	3.64	−1.0037	33.4584
30.08	2.09	−5.0197	7.0499
100.72	2.64	−10.0037	0.4260

Parameter values are: $K = 50.00$, interest rate $r = 0.045$, and dividend rate $\delta = 0.0000$. The calculation errors are lower than 10^{-6}

very close to K. In order to get a significantly different value, we need to take M, Y very small and C large, in order to have many positive jumps.

Remark 4.7 In [8], the behavior of $b(T)$ as the interest rate r goes to 0 was investigated. We can extend these results to our setting. In order to emphasize the dependence with respect to r, we write $b_r(t) = b(t)$ (for $0 \le t \le T$). Suppose that $\nu([a, +\infty)) > 0$ for all $a \in \mathbb{R}^+$. For r small enough, the condition $\int (e^x - 1)_+ \nu(dx) > r - \delta$ is satisfied, so that $b_r(T)$ is the unique solution of the equation

$$\varphi(x) = rK.$$

Since $\nu([a, +\infty)) > 0$ for all $a \in \mathbb{R}^+$, we have $\varphi(x) > 0$ for all $x \in (0, K)$, and since φ is nondecreasing, we have $\lim_{r \to 0} b_r(T) = 0$.

Note added in proof After this paper was accepted for publication, we received a preprint by E. Bayraktar and H. Xing, entitled "Analysis of the Optimal Exercise Boundary of American Oprions for Jump Diffusions", which contains related results and mentions a related article by C. Yang, L. Jiang and B. Bian "Free boundary and American options in a jump-diffusion model", European Journal of Applied Mathematics, 17(2006):95–127.

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References

1. Bensoussan, A., Lions, J.-L.: Impulse Control and Quasi Variational Inequalities. Wiley, New York (1987)
2. Bertoin, J.: Lévy Processes. Cambridge University Press, Cambridge (1998), reprinted edition
3. Boyarchenko, S., Levendorskiĭ, S.Z.: Non-Gaussian Merton–Black–Scholes Theory. World Scientific, River Edge (2002)
4. Carr, P., Geman, H., Madan, D.B., Yor, M.: The fine structure of returns: an empirical investigation. J. Bus. **75**, 305–332 (2002)
5. Cont, R., Tankov, P.: Financial Modelling with Jump Processes. Chapman and Hall/CRC, London (2004)
6. El Karoui, N.: Les aspects probabilistes du contrôle stochastique. In: Lecture Notes in Mathematics, vol. 876, pp. 72–238. Springer, Berlin (1981)
7. El Karoui, N., Lepeltier, J.P., Millet, A.: A probabilistic approach to the reduite in optimal stopping. Probab. Math. Stat. **13**, 97–121 (1992)
8. Levendorskiĭ, S.Z.: Pricing of the American put under Lévy processes. Int. J. Theor. Appl. Finance **7**, 303–336 (2004)
9. Merton, R.: Option pricing when the underlying stock returns are discontinuous. J. Financ. Econ. **3**, 125–144 (1976)
10. Pham, H.: Applications des méthodes probabilistes et de contrôle stochastique aux mathématiques financières. Thèse de doctorat à l'Université Paris Dauphine (1995)
11. Pham, H.: Optimal stopping, free boundary and American option in a jump-diffusion model. Appl. Math. Optim. **35**, 145–164 (1997)
12. Pham, H.: Optimal stopping of controlled jump diffusion processes: a viscosity solution approach. J. Math. Syst. Estim. Control **8**, 1–27 (1998)

13. Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge (1999)
14. Zhang, X.L.: Options américaines et modèles de diffusion avec sauts. C. R. Acad. Sci. Paris, Sér. 1 **317**, 857–862 (1993)
15. Zhang, X.L.: Analyse numérique des options américaines dans un modèle de diffusion avec sauts. Thèse de doctorat à l'ENPC (1994)
16. Zhang, X.L.: Numerical analysis of American option pricing in a jump-diffusion model. Math. Oper. Res. **22**, 668–690 (1997)