Lecture notes on: Dynamic Asset Allocation

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PART 1: Consumption-portfolio choice

♦ Introduction to standard consumption-portfolio choice problem

- Merton (1971):
  - Diffusion models
  - Dynamic programming
- Cox & Huang (1989, 1991), Karatzas, Lehoczky & Shreve (1987):
  - Ito processes
  - Probabilistic methods

♦ Outline

- Dynamic choice problem
- Basic valuation principles
- Equivalent static choice problem
- Optimal policies
- Examples

1.1 Consumption-portfolio choice: the diffusion model

⧫ Underlying structure

- Finite horizon $[0, T]$
- Brownian motion W, d-dimensional
- Information: filtration generated by W: $\mathcal{F}_{(\cdot)} = \{\mathcal{F}_t : t \in [0, T]\}$
- Probability space $(\Omega, \mathcal{F}, P)$ - P is physical measure

⧫ Financial market

- Risky assets: d stocks. Price of stock i, $i = 1, ..., d$, satisfies

$$dS_{it} = S_{it}\left[\left(\mu_i(Y_t, t) - \delta_i(Y_t, t)\right)dt + \sigma_i(Y_t, t)dW_t\right] \tag{1}$$

  – μ_i expected return, δ_i dividend yield, σ_i volatility coefficients $(1 \times d)$
  – depend on $k \times 1$ vector of state variables $Y = (Y_1, ..., Y_k)'$
  – Satisfy integrability conditions
  – Matrix σ assumed invertible at all times (i.e. all risks are hedgeable)

- Riskless asset

  – Money market account: pays interest at rate $r(Y_t, t)$
  – r is positive and depends on state variables
  – Satisfies integrability condition

⧫ State variables: $Y = (Y_1, ..., Y_k)'$

- Any variable affecting return components

  – Interest rate, market prices of risk, dividend-price ratio, firm size, sales
  – Evolution

$$dY_t = \mu^Y(Y_t, t)dt + \sigma^Y(Y_t, t)dW_t \tag{2}$$

  – $\mu^Y(Y_t, t)$ is $k \times 1$ vector of drift coeff., $\sigma^Y(Y_t, t)$ is $k \times d$ volatility matrix
  – Lipschitz+Growth conditions: existence of unique strong solution

♦ Consumption, portfolios and wealth

- Investor consumes and invests in the different assets available
  - Wealth X. Consumption c.
  - Portfolio π: $d \times 1$ vector of wealth fractions in stocks
  - Fraction in riskless asset is $1 - \pi_t' 1$
- Evolution of wealth:

$$dX_t = (X_t r_t - c_t)\,dt + X_t \pi_t' \left[(\mu_t - r_t \mathbf{1})\,dt + \sigma_t dW_t\right] \tag{3}$$

- Initial condition $X_0 = x$: amount of capital at initial date
- Assume integrability conditions

♦ Preferences

- Time-separable von Neumann-Morgenstern representation
  - Consumption-bequest plan (c, X_T) ranked according to

$$\mathbf{E}\left[\int_0^T u(c_v, v)dv + U(X_T, T)\right] \tag{4}$$

  - Instantaneous utility function $u : R_+ \times [0, T] \to R$
  - Bequest (terminal utility) function $U : R_+ \to R$
  - Strictly increasing, strictly concave, differentiable over domains
  - Various behavioral assumptions can be embedded in this setting:
    * Here assume Inada condition at 0 and ∞
    * $\lim_{c\to 0} u'(c, t) = \lim_{X\to 0} U'(X, T) = \infty$
    * $\lim_{c\to\infty} u'(c, t) = \lim_{X\to\infty} U'(X, T) = 0$ hold for all $t \in [0, T]$

- Example: constant relative risk aversion (CRRA)

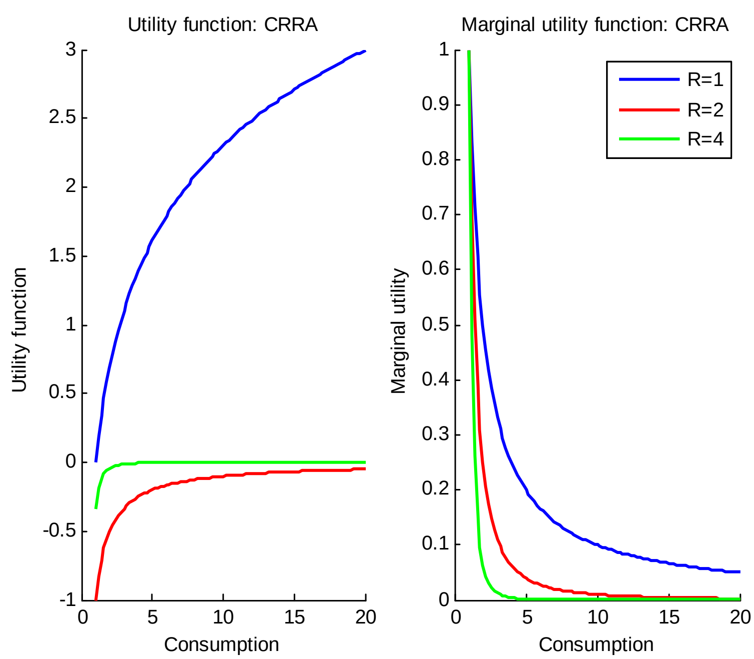
$$u(c,t) = a_t \begin{cases} \frac{1}{1-R} c^{1-R} & \text{for } R \neq 1, R > 0 \quad \text{Power utility} \\ \log(c) & \text{for } R = 1 \quad \text{Log utility} \end{cases}$$

  - a_t is subjective discount factor; assumed deterministic
  - Marginal utility

$$u'(c,t) = a_t \begin{cases} c^{-R} & \text{for } R \neq 1, R > 0 \quad \text{Power utility} \\ c^{-1} & \text{for } R = 1 \quad \text{Log utility} \end{cases}$$

  - Relative risk aversion

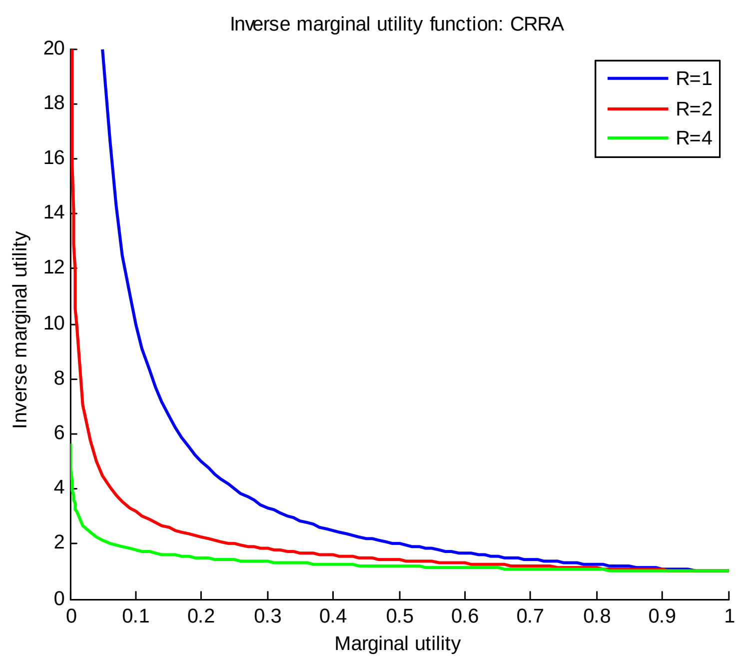
$$R(c) = -\frac{u''(c,t)\, c}{u'(c,t)} = \begin{cases} R & \text{for } R \neq 1, R > 0 \quad \text{Power utility} \\ 1 & \text{for } R = 1 \quad \text{Log utility} \end{cases}$$



- Under assumptions above inverse marginal utility functions exist and unique:
  - $I^u : R_+ \times [0,T] \to R_+$ solves $u'(I^u(y,t),t) = y$
  - $I^U : R_+ \to R_+$ solves $U'(I^U(y,T),T) = y$
  - Strictly decreasing
  - $\lim_{y\to 0} I^u(y,t) = \lim_{y\to 0} I^U(y,T) = \infty$ and $\lim_{y\to\infty} I^u(y,t) = \lim_{y\to\infty} I^U(y,T) = 0$

- Example: CRRA
  - Inverse marginal utility

$$I(y,t) = \begin{cases} \left(\frac{y}{a_t}\right)^{-1/R} & \text{for } R \neq 1, R > 0 \quad \text{Power utility} \\ \left(\frac{y}{a_t}\right)^{-1} & \text{for } R = 1 \quad \text{Log utility} \end{cases}$$



♦ Dynamic consumption-portfolio choice problem

$$\max_{(c,\pi,X_T)} E\left[\int_0^T u(c_v,v)dv + U(X_T,T)\right] \tag{5}$$

$$s.t. \qquad \begin{cases} dX_t = (r_tX_t - c_t)\,dt + X_t\pi_t'\left[(\mu_t - r_t\mathbf{1})\,dt + \sigma_t dW_t\right];\ X_0 = x \\ c_t \geq 0, t \in [0,T], \quad \text{and} \quad X_T \geq 0 \\ X_t \geq 0, , t \in [0,T] \end{cases}$$

- First eq. describes evolution of wealth given policy (c,π)
- Second captures physical restriction that consumption cannot be negative
- Last constraint is no-bankruptcy condition: wealth cannot be negative
- Optimization over consumption, terminal wealth (bequest) and portfolios

1.2 Valuation principles

♦ State prices

- Market price of risk:
  - $\theta_t \equiv \sigma_t^{-1}(\mu_t - r_t 1)$ where $1 = (1, ..., 1)'$ is d-dimensional vector
  - Premia per unit risk (price of Brownian motions) - Sharpe ratios
- State price density (SPD)

$$\xi_v \equiv \exp\left(-\int_0^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) ds - \int_0^v \theta_s' dW_s\right), v \in [0, T]$$

  - Stochastic discount factor for valuation at 0 of cash flows received at v
  - Marginal cost of consumption at time v
- Conditional state price density (CSPD)

$$\xi_{t,v} \equiv \exp\left(-\int_t^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) ds - \int_t^v \theta_s' dW_s\right) = \frac{\xi_v}{\xi_t}, v \in [t, T]$$

  - Stochastic discount factor for valuation at t of cash flows received at v

♦ Valuation

- Stocks

$$S_t = E_t \left[\int_t^T \xi_{t,v} D_v dv + \xi_{t,T} S_T \right]$$

  - Stock price is present value of future dividends
  - Dividends are discounted using risk-adjusted rates (implicit in θ)

- Contingent claim with payoff (f, F)

$$V_t = E_t \left[\int_t^T \xi_{t,s} f_v dv + \xi_{t,T} F_T \right]$$

  - Price of claim is present value of future cash flows
  - Cash flows discounted at same risk-adjusted rate

- Price behavior

  - Discounted cum-dividend prices are P-martingales

$$\xi_t S_t + \int_0^t \xi_v D_v dv = E_t \left[\int_0^T \xi_v D_v dv + \xi_T S_T \right]$$

  - Discounted ex-dividend prices are P-supermartingales (assuming $D > 0$)

$$\xi_t S_t = E_t \left[\int_t^T \xi_v D_v dv + \xi_T S_T \right] \geq E_t \left[\xi_T S_T \right]$$

1.3 Static consumption choice problem

♦ Static budget constraint

- Consumption plan (c, X_T) is budget feasible at x iff

$$E\left[\int_0^T \xi_v c_v dv + \xi_T X\right] \leq x. \tag{6}$$

- Budget set is set of consumption-bequest plans satisfying (6)
- Constraint (6) is static budget constraint:
  - Constraint on resource allocation, at zero, for all future times, states
  - Does not specify manner in which resources transferred over time
  - Market completeness ensures required transfers can be made

♦ Static consumption-portfolio choice problem

$$\max_{(c,X_T)} \mathbf{E}\left[\int_0^T u(c_v, v)dv + U(X_T, T)\right] \tag{7}$$

$$s.t. \quad \begin{cases} E\left[\int_0^T \xi_v c_v dv + \xi_T X\right] \leq x \\ c_t \geq 0, t \in [0,T] \quad \text{and} \quad X_T \geq 0. \end{cases} \tag{8}$$

- First constraint: static budget constraint
- Second: captures same physical restrictions as in dynamic problem
- Maximization is over consumption-bequest policies (c, X_T)

♦ Theorem 1.1: (Cox-Huang (1989, 1991) and Karatzas-Lehoczky-Shreve (1987))

- Suppose (c, π, X_T) solves dynamic consumption-portfolio choice problem. Then, (c, X_T) solves static problem
- Conversely, suppose (c, X_T) is a solution to the static problem. Then there exists a portfolio π such that (c, π, X_T) solves dynamic problem

♦ Remarks:

- Portfolio π financing (c, X_T) leads to wealth process

$$\xi_t X_t = x + E_t\left[\int_t^T \xi_v c_v dv + \xi_T X_T\right] - E\left[\int_0^T \xi_v c_v dv + \xi_T X_T\right]$$

- Assume cons.-bequest policy saturates budget: $E\left[\int_0^T \xi_v c_v dv + \xi_T X_T\right] = x$
  - Then wealth finances exactly PV future consumption at all times

$$\xi_t X_t = E_t\left[\int_t^T \xi_v c_v dv + \xi_T X_T\right] \equiv \xi_t V_t$$

    * Wealth is present value of future consumption
    * In particular $X_T = V_T$
  - Otherwise resources are left over after financing consumption

$$\xi_t X_t = \xi_t V_t + \left(x - E\left[\int_0^T \xi_v c_v dv + \xi_T X_T\right]\right)$$

- Optimal portfolio
  - If (c, X_T) solves static problem, optimal portfolio is $X\pi'\sigma = \xi^{-1}\phi' + X\theta'$ where ϕ is square integrable process representing martingale

$$M_t \equiv E_t\left[\int_0^T \xi_v c_v dv + \xi_T X_T\right] - E\left[\int_0^T \xi_v c_v dv + \xi_T X_T\right] = \int_0^t \phi_v' dW_v.$$

  - Martingale representation theorem shows existence of ϕ and π
  - Formula not very explicit. Structure of portfolio?

1.4 Optimal consumption-bequest policies

♦ Optimality conditions

- Complete market:
  - Every state contingent allocation can be attained by some port.
  - Investor free to select consumption state by state
  - No need to worry about means of transferring wealth across states-time
- State by state optimization: compare marginal cost and benefits
  - Marginal benefit of consumption at t is marginal utility $u'(c,t)$
  - Marginal benefit of bequest is $U'(X_T, T)$
  - Marginal cost of consumption at $t \in [0,T]$ is SPD
- First order conditions are

$$u'(c,t) = y\xi_t \tag{9}$$

$$U'(X_T,T) = y\xi_T \tag{10}$$

$$E\left[\int_0^T \xi_v c_v dv + \xi_T X_T\right] \leq x \tag{11}$$

♦ Theorem 1.2: Consumption-bequest policy (c^*, X_T^*) is optimal for the static problem (hence the dynamic problem), if and only if there exists a constant $y^* > 0$ such that (c^*, X_T^*, y^*) solves (9)-(11)

♦ Theorem 1.3:

- Optimal consumption and bequest policies

$$c_t^* = I^u\left(y^*\xi_t, t\right), t \in [0,T], \qquad X_T^* = I^U\left(y^*\xi_T, T\right)$$

  – where y^* is unique solution of non-linear equation

$$x = E\left[\int_0^T \xi_t I^u\left(y^*\xi_t, t\right)dt + \xi_T I^U\left(y^*\xi_T, T\right)\right].$$

- Optimal portfolio

$$X_t^*\pi_t^* = X_t^*\left(\sigma_t'\right)^{-1}\theta_t + \xi_t^{-1}\left(\sigma_t'\right)^{-1}\phi_t^*, t \in [0,T]$$

  – ϕ^* is d-dimensional, square-integrable and progressively meas. process
  – uniquely represents P-martingale

$$M_t = E_t\left[\int_0^T \xi_t c_t^* dt + \xi_T X_T^*\right] - E\left[\int_0^T \xi_t c_t^* dt + \xi_T X_T^*\right].$$

- Optimal wealth process

$$X_t^* = E_t\left[\int_t^T \xi_{t,v} c_v^* dt + \xi_{t,T} X_T^*\right], t \in [0,T]$$

- Value function

$$J_t^* = E_t\left[\int_t^T u\left(I^u\left(y^*\xi_t, t\right), t\right)dt + U\left(I^U\left(y^*\xi_T, T\right), T\right)\right], t \in [0,T]$$

1.5 Examples

♦ Examples: constant relative risk aversion

- $u(c,t) = a_t v_c(c)$, $U(X,T) = a_T v_x(X)$
- $a_t, t \in [0,T]$ is deterministic process with initial value $a_0 = 1$

♦ Example 1: Logarithmic utility, bequest functions (unit relative risk aversion)

- $v_c(e) = v_x(e) = \log(e)$
- Optimal consumption, bequest, wealth and value function J^* are

$$\begin{cases} c_t^* = \left(\frac{y^*\xi_t}{a_t}\right)^{-1} \qquad X_T = \left(\frac{y^*\xi_T}{a_T}\right)^{-1} \\ X_t^* = \left(\frac{y^*\xi_t}{a_t}\right)^{-1} m_t^{-1} \\ J_t^* = -\log\left(\frac{y^*\xi_t}{a_t}\right) a_t m_t^{-1} - E_t\left[\int_t^T a_v \log\left(\frac{\xi_{t,v}}{a_{t,v}}\right) dv + a_T \log\left(\frac{\xi_{t,T}}{a_{t,T}}\right)\right] \end{cases}$$

where

$$y^* = x^{-1} E\left[\int_0^T a_v dt + a_T\right]$$

$$m_t = \left(E_t\left[\int_t^T a_{t,v} dt + a_{t,T}\right]\right)^{-1}$$

- Alternatively

$$c_t^* = m_t X_t^*$$

$$J_t^* = (\log(m_t) + \log(X_t^*))\, a_t m_t^{-1} - E_t\left[\int_t^T a_v \log\left(\frac{\xi_{t,v}}{a_{t,v}}\right) dv + a_T \log\left(\frac{\xi_{t,T}}{a_{t,T}}\right)\right]$$

  - m_t is marginal propensity to consume out of wealth

♦ Construction:

- Inverse marginal functions: $I^u(y,t) = (y/a_t)^{-1}$ and $I^U(y,t) = (y/a_T)^{-1}$
- Candidate consumption-bequest functions:

$$c_v = I^u(y\xi_v, v) = \left(\frac{y\xi_v}{a_v}\right)^{-1} \qquad X_T = I^U(y\xi_T, T) = \left(\frac{y\xi_T}{a_T}\right)^{-1}.$$

- Budget constraint multiplier

$$\begin{aligned}
\Psi(y) &= E\left[\int_0^T \xi_v I^u(y\xi_v, v)dt + \xi_T I^U(y\xi_T, T)\right] \\
&= E\left[\int_0^T \xi_v \left(\frac{y\xi_v}{a_v}\right)^{-1} dt + \xi_T \left(\frac{y\xi_T}{a_T}\right)^{-1}\right] \\
&= y^{-1} E\left[\int_0^T a_v dt + a_T\right]
\end{aligned}$$

so that

$$(y^*)^{-1} = \frac{x}{E\left[\int_0^T a_v dt + a_T\right]}.$$

- Demand functions

$$c_v^* = I^u(y^*\xi_v, v) = \left(\frac{y^*\xi_v}{a_v}\right)^{-1} = \frac{x}{E\left[\int_0^T a_v dt + a_T\right]} \left(\frac{\xi_v}{a_v}\right)^{-1}$$

$$X_T^* = I^U(y^*\xi_T, T) = \left(\frac{y^*\xi_T}{a_T}\right)^{-1} = \frac{x}{E\left[\int_0^T a_v dt + a_T\right]} \left(\frac{\xi_T}{a_T}\right)^{-1}.$$

- Optimal wealth

$$\begin{aligned}
X_t^* &= E_t\left[\int_t^T \xi_{t,v} \left(\frac{y^*\xi_v}{a_v}\right)^{-1} dv + \xi_{t,T} \left(\frac{y^*\xi_T}{a_T}\right)^{-1}\right] \\
&= \left(\frac{y^*\xi_t}{a_t}\right)^{-1} E_t\left[\int_t^T \xi_{t,v} \left(\frac{\xi_{t,v}}{a_{t,v}}\right)^{-1} dv + \xi_{t,T} \left(\frac{\xi_{t,T}}{a_{t,T}}\right)^{-1}\right] \\
&= \left(\frac{y^*\xi_t}{a_t}\right)^{-1} E_t\left[\int_t^T a_{t,v} dv + a_{t,T}\right] \equiv \left(\frac{y^*\xi_t}{a_t}\right)^{-1} m_t^{-1}
\end{aligned}$$

- Feedback policies
  - Inverting wealth

$$\left(\frac{y^*\xi_t}{a_t}\right)^{-1} = m_t X_t^*$$

  - Optimal policies

$$c_t^* = \left(\frac{y^*\xi_t}{a_t}\right)^{-1} = m_t X_t^*$$

♦ Remarks:

- Consumption proportional to wealth
- Marginal propensity to consume does not depend on market coefficients (r, θ)
- Lifecycle behavior:
  - Marginal propensity to consume explodes as $t \to T$ if no bequest motive
  - Want to exhaust all resources as horizon approaches

♦ Example 2: Power utility, bequest functions (constant relative risk aversion)

- $v_c(e) = v_x(e) = (1-R)^{-1} e^{1-R},\ R > 0$
- Optimal policies

$$\begin{cases} c_t^* = \left(\frac{y^* \xi_t}{a_t}\right)^{-1/R} \text{ and } X_T = \left(\frac{y^* \xi_T}{a_T}\right)^{-1/R} \\ X_t^* = \left(\frac{y^* \xi_t}{a_t}\right)^{-1/R} m_t^{-1} \\ J_t^* = \frac{1}{1-R}\left(\frac{y^* \xi_t}{a_t}\right)^{\rho} a_t E_t\left[\int_t^T a_{t,v}^{1/R} \xi_{t,v}^{\rho} dv + a_{t,T}^{1/R} \xi_{t,T}^{\rho}\right] \end{cases}$$

where

$$y^* = x^{-R}\left(E\left[\int_0^T \xi_v^{\rho} a_v^{1/R} dv + \xi_T^{\rho} a_T^{1/R}\right]\right)^R$$

$$m_t = \left(E_t\left[\int_t^T a_{t,v}^{1/R} \xi_{t,v}^{\rho} dv + a_{t,T}^{1/R} \xi_{t,T}^{\rho}\right]\right)^{-1}$$

- Feedback form

$$c_t = m_t X_t$$

$$J_t^* = \frac{1}{1-R} X_t^{*1-R} a_t m_t^{-R}$$

.

♦ Consumption behavior:

- Consumption linear in wealth
- Market structure matters: dependence on (r, θ)
- Lifecycle behavior:
  - Horizon behavior similar to log utility
  - But dependence on state

- Assume constant coefficients β, r, θ

$$E_t\left[\xi_{t,v}^{\rho}\right] = \exp\left(-\left(\rho r + \frac{1}{2}\rho\left(1-\rho\right)\theta'\theta\right)(v-t)\right)$$

$$a_{t,v}^{1/R} E_t\left[\xi_{t,v}^{\rho}\right] = \exp\left(-\left(\frac{1}{R}\beta + \rho r + \frac{1}{2}\rho\left(1-\rho\right)\theta'\theta\right)(v-t)\right) \equiv \exp\left(-K\left(v-t\right)\right)$$

$$m_t = \left(E_t\left[\int_t^T a_{t,v}^{1/R}\xi_{t,v}^{\rho}dv + a_{t,T}^{1/R}\xi_{t,T}^{\rho}\right]\right)^{-1} = \left(\frac{1}{K}\left(1-\exp\left(-K\left(T-t\right)\right)\right) + \exp\left(-K\left(T-t\right)\right)\right)^{-1}$$

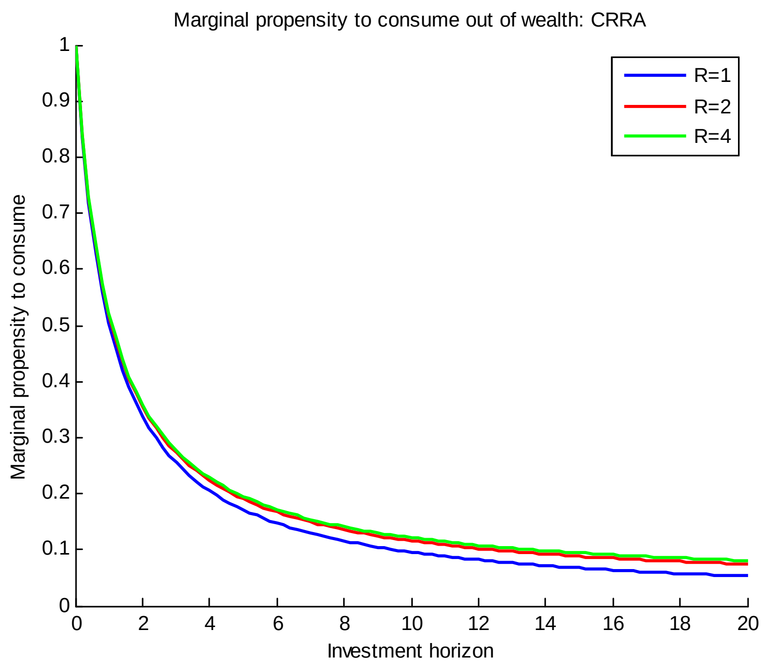


Figure 1: Marginal propensity to consume (CRRA). Parameter values: $\beta = 0.01$, $r = 0.06$, $\theta = 0.30$

1.6 Some extensions

♦ Failure of Inada condition at zero

- $u'(0,t) < \infty$ at $c = 0$
- Example: HARA
  - $u(c,t) = \frac{1}{1-R}(c+A)^{1-R}$ with $A \geq 0$
  - $u'(c,t) = (c+A)^{-R}$ so that $u'(0,t) = A^{-R}$
  - $u''(c,t) = -R(c+A)^{-R-1}$
  - $R(c) = -\frac{u''(c,t)c}{u''(c,t)} = \frac{R(c+A)^{-R-1}c}{(c+A)^{-R}} = R\frac{c}{c+A}$

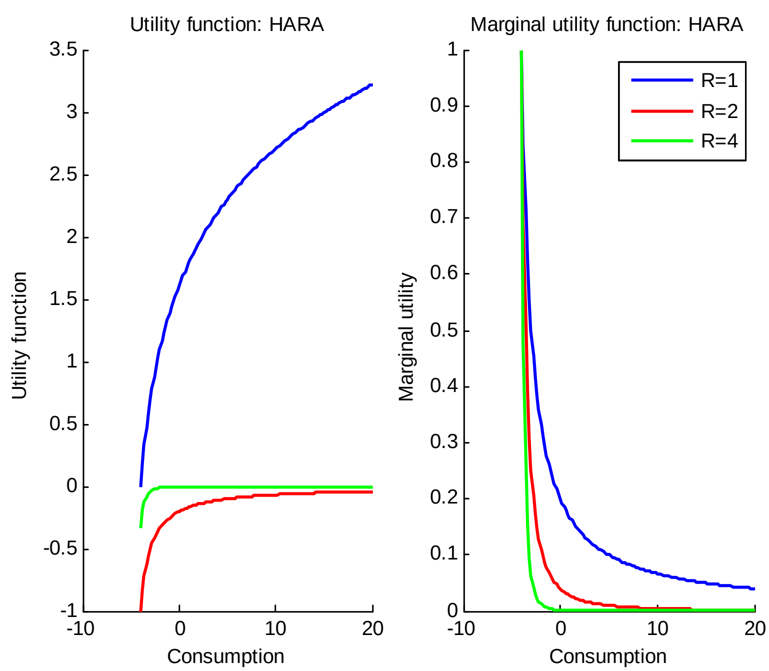
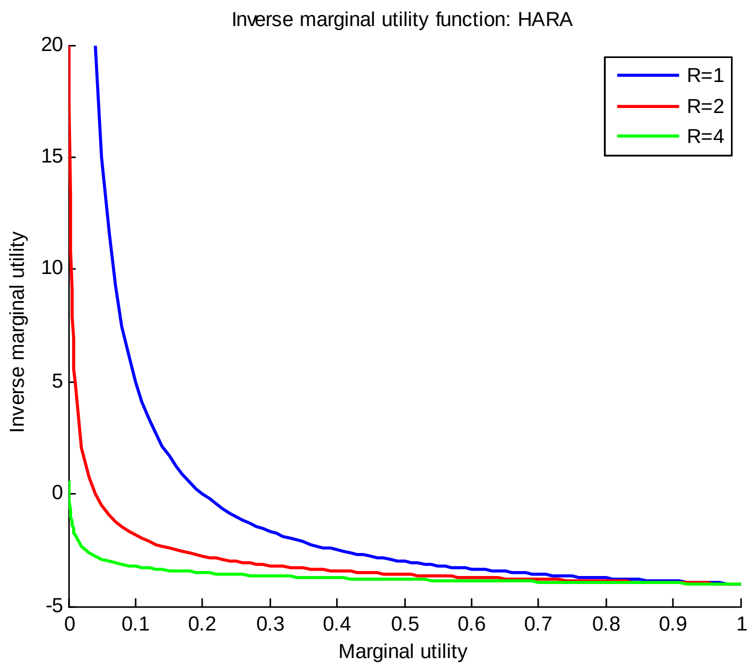


Figure 2: Utility and marginal utility for HARA with $A = 5$.

- Inverse marginal utility: $I^u(y) = y^{-1/R} - A$



- Optimal policy: $c_t = \max\{I^u(y^*\xi_t, t), 0\} = \max\left\{(y^*\xi_t)^{-1/R} - A, 0\right\}$

♦ Subsistence consumption, intolerance to shortfalls

- utility function

$$u(c,t) = \begin{cases} u(c-s,t) & \text{for } c \geq s \\ -\infty & \text{for } c < s \end{cases}$$

  - $s > 0$
  - $u'(0,t) = \infty$

- Example: HARA
  - $u(c-s,t) = \frac{1}{1-R}(c-s)^{1-R}$ for $c \geq s$
  - $u'(c-s,t) = (c-s)^{-R}$ so that $u'(0,t) = \infty$
  - $u''(c-s,t) = -R(c-s)^{-R-1}$
  - $R(c) = -\frac{u''(c-s,t)c}{u''(c-s,t)} = \frac{R(c-s)^{-R-1}c}{(c-s)^{-R}} = R\frac{c}{c-s}$

- Optimal policy: $c_t = I^u(y^*\xi_t, t) + s$

♦ Loss aversion and threshold effects

- Discontinuous derivative at some critical point(s)
- Asymmetric behavrior above and below threshold

PART 2: Introduction to Malliavin calculus

♦ Malliavin calculus is a calculus of variations for stochastic processes

- Applies to Brownian functionals: random variables and stochastic processes that depend on trajectories of Brownian motion
- Malliavin derivative measures impact of small change in trajectory of Brownian motion on value of Brownian functional
- Development of theory:
  - Malliavin, Stroock, Bismut,...
  - Existence and smoothness of densities
  - Reference: Nualart (1995)

♦ Outline

- Definition
- Riemann, Wiener and Ito integrals
- Clark-Ocone formula
- Chain rule
- Stochastic differential equations

2.1 Definition

♦ Smooth Brownian functionals

- Space of (smooth) functions: $C_p^{\infty}\left(R^{nd}\right)$
  - $f(\cdot): R^{nd} \rightarrow R$
  - Infinitely differentiable
  - Polynomial growth
- Wiener space generated by d-dimensional Brownian motion $W = (W_1, ..., W_d)'$
  - Each state of nature corresponds to a trajectory of BM
  - Set of states is space of trajectories
- Let $(t_1, ..., t_n)$ be a partition of $[0, T]$
  - Sample BM at points of this partition: $(W_{t_1}, ..., W_{t_n})$
  - Construct random variable

$$F(W) \equiv f\left(W_{t_1}, ..., W_{t_n}\right)$$

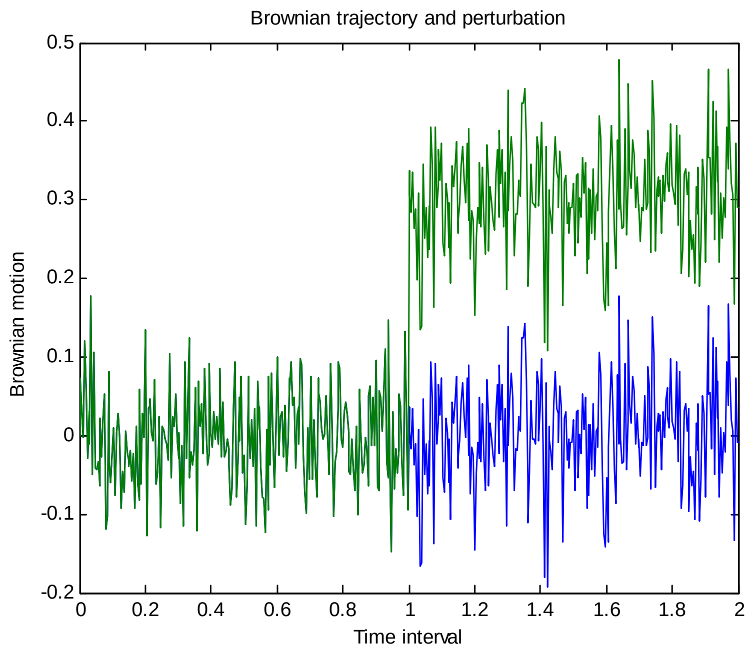
  - $f \in C_p^{\infty}\left(R^{nd}\right)$
  - F is smooth Brownian functional

♦ Examples: assume W is one-dimensional

- Quadratic function: W_T^2, $\sum_{j=1}^{n} W_{t_j}^2$
- Any polynomial: $\sum_{k=1}^{K} a_k W_T^k$, $\sum_{j=1}^{n}\left(\sum_{k=1}^{K} a_k W_{t_j}^k\right)$
- Stock price in Black-Scholes model: (limit of sequence of SBF)
  - $S_T = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right)$
  - Write $S_T = f(W_T)$ with $f(x) = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma x\right)$
  - S_T is (limit of) smooth Brownian functional (sampled at one point)

♦ Experiment:

- Perturbate trajectoy of BM from some time t onward
- Shift W by ε starting at t, where $t_k \leq t < t_{k+1}$ for some $k = 1, ..., d$



♦ Malliavin derivative of smooth Brownian functional (assume $d = 1$)

- MD at t of F is change in F due to a change in path of W starting at t
- MD of F at t is defined by

$$\mathcal{D}_t F(W) \equiv \left. \frac{\partial f \left(W_{t_1} + \varepsilon \mathbf{1}_{[t,\infty[}(t_1), ..., W_{t_n} + \varepsilon \mathbf{1}_{[t,\infty[}(t_n) \right)}{\partial \varepsilon} \right|_{\varepsilon=0} \tag{12}$$

$$= \lim_{\varepsilon \to 0} \frac{F(W + \varepsilon \mathbf{1}_{[t,\infty[}) - F(W)}{\varepsilon} \tag{13}$$

  – where $1_{[t,\infty[}$ is indicator of $[t, \infty)$ (i.e., $1_{[t,\infty[}(s) = 1$ for $s \in [t, \infty)$; 0 otherwise)
  – Compact notation

$$\mathcal{D}_t F(W) = \sum_{j=1}^{n} \partial_j f \left(W_{t_1}, ..., W_{t_k}, ..., W_{t_n} \right) \mathbf{1}_{[t,\infty[}(t_j) \tag{14}$$

  where $\partial_j f$ is derivative of f with respect to j^{th} argument of f

- MD of F is $\mathcal{D}F(W) = \{\mathcal{D}_t F(W) : t \in [0, T]\}$

♦ Example: Black-Scholes model

- Recall $S_T = f(W_T)$ with $f(x) = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma x\right)$
- Direct application of definition gives

$$
\begin{aligned}
\mathcal{D}_t S_T &= \partial f(W_T)\mathbf{1}_{[t,\infty[}(T) \\
&= \sigma S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma W_T\right)\mathbf{1}_{[t,\infty[}(T) = \sigma S_T \mathbf{1}_{[t,\infty[}(T)
\end{aligned}
$$

- Malliavin derivative is derivative with respect to W_T:
  - Perturbation of path of W from t onward affects S_T only through W_T
- Malliavin derivative at t of S_v

$$\mathcal{D}_t S_v = \sigma S_v \mathbf{1}_{[t,\infty[}(v)$$

♦ Multidimensional case: $d > 1$

- MD of F at t is now $1 \times d$-dimensional vector $\mathcal{D}_t F = (\mathcal{D}_{1t} F, ..., \mathcal{D}_{dt} F)$
- i^{th} coordinate $\mathcal{D}_{it} F$ measures impact of perturbation in W_i by ε starting at t
- If $t_k \leq t < t_{k+1}$ can write one-dimensional definition for this derivative

$$\mathcal{D}_{it} F = \sum_{j=k}^{n} \frac{\partial f}{\partial x_{ij}} (W_{t_1}, ..., W_{t_k}, ..., W_{t_n}) \, \mathbf{1}_{[t,\infty[}(t_j) \tag{15}$$

  - where $\partial f / \partial x_{ij}$ is derivative with respect to i^{th} component of j^{th} argument of f (i.e. derivative with respect to W_{it_j})

- MD of F is $\mathcal{D}F(W) = \{\mathcal{D}_t F(W) : t \in [0, T]\}$; d-dimensional (row) stoch. proc.

♦ Domain of Malliavin derivative operator

- MD exists for $F \in D^{1,2}$
- Completion of set of smooth Brownian functionals in norm

$$\| F \|_{1,2} = \left(E(F^2) + \mathbf{E} \left(\int_0^T \|\mathcal{D}_t F\|^2 \, dt \right) \right)^{\frac{1}{2}}$$

where $\|\mathcal{D}_t F\|^2 = \sum_i (\mathcal{D}_{it} F)^2$.

2.2 Malliavin derivatives of Riemann, Wiener, Ito integrals

♦ Wiener integral $F(W) = \int_0^T h(t)dW_t$, where $h(t)$ is fct of time and W is one-dim.

- Integration by parts: $F(W) = h(T)W_T - \int_0^T W_s dh(s)$
- Application of definition gives

$$\begin{aligned} F(W + \varepsilon \mathbf{1}_{[t,\infty[}) - F(W) &= h(T)\left(W_T + \varepsilon \mathbf{1}_{[t,\infty[}(T)\right) - \int_0^T \left(W_s + \varepsilon \mathbf{1}_{[t,\infty[}(s)\right) dh(s) \\ &\quad - \left(h(T)W_T - \int_0^T W_s dh(s)\right) \\ &= h(T)\varepsilon \mathbf{1}_{[t,\infty[}(T) - \int_0^T \varepsilon \mathbf{1}_{[t,\infty[}(s) dh(s) \\ &= \varepsilon \left(h(T) - \int_0^T \mathbf{1}_{[t,\infty[}(s) dh(s)\right) \\ &= \varepsilon \left(h(T) - \int_t^T dh(s)\right) \\ &= \varepsilon h(t). \end{aligned}$$

so that

$$\mathcal{D}_t F(W) = \lim_{\varepsilon \to 0} \frac{F(W + \varepsilon \mathbf{1}_{[t,\infty[}) - F(W)}{\varepsilon} = h(t) \tag{16}$$

- Conclusion: $D_t F = h(t)$
  - MD of F at t is volatility $h(t)$ of stochastic integral at t
  - Measures sensitivity of random variable F to Brownian shock at t

♦ Random Riemann integral with integrand depending on path of BM

- $F(W) \equiv \int_0^T h_s ds$ where h_s progressively measurable
- MD

$$\begin{aligned}\mathcal{D}_t F &= \lim_{\varepsilon\to 0} \frac{F(W+\varepsilon \mathbf{1}_{[t,\infty[}) - F(W)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \int_0^T \left(\frac{h_s(W+\varepsilon \mathbf{1}[t,\infty[) - h_s(W)}{\varepsilon} \right) ds = \int_t^T \mathcal{D}_t h_s ds\end{aligned}$$

♦ Ito integral

- $F(W) = \int_0^T h_s(W) dW_s$
- MD

$$\mathcal{D}_t F = h_t + \int_t^T \mathcal{D}_t h_s dW_s$$

♦ Malliavin derivatives of Wiener, Riemann, Ito integrals depending on multi-dimensional BM defined in same way (component by component)

2.3 Clark-Ocone formula

♦ Clark-Ocone formula:

- Any random variable $F \in D^{1,2}$ can be decomposed as

$$F = E[F] + \int_0^T E_t [\mathcal{D}_t F] \, dW_t \tag{17}$$

- Martingale closed by $F \in D^{1,2}$ (i.e. $M_t = E_t [F]$):
  - Take conditional expectations
  - $M_t = E[F] + \int_0^t E_s [\mathcal{D}_s F] \, dW_s$

♦ Remark

- Results can be used to show MD and conditional expectation commute
- For martingale $M_v = E_v [F]$ Malliavin derivative is $\mathcal{D}_t M_v = E_v [\mathcal{D}_t F]$
- Equivalently, $\mathcal{D}_t E_v [F] = E_v [\mathcal{D}_t F]$

2.4 Chain rule of Malliavin calculus

♦ In applications often need MD of function of path-dependent random variable

- Chain rule also applies in Malliavin calculus

♦ Let $G = g(F)$ where

- $F = (F_1, ..., F_n)$ is vector of random variables in $D^{1,2}$
- g is a differentiable function of F with bounded derivatives
- Malliavin derivative of $G = g(F)$ is

$$\mathcal{D}_t G = \mathcal{D}_t g(F) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(F) \mathcal{D}_t F_i$$

where $\frac{\partial g}{\partial x_i}(F)$ is derivative relative to the i^{th} argument of ϕ.

2.5 Stochastic differential equations

♦ Suppose state variable Y_t follows diffusion process

- $dY_t = \mu^Y(Y_t)dt + \sigma^Y(Y_t)dW_t$ where Y_0 given
  - Assume W one dimensional
  - Integral form

$$Y_t = Y_0 + \int_0^t \mu^Y(Y_s)ds + \int_0^t \sigma^Y(Y_s)dW_s.$$

- Taking Malliavin derivative on each side gives, for $s \geq t$,

$$\begin{aligned}\mathcal{D}_tY_s &= D_tY_0 + \int_t^s \partial\mu^Y \mathcal{D}_tY_v dv + \int_t^s \partial\sigma^Y \mathcal{D}_tY_v dW_v + \sigma(Y_t) \\ &= \int_t^s \partial\mu^Y \mathcal{D}_tY_v dv + \int_t^s \partial\sigma^Y \mathcal{D}_tY_v dW_v + \sigma(Y_t)\end{aligned}$$

  where second equality follows from $\mathcal{D}_tY_0 = 0$

- Conclusion: MD follows linear SDE

$$d(\mathcal{D}_tY_s) = \left[\partial\mu^Y(Y_s)ds + \partial\sigma^Y(Y_s)dW_s\right](\mathcal{D}_tY_s) \tag{18}$$

  subject to initial condition $\lim_{s\to t}\mathcal{D}_tY_s = \sigma^Y(Y_t)$

- Solution

$$\mathcal{D}_tY_s = \mathcal{D}_tY_t \times \exp\left(\int_t^s \left(\partial\mu^Y(Y_v) - \frac{1}{2}\left(\partial\sigma^Y(Y_v)\right)^2\right)dv + \int_t^s \partial\sigma^Y(Y_v)dW_v\right)$$

♦ Multidimensional case:

- If $\sigma^Y(Y_t)$ is $1 \times d$ vector (W is d-dimensional BM) same arguments apply
- Obtain (18) subject to initial condition $\lim_{s\to t}\mathcal{D}_tY_s = \sigma(Y_t)$
  - $\partial\sigma^Y(Y_s) \equiv (\partial\sigma_1^Y(Y_s), ..., \partial\sigma_d^Y(Y_s))$ is row vect: deriv. of components of $\sigma^Y(Y_s)$
- MD $\mathcal{D}_tY_s$ is $1 \times d$ row vector $\mathcal{D}_tY_s = (\mathcal{D}_{1t}Y_s, ..., \mathcal{D}_{dt}Y_s)$

PART 3: Optimal portfolios

♦ Determination of optimal portfolio (financing the consumption-bequest policy)

- Ocone and Karatzas (1991): Clark-Ocone formula
- Detemple, Garcia, Rindisbacher (2003): diffusion models - implementation

♦ Outline:

- Optimal portfolio formula
- Special cases and examples
- Implementation
- Example

3.1 The portfolio formula

♦ Summary:

- Optimal portfolio uniquely given by

$$X_t^* \pi_t^* = X_t^* \left(\sigma_t'\right)^{-1} \theta_t + \xi_t^{-1} \left(\sigma_t'\right)^{-1} \phi_t^* \tag{19}$$

  - ϕ^* is d-dimensional process representing martingale

$$M_t \equiv E_t \left[F_T^*\right] - E \left[F_T^*\right]$$
$$F_T^* \equiv \int_0^T \xi_t c_t^* dt + \xi_T X_T^*$$

  - (c^*, X_T^*) as given in Theorem 1.3

- For explicit formula it suffices to identify ϕ^* in terms of primitives (r, θ, u, U, T)
- Malliavin calculus is instrumental: Clark-Ocone formula

♦ Derivation:

- Assume $F_T^* \in D^{1,2}$
- Clark-Ocone formula gives

$$\phi_t^* = E_t \left[(\mathcal{D}_t F_T^*)' \right] \tag{20}$$

- Using rules of Malliavin calculus,

$$\begin{aligned} \mathcal{D}_t F_T^* &= \mathcal{D}_t \left(\int_0^T \xi_v I^u(y^* \xi_v, v) dt + \xi_T I^U(y^* \xi_T, T) \right) \\ &= \int_t^T \mathcal{D}_t \left(\xi_v I^u(y^* \xi_v, v) \right) dt + \mathcal{D}_t \left(\xi_T I^U(y^* \xi_T, T) \right) \\ &= \int_t^T \left(I^u(y^* \xi_v, v) + y^* \xi_v \partial_y I^u(y^* \xi_v, v) \right) \mathcal{D}_t \xi_v dv \\ &\quad + \left(I^U(y^* \xi_T, T) + y^* \xi_T \partial_y I^U(y^* \xi_T, T) \right) \mathcal{D}_t \xi_T \\ &\equiv \int_t^T Z^u(y^* \xi_v, v) \mathcal{D}_t \xi_v dv + Z^U(y^* \xi_T, T) \mathcal{D}_t \xi_T \end{aligned} \tag{21}$$

  where $\partial_y I^u(y^* \xi_v, v), \partial_y I^U(y^* \xi_T, T)$ are derivatives of $I^u(y^* \xi_v, v), I^U(y^* \xi_T, T)$ with respect to first argument

- MD of SPD: for all $v \geq t$

$$\begin{aligned}
\mathcal{D}_t\xi_v &= \mathcal{D}_t \exp\left(-\int_0^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) ds - \int_0^v \theta_s' dW_s\right) && \text{definition of SPD} \\
&= \xi_v \times \mathcal{D}_t \left(-\int_0^v \left(r_s + \frac{1}{2}\theta_s'\theta_s\right) ds - \int_0^v \theta_s' dW_s\right) && \text{chain rule} \\
&= -\xi_v \left(\int_t^v \left(\mathcal{D}_t r_s + \theta_s' \mathcal{D}_t\theta_s\right) ds + \int_t^v \left(dW_s\right)' \mathcal{D}_t\theta_s + \theta_t'\right) && \text{MD of Riemann, Ito int.} \\
&\equiv -\xi_v \left(H_{t,v}' + \theta_t'\right) && \text{def. of } H_{t,v} \qquad (22)
\end{aligned}$$

- Malliavin derivatives of r, θ
  - Chain rule: $\mathcal{D}_t r_s = \partial r\left(Y_s, s\right) \mathcal{D}_t Y_s$ and $\mathcal{D}_t \theta_s = \partial\theta\left(Y_s, s\right) \mathcal{D}_t Y_s$
  - Where $\mathcal{D}_t Y_s$ is derivative of solution of SDE

$$d\mathcal{D}_t Y_s = \left[\partial\mu^Y(s, Y_s)ds + \sum_{i=1}^{d} \partial\sigma_i^Y(s, Y_s)dW_{is}\right] \mathcal{D}_t Y_s; \quad \mathcal{D}_t Y_t = \sigma^Y(t, Y_t). \qquad (23)$$

    * Here $\partial f\left(Y\right)$ is $1 \times k$-gradient of function f with respect to Y

- Substituting (20)-(22) into (20) and (19)

$$\begin{aligned}
\phi_t^* &= E_t\left[\left(\int_t^T Z^u(y^*\xi_v, v)\mathcal{D}_t\xi_v dv + Z^U(y^*\xi_T, T)\mathcal{D}_t\xi_T'\right)'\right] \\
&= -E_t\left[\int_t^T Z^u(y^*\xi_v, v)\xi_v\left(H_{t,v} + \theta_t\right) dv + Z^U(y^*\xi_T, T)\xi_T\left(H_{t,T} + \theta_t\right)\right]
\end{aligned}$$

$$\begin{aligned}
\phi_t^* &= X_t^*\left(\sigma_t'\right)^{-1}\theta_t + \xi_t^{-1}\left(\sigma_t'\right)^{-1}\phi_t^* \\
&= X_t^*\left(\sigma_t'\right)^{-1}\theta_t \\
&\quad -\xi_t^{-1}\left(\sigma_t'\right)^{-1} E_t\left[\int_t^T Z^u(y^*\xi_v, v)\xi_v\left(H_{t,v} + \theta_t\right) dv + Z^U(y^*\xi_T, T)\xi_T\left(H_{t,T} + \theta_t\right)\right] \\
&= \left[X_t^* - E_t\left[\int_t^T Z^u(y^*\xi_v, v)\xi_{t,v} dv + Z^U(y^*\xi_T, T)\xi_{t,T}\right]\right]\left(\sigma_t'\right)^{-1}\theta_t \\
&\quad -\left(\sigma_t'\right)^{-1} E_t\left[\int_t^T Z^u(y^*\xi_v, v)\xi_{t,v}H_{t,v} dv + Z^U(y^*\xi_T, T)\xi_{t,T}H_{t,T}\right].
\end{aligned}$$

- Finally

$$\begin{aligned} & X_t^* - E_t \left[\int_t^T Z^u(y^* \xi_v, v) \xi_{t,v} dv + Z^U(y^* \xi_T, T) \xi_{t,T} \right] \\ = \; & -E_t \left[\int_t^T y^* \xi_v \partial_y I^u(y^* \xi_v, v) \xi_{t,v} dv + y^* \xi_T \partial_y I^U(y^* \xi_T, T) \xi_{t,T} \right] \end{aligned}$$

♦ Theorem 3.1:

- Optimal portfolio has decomposition $X_t^* \pi_t^* = X_t^* \left[\pi_{1t}^* + \pi_{2t}^*\right]$ where

$$\begin{aligned} X_t^* \pi_{1t}^* &= -E_t \left[\int_t^T y^* \xi_v \partial_y I^u (y^* \xi_v, v) \xi_{t,v} dv + y^* \xi_T \partial_y I^U (y^* \xi_T, T) \xi_{t,T} \right] (\sigma_t')^{-1} \theta_t \\ &= E_t \left[\int_t^T \xi_{t,v} \Gamma^u (c_v^*, v)\, dv + \xi_{t,T} \Gamma^U (X_T^*, T) \right] (\sigma_t')^{-1} \theta_t \end{aligned} \tag{24}$$

$$\begin{aligned} X_t^* \pi_{2t}^* &= -(\sigma_t')^{-1} E_t \left[\int_t^T Z^u (y^* \xi_v, v) \xi_{t,v} H_{t,v} dv + Z^U (y^* \xi_T, T) \xi_{t,T} H_{t,T} \right] \\ &= -(\sigma_t')^{-1} E_t \left[\int_t^T \xi_{t,v} \left(c_v^* - \Gamma^u (c_v^*, v)\right) H_{t,v} dv + \xi_{t,T} \left(X_T^* - \Gamma^U (X_T^*, T)\right) H_{t,T} \right] \end{aligned} \tag{25}$$

- MD of state variables, $D_t Y_s$, satisfies SDE(23)
- $\Gamma^u (c_v^*, v), \Gamma^U (X_T^*, T)$ are absolute risk tolerance measures

$$\Gamma^u (c, v) \equiv -\frac{u' (c, v)}{u'' (c, v)}, \qquad \Gamma^U (X, T) \equiv -\frac{U' (X, T)}{U'' (X, T)}$$

  – Evaluated at optimal consumption-bequest

♦ Remarks: two motives for investment

- First motive:
  – Tradeoff risk $\sigma\sigma'$ vs expected excess return $\mu - r1$: $(\sigma')^{-1} \theta = (\sigma\sigma')^{-1} (\mu - r\mathbf{1})$
  – Underlies mean-variance demand π_1
  – Originally identified by Markowitz (1952)
  – Still at core of practical implementations and financial advice
- Second motive:
  – Hedging motive: prompted by stochastic fluctuations in opportunity set (interest rate and market price of risk)
  – Underlies demand component π_2
  – Identified by Merton (1971)
  – Important aspect of optimal dynamic asset allocation policies

3.2 Special cases and examples

♦ Deterministic opportunity set (r, θ deterministic)

- Malliavin derivatives $D_t r_v = D_t \theta_v = 0$. Hedging demand vanishes $X_t^* \pi_{2t}^* = 0$
- Investment demand reduces to mean-variance term

$$X_t^* \pi_{1t}^* = E_t \left[\int_t^T \xi_{t,v} \Gamma^u \left(c_v^*, v \right) dv + \xi_{t,T} \Gamma^u \left(X_T^*, T \right) \right] \left(\sigma_t' \right)^{-1} \theta_t$$

- Irrespective of preferences
- Coefficient in MV demand is cost of optimal risk tolerance

♦ Stochastic opportunity set (r, θ stochastic)

- Dynamic hedging motive becomes relevant
- Signing hedges:
  - Suppose condition $\left[\left(\sigma_t' \right)^{-1} H_{t,v} \right]_i \geq 0$ for all $v \in [t, T]$
  - Hedging increases (decreases) holdings of asset i if risk tolerance exceeds (falls below) consumption and bequest
    * As $c_v^* - \Gamma^u \left(c_v^*, v \right) \leq 0$ and $X_T^* - \Gamma^U \left(X_T^*, T \right) \leq 0$
    * Can be restated in terms of relative risk aversion (Breeden (1979))

$$c_v^* - \Gamma^u \left(c_v^*, v \right) = \frac{c_v^*}{R^u \left(c_v^*, v \right)} \left(R^u \left(c_v^*, v \right) - 1 \right)$$

$$X_T^* - \Gamma^U \left(X_T^*, T \right) = \frac{X_T^*}{R^U \left(X_T^*, T \right)} \left(R^U \left(X_T^*, T \right) - 1 \right)$$

  - Condition on $H_{t,v}$ applies, in particular for IRH in one risky asset model
    * if interest rate negatively impacted by innovations, and
    * the stock market returns positively affected by innovations

♦ Constant relative risk aversion (Example 2) with subjective discount factor $a_t \equiv \exp(-\beta t)$ where β is a constant

- Optimal consumption policy $c_v^* = (y^* \xi_v / a_v)^{-1/R}$ and $X_T^* = (y^* \xi_T / a_T)^{-1/R}$
- Optimal portfolio $X_t^* \pi_t^* = X_t^* [\pi_{1t}^* + \pi_{2t}^*]$ where

$$X_t^* \pi_{1t}^* = \frac{X_t^*}{R} (\sigma_t')^{-1} \theta_t \tag{26}$$

$$X_t^* \pi_{2t}^* = -X_t^* \rho (\sigma_t')^{-1} \frac{E_t \left[\int_t^T \xi_{t,v}^{\rho} a_{t,v}^{1/R} H_{t,v} dv + \xi_{t,T}^{\rho} a_{t,T}^{1/R} H_{t,T} \right]}{E_t \left[\int_t^T \xi_{t,v}^{\rho} a_{t,v}^{1/R} dv + \xi_{t,T}^{\rho} a_{t,T}^{1/R} \right]} \tag{27}$$

with $\rho = 1 - 1/R$

- Details:
  - Consumption-bequest functions: $c_v^* = (y^* \xi_v / a_v)^{-1/R}$ and $X_T^* = (y^* \xi_T / a_T)^{-1/R}$
  - Substituting $\Gamma^u (c_v^*, v) = c_v^*/R$ and $\Gamma^U (X_T^*, T) = X_T^*/R$ in portfolio gives

$$X_t^* \pi_{1t}^* = \frac{1}{R} E_t \left[\int_t^T \xi_{t,v} c_v^* dv + \xi_{t,T} X_T^* \right] (\sigma_t')^{-1} \theta_t = \frac{1}{R} X_t^* \theta_t$$

$$\begin{aligned}
X_t^* \pi_{2t}^* &= -\rho (\sigma_t')^{-1} E_t \left[\int_t^T \xi_{t,v} c_v^* H_{t,v} dv + \xi_{t,T} X_T^* H_{t,T} \right] \\
&= -\rho (\sigma_t')^{-1} E_t \left[\int_t^T \xi_{t,v} \left(\frac{y^* \xi_v}{a_v} \right)^{-1/R} H_{t,v} dv + \xi_{t,T} \left(\frac{y^* \xi_T}{a_T} \right)^{-1/R} H_{t,T} \right] \\
&= -\left(\frac{y^* \xi_t}{a_t} \right)^{-1/R} \rho (\sigma_t')^{-1} E_t \left[\int_t^T \xi_{t,v} \left(\frac{\xi_{t,v}}{a_{t,v}} \right)^{-1/R} H_{t,v} dv + \xi_{t,T} \left(\frac{\xi_{t,T}}{a_{t,T}} \right)^{-1/R} H_{t,T} \right] \\
&= -\left(\frac{y^* \xi_t}{a_t} \right)^{-1/R} \rho (\sigma_t')^{-1} E_t \left[\int_t^T \xi_{t,v}^{\rho} a_{t,v}^{1/R} H_{t,v} dv + \xi_{t,T}^{\rho} a_{t,T}^{1/R} H_{t,T} \right].
\end{aligned}$$

  - Constant y^* eliminated by using wealth

$$\begin{aligned}
X_t^* &= E_t \left[\int_t^T \xi_{t,v} c_v^* dv + \xi_{t,T} X_T^* \right] \\
&= E_t \left[\int_t^T \xi_{t,v} \left(\frac{y^* \xi_v}{a_v} \right)^{-1/R} dv + \xi_{t,T} \left(\frac{y^* \xi_T}{a_T} \right)^{-1/R} \right] \\
&= \left(\frac{y^* \xi_t}{a_t} \right)^{-1/R} E_t \left[\int_t^T \xi_{t,v}^{\rho} a_{t,v}^{1/R} dv + \xi_{t,T}^{\rho} a_{t,T}^{1/R} \right]
\end{aligned}$$

to deduce

$$\left(\frac{y^*\xi_t}{a_t}\right)^{-1/R} = \frac{X_t^*}{E_t\left[\int_t^T \xi_{t,v}^{\rho} a_{t,v}^{1/R} dv + \xi_{t,T}^{\rho} a_{t,T}^{1/R}\right]}$$

- Properties:
  - Portfolio linear in wealth
  - Fraction of wealth invested depends on state (r, θ)

3.3 Implementation

♦ Computation of optimal portfolios:

- Structure of portfolios as conditional expectations suggests Monte Carlo
- Several possibilities for implementation: here method using formula above
- Monte Carlo Malliavin derivatives method - MCMD (DGR (2003))
- Two cases: depending on whether y^* is known or not

♦ Case 1: known multiplier

- Write $X_t^* \pi_{2t}^* = -(\sigma_t')^{-1} E_t[G_{t,T}]$ where $G_{t,T} \equiv G_{t,T}^c + G_{t,T}^x$, with

$$G_{t,s}^c \equiv \int_t^s \xi_{t,v} Z_1(y^* \xi_v, v) H_{t,v} dv \qquad \text{and} \qquad G_{t,T}^x \equiv \xi_{t,T} Z_2(y^* \xi_T, T) H_{t,T}. \tag{28}$$

- Write RV in hedges as joint system $V_{t,s} \equiv (Y_s, D_t Y_s, K_{t,s}, H_{t,s}, G_s^c)$, where

$$K_{t,v} \equiv \int_t^v \left(r_s + \frac{1}{2} \theta_s' \theta_s \right) ds + \int_t^v \theta_s' dW_s$$

$$H_{t,v}' \equiv \int_t^v \partial r(Y_s, s) \mathcal{D}_t Y_s ds + \int_t^v \theta_s' \partial \theta(Y_s, s) \mathcal{D}_t Y_s ds + \int_t^v dW_s' \cdot \partial \theta(Y_s, s) \mathcal{D}_t Y_s$$

$$\xi_{t,v} = \exp(-K_{t,v})$$

- By Ito's Lemma

$$dK_{t,s} = \left(r_s + \frac{1}{2} \theta_s' \theta_s \right) ds + \theta_s' dW_s \tag{29}$$

$$dH_{t,s}' = \partial r(Y_s, s) \mathcal{D}_t Y_s ds + (dW_s + \theta(Y_s, s) ds)' \, \partial \theta(Y_s, s) \mathcal{D}_t Y_s, \tag{30}$$

$$dG_{t,s}^c = \xi_{t,s} Z_1(y^* \xi_s, s) H_{t,s} ds \tag{31}$$

and $(Y_s, \mathcal{D}_t Y_s)$ satisfy SDEs

$$dY_t = \mu^Y(Y_t, t) dt + \sigma^Y(Y_t, t) dW_t \tag{32}$$

$$d\mathcal{D}_t Y_s = \left[\partial \mu^Y(s, Y_s) ds + \sum_{i=1}^d \partial \sigma_i^Y(s, Y_s) dW_{is} \right] \mathcal{D}_t Y_s; \qquad \mathcal{D}_t Y_t = \sigma^Y(t, Y_t). \tag{33}$$

- Simulate M trajectories of V using (29)-(31), (32)-(33)
  - Select discretization scheme (e.g., Euler, Milshtein, ...): N points in $[0, T]$
  - Simulate M trajectories of W along discretization. Construct traject. V
  - Get M estimates $\left\{V_{t,s}^{N,i} : s \in [t, T]\right\}, i = 1, ..., M$ of trajectories $\{V_{t,s} : s \in [t, T]\}$
  - From terminal values of simulated proc. construct M estimates of $G_{t,T}$
  - Averaging over these M values produces estimate of hedging demand

$$\widehat{X_t^* \pi_{2t}^*} = -\left(\sigma_t'\right)^{-1} \frac{1}{M} \sum_{i=1}^{M} G_{t,T}^{N,i}$$

♦ Case 2: y^* is unknown. Use two stage procedure:

- Stage 1: calculate y^* by simulation-iteration
  - Fix candidate multiplier y
  - Based on this choice simulate $\left(K_{0,s}, F_{0,s}^c\right)$ where $F_{0,s}^c = \int_0^s \xi_v I(y\xi_v, v) dv$
  - Obtain estimate of cost of consumption by taking average
  - If budget constraint fails raise y and repeat. Else reduce y
  - Repeat to desired precision
- Stage 2: proceed as described above
- Various schemes can be used to accelerate stage 1 (Newton-Raphson,...)

3.4 Example

♦ Model:

- One stock and riskless asset
- State variables (r, θ)
- Constant relative risk aversion

♦ Evolution of opportunity set

$$dr_t = \kappa_r(\overline{r} - r_t)\left(1 + \phi_r(\overline{r} - r_t)^{2\eta_r}\right)dt - \sigma_r r_t^{\gamma_r} dW_t, \qquad r_0 \text{ given} \tag{34}$$

$$d\theta_t = \left(\kappa_\theta(\overline{\theta} - \theta_t) + \mu_\theta^r(r_{t,}\theta_t)\right)dt + \sigma_\theta(\theta_t)dW_t, \qquad \theta_0 \text{ given,} \tag{35}$$

where W is one dimensional

$$\mu_\theta^r(r_t, \theta_t) \equiv \delta_r(\overline{r} - r_t)(\theta_l + \theta_t)\left(1 - \left(\frac{\theta_l + \theta_t}{\theta_l + \theta_u}\right)\right) \tag{36}$$

$$\sigma_\theta(\theta_t) = \sigma_\theta(\theta_l + \theta_t)^{\gamma_{1\theta}}\left(1 - \left(\frac{\theta_l + \theta_t}{\theta_l + \theta_u}\right)^{1-\gamma_{1\theta}}\right)^{\gamma_{2\theta}}. \tag{37}$$

- Coefficients
  - $(\kappa_r, \overline{r}, \phi_r, \eta_r, \sigma_r, \gamma_r, \kappa_\theta, \overline{\theta}, \eta_\theta, \sigma_\theta, \theta_l, \theta_u, \gamma_{1\theta}, \gamma_{2\theta})$ are constants
  - $(\kappa_r, \overline{r}, \kappa_\theta, \theta_l, \theta_u)$ are positive, and $\overline{\theta} \in (-\theta_l, \theta_u)$
  - Brownian motion W is unidimensional

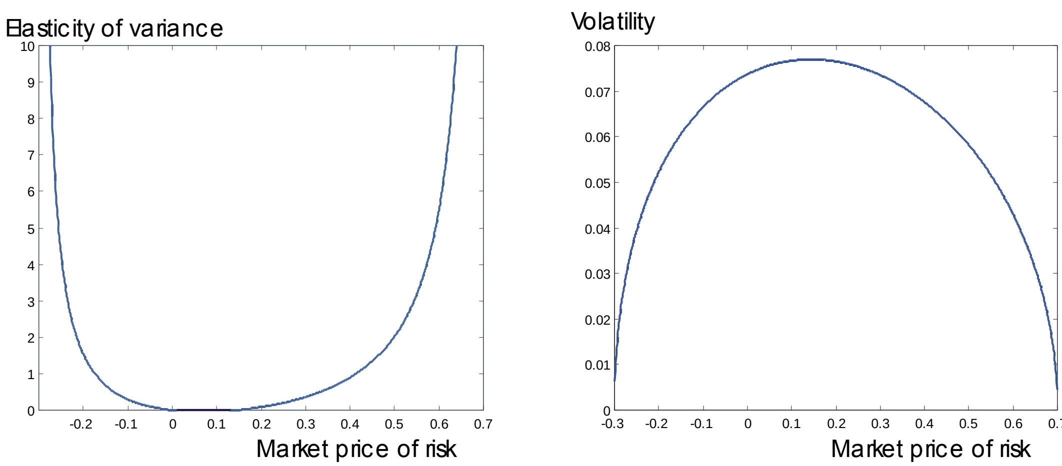
♦ Remarks:

- Interest rate process:
  - Mean reverting with constant elasticity of variance (NMRCEV), $2\gamma_r$
  - Nonlinear speed of mean reversion: $\phi_r(\overline{r} - r_t)^{2\eta_r}$

- Market price of risk process:
  - Mean reverting with hyperbolic elasticity of variance
  - Interest dependence in drift (MRHEVID)
  - Elasticity

$$\varepsilon(x) = -2\frac{x}{\theta_l + x}\left[\gamma_{1\theta} - \gamma_{2\theta}(1-\gamma_{1\theta})\frac{\left(\frac{\theta_l + x}{\theta_l + \theta_u}\right)^{1-\gamma_{1\theta}}}{1-\left(\frac{\theta_l + x}{\theta_l + \theta_u}\right)^{1-\gamma_{1\theta}}}\right].$$

Process stays between bounds



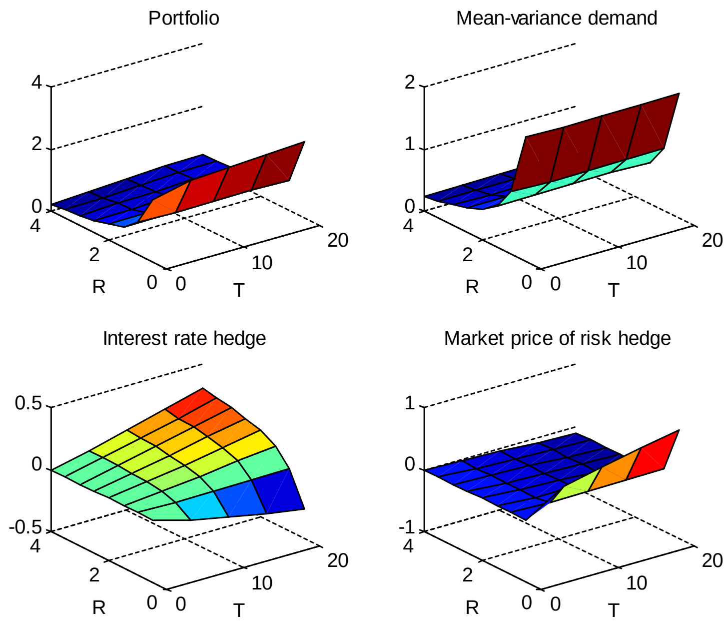
♦ Malliavin derivatives

$$d\mathcal{D}_t r_v = \left(\frac{\partial}{\partial r}\mu^r(r_v)dt - \frac{\partial}{\partial r}\sigma^r(r_v)dW_v\right)\mathcal{D}_t r_v, \qquad \mathcal{D}_t r_t = \sigma^r(r_t)$$

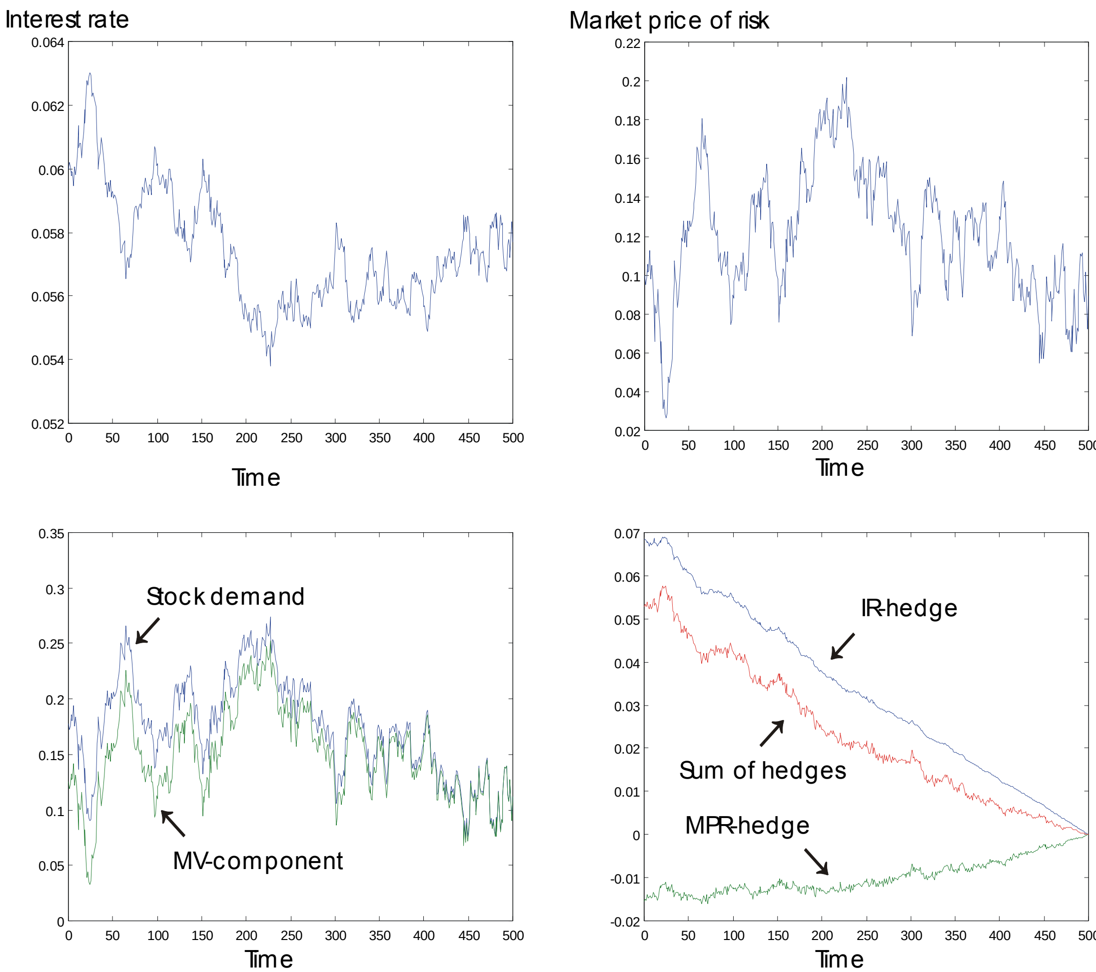
$$d\mathcal{D}_t\theta_v = \left(\frac{\partial}{\partial\theta}\mu_\theta(r_v,\theta_v)dv + \frac{\partial}{\partial\theta}\sigma_\theta(\theta_v)dW_v\right)\mathcal{D}_t\theta_v + \frac{\partial}{\partial r}\mu_\theta(r_v,\theta_v)\mathcal{D}_t r_v dv; \qquad \mathcal{D}_t\theta_t = \sigma_\theta(\theta_t)$$

♦ Parameter values (see DGR 2003)

♦ Implementation: portfolio components - risk aversion and horizon effects



♦ Dynamic behavior of portfolio components



PART 4: Optimal Portfolio and Bonds

♦ Alternative decomposition of portfolio

- Unobserved short rate: substitute information in term structure
- Portfolio behavior for long horizons: long run risk factors
- Portfolio and bond pricing models
- Detemple-Rindisbacher (2006)

♦ Outline

- Forward measure
- Optimal portfolio: utility of terminal wealth
- Optimal portfolio: intermediate consumption
- Diffusion models: implementation
- Deterministic forward density

4.1 Bond pricing and forward measure

♦ Forward measure:

- Pure discount bond with maturity $T \geq t$ has price: $B_t^T = E_t\left[\xi_{t,T}\right]$
- State price density in bond numeraire

$$Z_{t,T} \equiv \frac{\xi_{t,T}}{E_t\left[\xi_{t,T}\right]} = \frac{\xi_{t,T}}{B_t^T}$$

  - $Z_{t,T} > 0$ and $E_t\left[Z_{t,T}\right] = 1$
  - Use as density of new measure

- Forward T-measure
  - $dQ_t^T = Z_{t,T} dP$
  - Equivalent to P
  - $Z_{t,T}$ is forward T-density
  - Geman (1989), Jamshidian (1989)

♦ Pricing in bond numeraire

- Claim with payoff Y_T has price

$$V(t) = E_t\left[\xi_{t,T} Y_T\right] = E_t\left[\xi_{t,T}\right] E_t\left[\frac{\xi_{t,T}}{E_t\left[\xi_{t,T}\right]} Y_T\right] = B_t^T E_t^T\left[Y_T\right]$$

  - $E_t^T[\cdot] \equiv E_t\left[Z_{t,T} \cdot\right]$ is expectation under Q_t^T

- Price in bond numeraire

$$\frac{V(t)}{B_t^T} = E_t^T\left[Y_T\right] = E_t\left[Z_{t,T} Y_T\right]$$

- Density $Z_{t,T}$ is stochastic discount factor
  - Converting future cash flows into current values measured in bond units

♦ Theorem 4.1:

- The conditional state price density at time t is $\xi_{t,T} = B_t^T Z_{t,T}$
- The forward T-density is

$$Z_{t,T} \equiv \exp\left(\int_t^T \sigma^Z(s,T)'\, dW_s - \frac{1}{2}\int_t^T \sigma^Z(s,T)'\,\sigma^Z(s,T)\, ds\right) \tag{38}$$

  - Volatility $\sigma^Z(s,T) \equiv \sigma^B(s,T) - \theta_s$
  - $\sigma^B(s,T)' \equiv \mathcal{D}_s \log B_s^T$ is vol. of return on discount bond with maturity T

♦ Decomposition of SPD: $\xi_{t,T} = B_t^T Z_{t,T}$. Two parts

- Bond price
- Risk-adjusted SDF: applies to risky cash flows in bond numeraire

♦ Forward density formula:

- Volatility $-\sigma^Z(\cdot,T) \equiv \theta_\cdot - \sigma^B(\cdot,T)$
  - MPR in bond numéraire: forward market price of risk
- Cumulative standard deviation of the growth rate of the forward density

$$\Sigma(t,T) = \left(\int_t^T \sigma^Z(s,T)'\,\sigma^Z(s,T)\, ds\right)^{1/2}. \tag{39}$$

  - Measures risk to horizon T, in forward density
  - $\Sigma(t,T)$ is forward T-risk or forward risk

4.2 Optimal portfolio and long term bonds

♦ Theorem 4.2:

- Optimal wealth, for $t \in [0, T]$, is $X_t^* = B_t^T E_t \left[Z_{t,T} I \left(y^* \xi_t B_t^T Z_{t,T} \right)^+ \right]$.
- Portfolio has decomposition $\pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z$

$$X_t^* \pi_t^m = E_t^T \left[\Gamma_T^* 1_{\{I_T \geq 0\}} \right] B_t^T \left(\sigma_t' \right)^{-1} \theta_t \tag{40}$$

$$X_t^* \pi_t^b = \left(\sigma_t' \right)^{-1} \sigma^B (t, T) E_t^T \left[\left(X_T^* - \Gamma_T^* \right) 1_{\{I_T \geq 0\}} \right] B_t^T \tag{41}$$

$$X_t^* \pi_t^z = \left(\sigma_t' \right)^{-1} E_t^T \left[\left(X_T^* - \Gamma_T^* \right) 1_{\{I_T \geq 0\}} \mathcal{D}_t \log \left(Z_{t,T} \right) \right]' B_t^T. \tag{42}$$

  - $I_T \equiv I(y^* \xi_t B_t^T Z_{t,T})$
  - $E_t^T [\cdot] \equiv E_t [Z_{t,T} \ \cdot]$ is under forward T-measure.

♦ Interpretation:

- Mean-variance term π_t^m: as before
- Long term bond hedge π_t^b: fluctuations in price of horizon-matching bond
- Forward density hedge π_t^z: fluctuations in MPR in bond numeraire
- Shift focus from risk relative to short rate to risk relative to LT bond

♦ Additional remarks:

- Consistent with Preferred Habitat theory
  - Modigliani and Sutch
  - Investor naturally seeks LT bond with horizon-matching maturity
- Hedges
  - First hedge is static hedge (instantaneous fluct. in bond price)

– Forward density hedge is dynamic hedge (fluct. in opportunity set)

♦ Corollary 4.1: HARA utility function

$$U(x) = \begin{cases} \frac{1}{1-R}(x-A)^{1-R} & \text{if } x \geq A \\ -\infty & \text{if } x < A \end{cases}, \quad R > 0, A \gtreqless 0. \tag{43}$$

- When $A \geq 0$ optimal asset allocation is $\pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z$ with

$$X_t^* \pi_t^m = \frac{1}{R}\left(X_t^* - AB_t^T\right)\left(\sigma_t'\right)^{-1}\theta_t$$

$$X_t^* \pi_t^b = \left(\rho\left(X_t^* - AB_t^T\right) + AB_t^T\right)\left(\sigma_t'\right)^{-1}\sigma^B(t,T)$$

$$X_t^* \pi_t^z = \rho\left(X_t^* - AB_t^T\right)\left(\sigma_t'\right)^{-1} E_t^T\left[\frac{Z_{t,T}^{\rho-1}}{E_t^T\left[Z_{t,T}^{\rho-1}\right]}\mathcal{D}_t \log\left(Z_{t,T}\right)\right]'$$

where $\rho = 1 - 1/R$.

- When $A < 0$ portfolio components are as in Theorem 4.1 with

$$X_T^* = \left(\left(y^* \xi_t B_t^T Z_{t,T}\right)^{-1/R} + A\right)^+, \qquad \Gamma_T^* = \frac{1}{R}\left(X_T^* - A\right)$$

and $I_T \equiv \left(y^* \xi_t B_t^T Z_{t,T}\right)^{-1/R} + A$.

♦ Power utility $(A = 0)$:

- Knife edge property of log (Breeden (1979))
- Logarithmic investor: myopic
- More (less) RA than log holds (shorts) port. with highest correlation with LT bd
- More (less) RA than log holds (shorts) portfolio that hedges $\log(Z_{t,T})$

♦ HARA with $A > 0$: subsistence threshold

- Structure:
  - MV dem and forward density hedge proport. to excess wealth $X_t^* - AB_t^T$
  - Bond hedge affine in $X_t^* - AB_t^T$ with translation factor AB_t^T
- Explanation:
  - Decomposition of wealth:
    - Cost of financing threshold AB_t^T
    - Excess wealth $X_t^* - AB$
  - Portfolio financing excess wealth is proportional to $X_t^* - AB_t^T$
  - Portfolio financing cost of threshold is hedging port.; proport. to cost

4.3 Running consumption

♦ Model:

- Utility of intermediate consumption: $u(\cdot,\cdot) : D_u \times [0,T] \to R$
  - Strictly increasing, strictly concave, differentiable
  - Domain $D_u = [A_u, \infty) \subset R$ with A_u positive or negative
  - Inada: for all $t \in [0,T]$, $\lim_{c\to\infty} u'(c,t) = 0$, $\lim_{c\to A_u} u'(c,t) = \infty$
- Utility of terminal wealth $U : D_U \to R$
  - Strictly increasing, strictly concave and differentiable
  - Domain $D_U = [A_U, \infty) \subset R$
  - Inada: $\lim_{X\to\infty} U'(X) = 0$, $\lim_{X\to A_U} U'(X) = \infty$
- Initial wealth condition: $x > A_u^+ \left(\int_0^T B_0^v dv\right) + A_U^+ B_0^T$

♦ Theorem 4.3:

- Optimal consumption-bequest: $c_v^* = I^u\left(y^*\xi_t B_t^v Z_{t,v}, v\right)^+$ and $X_T^* = I^U\left(y^*\xi_t B_t^T Z_{t,T}\right)^+$
- Intermediate wealth satisfies

$$X_t^* = \int_t^T B_t^v E_t^v\left[c_v^*\right] dv + B_t^T E_t^T\left[X_T^*\right]$$

- Let $I_v^u \equiv I^u\left(y^*\xi_t B_t^v Z_{t,v}, v\right)$ and $I_T^U \equiv I\left(y^*\xi_t B_t^T Z_{t,T}\right)$
- Optimal portfolio has decomposition $\pi_t^* = \pi_t^m + \pi_t^b + \pi_t^z$ with

$$X_t^*\pi_t^m = \left(\int_t^T E_t^v\left[\Gamma_v^* 1_{\{I_v^u \geq 0\}}\right] B_t^v dv + E_t^T\left[\Gamma_T^* 1_{\left\{I_T^U \geq 0\right\}}\right] B_t^T\right)\left(\sigma_t'\right)^{-1}\theta_t$$

$$X_t^*\pi_t^b = \left(\sigma_t'\right)^{-1}\left(\int_t^T \sigma^B(t,v)\, B_t^v E_t^v\left[\left(c_v^* - \Gamma_v^*\right) 1_{\{I_v^u \geq 0\}}\right] dv + \sigma^B(t,T)\, B_t^T E_t^T\left[\left(X_T^* - \Gamma_T^*\right) 1_{\left\{I_T^U \geq 0\right\}}\right]\right)$$

$$X_t^*\pi_t^z = \left(\sigma_t'\right)^{-1}\left(\int_t^T E_t^v\left[\left(c_v^* - \Gamma_v^*\right) 1_{\{I_v^u \geq 0\}}\mathcal{D}_t \log Z_{t,v}\right] B_t^v dv + E_t^T\left[\left(X_T^* - \Gamma_T^*\right) 1_{\left\{I_T^U \geq 0\right\}}\mathcal{D}_t \log Z_{t,T}\right] B_t^T\right)$$

  - $Z_{t,v}$ is density of forward v-measure
    - Volatility $\sigma^Z(s,v) \equiv \sigma^B(s,v) - \theta_s$
    - $\sigma^B(s,v)' \equiv \mathcal{D}_s \log B_s^v$ is bond return volatility
  - $E_t^v[\cdot] \equiv E_t\left[Z_{t,v}\cdot\right]$ is under forward v-measure, $v \in [t,T]$.

♦ Interpretation:

- Mean-variance term, bond hedge, forward density hedge
- Bond hedge:
  - Coupon-paying bond
    - Coupon $C(v) \equiv E_t^v\left[\left(c_v^* - \Gamma_v^*\right) 1_{\{I_v^u \geq 0\}}\right]$ at $v \in [0,T)$
    - Bullet payment $F \equiv E_t^T\left[\left(X_T^* - \Gamma_T^*\right) 1_{\left\{I_T^U \geq 0\right\}}\right]$ at T

– Coupon bond price

$$B_t^T(C,F) \equiv \int_t^T B_t^v C(v)\,dv + B_t^T F$$

– Instantaneous coupon bond volatility (taking coupon as given)

$$\sigma\left(B_t^T(C,F)\right) B_t^T(C,F) = \int_t^T \sigma^B(t,v)\,B_t^v C(v)\,dv + \sigma^B(t,T)\,B_t^T F$$

– Hedge is positive if $c_v^* - \Gamma_v^* \geq 0$ for $v \in [0,T)$ and $X_T^* - \Gamma_T^* \geq 0$

♦ Corollary 4.2: HARA utilities with A_u, R_u for $u(c,t)$ and A_U, R_U for $U(x)$

- Assume $x \geq A_u^+ \int_0^T B_0^v dv + A_U^+ B_0^T$
  - Portfolio components given by formulas in Theorem 4.3 with

$$c_v^* = \left((y^* \xi_t)^{-1/R_u} \left(B_t^v Z_{t,v}\right)^{-1/R_u} + A_u\right)^+, \quad X_T^* = \left((y^* \xi_t)^{-1/R_U} \left(B_t^T Z_{t,T}\right)^{-1/R_U} + A_U\right)^+$$

$$I_v^u \equiv (y^* \xi_t)^{-1/R_u} \left(B_t^v Z_{t,v}\right)^{-1/R_u} + A_u \quad \text{and} \quad I_T^U \equiv (y^* \xi_t)^{-1/R_U} \left(B_t^T Z_{t,T}\right)^{-1/R_U} + A_U.$$

  - When $A_u, A_U \geq 0$ portfolio components take the form

$$X_t^* \pi_t^m = \left(\frac{1}{R_u}\left(\int_t^T (\Pi_t^v - A_u) B_t^v dv\right) + \frac{1}{R_U}\left(\Pi_t^T - A_U\right) B_t^T\right) (\sigma_t')^{-1} \theta_t$$

$$X_t^* \pi_t^b = (\sigma_t')^{-1} \left(\int_t^T \sigma^B(t,v) B_t^v \left(\rho_u \Pi_t^v + \frac{1}{R_u} A_u\right) dv + \sigma^B(t,T) B_t^T \left(\rho_U \Pi_t^T + \frac{1}{R_U} A_U\right)\right)$$

$$X_t^* \pi_t^z = (\sigma_t')^{-1} \left(\rho_u \int_t^T E_t^v \left[c_v^* \mathcal{D}_t \log Z_{t,v}\right] B_t^v dv + \rho_U E_t^T \left[X_T^* \mathcal{D}_t \log Z_{t,T}\right] B_t^T\right)'$$

    - $\Pi_t^v = E_t^v [c_v^*]$ is date t cost in bond numéraire of date v consumption
    - $\Pi_t^T = E_t^T [X_T^*]$ is date t cost in bond numéraire of terminal wealth
    - $\rho_u = 1 - 1/R_u,\ \rho_U = 1 - 1/R_U$.

♦ Coupon bond hedge:

- Coupon $C(v) = \rho_u \Pi_t^v + \frac{A_u}{R_u}$: affine in cost of date v consumption in bd numéraire
- Bullet payt $F = \rho_U \Pi_t^T + \frac{A_U}{R_U}$: affine in cost of terminal wealth in bd numéraire
- Can have positive coupon hedge $\rho_u \Pi_t^v + \frac{A_u}{R_u}$ & negative bullet hedge $\rho_U \Pi_t^T + \frac{A_U}{R_U}$

4.4 Diffusion models - implementation

♦ Model:

- Utility of terminal wealth (no intermediate consumption)
- Diffusion model:
  - Vector of state variables Y
  - Evolution of $\zeta_t' \equiv \left(\sigma^Z(t,T)', Y_t'\right)$

$$\begin{cases} d\sigma^Z(t,T) = \Phi(\zeta_t, t)\,dt + \Lambda(\zeta_t, t)\,dW_t \\ dY_t = \mu^Y(Y_t, t)\,dt + \sigma^Y(Y_t, t)\,dW_t \end{cases} \tag{44}$$

    with initial conditions $\sigma^Z(0,T)$ and Y_0

  - Functions $\Phi(\cdot,\cdot), \Lambda(\cdot,\cdot), \mu^Y(\cdot,\cdot), \sigma^Y(\cdot,\cdot)$ are continuously differentiable

♦ Theorem 4.5: (utility of terminal wealth)

- Malliavin derivative of log forward density

$$\mathcal{D}_t \log Z_{t,T} = \int_t^T \left(dW_s^T\right)' \mathcal{D}_t \sigma^Z(s,T) \tag{45}$$

where $\left(\mathcal{D}_t \sigma^Z(s,T), \mathcal{D}_t Y_s\right)$ satisfies linear SDE

$$\begin{cases} d\left(\mathcal{D}_t \sigma^Z(s,T)\right) = \left(A_1^Z ds + \sum_{j=1}^d \partial_1 \Lambda_j dW_{js}^T\right) \mathcal{D}_t \sigma^Z(s,T) + \left(A_2^Z ds + \sum_{j=1}^d \partial_2 \Lambda_j dW_{js}^T\right) \mathcal{D}_t Y_s \\ d\left(\mathcal{D}_t Y_s\right) = \left(A^Y ds + \sum_{j=1}^d \partial \sigma_j^Y dW_{js}^T\right) \mathcal{D}_t Y_s \end{cases} \tag{46}$$

  – Coefficients

$$A_1^Z \equiv \partial_1 \Phi + \sum_{j=1}^d \partial_1 \Lambda_j \sigma_j^Z, \qquad A_2^Z \equiv \partial_2 \Phi + \sum_{j=1}^d \partial_2 \Lambda_j \sigma_j^Z, \qquad A^Y \equiv \partial \mu^Y + \sum_{j=1}^d \sigma_j^Z \partial \sigma_j^Y$$

  – $\partial_i \Phi$, $\partial_i \Lambda$ are gradients with respect to i^{th} component of vector ζ in Φ, Λ

- Forward density

$$Z_{t,T} \equiv \exp\left(\int_t^T \sigma^Z(s,T)' dW_s^T + \frac{1}{2}\int_t^T \sigma^Z(s,T)' \sigma^Z(s,T)\, ds\right) \tag{47}$$

under bond numéraire, where $\left(\sigma^Z(t,T), Y_t\right)$ satisfies

$$\begin{cases} d\sigma^Z(t,T) = \left(\Phi(\zeta_t, t) + \Lambda(\zeta_t, t)\, \sigma^Z(t,T)\right) dt + \Lambda(\zeta_t, t)\, dW_t^T \\ dY_t = \left(\mu^Y(Y_t, t) + \sigma^Y(Y_t, t)\, \sigma^Z(t,T)\right) dt + \sigma^Y(Y_t, t)\, dW_t^T. \end{cases} \tag{48}$$

♦ Computation:

- Simulate relevant processes directly under forward measure
- Compute expectations by averaging over simulated values

4.5 Deterministic forward density volatility

♦ Assumption

- Forward density volatility $\sigma^Z(t,T)$ is a (nonstochastic) function of time
- Forward risk $\Sigma(t,T)$ is deterministic

♦ Corollary 4.3: (deterministic forward density vol)

- Optimal wealth

$$\frac{X_t^*}{B_t^T} = \int_{-\infty}^{d\left(U'(0\vee A),y^*\xi_t B_t^T\right)} I\left(y^*\xi_t B_t^T e^{\frac{1}{2}\Sigma(t,T)^2+\Sigma(t,T)z}\right) n(z)\,dz \equiv \chi\left(y^*\xi_t B_t^T\right) \tag{49}$$

$$d\left(U'(0\vee A),y^*\xi_t B_t^T\right) \equiv \frac{1}{\Sigma(t,T)}\left(\log\frac{U'(0\vee A)}{y^*\xi_t B_t^T} - \frac{1}{2}\Sigma(t,T)^2\right) \tag{50}$$

  – $\chi\left(y^*\xi_t B_t^T\right)$ is optimal wealth in bond numéraire
  – $n(z)$ is standard normal density
  – Inverting $\chi(\cdot)$ in (49) gives $y^*\xi_t B_t^T = \chi^{-1}\left(X_t^*/B_t^T\right)$

- Portfolio

$$X_t^*\pi_t^m = B_t^T K\left(\frac{X_t^*}{B_t^T},\Sigma(t,T)\right)(\sigma_t')^{-1}\theta_t \tag{51}$$

$$X_t^*\pi_t^b = (\sigma_t')^{-1}\sigma^B(t,T)\left(X_t^* - B_t^T K\left(\frac{X_t^*}{B_t^T},\Sigma(t,T)\right)\right) \tag{52}$$

$$X_t^*\pi_t^z = 0 \tag{53}$$

  – $K(\cdot,\cdot) \equiv E_t^T\left[\Gamma_T^* 1_{\{I_T\geq 0\}}\right]$: cost of optimal risk tol. in bd numéraire

$$K\left(\frac{X_t^*}{B_t^T},\Sigma(t,T)\right) = \int_{-\infty}^{d\left(U'(0\vee A),\chi^{-1}\left(\frac{X_t^*}{B_t^T}\right)\right)} \Gamma\left(I\left(\chi^{-1}\left(\frac{X_t^*}{B_t^T}\right)e^{\frac{1}{2}\Sigma(t,T)^2+\Sigma(t,T)z}\right)\right) n(z)\,dz \tag{54}$$

♦ HARA: $U'(0 \vee A) = (-A \vee 0)^{-R}$ and

$$\begin{aligned} \chi\left(y^*\xi_t B_t^T\right) &= \left(y^*\xi_t B_t^T\right)^{-1/R} e^{-\frac{1}{R}\rho\frac{1}{2}\Sigma(t,T)^2} N\left(d\left((-A \vee 0)^{-R}, y^*\xi_t B_t^T\right) + \frac{1}{R}\Sigma(t,T)\right) \\ &\quad + AN\left(d\left((-A \vee 0)^{-R}, y^*\xi_t B_t^T\right)\right) \end{aligned} \tag{55}$$

$$K\left(\frac{X_t^*}{B_t^T}, \Sigma(t,T)\right) = \frac{1}{R}\left(\frac{X_t^*}{B_t^T} - AN\left(d\left((-A \vee 0)^{-R}, \chi^{-1}\left(\frac{X_t^*}{B_t^T}\right)\right)\right)\right) \tag{56}$$

- $N(\cdot)$: cumulative normal distribution function

♦ Remarks:

- Forward market price of risk deterministic:
  - No reason to hedge
  - Forward density hedge null
- Components
  - Expressed in terms of optimal wealth and model coefficients
  - Truncated integrals of risk tolerance w.r.t. to normal random variate
- HARA utility
  - Risk tolerance affine in terminal wealth over domain where it is positive
  - Optimal wealth & port. components involve cumulative normal distrib.
  - Nonlinear wealth effects in portfolio components

♦ Proposition 4.1: (wealth effects)

- Derivative of cost of optimal risk tolerance $K(\cdot, \Sigma(t,T))$ w.r.t. X_t^*/B_t^T

$$K_1\left(\frac{X_t^*}{B_t^T}, \Sigma(t,T)\right) = \frac{\int_{-\infty}^{d(\cdot)} \Gamma'(\cdot)\,\Gamma(\cdot)\,n(z)\,dz + \Gamma(0 \vee A)\,n(d(\cdot))\,\frac{1}{\Sigma(t,T)}}{K\left(\frac{X_t^*}{B_t^T}, \Sigma(t,T)\right) + (0 \vee A)\,n(d(\cdot))\,\frac{1}{\Sigma(t,T)}}$$

  – $\Gamma'(\cdot), \Gamma(\cdot)$ evaluated at $I\left(\chi^{-1}\left(X_t^*/B_t^T\right)e^{\frac{1}{2}\Sigma(t,T)^2+\Sigma(t,T)z}\right)$
  – $d(\cdot) \equiv d\left(U'(0 \vee A), \chi^{-1}\left(X_t^*/B_t^T\right)\right)$

- Impact of wealth on portfolio share components

$$\frac{\partial \pi_t^m}{\partial X_t^*} \gtreqless 0 \Longleftrightarrow \left(K_1\left(\frac{X_t^*}{B_t^T}, \Sigma(t,T)\right)\frac{X_t^*}{B_t^T} - K\left(\frac{X_t^*}{B_t^T}, \Sigma(t,T)\right)\right)(\sigma_t')^{-1}\theta_t \gtreqless 0$$

$$\frac{\partial \pi_t^b}{\partial X_t^*} \gtreqless 0 \Longleftrightarrow -(\sigma_t')^{-1}\sigma^B(t,T)\left(K_1\left(\frac{X_t^*}{B_t^T}, \Sigma(t,T)\right)\frac{X_t^*}{B_t^T} - K\left(\frac{X_t^*}{B_t^T}, \Sigma(t,T)\right)\right) \gtreqless 0.$$

- Under the assumptions:

  – Absolute risk tol. is decreasing function $\left(\Gamma'(X) < 0\right)$
  – Relative risk tol. is increasing function $\left((\Gamma(X)/X)' > 0\right)$
  – MV share π_t^m decreases with wealth when $(\sigma_t')^{-1}\theta_t > 0$
  – Bond hedge share increases with wealth when $(\sigma_t')^{-1}\sigma^B(t,T) > 0$

♦ Arrow (1965): reasonable model for behavior

- Decreasing absolute risk tolerance
- Increasing relative risk tolerance

♦ In particular, if equities and bond with horizon-matching maturity marketed

- $(\sigma_t\sigma_t')^{-1}\sigma_t\sigma^B(t,T) = [0,1]'$
- Equity and bond shares

$$\pi_t^S = \frac{K\left(X_t^*/B_t^T, \Sigma(t,T)\right)}{X_t^*/B_t^T} m_t^S \tag{57}$$

$$\pi_t^B = \frac{K\left(X_t^*/B_t^T, \Sigma(t,T)\right)}{X_t^*/B_t^T} m_t^B + \left(1 - \frac{K\left(X_t^*/B_t^T, \Sigma(t,T)\right)}{X_t^*/B_t^T}\right) \tag{58}$$

with

$$(\sigma_t')^{-1}\theta_t \equiv \begin{bmatrix} m_t^S \\ m_t^B \end{bmatrix}$$

- Bond held for diversification and hedging
- Equities held exclusively for diversification
- When wealth increases:
  - MV part of bond share decreases while hedge part increases (if $m_t^B > 0$)
  - Bond increasingly held for hedging; diversification motive weakens
  - Bonds-to-equities ratio π_t^B/π_t^S increases

♦ Perspective:

- Flight-to-safety: substitution from stocks to bds during downturns or after losses
- Analysis shows flight-to-safety depends on risk attitudes
  - Under conditions stated wealth reductions imply decrease in BTE
  - Substitution away from bonds and into stocks!
  - Conventional wisdom inconsistent with Arrow's "reasonable" behavioral postulates

♦ Proposition 4.2: (forward risk effects)

- Derivative of $K\left(X_t^*/B_t^T,\cdot\right)$ wrt forward T-risk Σ has two parts $\left(K_2 = K_{21} + K_{22}\right)$

$$K_{21} = -\left(\int_{-\infty}^{d(\cdot)} \Gamma'(\cdot)\,\Gamma(\cdot)\,(\Sigma(t,T)+z)\,n(z)\,dz + \Gamma(0\vee A)\,n(d(\cdot))\left(\frac{d(\cdot)}{\Sigma(t,T)}+1\right)\right)$$

$$K_{22} = -K_1 \times \left(\int_{-\infty}^{d(\cdot)} \Gamma(\cdot)\,(\Sigma(t,T)+z)\,n(z)\,dz + (0\vee A)\,n(d(\cdot))\left(\frac{d(\cdot)}{\Sigma(t,T)}+1\right)\right)$$

  - K_{21} is direct impact of $\Sigma(t,T)$ keeping $\chi^{-1}\left(X_t^*/B_t^T\right)$ fixed
  - K_{22} is indirect effect through $\chi^{-1}\left(X_t^*/B_t^T\right)$
  - Impact of forward risk on portfolio share components

$$\frac{\partial\left(\pi_t^m/X_t^*\right)}{\partial\Sigma(t,T)} \gtreqless 0 \Longleftrightarrow (K_{21}+K_{22})\left(\sigma_t'\right)^{-1}\theta_t \gtreqless 0$$

$$\frac{\partial\left(\pi_t^b/X_t^*\right)}{\partial\Sigma(t,T)} \gtreqless 0 \Longleftrightarrow -\left(\sigma_t'\right)^{-1}\sigma^B(t,T)\,(K_{21}+K_{22}) \gtreqless 0.$$

  - Impact of investment horizon on portfolio share components:
    * Keeping wealth in bond numéraire X_t^*/B_t^T fixed
    * Identical to impact of forward risk

♦ Intuition: Investor averse to long run risk should shy away from risky long-lived assets when forward risk increases

- Aversion to forward T-risk
  - Negative impact of $\Sigma(t,T)$ on cost of optimal risk tolerance
  - $K_2 = K_{21} + K_{22} < 0$
- When $K_2 < 0$
  - Diversification part of port. shares decreases if $\left(\sigma_t'\right)^{-1}\theta_t > 0$: risk reduction
  - Static hedge part of port. shares increases if $\left(\sigma_t'\right)^{-1}\sigma^B > 0$: enhanced protection

♦ Bond versus equity choice:

- Assume $(\sigma_t')^{-1}\theta_t = \left[m_t^S, m_t^B\right]' > 0$ and $K_2 < 0$
- If long run risk increases:
  - Investor reduces fraction of wealth allocated to equities
  - Increases (reduces) share in the bond if $m_t^B < 1$ $\left(\text{if } m_t^B > 1\right)$
  - Unambiguously increases BTE ratio

♦ Aging effects (horizon effects):

- As horizon increases (age decreases) forward risk increases
- Suppose wealth in bond numéraire held constant. Horizon effects are then same as forward risk effects
- Standard advice:
  - Increase BTE when individuals age
  - Perception that "stocks are for the long run"
    - Siegel (1998)
    - Large magnitude of long horizon Sharpe ratios for stocks wrt bds
- Analysis above shows optimal behavior critically depends on preferences
  - Investors averse to forward T-risk will actually increase their BTE in response to increase in risk induced by longer horizon
  - Younger investors of this sort will find it optimal to tilt their risky allocation toward bonds, not toward equities as recommended

PART 5: Applications

♦ Questions of interest:

- Extreme risk aversion:
  - Impact on portfolio
  - Preferences for risky assets (stocks and bonds)
- Long run portfolios
  - Investors caring about distant horizons (pension plans, institutions,...)
  - How to invest?

5.1 Extreme risk aversion

♦ Extreme risk aversion:

- Absolute risk aversion goes to infinity
- Absolute risk tolerance goes to zero
- $(\Gamma_u(z,v), \Gamma_U(z)) \to (0,0)$ for all $z \in D$ and all $v \in [0,T]$

♦ Extreme behavior can take various forms:

- More intense in certain maturity ranges
- Consumption & bequest preferences provide natural classification of behavior
- Behavior of ratio: $\Gamma_u(z_1,v)/\Gamma_U(z_2)$

♦ Proposition 5.1:

- Assume risk tolerance measures vanish
  - $(\Gamma_u(z,v), \Gamma_U(z)) \to (0,0)$ for all $z \in D$ and all $v \in [0,T]$
  - Ratios of risk tolerance measures

$$\frac{\Gamma_u(z_1,v)}{\Gamma_U(z_2)} \to k \quad \text{for all } z_1, z_2 \in \mathbb{D} \text{ and all } v \in [0,T]$$

$$\frac{\Gamma_u(z_1,v_1)}{\Gamma_u(z_2,v_2)} \to 1 \quad \text{for all } z_1, z_2 \in \mathbb{D} \text{ and all } v_1, v_2 \in [0,T]$$

    * for some constant $k \in [0,+\infty)$

- Optimal allocation in the limit:
  - Coupon-paying bond with constant coupon C and face value F

$$C = \frac{x}{\int_0^T B_0^v dv + B_0^T/k} \qquad \text{and} \qquad F = \frac{x}{\int_0^T B_0^v dvk + B_0^T}.$$

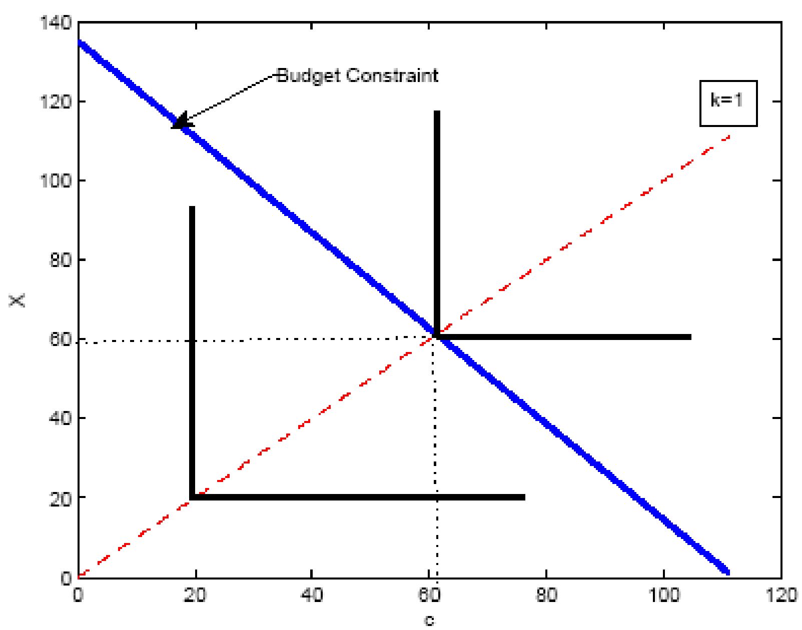
  - If $k = 0$ exclusive preference for pure discount bd: $(C,F) = \left(0, x/B_0^T\right)$
  - If $k \to \infty$ preference for a pure coupon bond: $(C,F) = \left(x/\int_0^T B_0^v dv, 0\right)$

♦ Limit habitat preferences are striking

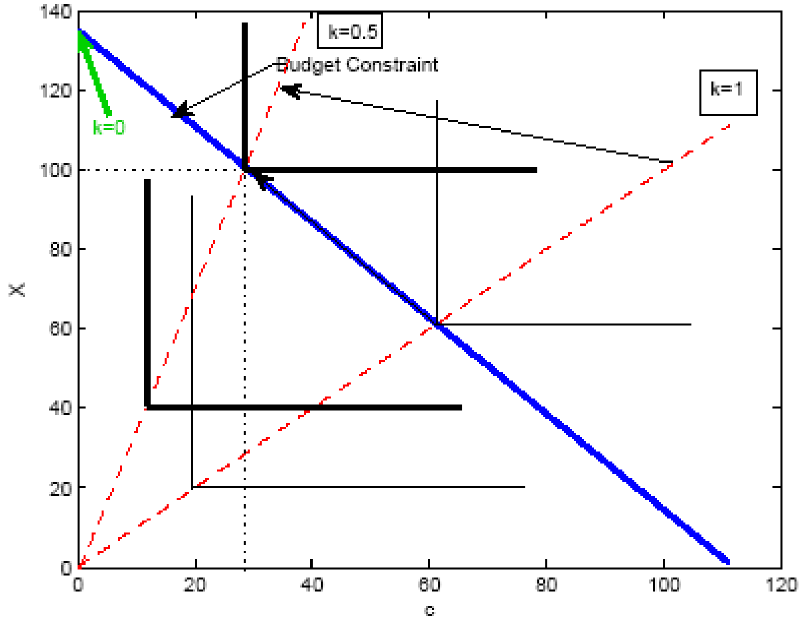
- Natural conjecture: more extreme RA determines preferred instrument
- Reverse holds
  - Least extreme drives habitat
  - More weight on maturities where risk tolerance greater

♦ Reason:

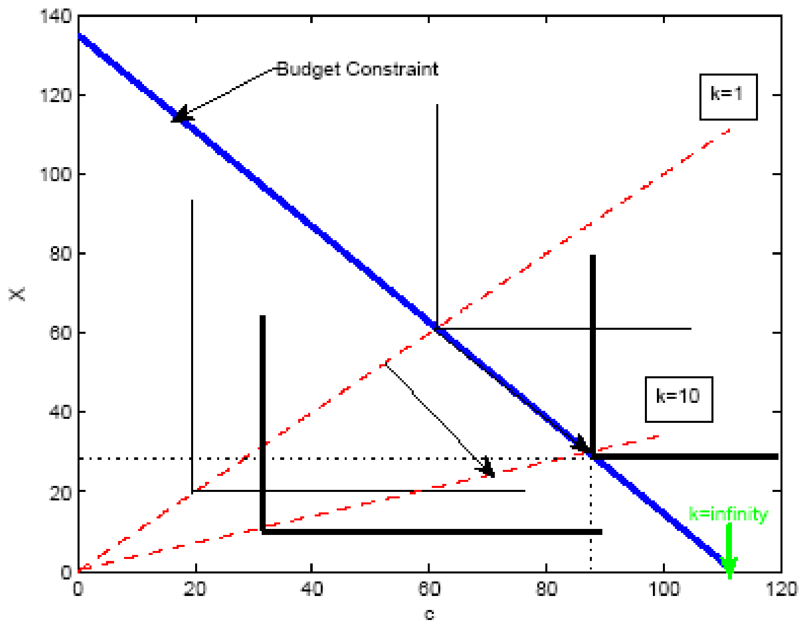
- When absolute risk tolerances vanish investor seeks perfect smoothing
- Preference for certainty: constant consumption and terminal wealth
- With vNM preferences:
  - Vanishing risk tolerance implies vanishing elasticity of intertemp. substit.
  - Limit preferences, in (C, F) plane, induce Leontief indifference curves
  - Engle curves:
    * Relate demand for C, F to income
    * Keeping prices B_t^T and $\int_t^T B_t^v dv$ constant
    * Slope k
- If k is finite solution is interior as both income elast of C & F are finite



- If $k = 0$ Engle curves horizontal; income elast of consumption fct. null



- If $k \to \infty$ Engle curves vertical; income elast. of bequest fct. null



- Income elasticity behavior explains choice of preferred habitat

♦ Remark:

- Wachter (2003) special case with utility over bequest and Ito prices
- Finds preferred habitat when relative RA goes to infinity: pure discount bd

5.2 Portfolio turnpike theorems: asymptotic portfolios

♦ Market:

- Preferences: vNM with utility over terminal wealth
- Financial market: equities, long term bonds and money market account

$$\begin{bmatrix} dS_t/S_t \\ dB_t^T/B_t^T \end{bmatrix} = \begin{bmatrix} \mu_t^S - \delta_t \\ \mu_t^B \end{bmatrix} dt + \begin{bmatrix} \sigma_t^S & 0 \\ \sigma^B(t,T)\,\varrho(t,T) & \sigma^B(t,T)\sqrt{1-\varrho(t,T)^2} \end{bmatrix} \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix}$$

  - $\sigma^B(t,T)$ instantaneous bond return volatility
  - $\varrho(t,T)$ instantaneous correlation between bond and equities return

♦ Proposition 5.1: (Gaussian bond return models)

- Deterministic forward market price of risk $-\sigma^Z(t,T)$
- Limits

$$\lim_{T\uparrow\infty} B_t^T = 0, (P - a.s.) \quad (normal\ market) \tag{59}$$

$$\lim_{T\uparrow\infty} \varrho(t,T) = \varrho^L(t) \in (-1,1) \tag{60}$$

$$\lim_{T\uparrow\infty} \sigma^B(t,T) = \sigma^{B,L}(t) \in [-\infty,+\infty] \tag{61}$$

(where $\varrho^L(t), \sigma^{B,L}(t)$ are deterministic)

- If positive part of inverse marginal utility has MD (i.e., $I(y^*\xi_T)^+ \in D^{1,2}$) and marginal utility U' varies regularly at infinity with exponent $-R^L$, i.e.,

$$\lim_{x\uparrow\infty} \frac{U'(ax)}{U'(x)} = a^{-R^L}, \quad for\ all\ a > 0$$

then long run optimal portfolio is given by

$$\left(\frac{\pi_t^S}{X_t^*}\right)_L \equiv \lim_{T\to\infty} \frac{\pi_t^S}{X_t^*} = \frac{1}{R^L}\left(\frac{\theta_{1t}}{\sigma_t^S} - \gamma^L(t)\frac{\theta_{2t}}{\sigma_t^S}\right)$$

$$\left(\frac{\pi_t^B}{X_t^*}\right)_L \equiv \lim_{T\uparrow\infty} \frac{\pi_t^B}{X_t^*} = \begin{cases} sign(\theta_{2t}) \times \infty & if\ \sigma^{B,L}(t) = 0 \\ 1 - \frac{1}{R^L} & if\ |\sigma^{B,L}(t)| = +\infty \\ \frac{1}{R^L}\left(\frac{\theta_{2t}}{\sigma^{B,L}(t)\sqrt{1-\rho^L(t)}}\right) + 1 - \frac{1}{R^L} & otherwise \end{cases}$$

where $\gamma^L(t) \equiv \varrho^L(t)/\sqrt{1-\varrho^L(t)^2}$. The long run bond-to-equities ratio is

$$e_t^L \equiv \frac{\left(\frac{\pi_t^B}{X_t^*}\right)_L}{\left(\frac{\pi_t^S}{X_t^*}\right)_L} = \begin{cases} sign\left(\frac{\theta_{2t}}{\sigma_t^S}\right) \times sign\left(\theta_{1t} - \gamma^L(t)\theta_{2t}\right) \times \infty & if\ \sigma^{B,L}(t) = 0 \\ \left(R^L - 1\right)\left(\frac{\theta_{1t}}{\sigma_t^S} - \gamma^L(t)\frac{\theta_{2t}}{\sigma_t^S}\right)^{-1} & if\ |\sigma^{B,L}(t)| = +\infty \\ \sigma_t^S\left(\frac{\theta_{2t}}{\sigma^{B,L}(t)\sqrt{1-\varrho^L(t)^2}} + R^L - 1\right)\left(\theta_{1t} - \gamma^L(t)\theta_{2t}\right)^{-1} & otherwise \end{cases}$$

♦ Assumptions: markets

- Condition (59): normal market - Dybvig, Rogers and Back (1999)
- Condition (60): markets are complete and non-degenerate in the limit
- Condition (61): limit of bond volatilities exists, but may take infinite values

♦ Assumptions: preferences

- Regularly varying marginal util. behaves like CRRA as wealth becomes large
- HARA utility: regular variation with coefficient $-R$ at infinity

$$U(x) = \frac{1}{1-R}(x-A)^{1-R}, \quad U'(x) = (x-A)^{-R}, \quad A > 0$$

$$\lim_{x\uparrow\infty} \frac{U'(ax)}{U'(x)} = \lim_{x\uparrow\infty} \frac{(ax-A)^{-R}}{(x-A)^{-R}} = a^{-R}, \quad \text{for all } a > 0$$

- Mixtures of power utilities: regular variation with exponent $-R_1$ at infiniti

$$U(x) = \sum_{k=1}^{K} \frac{1}{1-R_k} x^{1-R_k}, \quad U'(x) = \sum_{k=1}^{K} x^{-R_k}, \quad 0 < R_1 < ... < R_K$$

$$\lim_{x\uparrow\infty} \frac{U'(ax)}{U'(x)} = \lim_{x\uparrow\infty} \frac{\sum_{k=1}^{K}(ax)^{-R_k}}{\sum_{k=1}^{K} x^{-R_k}} = a^{-R_1}, \quad \text{for all } a > 0$$

- Mixtures of HARA: regular variation with exponent $-R_1$

$$U(x) = \sum_{k=1}^{K} \frac{1}{1-R_k}(x-A_k)^{1-R_k}, \quad U'(x) = \sum_{k=1}^{K}(x-A_k)^{-R_k}, \quad 0 < R_1 < ... < R_K$$

with $A_k > 0, k = 1, ..., K$, satisfy

$$\lim_{x\uparrow\infty} \frac{U'(ax)}{U'(x)} = \lim_{x\uparrow\infty} \frac{\sum_{k=1}^{K}(ax-A_k)^{-R_k}}{\sum_{k=1}^{K}(x-A_k)^{-R_k}} = a^{-R_1}, \quad \text{for all } a > 0$$

- Sums, products, compositions of RV functions are RV (Seneta (1976))

♦ Portfolio behavior:

- No forward density hedge: MPR in bd numeraire deterministic
- Demand for equities is pure mean-variance
- Demand for bonds depends on bond volatility
  - Bond vol null: bond demand goes to infinity
  - Bond vol infinite: mean-variance demand vanishes, bond hedge remains
  - Otherwise: combination of these two motives

♦ Remarks: relation to literature

- Financial market: covers most models examined for long run behavior
- Long run risk models: Bansal & Yaron (2004), Alvarez & Jermann (2005)
- Portfolio turnpike models:
  - Huberman-Ross (1983), Theorem 2 of Dybvig-Rogers-Back (1999)
  - Asset returns serially independent and interest rate non-random
- Here interest rates can be random
  - Results identify limit portfolio explictly

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