Part III: Time Consistency

Georg Ch. Pflug

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Time consistency for final processes

We consider a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\boldsymbol{\mathcal{F}} \in \mathcal{F}$. Let

$$\mathcal{A}_2(\cdot|\mathcal{F}_1) : L_p(\Omega, \mathcal{F}, P) \rightarrow L_{p'}(\Omega, \mathcal{F}_1, P)$$

a conditional acceptability-type mapping and let

$$\mathcal{A}_1(\cdot)$$

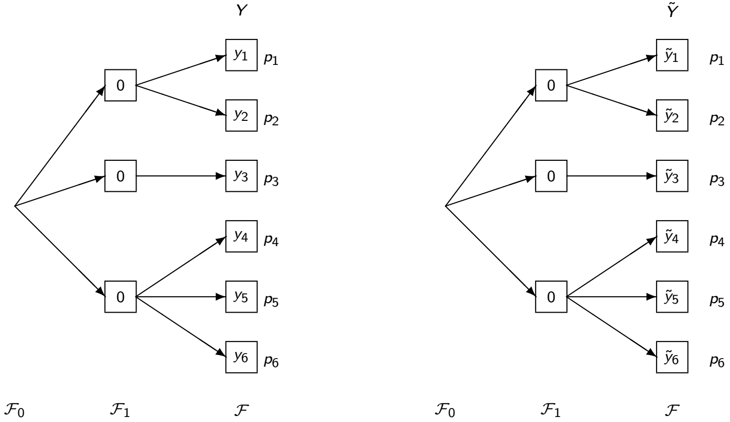
be an unconditional acceptability measure. Typically, but not necessarily, $\mathcal{A}_1$ is the unconditional counterpart of $\mathcal{A}_2(\cdot|\mathcal{F}_1)$. Notice that we consider now just one final profit&loss variable Y and not a full profit&loss process.

Definition. (Artzner at al. 2007). The pair $\mathcal{A}_1(\cdot)$, $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ is called *time consistent*, if for all $X, Y \in L_p(\Omega, \mathcal{F}, P)$ the implication

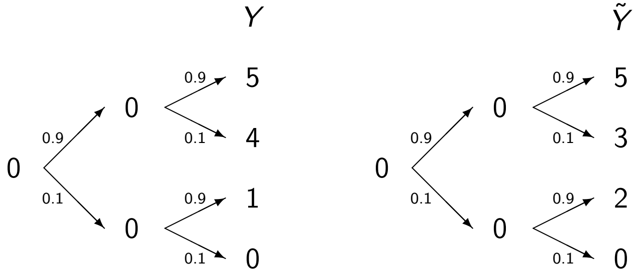
$$\mathcal{A}_2(Y|\mathcal{F}_1) \leq \mathcal{A}_2(\tilde{Y}|\mathcal{F}_1) \; a.s. \implies \mathcal{A}_1(Y) \leq \mathcal{A}_1(\tilde{Y})$$

holds.

Illustration



$\mathbb{A}$V@R is not time-consistent.



$$\mathbb{A}\mathrm{V@R}_{0.1}(Y|\mathcal{F}_1) = (4;0) \geq (3;0) = \mathbb{A}\mathrm{V@R}_{0.1}(\tilde{Y}|\mathcal{F}_1)$$

while

$$\mathbb{A}\mathrm{V@R}_{0.1}(Y) = 0.9 < 1.8 = \mathbb{A}\mathrm{V@R}_{0.1}(\tilde{Y}).$$

Definition. A pair $\mathcal{A}_1(\cdot)$, $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ is called *acceptance consistent*, if for all $Y \in L_p(\Omega, \mathcal{F}, \mu)$ the implication

$$\operatorname{ess\,inf} \mathcal{A}_2(Y|\mathcal{F}_1) \leq \mathcal{A}_1(Y)$$

holds. It is called *rejection consistent*, if

$$\operatorname{ess\,sup} \mathcal{A}_2(Y|\mathcal{F}_1) \geq \mathcal{A}_1(Y).$$

(see e.g. Weber, 2006).

Remark. Acceptance consistency is equivalent to:
For all q

$$\mathcal{A}_2(Y|\mathcal{F}_1) \geq q \, a.s. \Longrightarrow \mathcal{A}_1(Y) \geq q$$

(*if each conditional $Y|\mathcal{F}_1$ is accepted, then also Y is accepted*).

Rejection consistency is equivalent to: for all q

$$\mathcal{A}_2(Y|\mathcal{F}_1) \leq q \, a.s. \Longrightarrow \mathcal{A}_1(Y) \leq q$$

(*if each conditional $Y|\mathcal{F}_1$ is rejected, then also Y is rejected*).
Proposition. If $\mathcal{A}_1(0) = 0$ and $\mathcal{A}_2(0|\mathcal{F}_1) = 0$ *a.s.* and $\mathcal{A}_1(\cdot)$, $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ are translation equivariant then time consistency implies acceptance and rejection consistency.

Definition. A pair $\mathcal{A}_1(\cdot)$, $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ is called

(i) *compound convex* , if for all $Y \in dom\mathcal{A}$

$$\mathcal{A}_1(Y) \leq \mathbb{E}\left(\mathcal{A}_2(Y|\mathcal{F}_1)\right).$$

(ii) *compound concave*, if for all $Y \in dom\mathcal{A}$

$$\mathcal{A}_1(Y) \geq \mathbb{E}\left(\mathcal{A}_2(Y|\mathcal{F}_1)\right).$$

If both properties hold, we call the pair *compound linear*.

Remark.
For version-independent conditional functionals, the definition is modified as follows:

Definition.

(i) $\mathcal{A}$ is called *compound convex*, if for all $K(\cdot|v)$, $G(v)$

$$\mathcal{A}\{K \circ G\} \leq \int \mathcal{A}\{K(\cdot|v)\}\, \mathrm{d}G(v).$$

(ii) $\mathcal{A}$ is called *compound concave*, if for all $K(\cdot|v)$, $G(v)$

$$\mathcal{A}\{K \circ G\} \geq \int \mathcal{A}\{K(\cdot|v)\}\, \mathrm{d}G(v).$$

Here $K \circ G = \int K(\cdot|v)\, \mathrm{d}G(v)$.

Theorem.

(i) compound convexity implies rejection consistency.

(ii) compound concavity implies acceptance consistency.

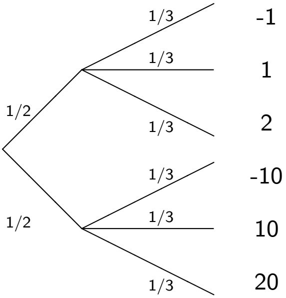
Theorem. Let $\boldsymbol{\mathcal{F}} = (\mathcal{F}_0, \ldots, \mathcal{F}_T)$ be a filtration.

(i) $\mathcal{A}(\cdot|\cdot)$ is compound convex, iff $Y_t = \mathcal{A}(Y|\mathcal{F}_t)$ is a submartingale.

(ii) $\mathcal{A}(\cdot|\cdot)$ is compound concave, iff $Y_t = \mathcal{A}(Y|\mathcal{F}_t)$ is a supermartingale.

The $\mathbb{A}$V@R is compound convex, hence rejection consistent.

$\mathbb{A}$V@R is not acceptance consistent.



$$\mathbb{A}\text{V@R}_{2/3}(Y|\mathcal{F}_1) = 0, \qquad \mathbb{A}\text{V@R}_{2/3}(Y) = -2.$$

Definition. (Artzner (2008), Kupper (2008), Jobert (2000)) A pair $\mathcal{A}_1(\cdot)$, $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ is called *recursive*, if for all $Y \in L_p(\Omega, \mathcal{F}, \mu)$ the equation

$$\mathcal{A}_1(Y) = \mathcal{A}_1\left(\mathcal{A}_2(Y|\mathcal{F}_1)\right)$$

holds.
Of special interest are version-independent conditional functionals, which are auto-recursive (i.e. for which $\mathcal{A}_1(\cdot) = \mathcal{A}_2(\cdot|\mathcal{F}_0)$).
Examples.

- EC-functionals (i.e. functionals of the form $\mathbb{E}[\mathcal{A}(Y|\mathcal{F}_1)]$) are recursive.
- The entropic functional is auto-recursive.

$$\begin{aligned} & -\frac{1}{\gamma}\log\mathbb{E}[\exp(-\gamma[-\frac{1}{\gamma}\log\mathbb{E}[\exp(-\gamma Y)|\mathcal{F}_1)]])] \\ = \; & -\frac{1}{\gamma}\log\mathbb{E}[\mathbb{E}[\exp(-\gamma)|\mathcal{F}_1]] = -\frac{1}{\gamma}\log\mathbb{E}[\exp(-\gamma Y)] \end{aligned}$$

- The $\mathbb{A}$V@R is not auto-recursive.

The relation between time consistency and recursivity

Theorem. (Artzner et. al., 2007) A pair $\mathcal{A}_1(\cdot)$, $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ with translation equivariant $\mathcal{A}(\cdot|\mathcal{F}_1)$, the property $\mathcal{A}(0|\mathcal{F}_1) = 0$ and monotonic $\mathcal{A}(\cdot)$ is time consistent if and only if it is recursive.
Proof. Let the pair be recursive and let $\mathcal{A}_2(Y|\mathcal{F}_1) \leq \mathcal{A}_2(\tilde{Y}|\mathcal{F}_1)$.
Then, by monotonicity,
$\mathcal{A}_1(Y) = \mathcal{A}_1(\mathcal{A}_2(Y|\mathcal{F}_1)) \leq \mathcal{A}_1(\mathcal{A}_2(\tilde{Y}|\mathcal{F}_1)) = \mathcal{A}_1(\tilde{Y})$.
Conversely, let the pair be time consistent. By assumption,

$$\mathcal{A}_2(\mathcal{A}_2(Y|\mathcal{F}_1)|\mathcal{F}_1) = \mathcal{A}_2(\mathcal{A}_2(Y|\mathcal{F}_1) + 0|\mathcal{F}_1) = \mathcal{A}_2(Y|\mathcal{F}_1) + 0.$$

Setting $\tilde{Y} = \mathcal{A}_2(Y|\mathcal{F}_1)$ and using the time consistency, leads to

$$\mathcal{A}_1(\tilde{Y}) = \mathcal{A}_1(\mathcal{A}_2(Y|\mathcal{F}_1)) = \mathcal{A}_1(Y),$$

which is the equation of recursivity.

Theorem. (Kupper and Schachermayer, 2008) Suppose that the pair $\mathcal{A}_1(\cdot)$ and $\mathcal{A}(\cdot|\mathcal{F}_t)$ is recursive for a sequence $\mathcal{F}_t, t = 1, 2, \ldots$ of σ-algebras, such that $\mathcal{A}_1$ is strictly monotonic, version independent and satisfies $\mathcal{A}_1(c) = c$. If moreover all $\mathcal{F}_t$'s are atomless and one may construct a sequence of independent Bernoulli random variables adapted to $(\mathcal{F}_t)$, then $\mathcal{A}_0(Y)$ must be of the form $U^{-1}[\mathbb{E}(U(Y))]$ for some utility function U. If $\mathcal{A}$ is translation-equivariant, it must be the entropic functional.

Theorem. For a time consistent pair $\mathcal{A}_1(\cdot)$, $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ with translation equivariant $\mathcal{A}_2(\cdot|\mathcal{F}_1)$ and monotonic $\mathcal{A}_1(\cdot)$, strictness (i.e. $\mathcal{A}_2(X) \leq \mathbb{E}(X)$) implies compound convexity.

Theorem. Compound-linearity (i.e. compound convexity and compound concavity together) implies time consistency.

Enforcing time consistency by composition (nesting)

Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\boldsymbol{\mathcal{F}} = (\mathcal{F}_0, \ldots, \mathcal{F}_T)$ of σ-fields $\mathcal{F}_t$, $t = 0, ..., T$, with $\mathcal{F}_T = \mathcal{F}$ be given. Let $\mathcal{Y}_t := L_p(\mathcal{F}_t)$ for $t = 1, \ldots, T$ and some $p \in [1, +\infty)$. Let, for each $t = 1, \ldots, T$, conditional acceptability mappings $\mathcal{A}_{t-1} := \mathcal{A}(\cdot\,|\mathcal{F}_{t-1})$ from $\mathcal{Y}_T$ to $\mathcal{Y}_{t-1}$ be given. Introduce a multi-period probability functional $\mathcal{A}$ on $\mathcal{Y} := \times_{t=1}^{T} \mathcal{Y}_t$ by compositions of the conditional acceptability mappings $\mathcal{A}_{t-1}$, $t = 1, \ldots, T$, namely,

$$\begin{aligned} \mathcal{A}(Y; \boldsymbol{\mathcal{F}}) &:= \mathcal{A}_0[Y_1 + \cdots + \mathcal{A}_{T-2}[Y_{T-1} + \mathcal{A}_{T-1}(Y_T)]\cdot] \\ &= \mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{T-1}(\sum_{t=1}^{T} Y_t) \end{aligned}$$

for every $Y_t \in \mathcal{Y}_t$. (Ruszczynski and Shapiro, 2006). Notice that these functionals are recursive in a trivial way.

The nested $\mathbb{A}\text{V@R}$

Example. Consider the conditional Average Value-at-Risk (of level $\alpha \in (0,1]$) as conditional acceptability mapping

$$\mathcal{A}_{t-1}(Y_t) := \mathbb{A}\text{V@R}_\alpha(\cdot\,|\mathcal{F}_{t-1})$$

for every $t = 1, \dots, T$. Then the multi-period probability functional

$$n\mathbb{A}\text{V@R}_\alpha(Y;\boldsymbol{\mathcal{F}}) = \mathbb{A}\text{V@R}_\alpha(\cdot\,|\mathcal{F}_0) \circ \cdots \circ \mathbb{A}\text{V@R}_\alpha(\cdot\,|\mathcal{F}_{T-1})(\sum_{t=1}^{T} Y_t)$$

satisfies (MA0), (MA1'), (MA2), (MA3). It is called the *nested Average Value-at-Risk*.

Proposition. Suppose that for every t the conditional acceptability functional $\mathcal{A}_t(\cdot|\mathcal{F}_t)$ maps $L_p(\mathcal{F}_t)$ to $L_p(\mathcal{F}_{t-1})$ and is defined by

$$\begin{aligned}\mathcal{A}_t(Y|\mathcal{F}_t) &= \mathbf{inf}\{\mathbb{E}(Y\ Z|\mathcal{F}_t) - \mathcal{A}_t^+(Z|\mathcal{F}_t) : Z \geq 0, \\ &\qquad \mathbb{E}(Z|\mathcal{F}_t) = 1, Z \in \mathcal{Z}_t(\mathcal{F}_t)\}.\end{aligned}$$

Then the nested acceptability functional $\mathcal{A}(Y;\boldsymbol{\mathcal{F}}) = \mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{T-1}(\sum_{t=1}^T Y_t)$ has the dual representation

$$\begin{aligned}&\mathcal{A}(Y;\boldsymbol{\mathcal{F}}) = \inf\{\mathbb{E}[(Y_1 + \cdots + Y_T)M_T] - \sum_{t=1}^T \mathbb{E}[\mathcal{A}_t^+(Z_t|\mathcal{F}_t)M_{t-1}] : \\ &\mathbb{E}(Z_t|\mathcal{F}_t) = 1, Z_t \geq 0, Z_t \in \mathcal{Z}_t(\mathcal{F}_t)\}\end{aligned}$$

where $M_t = \prod_{s=1}^t Z_t$ and $M_0 = 1$. Notice that the supergradients (M_t) must be martingales w.r.t. $\boldsymbol{\mathcal{F}}$ with $\mathbb{E}(|M_t|^q) < \infty$.

The entropic functional

The nested entropic acceptability functional is $\mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{T-1}(Y)$ with $\mathcal{A}_t(Y) = -\frac{1}{\gamma} \log \mathbb{E}[\exp(-\gamma Y)|\mathcal{F}_t]$, for Y nonnegative and nonvanishing. Recall that the dual representation of $\mathcal{A}_t$ is

$$\mathcal{A}_t(Y|\mathcal{F}_1) = \textbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_t) + \frac{1}{\gamma}\mathbb{E}(Z \log Z|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = 1, Z \geq 0\}.$$

Here $0 \log 0$ is defined as 0. The nested entropic acceptability functional has the representation $\mathcal{A}(Y; \boldsymbol{\mathcal{F}}) =$

$$\inf\{\mathbb{E}[(\sum_{t=1}^{T} Y_t) \prod_{s=1}^{T} Z_s] + \sum_{t=1}^{T} \mathbb{E}[\mathbb{E}(Z_t \log Z_t|\mathcal{F}_t) \prod_{s=1}^{t-1} Z_s] : \mathbb{E}(Z_t|\mathcal{F}_t) = 1, Z_t >$$

$$= \inf\{\mathbb{E}[(\sum_{t=1}^{T} Y_t)M] + \mathbb{E}[M \log M] : \mathbb{E}(M) = 1, M > 0\}.$$

The nested entropic functional collapses to the unconditional entropic functional.

The nested $\mathbb{A}$V@R

Example. The nested $\mathbb{A}$V@R has the following dual representation:

$$n\mathbb{A}\mathrm{V@R}_\alpha(Y;\mathcal{F}) = \inf\{\mathbb{E}[(Y_1 + \cdots + Y_T)M_T] : 0 \le M_t \le \frac{1}{\alpha}M_{t-1},$$
$$\mathbb{E}(M_t|\mathcal{F}_{t-1}) = M_{t-1}, M_0 = 1, t = 1, \ldots, T\}.$$

The nested average value-at-risk $n\mathbb{A}$V@R is given by a linear stochastic optimization problem containing functional constraints.

A comparison

- **Composition**. $\mathcal{A}(Y;\mathcal{F}) = \mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{T-1}(\sum_{t=1}^T Y_t)$ has the dual representation

$$\mathcal{A}(Y;\mathcal{F}) = \inf\{\mathbb{E}[(\sum_{t=1}^T Y_t)M_T] - \sum_{t=1}^T \mathbb{E}[\mathcal{A}_t^+(Z_t|\mathcal{F}_t)M_{t-1}] :$$
$$\mathbb{E}(Z_t|\mathcal{F}_t) = 1, Z_t \geq 0, Z_t \in \mathcal{Z}_t(\mathcal{F}_t)\}$$

  where $M_t = \prod_{s=1}^t Z_t$ and $M_0 = 1$. Notice that the supergradients (M_t) must be martingales w.r.t. $\mathcal{F}$ with $\mathbb{E}(|M_t|^q) < \infty$.
- **Separable EC**. $\sum_{t=1}^T \mathbb{E}[\mathcal{A}_t(Y_t|\mathcal{F}_{t-1})] =$

$$\inf\{\sum_{t=1}^T \mathbb{E}[Y_t Z_t] - \sum_{t=1}^T \mathbb{E}[\mathcal{A}_t^+(Z_t|\mathcal{F}_t)] : Z_t \lhd \mathcal{F}_t, Z_t \geq 0, \mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1$$

$$= \sum_{t=1}^T \inf\{\mathbb{E}[Y_t Z_t] - \mathbb{E}[\mathcal{A}_t^+(Z_t|\mathcal{F}_t)] : Z_t \lhd \mathcal{F}_t, Z_t \geq 0, \mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1\}$$

Time consistent decisions ?

Let a stochastic multistage decision problem be given, which is defined on the basis of a tree process $\nu = (\nu_1, \ldots, \nu_T)$. Let $\mathbb{P}$ be the probability governing the tree process. Let $\mathbb{P}^{\nu_t = z}$ be the conditional distribution of the tree process, given that the value of ν_t is z. The solution is called time-consistent, if the solutions of the original problem and the conditional problems (when the decisions at times $1, \ldots, t-1$ are kept fixed) coincide on the subtree of $\nu_t = z$.

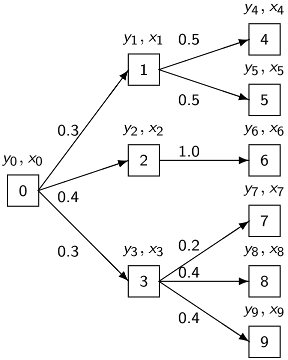
Proposition. If the objective is a nested acceptability functional (and no other constraints are present), then the decision problem leads to time consistent decisions.

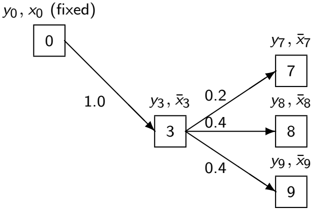
Theorem. (Kreps and Porteus, 1978). Under some consistency axioms, including a condition on "linearity of decisions w.r.t. probabilities, time consistent decision problem are those for which the objective is the expected utility.

y_i: values of the scenario process

x_i: optimal decisions

$\boxed{i}$: node numbers

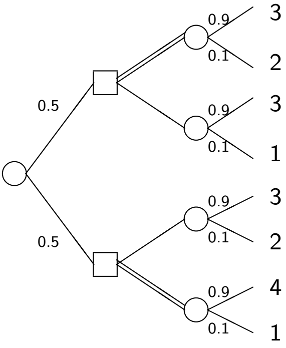




A full problem and the conditional problem "given node 3". The decision problem is time-consistent, if $x_i = \bar{x}_i$, for all nodes, which are in the subtree of the conditioning node.

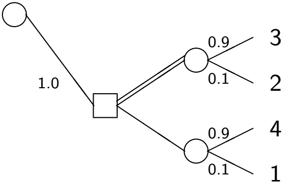
↯ Time inconsistency appears in a natural way in optimality problems. We want to find

$$\max\{\mathbb{E}(Y) : \mathbb{A}V@R_{0.05}(Y) \geq 2\} \qquad \text{or} \qquad \max \mathbb{E}(Y) + \mathbb{A}V@R_{0.05}(Y).$$



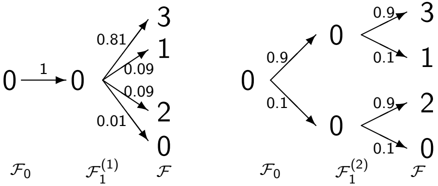
double line = optimal decision

The conditional problem given the first node:



The paradoxon disappears, if the objective is a nested functional, e.g. the nested $\mathbb{A}$V@R or the entropic functional.

Time consistency contradicts information monotonicity.



In both examples, the final income Y is the same, but in the right example, the filtration is finer. One calculates

$$\mathbb{A}\mathrm{V@R}_{0.1}[\mathbb{A}\mathrm{V@R}_{0.1}(Y|\mathcal{F}_1^{(1)})] = 0.9 > 0 = \mathbb{A}\mathrm{V@R}_{0.1}[\mathbb{A}\mathrm{V@R}_{0.1}(Y|\mathcal{F}_1^{(2)})].$$

Notice that

$$\mathbb{E}[\mathbb{A}\mathrm{V@R}_{0.1}(Y|\mathcal{F}_1^{(1)})] = \mathbb{E}[\mathbb{A}\mathrm{V@R}_{0.1}(Y|\mathcal{F}_1^{(2)})] = 0.9.$$