New insights into exponential utility indifference valuation

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based on joint work with **Christoph Frei** (ETH Zürich)

Original problem

Financial problem

- **Basic problem:** valuation of non-attainable contingent claims in incomplete financial markets.
- **Basic idea:** value by **utility indifference**, i.e., define time t **seller value** b_t of time T payoff B implicitly via

$$\operatorname*{ess\,sup}_{\pi} E\left[U\left(x_t + \int_t^T \pi_r \, dS_r \right) \middle| \mathcal{F}_t \right]$$
$$= \operatorname*{ess\,sup}_{\pi} E\left[U\left(x_t + b_t + \int_t^T \pi_r \, dS_r - B \right) \middle| \mathcal{F}_t \right].$$

- **Exponential** utility indifference valuation: $U(x) = -e^{-\alpha x}$.
- Why exponential?
  - For expected utility, problem seems intractable for other U.
  - With exponential U, can even obtain fairly explicit results.

Mathematical problem

- So: need to understand in detail **dynamic value process** for problem of maximising expected utility from terminal wealth with **random endowment**, i.e., the process

$$V_t^B := V_t^B(0) := \operatorname*{ess\,sup}_{\pi} E\left[U\left(\int_t^T \pi_r \, dS_r - B \right) \middle| \mathcal{F}_t \right].$$

- Then **exponential utility indifference seller value** is

$$b_t = \frac{1}{\alpha} \log \frac{V_t^B}{V_t^0}.$$

- **Many references . . .**

References . . .

- Becherer (2003, 2006)
- Biagini/Frittelli (2005)
- Biagini/Frittelli/Grasselli/Hurd (2008)
- Delbaen/Grandits/Rheinländer/Samperi/S/Stricker (2002)
- Frei/S (2008a,b)
- Grasselli (2007)
- Grasselli/Hurd (2007)
- Henderson (2002)
- Hobson (1994)
- Henderson/Hobson (2002, 2004)

... references ...

- Hu/Imkeller/Müller (2005)
- Kabanov/Stricker (2002)
- Leung/Sircar (2008)
- Mania/S (2005)
- Monoyios (2006)
- Morlais (2007)
- Musiela/Zariphopoulou (2004)
- Rouge/El Karoui (2000)
- Tehranchi (2004)
- Zariphopoulou (2001)

. . . references

- (other missing links)
- . . .

Examples and motivation

A simple Markovian example

- Discounted price S and **factor/nontraded asset** Y given by

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sigma S_t\,dW_t, \\ dY_t &= b(t, Y_t)\,dt + a(t, Y_t)\,d\overline{W}_t; \end{aligned}$$

  correlated Brownian motions $W, \overline{W}$ with $d\langle W, \overline{W}\rangle_t = \rho\,dt$.
- Payoff is $B = g(Y_T)$.
- **Intuition:** Payoff depends on "asset" Y, but can only use correlated asset S for trading and hedging.
- Typical examples:
  - valuation of executive stock options (ESOs)
  - valuation of weather derivatives
  - ...

PDE approach

- All is **Markovian**; hence, writing $X_t = X_t^\pi$ for wealth from self-financing strategy π, expect, for some function $v(t,x,y)$,

$$V_t^B = v(t, X_t, Y_t).$$

- Exponential utility is multiplicative; so guess **separable** form

$$v(t,x,y) = U(x)F(t,y).$$

- **Formal** derivation of **HJB equation** gives

$$0 = F_t + \max_\pi \left(\tfrac{1}{2}\sigma^2\pi^2\alpha^2 F + \pi(-\rho\sigma a\alpha F_y - \mu\alpha F)\right) + \tfrac{1}{2}a^2 F_{yy} + bF_y$$

  with boundary condition $F(T,y) = g(y)$.
- Formal maximisation gives **nonlinear PDE**

$$0 = F_t + \tfrac{1}{2}a^2 F_{yy} + bF_y - \tfrac{1}{2}\frac{(\rho\sigma a F_y + \mu F)^2}{\sigma^2 F} :$$ **how to solve?**

PDE transformation

- Using **clever power transformation**

$$F(t,y) = f(t,y)^{1/(1-\rho^2)}$$

magically reduces nonlinear PDE to linear, solvable PDE; find

$$V_t^B = F(t, Y_t) = -\left(E_{\widehat{P}}\left[\left(e^{\alpha B - \frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)} \right)^{1/\delta} \middle| Y_t \right] \right)^{\delta} \qquad (1)$$

with **minimal martingale measure** $\widehat{P}$ and **distortion power**

$$\delta = \frac{1}{1-\rho^2} \qquad \longleftarrow \textbf{ remember this!}$$

- **Why** does this work**?** PDE techniques give no insight (to us).
- **References:** Henderson/Hobson, Musiela/Zariphopoulou, . . .

A first generalisation

- PDEs are not needed: consider $dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_t$; **correlated** Brownian motions $W, \overline{W}$ with $d\left\langle W, \overline{W} \right\rangle_t = \rho \, dt$.
- **Tehranchi:** no explicit factor Y;
  - μ, σ are $\mathbb{F}^{\overline{W}}$**-predictable**;
  - B is $\mathcal{F}_T^{\overline{W}}$**-measurable**.
- Then again, with $\delta = \frac{1}{1-\rho^2}$,

$$V_t^B = -\left(E_{\widehat{P}}\left[\left(\exp\left(\alpha B - \frac{1}{2}\int_t^T \frac{\mu_r^2}{\sigma_r^2} \, dr \right) \right)^{1/\delta} \middle| \mathcal{F}_t^{\overline{W}} \right] \right)^{\delta}. \quad (2)$$

- Correlation ρ is still **constant**.
- Technique: **clever Hölder-type inequality**; again gives no genuine insight (to us).

Further generalisations

- **Frei/S:** S and Y both Itô processes;
  - Sharpe ratio μ/σ is $\mathbb{F}^{\overline{W}}$**-predictable**;
  - payoff B is $\mathcal{F}_T^{\overline{W}}$**-measurable**;
  - **stochastic correlation** ρ is $\mathbb{F}^{\overline{W}}$**-predictable**.
- Then, compare (2),

$$V_t^B = -\left(E_{\widehat{P}}\left[\left(\exp\left(\alpha B - \frac{1}{2}\int_t^T \frac{\mu_r^2}{\sigma_r^2}\,dr\right)\right)^{1/\delta} \middle| \mathcal{F}_t^{\overline{W}}\right]\right)^{\delta}\Bigg|_{\delta=\delta_t(\omega)} \tag{3}$$

for some **random** $\mathcal{F}_t$-measurable δ_t satisfying

$$\inf_{s\in[t,T]} \frac{1}{\|1-\rho_s^2\|_{L^\infty}} \le \delta_t \le \sup_{s\in[t,T]} \left\| \frac{1}{1-\rho_s^2} \right\|_{L^\infty} \quad \longleftarrow \quad \textbf{note: } \delta \approx \frac{1}{1-\rho^2}$$

- Proofs via martingale arguments, which seem (to us) more transparent; but still no full understanding; in particular, why measurability conditions?
- Extends to **multidimensional** Itô process setting; then role of bounds on ρ is played by bounds on minimal and maximal (random) eigenvalues of instantaneous correlation matrix.
- Extensions to **sharper inequalities via BSDE techniques** are recent work (with **C. Frei** and **S. Malamud**).
- **Completely general semimartingale** S; gives (at last) understanding.
- Key is good expression for

$$V_t^B := V_t^B(0) := \operatorname*{ess\,sup}_{\pi} E\left[U\left(\int_t^T \pi_r \, dS_r - B\right)\middle|\mathcal{F}_t\right].$$

General ideas for static case

The static case ($t = 0$) with $B \equiv 0$

- **Basic problem:** for exponential utility $U(x) = -e^{-\alpha x}$, solve

$$E\left[U\left(X_T^\pi\right)\right] = E\left[U\left(\int_0^T \pi_r \, dS_r\right)\right] = \max_\pi !$$

- Equivalently: $E_P\left[\exp\left(-\alpha \int_0^T \pi_r \, dS_r\right)\right] = \min_\pi !$
- **Simplest case:** if S is a P-**martingale**, then by Jensen

$$E_P\left[\exp\left(-\alpha \int_0^T \pi_r \, dS_r\right)\right] \geq \exp\left(-\alpha E_P\left[\int_0^T \pi_r \, dS_r\right]\right) = 1;$$

equality holds for $\pi \equiv 0$, and **value** is 1.

- In general, take an **EMM Q for S** and write

$$E_P\left[\exp\left(-\alpha\int_0^T \pi_r\,dS_r\right)\right] = E_Q\left[\frac{dP}{dQ}\exp\left(-\alpha\int_0^T \pi_r\,dS_r\right)\right] = \min_{\pi}!$$

- Can use **same trick as above** if EMM Q has **special form**

$$\frac{dQ^*}{dP} = \exp\left(c^* + \int_0^T \zeta_r^*\,dS_r\right). \tag{4}$$

- Then optimal strategy is $\pi = -\frac{1}{\alpha}\zeta^*$ and optimal **value** is

$$V_0^0 = -\exp(-c^*) \qquad \longleftarrow \textbf{note: still for } B \equiv 0$$

- **Key point:** Requirement (4) on density of Q^* already identifies Q^* as **minimal entropy martingale measure** for S.

The static case with random endowment

- Next **(intermediate) problem:** for **claim** B and exponential utility $U(x) = -e^{-\alpha x}$, solve

$$E\left[U\left(X_T^\pi - B\right)\right] = \max_\pi !$$

- Equivalently: reduce to above problem structure via

$$\begin{aligned} E_P\left[\exp\left(-\alpha\int_0^T \pi_r\,dS_r + \alpha B\right)\right] &= C\,E_{P_B}\left[\exp\left(-\alpha\int_0^T \pi_r\,dS_r\right)\right] \\ &= \min_\pi !, \end{aligned}$$

where we exploit multiplicative structure of exponential utility to introduce

$$\frac{dP_B}{dP} = C^{-1}\,e^{\alpha B}.$$

Solution for static case

- So earlier **solution recipe** changes as follows:

  1) find MEMM Q_B^* for S under P_B, i.e., $Q_B^* = \underset{Q}{\operatorname{argmin}}\, H(Q|P_B)$.

  2) find constant $c^{*,B}$ and integrand $\zeta^{*,B}$ in representation

$$\frac{dQ_B^*}{dP} = \exp\left(c^{*,B} + \int_0^T \zeta_r^{*,B}\, dS_r\right).$$

  3) optimal strategy is $\pi = -\frac{1}{\alpha}\zeta^{*,B}$; optimal **value** is

$$V_0^B = -\exp\left(-c^{*,B} + \log E_P[e^{\alpha B}]\right).$$

Conditional formulation

The conditional problem

- **Dynamic problem:** for exponential utility $U(x) = -e^{-\alpha x}$, solve at time t

$$E\left[U\left(X^{\pi}_{t,T} - B\right)\,\middle|\,\mathcal{F}_t\right] = \max_{\pi}!$$

- As before, use change of measure to Q with density process $Z^Q = \mathcal{E}(N^Q)$ to rewrite problem as

$$E_P\left[\exp\left(-\alpha\int_t^T \pi_r\,dS_r + \alpha B\right)\,\middle|\,\mathcal{F}_t\right]$$

$$= E_Q\left[\frac{\mathcal{E}(N^Q)_t}{\mathcal{E}(N^Q)_T}\exp\left(-\alpha\int_t^T \pi_r\,dS_r + \alpha B\right)\,\middle|\,\mathcal{F}_t\right] = \min_{\pi}!$$

Repeating the idea

- Since we can take out $\mathcal{F}_t$-measurable factors, we want, **for easy solution**

$$e^{\alpha B}\frac{\mathcal{E}(N^Q)_t}{\mathcal{E}(N^Q)_T} = \text{something } \mathcal{F}_t\text{-measurable}$$
$$\times \exp(\text{a stochastic integral of } S \text{ from } t \text{ to } T),$$

i.e.,

$$e^{\alpha B} = \frac{\mathcal{E}(N^Q)_T}{\mathcal{E}(N^Q)_t} \times \exp\left(\int_t^T \zeta_r \, dS_r\right) \times \kappa_t .$$

- Take log and simplify a little to get ...

The fundamental entropy representation

- ... the **fundamental representation** we want, namely

$$B = \frac{1}{\alpha} \log \frac{\mathcal{E}(N^B)_T}{\mathcal{E}(N^B)_t} + \int_t^T \eta_r^B \, dS_r + k_t^B \quad (5)$$

$$= \frac{1}{\alpha} \log \mathcal{E}(N^B)_T + \int_0^T \eta_r^B \, dS_r + k_0^B \quad (6)$$

$$+ k_t^B - k_0^B - \int_0^t \eta_r^B \, dS_r - \frac{1}{\alpha} \log \mathcal{E}(N^B)_t. \quad (7)$$

- (5) is a **non-standard BSDE**.
- (6) shows the $FER(B)$.
- (7) shows how to express k^B in terms of **triple** (N^B, η^B, k_0^B).

The general dynamic solution

- Additional **requirements** on $FER(B)$:
  - N^B is local P-martingale null at 0;
  - $\mathcal{E}(N^B)$ is positive P-martingale;
  - S is under $P(N^B)$ local martingale, i.e., $P(N^B)$ is EMM for S.
- In above terms, **solution** is straightforward:
  - the optimal strategy is $\pi = \eta^B$;
  - the **optimal value** at time t is

$$V_t^B = V_t^B(0) = -e^{\alpha k_t^B};$$

  - the **utility indifference seller value** is $b_t = k_t^B - k_t^0$.
- So: **key** is understanding structure of process k^B.

Theory: Main results

Precise model

- S is general semimartingale (need **not** be locally bounded).
- Claim B satisfies $E_P\left[e^{\alpha B}\right] < \infty$.
- There exist loss variables (in the sense of Biagini/Frittelli 2005) for B and for 0 (i.e., under P_B and under $P = P_0$):
  - $W \geq 1$;
  - $E_R\left[e^{cW}\right] < \infty$ for all $c > 0$ and $R \in \{P_B, P\}$;
  - there are $\beta^i \in L(S^i)$, never 0, with $\left|\int \beta^i \, dS^i\right| \leq W$.
- $\mathbb{P}_B^{e,f}$, assumed $\neq \emptyset$, denotes the set of all $Q \approx P$ such that $H(Q|P_B) < \infty$ and S is a σ-martingale under Q.

$FER(B)$

- By **definition**, **fundamental entropy representation** $FER(B)$ exists if

$$B = \frac{1}{\alpha} \log \mathcal{E}(N^B)_T + \int_0^T \eta_r^B \, dS_r + k_0^B$$

where

- N^B is a local P-martingale null at 0; $\mathcal{E}(N^B)$ is a positive P-martingale; and S is under $P(N^B)$ a σ-martingale;
- η^B is in $L(S)$ with $\int_0^T \eta_r^B \, dS_r$ in $L^1(P(N^B))$;
- k_0^B is a constant.

$FER^*(B)$

- An $FER(B)$,

$$B = \frac{1}{\alpha} \log \mathcal{E}(N^B)_T + \int_0^T \eta_r^B \, dS_r + k_0^B,$$

is called an $FER^*(B)$ if in addition

  - $\int_0^T \eta_r^B \, dS_r$ is Q-integrable with Q-expectation ≤ 0 for every $Q \in \mathbb{P}_B^{a,f}$;
  - $\int \eta^B \, dS$ is a martingale under $P(N^B)$.

- **Idea: good representation** with good integrability properties.

First main result

- **Theorem 1:**
  $FER(B)$ exists $\Longleftrightarrow$ $FER^*(B)$ exists $\Longleftrightarrow$ $\mathbb{P}_B^{e,f} \neq \emptyset$.
- **Comments:**
  - not surprising; but very general and neat.
  - can be viewed as **existence result for non-standard BSDE**.

- **Proposition 2:**
  Necessary and sufficient conditions for a given $FER(B)$ to be the (unique) $FER^*(B)$.
- **Comments:**
  - happens iff $P(N^B)$ **equals the MEMM** Q_B^* for S and P_B, and $\int \eta^B \, dS$ is a $P(N^B)$-martingale.
  - example shows that multiple $FER(B)$ may exist.

Second main result

- **Theorem 3:**
  Optimal value for conditional problem at time t is given by

$$V_t^B = \operatorname*{ess\,sup}_\pi E_P\left[-\exp\left(-\alpha\int_t^T \pi_r\,dS_r + \alpha B\right)\middle|\mathcal{F}_t\right] = -e^{\alpha k_t^B}.$$

- **Comments:**
  - not surprising, but again very general and neat.
  - gives **indifference value** at time t as $b_t = k_t^B - k_t^0$.
  - gives **non-standard BSDE** for process k^B as

$$k_t^B = B - \int_t^T \eta_r^B\,dS_r - \frac{1}{\alpha}\log\frac{\mathcal{E}(N^B)_T}{\mathcal{E}(N^B)_t}. \qquad (8)$$

- Equation (8) is **key to understanding** distortion formulas!

Computations help understanding

Rewriting the dynamic value process

$$-V_t^B = e^{\alpha k_t^B} \overset{(8)}{\underset{\uparrow}{=}} \exp\left(\alpha B - \alpha \int_t^T \eta_r^B \, dS_r + \log \mathcal{E}(N^B)_{t,T}\right)$$

$$= \frac{\exp\left(\alpha B + \int_t^T \varphi_r \, dS_r\right)}{Z_{t,T}^Q} \frac{Z_{t,T}^Q}{\mathcal{E}(N^B)_{t,T}} \exp\left(-\int_t^T (\alpha\eta_r^B + \varphi_r)\, dS_r\right)$$

$$=: \Psi_t^B \frac{Z_{t,T}^Q}{\mathcal{E}(N^B)_{t,T}} \exp\left(-\int_t^T (\alpha\eta_r^B + \varphi_r)\, dS_r\right)$$

- Now **estimate** in two ways:
  - $e^{-\alpha k_t^B} E_Q[\Psi_t^B | \mathcal{F}_t] = \dots$ $\longleftarrow$ log **outside cond. exp.**
  - $\alpha k_t^B = E_Q[\log(-V_t^B) | \mathcal{F}_t] = \dots$ $\longleftarrow$ log **inside**

First estimation

- **First estimate:**

$$\begin{aligned} & e^{-\alpha k_t^B} E_Q[\Psi_t^B | \mathcal{F}_t] \\ &= E_Q\left[\frac{\mathcal{E}(N^B)_{t,T}}{Z^Q_{t,T}} \exp\left(- \int_t^T (\alpha \eta_r^B + \varphi_r)\, dS_r \right) \middle| \mathcal{F}_t \right] \\ &= E_{Q_B^*}\left[\exp\left(- \int_t^T (\alpha \eta_r^B + \varphi_r)\, dS_r \right) \middle| \mathcal{F}_t \right] \geq 1 \end{aligned}$$

by Bayes and Jensen, if both stochastic integrals are Q_B^*-martingales.

- So: $$\alpha k_t^B \leq \log E_Q[\Psi_t^B | \mathcal{F}_t].$$

Second estimation

- **Second estimate:**

$$\alpha k_t^B = E_Q\left[\log\Psi_t^B - \int_t^T (\alpha\eta_r^B + \varphi_r)\,dS_r - \log\frac{\mathcal{E}(N^B)_{t,T}}{Z_{t,T}^Q}\,\middle|\,\mathcal{F}_t\right]$$
$$= E_Q[\log\Psi_t^B|\mathcal{F}_t] - 0 + E_Q\left[-\log\frac{\mathcal{E}(N^B)_{t,T}}{Z_{t,T}^Q}\,\middle|\,\mathcal{F}_t\right],$$

  if both stochastic integrals are Q-martingales; and last term is ≥ 0 by Bayes and Jensen since $\mathcal{E}(N^B)$ is P-martingale.

- So: $$\alpha k_t^B \geq E_Q[\log\Psi_t^B|\mathcal{F}_t].$$

A simple general fact

- **General fact:**

$$E_Q[\log R|\mathcal{G}] \le Y \le \log E_Q[R|\mathcal{G}]$$

implies that

$$Y = \log \left(E_Q \left[R^{\frac{1}{\delta}} \middle| \mathcal{G} \right] \right)^{\delta} \bigg|_{\delta = \delta^R(\omega)}$$

for some random $\mathcal{G}$-measurable δ^R.

- **Indeed:** $(\omega, \delta) \mapsto f(\omega, \delta) := \log \left(E_Q \left[R^{\frac{1}{\delta}} \middle| \mathcal{G} \right](\omega) \right)^{\delta}$ is P-a.s. continuous and decreasing in δ on $[1, \infty)$, with P-a.s.

$$\lim_{\delta \to 1} f(\omega, \delta) = \log E_Q[R|\mathcal{G}], \quad \lim_{\delta \to \infty} f(\omega, \delta) = E_Q[\log R|\mathcal{G}].$$

So can **interpolate** to get result.

Explaining the distortions

Third main result

- **Theorem 4: Interpolation expression:**

$$k_t^B(\omega) = \frac{1}{\alpha} \log \left(E_Q \left[|\Psi_t^B|^{1/\delta} \,\middle|\, \mathcal{F}_t \right] (\omega) \right)^{\delta} \Bigg|_{\delta=\delta_t^B(\omega)} \qquad (9)$$

for some $\mathcal{F}_t$-measurable δ_t^B, provided that

$$\Psi_t^B := \frac{\exp\left(\alpha B + \int_t^T \varphi_r \, dS_r \right)}{Z_T^Q / Z_t^Q}$$

is bounded away from 0 and ∞ for some $Q \in I\!P_B^{e,f}$ and some $\varphi \in L(S)$ such that $\int \varphi \, dS$ is both a Q- and a Q_B^*-martingale.

Is this any good?

- **Comments** on Theorem 4:
  - **general version** of PDE distortion power transformation!
  - **explains distortion via interpolation.**
  - compare (1), (2), (3).
- **Question:** how to find some $Q \in \mathbb{P}_B^{e,f}$ and some $\varphi \in L(S)$ such that

$$\Psi_t^B := \frac{\exp\left(\alpha B + \int_t^T \varphi_r \, dS_r\right)}{Z_T^Q / Z_t^Q}$$

is bounded away from 0 and ∞ and $\int \varphi \, dS$ is both a Q- and a Q_B^*-martingale? **Hopeless?**

- **Answer: No!** Can find Q and φ explicitly if S is **continuous**.

The case of continuous S

- Suppose S is **continuous**; so $S = S_0 + M + \int \lambda \, d\langle M \rangle$.
- Take $Z^Q := \mathcal{E}\left(-\int \lambda \, dM\right) = \exp\left(-\int \lambda \, dM - \frac{1}{2}\int \lambda^2 \, d\langle M \rangle\right)$, the density process of the **minimal martingale measure**.
- Choose $\varphi = -\lambda$; so $\exp(\int \varphi \, dS) = Z^Q \exp\left(-\frac{1}{2}\int \lambda^2 \, d\langle M \rangle\right)$.
- Then

$$\Psi_t^B := \frac{\exp\left(\alpha B + \int_t^T \varphi_r \, dS_r\right)}{Z_T^Q / Z_t^Q} = \exp\left(\alpha B - \frac{1}{2}\int_t^T \lambda_r^2 \, d\langle M \rangle_r\right).$$

- **Example:** in Itô process model, $\int_t^T \lambda_r^2 \, d\langle M \rangle_r = \int_t^T \frac{\mu_r^2}{\sigma_r^2} \, dr$: recovers with (9) earlier results in (1), (2) and (3).

Getting back to correlation

- Note: in general, δ is not further specified (lies between 1 and ∞).
- On the other hand: often $\delta \approx \frac{1}{1-\rho^2}$.
- **Brownian setting:**
  - can derive $FER(B)$ in (6) from **predictable representation property in** filtration $\mathbb{F}^{\overline{W}}$ of $\overline{W}$;
  - this needs **measurability conditions** on $\frac{\mu}{\sigma}$, B and ρ;
  - allows to estimate k^B in terms of ρ;
  - hence can interpolate in concrete ρ instead of abstract δ;
  - hence get distortion formula in terms of (bounds on) correlation.
- More details in **Frei/S (2008a,b)**.

The end (for now ...)

Thank you for your attention !

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http://www.math.ethz.ch/∼frei