Dynamic Correlation Hedging in Copula Models for Portfolio Selection

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Outline

Motivation and objectives

The model for asset prices - accounting for extreme dependencies through:

- Tail dependence
- Observable factors driving the dynamics of asset correlation

The portfolio problem:

- Market price of risk hedging demands due to tail dependence
- Correlation hedging demands due to observable factors

Conclusion

Asymmetries and downside risk

- Probability that assets in a portfolio will jointly decline
  - Correlation? Tail events (extreme moves) ask for different dependence measures

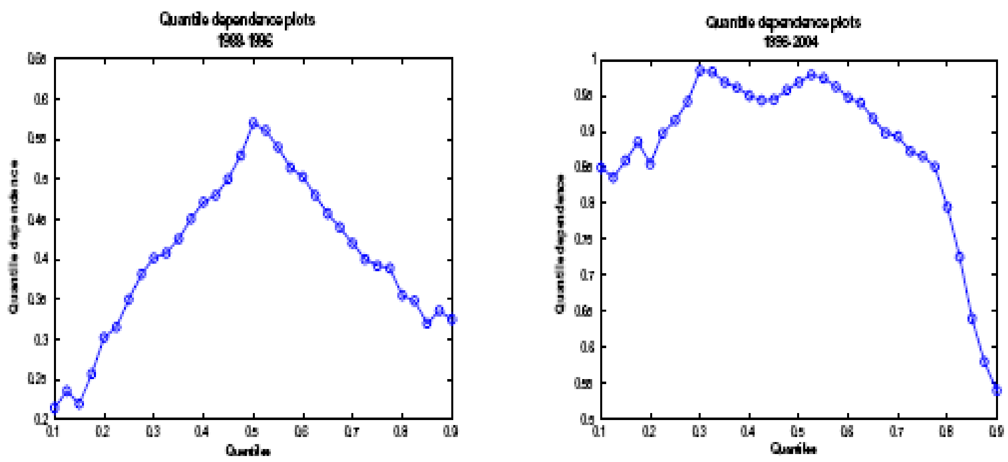
- Asymmetries:
  - Univariate case: skewness
  - Multivariate case: widespread evidence that correlations are higher in extreme market downturns than in extreme market upturns
    - Longin and Solnik (2001), Ang and Chen (2002), Poon, Rockinger, Tawn (2004)
  - Theoretical justification of this empirical fact: REE model, Ribeiro and Veronesi (2002)

- Portfolio choice implications
  - Beyond mean-variance: investors' sensitivity to downside risk - aversion to extreme negative returns
  - More than myopic behaviour: hedging terms that shift the portfolio composition under extremal dependence

Evidence of dependence asymmetry

A 'near' tail dependence measure (Coles, Currie and Tawn, 1999):

the probability that one variable exceeds a certain quantile given that the other has exceeded it:



Plots of quantile dependence for the de-trended log-prices of S&P500 vs. NASDAQ for the 1988-1996 and 1996-2004 subperiods

Objectives

- Propose a model that is able to accommodate an extremal dependence structure
  - in two methodologically distinct ways:
    - static (tail dependence) vs. dynamic dependence (DCC) with observable factors driving it
  - ... that also models in a tractable way univariate asset return properties
  - ... while keeping a continuous time complete market setup for tractable portfolio solutions

- Examine its effect on portfolio choice and isolate intertemporal hedging demands, including those for correlation hedging
  - Detect changes in portfolio composition: expect a shift towards the risk-free asset in turmoil periods
  - Determine the loss in terms of wealth resulting from disregarding dependence during extreme return realizations
  - Determine the impact on the hedging terms of observable factors that can drive dependence between the assets in the portfolio

Related literature

- Modeling comovement asymmetries
  - GARCH-copula (Jondeau and Rockinger (2002,2005), Patton (2004))
  - Regime-Switching (Ang and Chen (2002), Ang and Bekaert (2002), Chesnay and Jondeau (2001))
  - Systemic jumps (Das and Uppal (2003))
  - Stock return correlations and the phase of the business cycle: Ledoit et al. (2003), Erb et al. (1994)
- Stationary diffusion
  - Univariate process based on the GH distribution: Eberlein and Keller (1995), Rydberg (1999), Bibby and Sorensen (2003)
  - Multivariate process using copula functions: Kunz (2002) – multivariate CIR process
- Portfolio choice
  - Unconditional allocation (Patton (2004))
  - Conditional allocation and the hedging demands (Ang and Bekaert (2003), Das and Uppal (2004), Liu, Longstaff, and Pan (2003))
  - Correlation hedging: Buraschi et al. (2007)
- Solution methodology:
  - Monte Carlo with Malliavin Derivatives: Detemple, Garcia and Rindisbacher (2003)

Contribution

- Propose a model for asset prices that is able to account for extremal dependence
- Solve for the optimal portfolio in the presence of tail dependence for a general utility function specification
- Examine the intertemporal portfolio hedging terms induced by a possibly asymmetric dependence structure and the correlation hedging demands induced by observable factors
- Impact of copulas on the risk management of asset portfolios in a dynamic framework

The model: tools

Modeling the dependence structure and the concept of copulas

Main question: the effect of the dependence structure on portfolio hedging terms

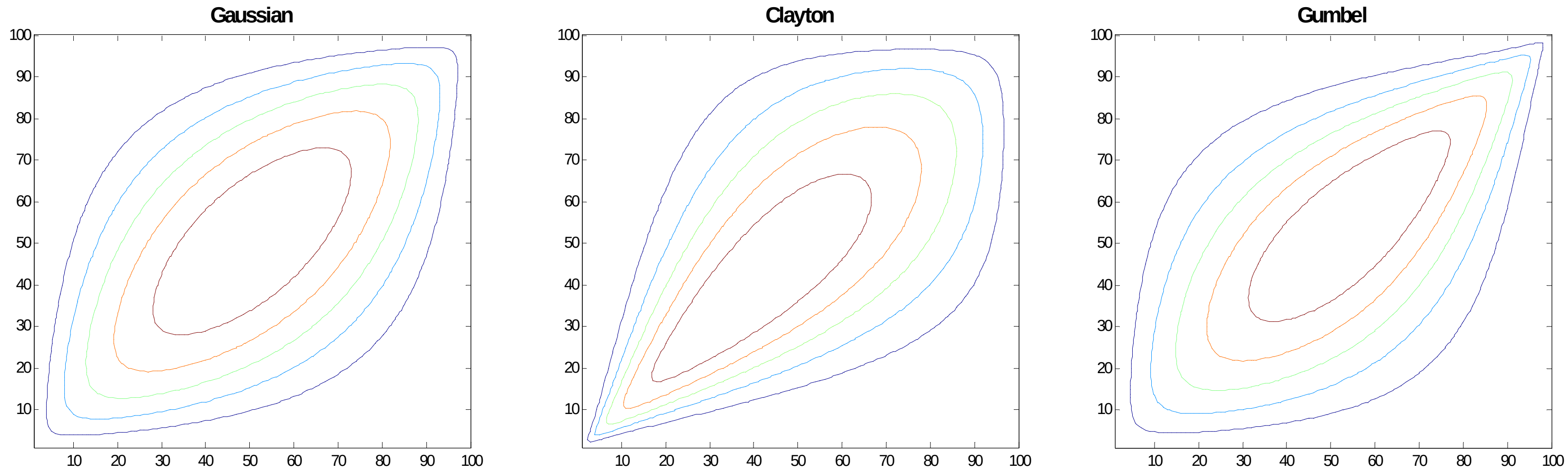
$\Rightarrow$ Isolate the effect of **marginals** (ex. fat tails) from that of the **dependence structure** (ex. asymmetric tail dependence) through the use of copulas:

$$C : [0,1]^n \rightarrow [0,1]$$
$$F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n))$$

$$C(u_1,\ldots,u_n) = F\left(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)\right)$$
$$c(F_1(x_1),\ldots,F_n(x_n)) \times \prod_{i=1}^{n} f_i(x_i) = f(x_1,\ldots,x_n)$$

Tail dependence and copula functions

Distributions with N(0,1) marginals and different copulas with corr = 0.7



Substantially different tail behaviour for the same correlation parameter!

- Upper tail dependence $$\lambda_U = \lim_{u\to 1} \Pr\left(Y > F_Y^{-1}(u) \middle| X > F_X^{-1}(u)\right) = \lim_{u\to 1} \frac{\left(1-2u+C(u,u)\right)}{1-u}$$
- Lower tail dependence $$\lambda_L = \lim_{u\to 0} \Pr\left(Y < F_Y^{-1}(u) \middle| X < F_X^{-1}(u)\right) = \lim_{u\to 0} \frac{C(u,u)}{u}$$

The model for stock prices

Incorporating tail dependence

$$S_{it} = S_{0t} \exp\left(k_i t + X_{it}\right) \qquad , i = 1 \ldots d$$

$$\text{where} \quad dX_t = \mu\left(X_t, F_t\right) dt + \Lambda\left(X_t, F_t\right) dW_t$$

- A simple analogy with GBM
- Incorporate thick tails and dependence in extreme realizations in the stationary distribution of the state variable process
- Significance from the perspective of an investor with a long-term investment horizon

The model for stock prices

Incorporating tail dependence (cont.)

- Need a link between the stationary distribution of the process and its diffusion specification (Chen, Hansen, Scheinkman (2005)):

$$dX_t = \tilde{\mu}(X_t, F_t)\,dt + \tilde{\Lambda}(X_t, F_t)\,dW_t$$

$$\tilde{\mu}_{ij} = \frac{1}{2q}\sum_{i=1}^{d}\frac{\partial\left(v_{ij}q\right)}{\partial x_i}$$

$$\Sigma = \tilde{\Lambda}\tilde{\Lambda}' \quad \text{with entries } v_{ij}$$

- Thick tails through the marginals and tail dependence through the copula specification of the stationary density:

$$q(x_1,\dots,x_n) \equiv \tilde{c}(x_1,\dots,x_n)\prod_{i=1}^{n}\tilde{f}^{\,i}(x_i)$$

- Conditional volatility and correlation dynamics:

$$v_{ij} = \rho_{ij}\sigma_i^X\sigma_j^X$$

$$\sigma_i^X = \sigma_i\left[\tilde{f}^{\,i}(x_i)\right]^{-\frac{1}{2}\kappa_i}$$

Distributional assumptions

- **Marginal behaviour:** semi-heavy tails through the GH specification

$$f(x) \sim |x|^{\lambda-1} \exp\{(\mp\alpha + \beta)x\} \quad x \to \pm\infty$$

- **Dependence:** through the copula specification
  - *Gaussian copula*: no tail dependence
  - *Student's t copula*: symmetric tail dependence
  - *Gaussian – Symmetrized Joe-Clayton mixture copula*: asymmetric tail dependence

Conditional correlation dynamics

- The dynamics of conditional correlation

$$dX_t = \tilde{\mu}(X_t, F_t)\,dt + \tilde{\Lambda}(X_t, F_t)\,dW_t^X$$

$$\text{entries of } \tilde{\Sigma} = \tilde{\Lambda}\tilde{\Lambda}' : \tilde{\nu}_{ij}(X_t, F_t) = \tilde{\Upsilon}_{ij}(X_t, F_t)\,\sigma_i^X(X_t)\,\sigma_j^X(X_t)$$

$$\text{conditional correlation: } \Upsilon_{ij}(X_t, F_t) = \mathrm{A}\left(h_{ij}(X_t, F_t)\right)$$

Correlation hedging demands implied by observable factors:

- Macroeconomic conditions (CFNAI index)
- Market-wide volatility (the VIX)

Conditional correlation dynamics

- Dynamic conditional correlation
  - Case A: $h_{ij}(X_t) = \gamma_{ij,0} + \gamma_{ij,1} \max\left(\sigma_1{}^X(X_t), \dots, \sigma_d{}^X(X_t)\right) + \gamma_{ij,2} \prod_{k=1}^{d} \tilde{F}(X_{kt})$
  - Case B: $h_{ij}(X_t, F_t) = \gamma_{ij,0} + \gamma_{ij,1} F_t^{VIX} + \gamma_{ij,2} \prod_{k=1}^{d} \tilde{F}(X_{kt}) + \gamma_{ij,3} F_t^{CFNAI}$
  - Case C: $h_{ij}(F_t) = \gamma_{ij,0} + \gamma_{ij,1} F_t^{VIX} + \gamma_{ij,3} F_t^{CFNAI}$
- Benchmark case: CCC $\gamma_{ij,1} = \gamma_{ij,2} = \gamma_{ij,3} = 0$

Portfolio choice in the presence of extremal dependence

- The investor's problem $\max_{\alpha} U(\omega_T) \equiv E\left[u(\omega_T)\right]$

  Evolution of wealth equation:

  $$d\omega_t = r_t\omega_t dt + \omega_t\alpha_t^{'}\left[(\mu_t - r_t 1)dt + \Lambda_t dW\right]_t \qquad ,\omega_0 = \bar{\omega}$$

- Utility function: HARA (ex. Cox and Huang, 1989)

  $$u(x) = \frac{1}{1-R}(x+B)^{1-R}$$

  – Intolerance towards wealth shortfalls: infinite risk aversion when wealth approaches a lower boundary

Portfolio choice in the presence of extremal dependence

The portfolio decomposition formula:

Explicit hedging demands in terms of conditional expectations of the state variables and their Malliavin derivatives

$$\alpha_t = \left(\Lambda_t\left(Y_t\right)'\right)^{-1}\left[\frac{1}{R(\omega_t)}\theta(Y_t)MV(Y_t,\omega_t) - IRH(Y_t,\omega_t) - MPRH(Y_t,\omega_t)\right]$$
$$= \alpha_t^{MV} + \alpha_t^{IRH} + \alpha_t^{MPRH}$$

- Mean-variance demand $MV(t,Y_t,\omega_t) \equiv E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\frac{R(\omega_t)}{R(\omega_T)}I_{\omega_T>0}\right]$
- Interest rate hedge $IRH(t,Y_t,\omega_t)' \equiv E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\left(1-R(\omega_T)^{-1}\right)I_{\omega_T>0}\int_t^T D_t r_s ds\right]$
- Market price of risk hedge $MPRH(t,Y_t,\omega_t)' \equiv E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\left(1-R(\omega_T)^{-1}\right)I_{\omega_T>0}\int_t^T (dW_s+\theta_s ds)' D_t\theta_s\right]$

Effect of the dependence structure: in the MPR hedge through the process of the market price of risk and its Malliavin derivative

Correlation hedging demands

- MPR hedging term:

$$H_t^{\Theta} = \int_t^T \Psi_s D_t Y_s$$

$$\text{where } \Psi_s = \left(dW_s + \Theta\left(s, Y_s\right) ds\right)' \partial_2 \Theta\left(s, Y_s\right)$$

$$H_{t,T,i}^{\Theta} = \int_t^T \left(\Psi_{1,s} D_{i,t} X_{1,s} + \ldots + \Psi_{d,s} D_{i,t} X_{d,s}\right) + \int_t^T \left(\Psi_{d+1,s} D_{i,t} F_s^{VIX} + \Psi_{d+2,s} D_{i,t} F_s^{CFNAI}\right)$$

- Correlation hedging demands due to observable factors

$$V_{t,T,i}^{\Theta} = \int_t^T \Psi_{d+1,s} D_{i,t} F_s^{VIX}$$

$$M_{t,T,i}^{\Theta} = \int_t^T \Psi_{d+2,s} D_{i,t} F_s^{CFNAI}$$

Correlation hedging demands

- Modeling no tail dependence in the stationary distribution of the state variables would render the **conditional correlation specification** solely responsible for reproducing increased dependence in bad states
  - With or without observed factors

- Letting conditional correlation be constant opens the second channel of reproducing the stylized fact through the **stationary distribution** only -> portfolio impact of unconditional dependence beyond that induced by correlation hedging

- Examine the behavior of the hedging demands in all alternative scenarios: are the two channels of reproducing dependence in bad states leading to similar results in terms of:
  - Magnitude
  - Certainty equivalent cost

The cost of ignoring extremal dependence asymmetries

- The certainty equivalent (CEQ) cost:

  Compare alternative strategies on the basis of the CEQ cost: the additional wealth required by the investor in order to use a suboptimal portfolio strategy

$$E_0\left[u\left(\omega_T^* \middle| \omega_0 = 1\right)\right] = E_0\left[u\left(\omega_T \middle| \omega_0 = \bar{\omega}\right)\right]$$

For a CRRA investor: $\bar{\omega} = \left\{ E_0\left[\xi_T^{*\,1-\frac{1}{\gamma}} \right] / E_0\left[\xi_T^{\,1-\frac{1}{\gamma}} \right] \right\}^{\frac{\gamma}{1-\gamma}}$

- No tail dependence vs. asymmetric tail dependence
- Constant vs. dynamic conditional correlation with observable factors

Bivariate application: S&P500 vs. NASDAQ

- Portfolio assets:
  - 2 risky funds

$$dS_{1t} = S_{1t}\left\{\mu_1^{S}\left(X_t, F_t\right)dt + \sigma_1^{X}\left(X_t\right)dW_{1t}^{X}\right\}$$

$$dS_{2t} = S_{2t}\left\{\mu_2^{S}\left(X_t, F_t\right)dt + \Upsilon\left(X_t, F_t\right)\sigma_1^{X}\left(X_{tt}\right)dW_{1t}^{X} + \sqrt{1-\Upsilon\left(X_t, F_t\right)^2}\,\sigma_2^{X}\left(X_{tt}\right)dW_{2t}^{X}\right\}$$

  - A long term pure discount bond

$$dY_t^r = \kappa_r\left(\theta^r - Y_t^r\right)dt + \sigma_r\sqrt{Y_t^r}\,dW_t^r$$

$$B\left(t,T\right) = \exp\left\{a\left(T-t\right) + b\left(T-t\right)Y_t^r\right\}$$

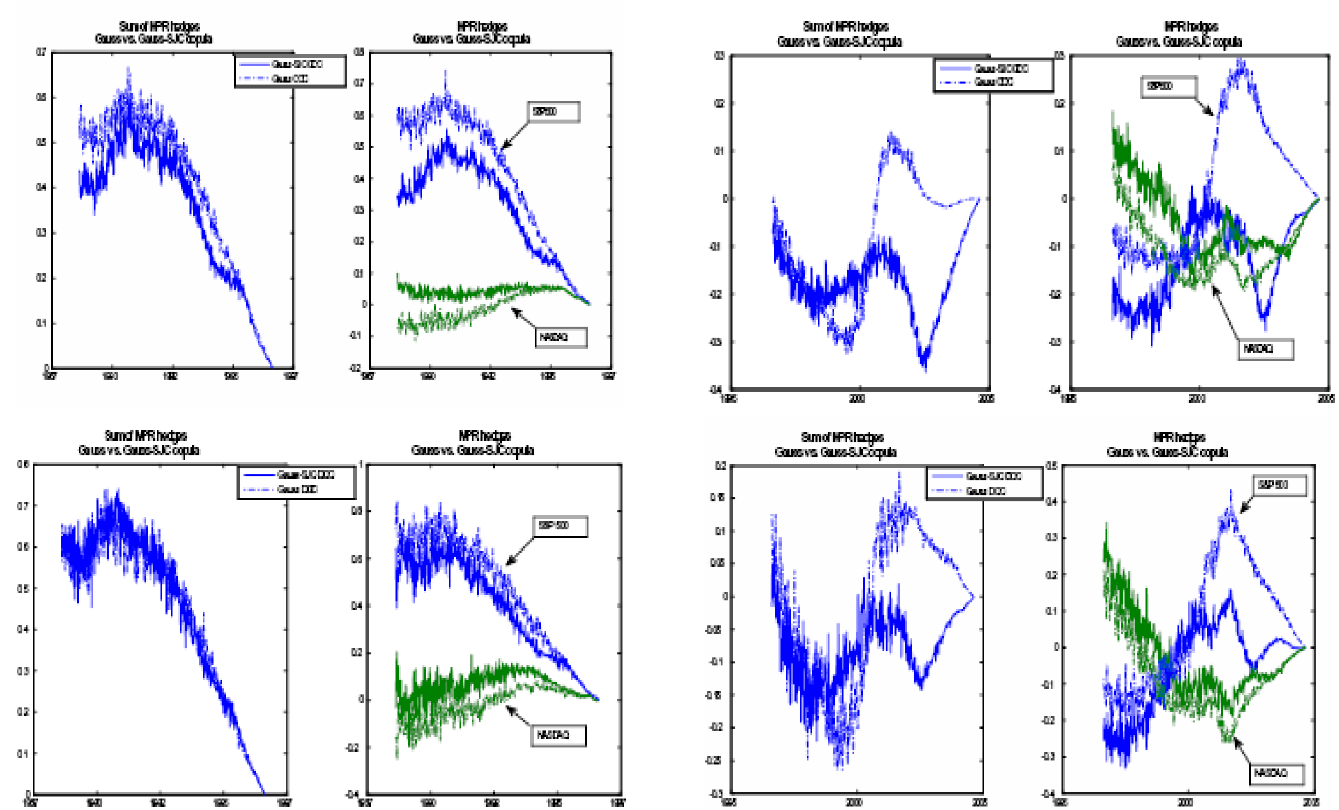
  - Cash

- **The long term bond is solely responsible for hedging away the source of risk related to the short rate**
- **The intertemporal hedging demand for the two risky funds is comprised by the market price of risk hedges**

Hedging demands along realized paths of the state variables

The portfolio effect of tail dependence in the unconditional distribution

1988-1996 | 1996-2004



CCC

DCC

Simulations: correlation hedging

Panel A. Gaussian-SJC CCC diffusion $(\gamma_1 = \gamma_2 = \gamma_3 = 0)$

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.1056	0.0614	0.0442	-	-	0.8623	0.6588	0.2035	-	-
CRRA, γ=10	0.0643	0.0370	0.0273	-	-	0.7341	0.5807	0.1534	-	-
HARA, γ=5,b=-0.2	0.0873	0.0505	0.0368	-	-	0.8201	0.6315	0.1885	-	-
HARA, γ=10,b=-0.2	0.0539	0.0307	0.0232	-	-	0.7113	0.5661	0.1451	-	-

Panel B. Gaussian-SJC DCC diffusion with only VIX driving conditional correlation $(\gamma_2 = \gamma_3 = 0)$

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.1060	0.0613	0.0446	-	0.4368	0.8617	0.6575	0.2041	-	0.4103
CRRA, γ=10	0.0646	0.0370	0.0276	-	0.2815	0.7348	0.5809	0.1539	-	0.1492
HARA, γ=5,b=-0.2	0.0878	0.0505	0.0372	-	0.3693	0.8203	0.6314	0.1890	-	0.3392
HARA, γ=10,b=-0.2	0.0541	0.0307	0.0234	-	0.2429	0.7112	0.5657	0.1455	-	0.1078

Panel C. Gaussian-SJC DCC diffusion with only CFNAI driving conditional correlation $(\gamma_1 = \gamma_2 = 0)$

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0353	0.0620	-0.0267	-0.0673	-	0.8030	0.7259	0.0771	-0.0521	-
CRRA, γ=10	0.0204	0.0379	-0.0174	-0.0403	-	0.7356	0.6523	0.0833	0.0024	-
HARA, γ=5,b=-0.2	0.0285	0.0512	-0.0227	-0.0558	-	0.7798	0.7021	0.0777	-0.0373	-
HARA, γ=10,b=-0.2	0.0160	0.0313	-0.0153	-0.0338	-	0.7215	0.6371	0.0844	0.0101	-

Simulations: tail dependence effect

Panel D. Gaussian DCC diffusion

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0557	0.0652	-0.0094	-0.0469	0.5440	0.9605	0.7419	0.2187	-0.0127	0.1512
CRRA, γ=10	0.0374	0.0431	-0.0057	-0.0277	0.3209	0.8905	0.6694	0.2211	0.0237	-0.2819
HARA, γ=5,b=-0.2	0.0476	0.0555	-0.0078	-0.0387	0.4491	0.9353	0.7167	0.2186	-0.0026	0.0358
HARA, γ=10,b=-0.2	0.0321	0.0372	-0.0052	-0.0232	0.2673	0.8752	0.6559	0.2193	0.0292	-0.3400

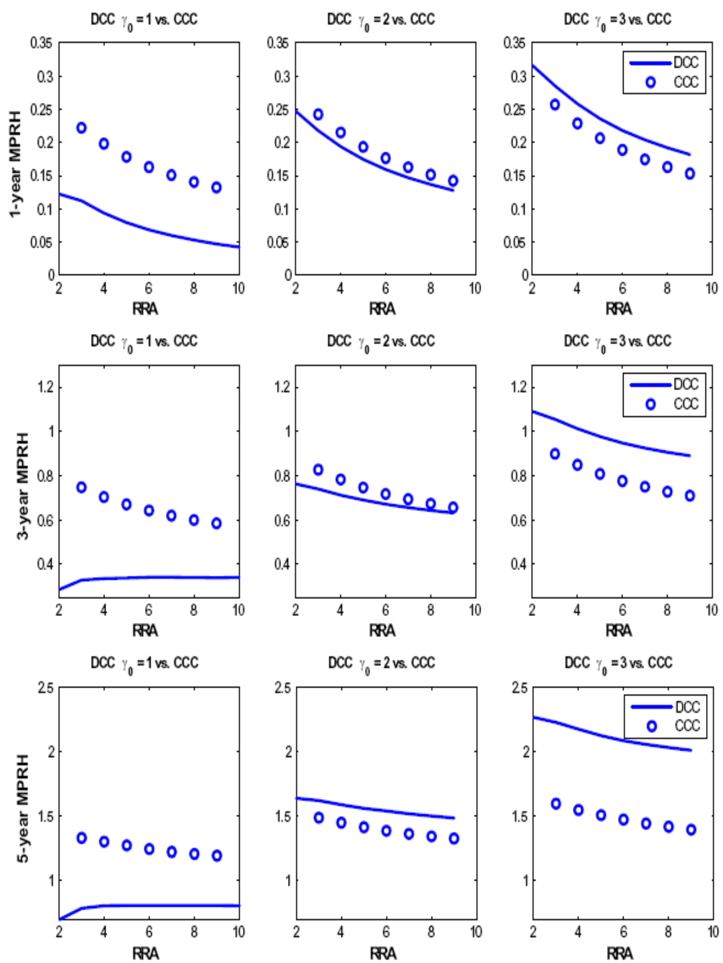
Panel E. Student's t DCC diffusion

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0442	0.0676	-0.0235	-0.0487	-0.1939	1.0418	0.7998	0.2419	-0.0670	-0.2690
CRRA, γ=10	0.0280	0.0424	-0.0145	-0.0283	-0.1128	0.9319	0.7114	0.2205	-0.0245	-0.1000
HARA, γ=5,b=-0.2	0.0364	0.0562	-0.0198	-0.0400	-0.1596	1.0045	0.7699	0.2346	-0.0538	-0.2182
HARA, γ=10,b=-0.2	0.0237	0.0362	-0.0125	-0.0236	-0.0940	0.9082	0.6915	0.2167	-0.0187	-0.0771

Panel F. Gaussian-SJC DCC diffusion

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0268	0.0594	-0.0326	-0.0652	0.3782	0.8419	0.7265	0.1154	-0.0245	0.1429
CRRA, γ=10	0.0140	0.0356	-0.0216	-0.0386	0.2243	0.7708	0.6500	0.1208	0.0226	-0.1317
HARA, γ=5,b=-0.2	0.0206	0.0484	-0.0277	-0.0538	0.3129	0.8166	0.7007	0.1159	-0.0110	0.0664
HARA, γ=10,b=-0.2	0.0107	0.0297	-0.0190	-0.0323	0.1885	0.7552	0.6350	0.1202	0.0303	-0.1761

Simulations: the impact of the correlation level



Dynamic correlation-induced portfolio hedging terms:

the Gaussian-SJC diffusion with DCC vs. CCC for different correlation levels

The CEQ cost of disregarding tail dependence

Panel A. The cost of disregarding tail dependence

	(Gaussian alternative, DCC)			(Gaussian alternative, CCC)		
	HARA b=-0.2	CRRA b=0	HARA b=0.2	HARA b=-0.2	CRRA b=0	HARA b=0.2
$\gamma = 2$	1.3153	1.5158	1.7162	3.2467	3.8692	4.4916
$\gamma = 4$	0.6384	0.7438	0.8492	1.1366	1.4361	1.7357
$\gamma = 6$	0.3912	0.4619	0.5326	0.4602	0.6562	0.8523
$\gamma = 8$	0.2658	0.3189	0.3719	0.1301	0.2757	0.4212
$\gamma = 10$	0.1902	0.2327	0.2751	-0.0650	0.0507	0.1664

Panel B. The cost of disregarding asymmetric tail dependence

	(Student's t alternative, DCC)			(Student's t alternative, CCC)		
	HARA b=-0.2	CRRA b=0	HARA b=0.2	HARA b=-0.2	CRRA b=0	HARA b=0.2
$\gamma = 2$	0.1886	0.1696	0.1506	0.5891	0.6486	0.7081
$\gamma = 4$	0.4271	0.4416	0.4561	0.4755	0.5176	0.5597
$\gamma = 6$	0.4259	0.4403	0.4546	0.3960	0.4260	0.4559
$\gamma = 8$	0.4121	0.4245	0.4369	0.3509	0.3740	0.3970
$\gamma = 10$	0.3999	0.4106	0.4213	0.3224	0.3411	0.3598

The CEQ cost of disregarding dynamic conditional correlation with observable factors

Panel A. The cost of disregarding DCC (CCC alternative)

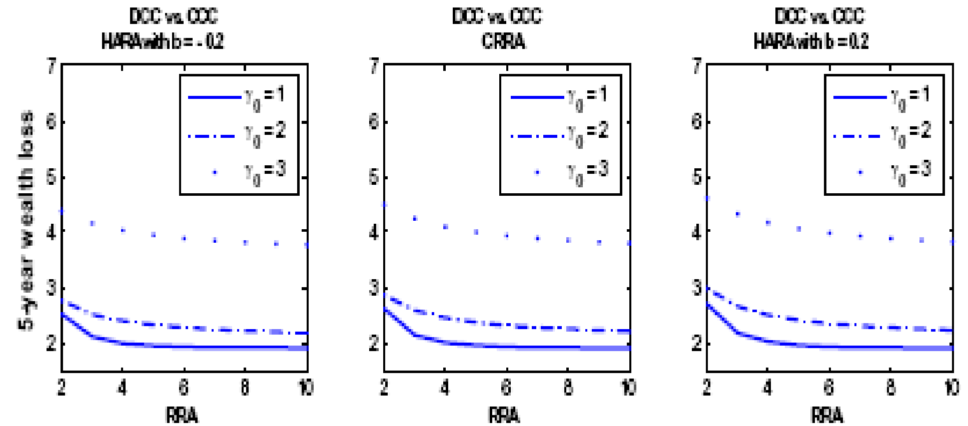
	HARA, $b=-0.2$	CRRA	HARA, $b=0.2$
$\gamma=2$	2.3054	2.4039	2.5024
$\gamma=4$	1.8987	1.9369	1.9751
$\gamma=6$	1.7983	1.8216	1.8449
$\gamma=8$	1.7538	1.7706	1.7873
$\gamma=10$	1.7289	1.7419	1.7549

Panel B. The cost of disregarding the CFNAI factor (DCC with $\gamma_2=0$ alternative)

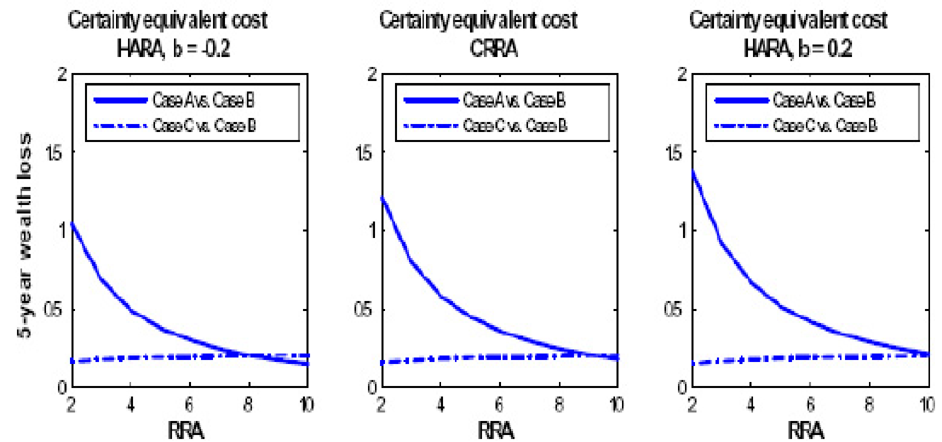
	HARA, $b=-0.2$	CRRA	HARA, $b=0.2$
$\gamma=2$	2.4273	2.5533	2.6792
$\gamma=4$	1.9309	1.9832	2.0355
$\gamma=6$	1.7988	1.8315	1.8643
$\gamma=8$	1.7384	1.7622	1.7860
$\gamma=10$	1.7039	1.7226	1.7413

The CEQ cost of disregarding dynamic conditional correlation with observable factors

For varying correlation levels:



Latent vs. observable factors:



Conclusion

- The portfolio solution methodology allows us to isolate:
  - correlation hedging demands due to observable factors
  - the impact of tail dependence on market price of risk hedging terms

- Correlation hedging demands and intertemporal demands due to high level of tail dependence have a distinct impact on the optimal portfolio behavior:
  - both in terms of portfolio composition
  - and economic significance

- Extensions
  - Changing copula composition conditional upon observable factors
  - Dependence between bond and stock dynamics