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Errors from discrete hedging in exponential Lévy models: the L^2 approach.

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Abstract We analyze the errors arising from discrete rebalancing of the hedging portfolio in exponential Lévy models, and establish the rates at which the expected squared discretization error goes to zero when the length of the rebalancing step decreases. Different hedging strategies and option pay-offs are considered. The case of digital options is studied in detail, and it turns out that in this case quadratic hedging produces different rates from the usual delta hedging strategy and that for both strategies the rates of convergence depend on the Blumenthal-Getoor index of the process.

Keywords exponential Lévy models · quadratic hedging · delta hedging · discretization error · L^2 convergence · digital options

Mathematics Subject Classification (2000) 60F25 · 60G51 · 91B28

JEL Classification G13

1 Introduction

We study the problem of hedging an option with a discretely rebalanced portfolio in an exponential Lévy model. This setting corresponds to an incomplete market and therefore gives rise to two kinds of hedging errors. The market incompleteness error is the difference between the option's pay-off and the theoretical hedging portfolio which assumes continuous rebalancing. This error and its minimization has been analyzed in several papers in the context of

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exponential Lévy models [17,8]. In this study we therefore focus on the discretization error, defined as the difference between the theoretical continuously rebalanced portfolio and the discretely rebalanced one. Assuming the interest rate to be zero, and letting T_i, $i = 0, \dots, n$ denote the portfolio readjustment dates, the discretization error is defined by

$$\varepsilon_T = \int_0^T F_{t-} dS_t - \int_0^T F_{\underline{\eta}(t)} dS_t,$$

where F denotes the continuous-time strategy, S the stock price and $\underline{\eta}(t) := \sup\{T_i : T_i < t\}$ is the last rebalancing date before t.

The error from discrete-time hedging and the related problem of approximating a stochastic integral with a Riemann sum has been analyzed by several authors in the context of diffusion models or continuous Itô processes. Bertsimas, Kogan and Lo [4] and later Hayashi and Mykland [16] gave the conditions under which the renormalized hedging error $\sqrt{n}\varepsilon_T$ converges weakly to a nondegenerate limiting distribution as the number of discretization dates n goes to infinity. The rate of L^2 convergence of the discretization error to zero was analyzed by Zhang [23], who showed that for European Call and Put options $nE[\varepsilon_T^2]$ converges to a nonzero finite limit as $n \to \infty$ and by Gobet and Temam [14], who studied irregular pay-offs and showed in particular that for digital options $\sqrt{n}E[\varepsilon_T^2]$ converges to a nondegenerate limit. Geiss [13,12], showed that for irregular pay-off functions the convergence rate of n rather than $\sqrt{n}$ may be recovered by taking a non-equidistant (but deterministic) time net, where the rebalancing frequency increases as the option approaches expiry.

In the context of discontinuous processes, the limiting behavior of the discretization error was studied in [22] from the point of view of weak convergence, and it was shown in particular that if the underlying process has no diffusion component, under some technical conditions, $\sqrt{n}\varepsilon_T \to 0$ in probability as $n \to \infty$. However, in financial applications the risk is more commonly measured by an L^2 criterion. In this paper we therefore concentrate on the rate of L^2 convergence of the discretization error to zero, and we show in particular that $\lim_{n\to 0} nE[\varepsilon_T^2] > 0$ in all cases (this limit may be infinite). This means that for pure-jump Lévy processes, the rate of L^2 convergence is different from the rate of convergence in probability. This phenomenon is not encountered in diffusion models, and is explained by the fact that the big jumps do not contribute to the rate of convergence in probability, while they do contribute to the rate of L^2 convergence.

In this paper, we suppose that the rebalancing dates are equidistant. It is possible that the convergence rates for options with irregular pay-offs may be improved by taking non-equidistant dates as in [13], but in many practical situations such non-equidistant time grids cannot be used (for example, when one needs to hedge a portfolio of options on the same underlying with different expiry dates).

In the rest of this section, we give an overview of the paper and summarize the most important results.

First, in section 2, we recall the Fourier transform approach to option pricing in exponential Lévy models, and derive the corresponding expressions for the two strategies that we study and compare in this paper, that is, the delta hedging strategy and the quadratic hedging strategy. We show that these strategies can be represented as stochastic integrals of some processes which can also be computed by Fourier transforms.

Next, in section 3, we establish a general criterion for the L^2 convergence with a given rate of the error from discrete hedging, depending on the coefficients of the integral representation of the hedging strategy (Theorem 3.1). The convergence can take place in two different regimes: the *regular regime* when the expected squared error decays proportionally to the rebalancing step h and the *irregular regime* when the convergence is slower. In the regular regime, the entire trajectory from inception to the maturity of the option contributes to the hedging error, and in the irregular regime the main contribution comes from the period just before the option attains its maturity.

In section 4, we apply the general criterion to the specific hedging strategies and show that for options with sufficiently regular pay-offs, including European calls and puts, under a set of assumptions satisfied by all exponential Lévy models used in the literature, the convergence takes place in the regular regime for both delta hedging (Theorem 4.1) and the quadratic hedging (Theorem 4.3).

The situation becomes completely different for options with irregular pay-offs, such as digitals. In this case, the convergence rate depends on the fine properties of the Lévy measure near zero, and an additional assumption on the structure of the Lévy measure is necessary in order to compute the convergence rate explicitly. We therefore assume that the small jumps have stable-like behavior with index α (which is the case, e.g. for the CGMY model with $\alpha = Y$ or for the NIG model with $\alpha = 1$) and characterize the convergence rates for quadratic hedging and delta hedging depending on α. For the quadratic hedging strategy the convergence takes place in the regular regime for $\alpha \leq \frac{3}{2}$ (Theorem 4.5) whereas the rate is decreasing in α for $\alpha > \frac{3}{2}$ (Theorem 4.6). For the delta hedging strategy, the convergence rate is increasing in α for $\alpha > 1$ (Theorem 4.4) and no power-like rate can be identified for $\alpha \leq 1$ (this means that there is either no convergence or that the convergence is slower than any power of h). The dependence of the theoretical rates on α is illustrated in Figure 1.1. Note that for $\alpha \to 2$, both rates converge to $h^{\frac{1}{2}}$, the value which was identified by Gobet and Temam [14] in the Black-Scholes setting. These findings make it clear that although for options with regular pay-offs delta hedging and quadratic hedging strategies may be quite close (see e.g. [17, section 5]), in the case of irregular pay-offs, the delta hedging strategy, because it involves differentiation of the option price function, suffers from much larger discretization errors than quadratic hedging.

To further illustrate the case of digital pay-offs, we performed a simulation study in the CGMY model. Figure 1.2, left graph, shows the convergence of the discretization error of the quadratic hedging strategy to zero as the number of rebalancing dates increases (and h decreases) for different values of α. We see

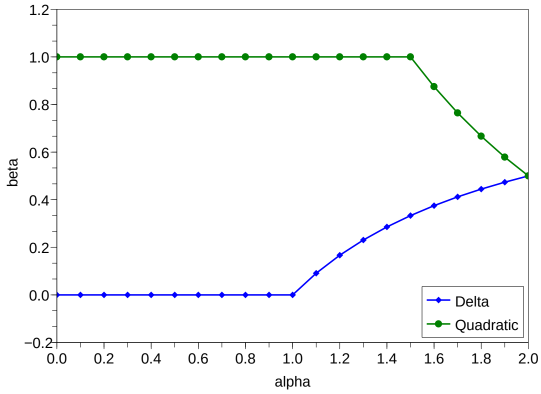

Fig. 1.1 Convergence rate of the expected squared discretization error to zero as function of the stability index α for a digital option. The rate is given by $r(h) = h^{\beta}$, where β is plotted in the graph.

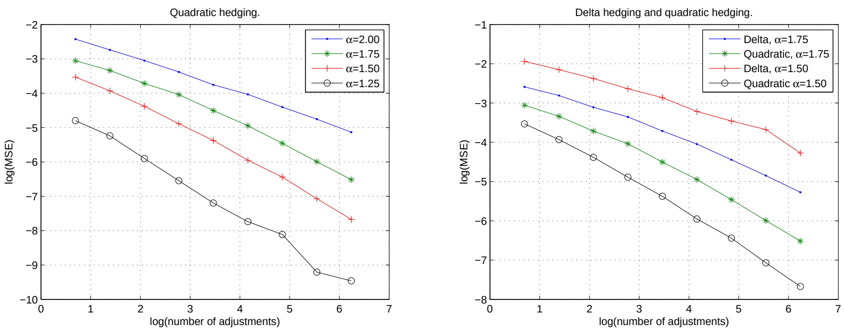

Fig. 1.2 Convergence of the discretization error to zero in the case of hedging of a digital option in the CGMY model. Left: quadratic hedging. Right: delta hedging vs. quadratic hedging.

that as α decreases from 2 to 1.5, the convergence rate clearly improves, while between $\alpha = 1.5$ and $\alpha = 1.25$ the two curves appear parallel. The right graph of Figure 1.2 compares the convergence rate of the quadratic hedging strategy to that of the delta hedging strategy. Here it is clear that the error from delta hedging always converges to zero slower than in the quadratic hedging case, and the convergence rate becomes worse as α increases.

2 Pricing and hedging in exponential Lévy models

Standard notation and basic assumptions We now introduce the common notation for the rest of the paper. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, let the stock price be modeled by $S_t = e^{X_t}$ where X is a Lévy process with characteristic triple (a^2, ν, γ). The assumption that $S_0 = 1$ is with no loss of generality. Since we study the L^2 hedging error, we will always suppose that S is square integrable. The characteristic function of X is denoted by ϕ_t and the characteristic exponent by ψ: $E[e^{iuX_t}] \equiv \phi_t(u) \equiv e^{t\psi(u)}$. The process S can be written in the form

$$S_t = 1 + \int_0^t bS_u du + \int_0^t aS_u dW_u + \int_0^t S_{u-} \int_{\mathbb{R}} (e^z - 1)\tilde{J}(du \times dz)\,,$$

where W is a standard Brownian motion, $\tilde{J}$ a compensated Poisson random measure with intensity measure $dt \times \nu$ and $b := \gamma + \frac{1}{2}a^2 + \int_{\mathbb{R}}(e^z - 1 - z1_{|z|\le 1})\nu(\mathrm{d}z)$. Furthermore, we denote $A := a^2 + \int_{\mathbb{R}}(e^z - 1)^2\nu(dz)$.

We assume that there exists a risk-neutral probability $Q \sim P$, such that the prices of all assets are martingales under Q (the interest rate is assumed to be zero). Moreover, we assume that X is a Lévy process under Q with characteristic exponent $\bar{\psi}$, characteristic function $\bar{\phi}_t$ and Lévy measure $\bar{\nu}$.

Option pricing Consider a European option with pay-off $G(S_T)$ at time T and denote by g its log-payoff function: $G(e^x) \equiv g(x)$. Prices of European options in exponential Lévy models can be computed directly from the risk-neutral characteristic function $\bar{\phi}$.

Proposition 2.1

(i) Suppose that there exists $R \in \mathbb{R}$ such that

$$g(x)e^{-Rx} \quad \textit{has finite variation on } \mathbb{R}, \tag{2.1}$$

$$g(x)e^{-Rx} \in L^1(\mathbb{R}), \tag{2.2}$$

$$E^Q[e^{RX_{T-t}}] < \infty \quad \textit{and} \quad \int_{\mathbb{R}} \frac{|\bar{\phi}_{T-t}(u - iR)|}{1 + |u|} du < \infty. \tag{2.3}$$

Then

$$C(t, S_t) := E^Q[G(S_T)|\mathcal{F}_t] = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR)\bar{\phi}_{T-t}(-u - iR)S_t^{R-iu} du, \tag{2.4}$$

where

$$\hat{g}(u) := \int_{\mathbb{R}} e^{iux} g(x) dx$$

and moreover

$$|\hat{g}(u + iR)| \le \frac{C}{1 + |u|}, \quad u \in \mathbb{R} \tag{2.5}$$

for some $C > 0$.

(ii) Suppose that g is differentiable and there exists $R \in \mathbb{R}$ such that

$$g'(x)e^{-Rx} \quad \text{has finite variation on } \mathbb{R}, \tag{2.6}$$
$$g'(x)e^{-Rx} \in L^1(\mathbb{R}), \tag{2.7}$$
$$g(x)e^{-Rx} \in L^1(\mathbb{R}), \tag{2.8}$$
$$E^Q[e^{RX_{T-t}}] < \infty. \tag{2.9}$$

Then the representation (2.4) holds and

$$|\hat{g}(u+iR)| \le \frac{C}{1+|u|^2}, \quad u \in \mathbb{R} \tag{2.10}$$

for some $C > 0$.

For the proof, see [21].

For the first part of the above proposition, a typical example is the digital option with pay-off $G(S_T) = 1_{S_T \ge K}$. In this case for all $R > 0$ conditions (2.1) and (2.2) are satisfied and

$$\hat{g}(u+iR) = \frac{K^{iu-R}}{R-iu}.$$

For the second part, consider the European call option with pay-off $G(S_T) = (S_T - K)^+$. In this case, conditions (2.1) and (2.2) are satisfied for all $R > 1$ and

$$\hat{g}(u+iR) = \frac{K^{iu+1-R}}{(R-iu)(R-1-iu)}.$$

In any case, conditions (2.1) and (2.2) imply $|G(S)| \le CS^R$ for some $C > 0$ and all $S > 0$.

In this paper, we study the behavior of the discretization error for the commonly used hedging strategies: the delta hedging strategy and the quadratic hedging strategy. Our method is based on the integral representation for the strategy F of the form

$$F_t = F_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u + \int_0^t \int_{\mathbb{R}} \gamma_{u-}(z)\tilde{J}(du \times dz), \quad \forall t < T. \tag{2.11}$$

Below we show how this representation can be obtained for the strategies we are interested in.

Delta hedging The delta hedging strategy is the classical hedging strategy inherited from the Black-Scholes model and given by $F_t = \frac{\partial C(t,S_t)}{\partial S}$. It is not optimal in exponential Lévy models but is nevertheless commonly used by market practitioners.

Proposition 2.2 (Delta hedging) *Let the conditions* (2.1), (2.2) *and* (2.3) *for all $t < T$ be satisfied, and assume that*

$$\int_{\mathbb{R}} |\hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)(R-iu)|du < \infty, \quad \forall t < T. \tag{2.12}$$

Then the delta hedging strategy is given by

$$F_t = \frac{\partial C(t,S_t)}{\partial S} = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)(R-iu)S_t^{R-iu-1}du. \tag{2.13}$$

Assume in addition

$$\int_{|x|>1} e^{2(R-1)x}\nu(dx) < \infty\,. \tag{2.14}$$

Then the representation (2.11) *holds for F with*

$$\begin{aligned}\mu_t = \frac{1}{2\pi}\int_{\mathbb{R}} &\hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)(R-iu)S_t^{R-1-iu}\\ &\times(\psi(-u-i(R-1))-\bar{\psi}(-u-iR))du\,,\end{aligned} \tag{2.15}$$

$$\sigma_t = \frac{a}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)(R-iu)(R-1-iu)S_t^{R-1-iu}du\,, \tag{2.16}$$

$$\gamma_t(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)(R-iu)S_t^{R-1-iu}(e^{(R-1-iu)z}-1)du\,. \tag{2.17}$$

Proof The expression (2.13) is deduced directly from (2.4) using the dominated convergence theorem and the condition (2.12). The martingale representation follows by applying Lemma B.1 with $f(u) := \frac{1}{2\pi}\hat{g}(u+iR)(R-iu)$ and $R' = R-1$. □

Quadratic hedging under the martingale probability Quadratic hedging in the literature comes in three different flavors: one can (i) minimize the global L^2 hedging error computed under the martingale probability (as in [11] and many subsequent papers); (ii) minimize the local variation of the hedging portfolio under the historical probability (as in e.g. [10]) or (iii) minimize the global L^2 hedging error under the historical probability (as in [17,6]). In this paper we choose the martingale approach, that is, we minimize

$$E^Q\left[\left(G(S_T) - C(0,S_0) - \int_0^T F_t dS_t\right)^2\right].$$

This approach is the simplest of the three and thus enables us to explain the main ideas and insights in a less technical setting; some remarks on the behavior of the discretization error under other quadratic hedging approaches are given at the end of this section. See also [8] for some arguments towards using this strategy in practice rather than minimizing the quadratic hedging error under the historical measure. The solution to this minimization problem is given by the Kunita-Watanabe decomposition (see e.g. [19, page 181]), and can be explicitly written as (see [6])

$$F_t = \frac{d\langle C, S\rangle_t^Q}{d\langle S, S\rangle_t^Q},$$

where we denote $C_t := C(t, S_t)$.

Proposition 2.3 *Assume* (2.1)–(2.2)*;* (2.3) *for all* $t < T$ *and*

$$\int_{|x|>1} e^{2(Rx\vee x)}\bar{\nu}(dx) < \infty. \tag{2.18}$$

Then the optimal quadratic hedging strategy under the martingale probability is given by

$$F_t = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)S_t^{R-iu-1}\Upsilon(u)du \tag{2.19}$$

$$\textit{where}\quad \Upsilon(u) = \frac{\bar{\psi}(-u-i(R+1)) - \bar{\psi}(-u-iR) - \bar{\psi}(-i)}{\bar{\psi}(-2i) - 2\bar{\psi}(-i)}. \tag{2.20}$$

Assume in addition that

$$\int_{|x|>1} e^{2(R-1)x}\nu(dx) < \infty.$$

Then the representation (2.11) *holds for* F *with*

$$\mu_t = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)\Upsilon(u)S_t^{R-1-iu} \times (\psi(-u-i(R-1)) - \bar{\psi}(-u-iR))du\,, \tag{2.21}$$

$$\sigma_t = \frac{a}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)\Upsilon(u)(R-1-iu)S_t^{R-1-iu}du\,, \tag{2.22}$$

$$\gamma_t(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)\Upsilon(u)S_t^{R-1-iu}(e^{(R-1-iu)z}-1)du\,. \tag{2.23}$$

Proof The first part of this result (expression (2.19) for the optimal strategy) is proved in [21, Proposition 7]; under slightly different conditions this result also follows from the general theorem in [17].

To obtain the martingale representation (2.21)–(2.23), we apply, once again, Lemma B.1, with $R' = R-1$ and $f(u) = \frac{1}{2\pi}\hat{g}(u+iR)\Upsilon(u)$. The validity of condition (B.2) follows from Lemma A.4, assumption (2.3) and assumption (2.18). □

Heuristic comments on other quadratic hedging approaches The locally risk minimizing or Föllmer-Schweizer strategy is the process F^{FS} appearing in the so-called Föllmer-Schweizer decomposition of the terminal pay-off:

$$G(S_T) = \int_0^T F_t^{FS} dS_t + L_T,$$

where L is a square integrable martingale orthogonal to the martingale part of S (under the historical measure P). The Föllmer-Schweizer decomposition for processes with independent increments is computed in [17] under assumptions on the pay-off function which are slightly different from ours, and with our notation takes a very similar form to equation (2.19):

$$F_t^{FS} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR)\phi_{T-t}(-u - iR) S_t^{R-iu-1} \Upsilon(u) e^{-\psi(-i)\Upsilon(u)(T-t)} du \tag{2.24}$$

$$\text{where} \quad \Upsilon(u) = \frac{\psi(-u - i(R+1)) - \psi(-u - iR) - \psi(-i)}{\psi(-2i) - 2\psi(-i)}. \tag{2.25}$$

Since, by Lemma A.4, $\Upsilon(u)$ is small compared to $\psi(u)$ for sufficiently large u, $e^{-\psi(-i)\Upsilon(u)(T-t)}$ is a small correction to $\phi_{T-t}(-u - iR)$, and therefore it is clear that the results of section 4 will also hold for the Föllmer-Schweizer strategy, under appropriately modified assumptions.

The *variance-optimal* hedging strategy minimizes the global L^2 hedging error under the historical probability:

$$E^P \left[\left(G(S_T) - V_0 - \int_0^T F_t dS_t \right)^2 \right].$$

In [17] it is shown that this strategy admits a feedback representation:

$$F_t = F_t^{FS} + \frac{\lambda}{S_{t-}} \left(L_t - V_0 - \int_0^{t-} (F_{s-} - F_{s-}^{FS}) dS_s \right).$$

Therefore, F_t is not a deterministic function of t and S_t alone, does not admit a simple Fourier representation like (2.24) or (2.19) and is not directly covered by our methods. We plan to address this strategy in future research.

3 Errors from discrete hedging: general result

Since continuously rebalancing one's portfolio is unfeasible in practice, we assume that the hedging portfolio is rebalanced at equally spaced dates $T_i = iT/n$, $i = 0, \ldots, n-1$, and denote by h the distance between the rebalancing dates: $h := T/n$. For $t \in (0, T]$ we denote by $\underline{\eta}(t)$ the rebalancing date immediately before t and by $\overline{\eta}(t)$ the rebalancing date immediately after t:

$$\underline{\eta}(t) = \sup\{T_i, T_i < t\}, \qquad \overline{\eta}(t) = \inf\{T_i, T_i \geq t\}.$$

The trading strategy is therefore piecewise constant and is assumed to be given by $F_{\underline{\eta}(t)}$, where (F_t) is the 'ideal' continuous-time hedging strategy that the agent would use if continuous rebalancing were possible. The value of the hedging portfolio at time t is $V_0 + \int_0^t F_{s-} dS_s$ with continuous hedging and $V_0 + \int_0^t F_{\eta(s)} dS_s$ with discrete hedging. F^h_t denotes the left-continuous difference between the continuously rebalanced strategy and the discretely rebalanced one: $F^h_t := F_{t-} - F_{\underline{\eta}(t)}$. We study the L^2 convergence to 0, when $h \to 0$, of the difference between discrete and continuous hedging portfolio

$$\int_0^T (F_{t-} - F_{\underline{\eta}(t)}) dS_t \equiv \int_0^T F^h_t dS_t.$$

Choose a function $r(h) : (0,\infty) \to (0,\infty)$ with $\lim_{h\downarrow 0} r(h) = 0$ (the rate of convergence to zero of the hedging error). We shall see that under suitable assumptions $E[(\int_0^T F^h_t dS_t)^2 / r(h)]$ converges to a finite nonzero limit when $h \downarrow 0$.

Theorem 3.1 *Assume that the hedging strategy F is of the form* (2.11) *and*

$$\lim_{h\downarrow 0} \frac{h}{r(h)} E\left[\int_0^T S_t^2 (\overline{\eta}(t) - t)\left(\mu_t^2 + \sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z)\nu(dz)\right) dt\right] = 0\,. \quad (3.1)$$

Then

$$\lim_{h\downarrow 0} \frac{1}{r(h)} E\left[\left(\int_0^T F^h_t dS_t\right)^2\right] = \lim_{h\downarrow 0} \frac{A}{r(h)} E\left[\int_0^T S_t^2 (\overline{\eta}(t) - t)\left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z) e^{2z}\nu(dz)\right) dt\right], \quad (3.2)$$

whenever the limit on the right-hand side exists.

Corollary 3.2 *Assume that* (3.1) *is satisfied and*

$$E\left[\int_0^T S_t^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z) e^{2z}\nu(dz)\right) dt\right] < \infty. \quad (3.3)$$

Then

$$\lim_{h\downarrow 0} \frac{1}{h} E\left[\left(\int_0^T F^h_t dS_t\right)^2\right] = \frac{A}{2} E\left[\int_0^T S_t^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z) e^{2z}\nu(dz)\right) dt\right].$$

If the condition (3.3) is not satisfied, then clearly the limit in (3.2) can only exist with a convergence rate worse than $r(h) = h$. Therefore the best possible convergence rate which can be obtained with Theorem 3.1, and which is realized for regular strategies, is $r(h) = h$. However, worse rates may arise in the presence of irregular pay-offs. In the following, we will refer to the situation when (3.3) is satisfied and $r(h) = h$ as *regular regime* and to the other situations as *irregular regime*.

Proof (Proof of Corollary 3.2) The proof is very similar to that of the Riemann-Lebesgue lemma. Let

$$g_h(t) := \frac{\bar{\eta}(t) - t}{h}, \quad f(t) := E\left[S_t^2\left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z) e^{2z} \nu(dz)\right)\right].$$

For any piecewise constant function $u : [0, T] \to \mathbb{R}$, we clearly have

$$\lim_{h \downarrow 0} \int_0^T g_h(t) u(t) dt = \frac{1}{2} \int_0^T u(t) dt.$$

Let $(f_n)_{n \geq 1}$ be a sequence of piecewise constant functions satisfying $f_n(t) \leq f_{n+1}(t) \leq f(t)$ and $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in [0, T]$. Then, by monotone convergence, since $|g_h| \leq 1$,

$$\lim_{n \to \infty} \int_0^T (f(t) - f_n(t)) g_h(t) dt = 0,$$

uniformly on h, which proves that

$$\lim_{n \to \infty} \int_0^T f(t) g_h(t) dt = \frac{1}{2} \int_0^T f(t) dt.$$

□

Proof (Proof of Theorem 3.1) We define two auxiliary probability measures P^1 and P^2 by

$$\frac{dP^1}{dP}|_{\mathcal{F}_t} = \frac{e^{X_t}}{e^{t\psi(-i)}} \quad \text{and} \quad \frac{dP^2}{dP}|_{\mathcal{F}_t} = \frac{e^{2X_t}}{e^{t\psi(-2i)}}.$$

Under the measure P^1, the process $(W_t^{(1)})$ defined by $W_t^{(1)} = W_t - at$ is a standard Brownian motion and

$$\tilde{J}^{(1)}(dt \times dz) = \tilde{J}(dt \times dz) - dt \times (e^z - 1)\nu(dz)$$

is a compensated Poisson random measure, and similarly, under P^2, the prices $W_t^{(2)} = W_t - 2at$ is a standard Brownian motion and

$$\tilde{J}^{(2)}(dt \times dz) = \tilde{J}(dt \times dz) - dt \times (e^{2z} - 1)\nu(dz)$$

is a compensated Poisson random measure. Therefore, the drift of F under P^1 and P^2 is given, respectively, by

$$\mu_t^{(1)} = \mu_t + a\sigma_t + \int_{\mathbb{R}} \gamma_t(z)(e^z - 1)\nu(dz), \tag{3.4}$$

$$\mu_t^{(2)} = \mu_t + 2a\sigma_t + \int_{\mathbb{R}} \gamma_t(z)(e^{2z} - 1)\nu(dz). \tag{3.5}$$

The hedging error satisfies

$$\frac{1}{r(h)}E\left[\left(\int_0^T F_t^h dS_t\right)^2\right] = \frac{1}{r(h)}E\left[\left(\int_0^T F_t^h S_t^m\right)^2\right]$$
$$+\frac{b^2}{r(h)}E\left[\left(\int_0^T F_t^h S_t dt\right)^2\right] + \frac{2b}{r(h)}E\left[\int_0^T F_t^h S_t dt \times \int_0^T F_t^h dS_t^m\right], \tag{3.6}$$

where S^m denotes the martingale part of S. The first term in the right-hand side satisfies

$$\frac{1}{r(h)}E\left[\left(\int_0^T F_t^h S_t^m\right)^2\right] = \frac{A}{r(h)}E\left[\int_0^T (F_t^h)^2 S_t^2 dt\right]$$

if the latter expectation is finite. Passing to the probability P^2, we get

$$\frac{A}{r(h)}E\left[\int_0^T (F_t^h)^2 S_t^2 dt\right] = \frac{A}{r(h)}\int_0^T e^{t\psi(-2i)}E^{P^2}[(F_t^h)^2]dt.$$

The expectation under the integral sign can be decomposed as follows:

$$E^{P^2}[(F_t^h)^2] = E^{P^2}\left[\left(\int_{\underline{\eta}(t)}^t \mu_s^{(2)} ds\right)^2\right]$$
$$+E^{P^2}\left[\left(\int_{\underline{\eta}(t)}^t \sigma_s dW_s^{(2)} + \int_{\underline{\eta}(t)}^t \int_{\mathbb{R}} \gamma_{s-}(z)\tilde{J}^{(2)}(ds\times dz)\right)^2\right]$$
$$+E^{P^2}\left[\left(\int_{\underline{\eta}(t)}^t \sigma_s dW_s^{(2)} + \int_{\underline{\eta}(t)}^t \int_{\mathbb{R}} \gamma_{s-}(z)\tilde{J}^{(2)}(ds\times dz)\right)\int_{\underline{\eta}(t)}^t \mu_s^{(2)} ds\right]. \tag{3.7}$$

The second term in the right-hand side above satisfies

$$E^{P^2}\left[\left(\int_{\underline{\eta}(t)}^t \sigma_s dW_s^{(2)} + \int_{\underline{\eta}(t)}^t \int_{\mathbb{R}} \gamma_{s-}(z)\tilde{J}^{(2)}(ds\times dz)\right)^2\right]$$
$$= E^{P^2}\left[\int_{\underline{\eta}(t)}^t \left(\sigma_s^2 + \int_{\mathbb{R}} \gamma_s^2(z) e^{2z}\nu(dz)\right)\right]$$

and its integral gives, using integration by parts and switching back to the probability P,

$$\frac{A}{r(h)}\int_0^T e^{t\psi(-2i)}E^{P^2}\left[\int_{\underline{\eta}(t)}^t\left(\sigma_s^2+\int_{\mathbb{R}}\gamma_s^2(z)e^{2z}\nu(dz)\right)\right]dt$$
$$=\frac{A}{r(h)}\int_0^T dtE^{P^2}\left[\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right]\int_t^{\overline{\eta}(t)}e^{s\psi(-2i)}ds$$
$$=\frac{A(1+O(h))}{r(h)}E\left[\int_0^T S_t^2(\overline{\eta}(t)-t)\left(\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right)\right],$$

which converges to the same limit as (3.2). In view of this result and of the fact that the cross terms in (3.6) and (3.7) can be estimated using the Cauchy-Schwartz inequality, to prove the theorem it remains to show that, under the assumptions and when the limit in the right-hand side of (3.2) exists,

$$\lim_{h\downarrow 0}\frac{1}{r(h)}\int_0^T e^{t\psi(-2i)}E^{P^2}\left[\left(\int_{\underline{\eta}(t)}^t\mu_s^{(2)}ds\right)^2\right]dt=0 \tag{3.8}$$

$$\text{and}\quad \lim_{h\downarrow 0}\frac{1}{r(h)}E\left[\left(\int_0^T F_t^hS_tdt\right)^2\right]=0. \tag{3.9}$$

Proof of (3.8) The expression under the lim sign satisfies

$$\frac{1}{r(h)}\int_0^T e^{t\psi(-2i)}E^{P^2}\left[\left(\int_{\underline{\eta}(t)}^t\mu_s^{(2)}ds\right)^2\right]dt$$
$$\leq\frac{h}{r(h)}\int_0^T e^{t\psi(-2i)}E^{P^2}\left[\int_{\underline{\eta}(t)}^t(\mu_s^{(2)})^2ds\right]dt$$
$$=\frac{h(1+O(h))}{r(h)}\int_0^T(\overline{\eta}(t)-t)E\left[S_t^2(\mu_t^{(2)})^2\right]dt$$
$$\leq\frac{Ch}{r(h)}\int_0^T(\overline{\eta}(t)-t)E\left[S_t^2\mu_t^2+S_t^2\sigma_t^2+S_t^2\left(\int_{\mathbb{R}}\gamma_t(z)(e^{2z}-1)\nu(dz)\right)^2\right]dt.$$

for some constant $C<\infty$, where the last estimate follows from (3.5). By the Jensen inequality (for $|z|>1$) and the Cauchy-Schwartz inequality (for $|z|\leq 1$),

$$\left(\int_{\mathbb{R}}\gamma_t(z)(e^{2z}-1)\nu(dz)\right)^2\leq 2\int_{|z|\leq 1}(e^{2z}-1)^2\nu(dz)\int_{|z|\leq 1}\gamma_t^2(z)\nu(dz)$$
$$+2\int_{|z|>1}|e^{2z}-1|\nu(dz)\int_{|z|>1}\gamma_t^2(z)|e^{2z}-1|\nu(dz)$$
$$\leq C\int_{\mathbb{R}}\gamma_t^2(z)(1+e^{2z})\nu(dz).$$

The limit (3.8) now follows from the assumptions of the theorem and the existence of the limit (3.2).

Proof of (3.9) The error term in (3.9) can be rewritten as

$$\begin{aligned}
&\frac{1}{r(h)}E\left[\left(\int_0^T F_t^h S_t dt\right)^2\right] \\
&= \frac{2}{r(h)}\sum_{i=1}^n \int_{T_{i-1}}^{T_i} dt \int_t^{T_i} dsE[(F_t - F_{T_{i-1}})(F_s - F_{T_{i-1}})S_t S_s] \\
&\quad + \frac{2}{r(h)}\sum_{1\leq i<j\leq n} \int_{T_{i-1}}^{T_i} dt \int_{T_{j-1}}^{T_j} dsE[(F_t - F_{T_{i-1}})(F_s - F_{T_{j-1}})S_t S_s] \\
&= \frac{2}{r(h)}\sum_{i=1}^n \int_{T_{i-1}}^{T_i} dt \int_t^{T_i} dse^{(s-t)\psi(-i)}E[(F_t - F_{T_{i-1}})^2 S_t^2] \\
&\quad + \frac{2}{r(h)}\sum_{i=1}^n \int_{T_{i-1}}^{T_i} dt \int_t^{T_i} dse^{s\psi(-i)}E^{P^1}[(F_t - F_{T_{i-1}})(F_s - F_t)S_t] \\
&\quad + \frac{2}{r(h)}\sum_{1\leq i<j\leq n} \int_{T_{i-1}}^{T_i} dt \int_{T_{j-1}}^{T_j} dse^{s\psi(-i)}E^{P^1}[(F_t - F_{T_{i-1}})(F_s - F_{T_{j-1}})S_t] \\
&= \frac{O(h)}{r(h)}\int_0^T dtE[(F_t - F_{\underline{\eta}(t)})^2 S_t^2] \\
&\quad + \frac{2}{r(h)}\int_0^T dt \int_t^T dse^{s\psi(-i)}E^{P^1}[(F_t - F_{\underline{\eta}(t)})(F_s - F_{\underline{\eta}(s)\vee t})S_t].
\end{aligned}$$

The first term in the last line converges to zero by the first part of the proof. To compute the second term, introduce conditional expectation with respect to $\mathcal{F}_{\underline{\eta}(s)\vee t}$:

$$E^{P^1}[(F_s - F_{\underline{\eta}(s)\vee t})|\mathcal{F}_{\underline{\eta}(s)\vee t}] = E^{P^1}\left[\int_{\underline{\eta}(s)\vee t}^s \mu_u^{(1)} du \Big| \mathcal{F}_{\underline{\eta}(s)\vee t}\right].$$

The fact that the local martingale part of F has zero expectation can be justified using (3.2). Finally, we get

$$\left|\frac{2}{r(h)}\int_0^T dt\int_t^T ds e^{s\psi(-i)}E^{P^1}[(F_t-F_{\underline{\eta}(t)})(F_s-F_{\underline{\eta}(s)\vee t})S_t]\right|$$
$$=\left|\frac{2}{r(h)}\int_0^T dt\int_t^T ds e^{s\psi(-i)}E^{P^1}\left[(F_t-F_{\underline{\eta}(t)})S_t\int_{\underline{\eta}(s)\vee t}^s \mu^{(1)}_u du\right]\right|$$
$$=\left|\frac{2}{r(h)}\int_0^T dt\int_t^T dsE[\mu^{(1)}_s(F_t-F_{\eta(t)})S_tS_s]\int_s^{\overline{\eta}(s)}e^{(u-s)\psi(-i)}du\right|$$
$$\leq \frac{2}{r(h)}\int_0^T dtE[(F_t-F_{\eta(t)})^2S_t^2]^{\frac12}\int_0^T dsE[(\mu^{(1)}_s)^2S_s^2]^{\frac12}\int_s^{\overline{\eta}(s)}e^{(u-s)\psi(-i)}du$$
$$\leq C\left(\frac{1}{r(h)}\int_0^T dtE[(F_t-F_{\eta(t)})^2S_t^2]\right)^{\frac12}$$
$$\times\left(\frac{1}{r(h)}\int_0^T dsE[(\mu^{(1)}_s)^2S_s^2]\left\{\int_s^{\overline{\eta}(s)}e^{(u-s)\psi(-i)}du\right\}^2\right)^{\frac12},$$

where the last implication follows from the Jensen inequality. The first factor in the in the right-hand side above was shown to be bounded in the beginning of the proof. As for the second factor,

$$\frac{1}{r(h)}\int_0^T dsE[(\mu^{(1)}_s)^2S_s^2]\left\{\int_s^{\overline{\eta}(s)}e^{(u-s)\psi(-i)}du\right\}^2$$
$$=\frac{O(h)}{r(h)}\int_0^T ds(\overline{\eta}(s)-s)E[(\mu^{(1)}_s)^2S_s^2],$$

which can be shown to converge to zero in the same way as we did in the proof of (3.8). □

4 Convergence rates of specific strategies

We start by introducing a set of assumptions on the Lévy measure ν of X, which will be used in different theorems later in this section. In theorems dealing with the delta-hedging strategy we require:

$$\int_{|x|>1}e^{Rx}\bar{\nu}(dx)<\infty,\quad \int_{|x|>1}e^{2(Rx\vee x)}\nu(dx)<\infty,\quad \int_{|x|>1}e^{2(R-1)x}\nu(dx)<\infty. \tag{4.1}$$

The first condition guarantees the integrability of the option payoff under Q (recall that pay-off function satisfies $|G(S)|\leq CS^R$), the second ensures the square integrability of the option price and the stock price under P, and the last condition allows to construct a martingale-drift representation for the strategy.

For analyzing the quadratic hedging under the martingale probability we require the stock price and the option pay-off to be square integrable under Q as well:

$$\int_{|x|>1} e^{2(Rx\vee x)}\bar{\nu}(dx) < \infty, \quad \int_{|x|>1} e^{2(Rx\vee x)}\nu(dx) < \infty,$$
$$\int_{|x|>1} e^{2(R-1)x}\nu(dx) < \infty. \tag{4.2}$$

The following alternative assumptions determine the decay properties of characteristic function of X at infinity.

(H1) The Lévy measure ν is of the form $\nu = \nu_0 + \nu_1$ where ν_0 is a finite measure on $\mathbb{R}$ and ν_1 has a positive density of the form

$$\nu_1(x) = \frac{k(x)}{|x|},$$

where the function k is right-continuous and increasing on $(-\infty, 0)$ and left-continuous and decreasing on $(0,\infty)$.

(H2-α) The Lévy measure ν satisfies

$$\limsup_{r\to 0} r^{\alpha-2}\int_{[-r,r]} x^2\nu(dx) > 0.$$

(H3-α) The Lévy measure ν satisfies

$$\int_{[-1,1]} |x|^\alpha \nu(dx) < \infty.$$

(H4-α) The Lévy measure ν has a density satisfying

$$\nu(x) = \frac{f(x)}{|x|^{1+\alpha}}, \quad \lim_{x\to 0+} f(x) = f_+, \quad \lim_{x\to 0-} f(x) = f_-$$

for some constants $f_- > 0$ and $f_+ > 0$.

The assumption H1 guarantees at least power-law decay of the characteristic function at infinity (see Lemma A.1 in the Appendix). It is satisfied by most parametric infinite intensity processes used in financial modeling: for the variance gamma [18] and CGMY [5] processes this is immediately clear by looking at the Lévy measure while for the normal inverse Gaussian process [2] and the generalized hyperbolic distribution [9] it follows from the self-decomposability of these distributions shown in [15] and the characterization of self-decomposable distributions in [20, Chapter 3].

The assumptions H2-α, H3-α and H4-α with $0 < \alpha < 2$ characterize different aspects of stable-like behavior of small jumps of the Lévy process. They are satisfied by the CGMY process (with $\alpha = Y$), the normal inverse Gaussian process (with $\alpha = 1$) and the generalized hyperbolic distribution (with $\alpha = 1$ in general; see [7, pages 125–126]). They are not satisfied by the variance gamma process. It is clear that the assumption H4-α implies H2-α and H3-α.

We start our analysis with the delta hedging strategy (Proposition 2.2). To obtain the convergence rate $r(h) = h$, we need to impose stronger regularity conditions on the option's pay-off, which exclude discontinuities. In the following theorem and its proof, we use the notation of Proposition 2.2.

Theorem 4.1 (Delta hedging, european-like options) *Let the pay-off function and the Lévy process satisfy the conditions* (2.6)–(2.8) *and* (4.1) *for some $R \in \mathbb{R}$ and assume that one of the three alternative conditions holds:*

- *ν satisfies the assumption H1 and $a = 0$;*
- *ν satisfies the assumption H2-α with $\alpha \in (0, 2)$ and $a = 0$;*
- *$a > 0$.*

Let the hedging strategy be given by Proposition 2.2. Then

$$\lim_{h \downarrow 0} \frac{1}{h} E\left[\left(\int_0^T F_t^h \, dS_t\right)^2\right] = \frac{A}{2} E\left[\int_0^T S_t^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t(z) e^{2z} \nu(dz)\right) dt\right]. \tag{4.3}$$

Remark 4.2 As will become clear from the proof, the limiting renormalized discretization error appearing in the right-hand side of (4.3) can be evaluated via a two-dimensional integral.

$$\begin{aligned}
&\lim_{h \downarrow 0} \frac{1}{h} E\left[\left(\int_0^T F_t^h \mathrm{d}S_t\right)^2\right] \\
&= \frac{A}{8\pi^4} \int_{\mathbb{R}^2} du_1 \, du_2 (\phi_T(-u_1 - u_2 - 2iR) - \bar{\phi}_T(-u_1 - iR)\bar{\phi}_T(-u_2 - iR)) \\
&\quad \times \hat{g}(u_1 + iR)\hat{g}(u_2 + iR) f(u_1, u_2), \\
&\text{where} \quad f(u_1, u_2) = (R - iu_1)(R - iu_2) \\
&\times \frac{\psi(-u_1 - u_2 - 2iR) - \psi(-u_1 - i(R+1)) - \psi(-u_2 - i(R+1)) + \psi(-2i)}{\psi(-u_1 - u_2 - 2iR) - \bar{\psi}(-u_1 - iR) - \bar{\psi}(-u_2 - iR)}.
\end{aligned}$$

A similar expression can easily be obtained in the case of the quadratic hedging strategy, analyzed below in Theorem 4.3. These expression remain relatively inexplicit, because in the present regular case the entire hedging period contributes to the discretization error. In the case of irregular pay-offs we shall see that the main contribution comes only from the interval of vanishing length (as $h \to 0$) immediately preceding the maturity of the option, and the constants will therefore become more explicit. In any case, our goal in this paper is not to compute the hedging error explicitly but rather to gain an understanding of its behavior as the rebalancing step tends to zero.

Proof Step 1. From Lemma A.1 or, under the condition $a > 0$, directly from the form of the characteristic function, and from Lemma A.6 it follows that

$$\int_{\mathbb{R}} \frac{|\bar\phi_{T-t}(u - iR)|}{1 + |u|} du < \infty, \quad \forall t < T,$$

and therefore Proposition 2.2 holds.

With μ, σ and γ as in (2.15)–(2.17), define

$$I_1(t) := E[S_t^2 \mu_t^2]\,, \qquad I_2(t) := E[S_t^2 \sigma_t^2]\,, \tag{4.4}$$

$$I_3(t) := E[S_t^2 \int_{\mathbb{R}} \gamma_t^2(z)\nu(\mathrm{d}z)]\,, \qquad I_4(t) := E[S_t^2 \int_{\mathbb{R}} \gamma_t^2(z) e^{2z}\nu(\mathrm{d}z)]\,. \tag{4.5}$$

Suppose that we can show that $\int_0^T I_i(t)\mathrm{d}t < \infty$ for $i \in \{1, 2, 3, 4\}$. Then assumption (3.3) of Corollary 3.2 is satisfied and assumption (3.1) of Theorem 3.1 is satisfied as well ($r(h) = h$). Therefore, by an application of Corollary 3.2 the proof is complete.

Step 2. From equations (2.15)–(2.17) and the bound (2.10),

$$I_i(t) \le C \int_{\mathbb{R}^2} \frac{|f_i(u_1, u_2)\bar\phi_{T-t}(-u_1 - iR)\bar\phi_{T-t}(-u_2 - iR)\phi_t(-u_1 - u_2 - 2iR)|}{(|u_1| + 1)(|u_2| + 1)} du_1 du_2$$

for some $C > 0$, where

$$\begin{aligned}
f_1(u_1, u_2) &= (\psi(-u_1 - i(R - 1)) - \bar\psi(-u_1 - iR)) \\
&\qquad \times (\psi(-u_2 - i(R - 1)) - \bar\psi(-u_2 - iR)),\\
f_2(u_1, u_2) &= a^2(R - 1 - iu_1)(R - 1 - iu_2),\\
f_3(u_1, u_2) &= \int_{\mathbb{R}} (e^{(R-1-iu_1)z} - 1)(e^{(R-1-iu_2)z} - 1)\nu(\mathrm{d}z)\\
&= \psi(-u_1 - u_2 - 2i(R - 1)) - \psi(-u_1 - i(R - 1))\\
&\qquad - \psi(-u_2 - i(R - 1)) - f_2(u_1, u_2)
\end{aligned}$$

and

$$\begin{aligned}
f_4(u_1, u_2) &= \int_{\mathbb{R}} e^{2z}(e^{(R-1-iu_1)z} - 1)(e^{(R-1-iu_2)z} - 1)\nu(\mathrm{d}z)\\
&= \psi(-u_1 - u_2 - 2iR) - \psi(-u_1 - i(R + 1))\\
&- \psi(-u_2 - i(R + 1)) + \psi(-2i) - a^2(R - 1 - iu_1)(R - 1 - iu_2)
\end{aligned}$$

From Lemmas A.6 and A.4,

$$|f_i(u_1, u_2)| \le C(1 + \sqrt{|\Re\psi(u_1)|})(1 + \sqrt{|\Re\psi(u_2)|}) \tag{4.6}$$

for some $C < \infty$ and $i \in \{1, 2, 3, 4\}$. Corollary A.5 and Lemma A.6 then imply that $I_i(t) \le J(t)$, where the function J is defined by

$$J(t) = C \int_{\mathbb{R}^2} \frac{(1 + \sqrt{|\Re\psi(u_1)|})(1 + \sqrt{|\Re\psi(u_2)|})}{(1 + |u_1|)(1 + |u_2|)} \times e^{c(\Re\psi(u_1+u_2)t + \Re\psi(u_1)(T-t) + \Re\psi(u_2)(T-t))} \mathrm{d}u_1 \mathrm{d}u_2$$

for some constants $C > 0$ and $c > 0$ (which will later change from line to line). It remains to be shown that

$$\int_0^T J(t)\mathrm{d}t < \infty\,, \tag{4.7}$$

and the theorem will be proved.

Step 3. Assume first that ν satisfies H1. The change of variables $u_1 + u_2 = v_1$ and $u_1 - u_2 = v_2$ together with (A.9) and Lemma A.3 yields

$$J(t) \le C \int_{\mathbb{R}^2} \frac{(1 + \sqrt{|\Re\psi((v_1 + v_2)/2)|})(1 + \sqrt{|\Re\psi((v_1 - v_2)/2)|})}{(1 + |v_1 + v_2|)(1 + |v_1 - v_2|)} \times e^{c(\Re\psi(v_1)T + \Re\psi(v_2)(T-t))} \mathrm{d}v_1 \mathrm{d}v_2\,.$$

Now by (A.10)

$$J(t) \le C \int_{\mathbb{R}^2} \frac{1 + |\Re\psi(v_1/2))| + |\Re\psi(v_2/2)|}{(1 + |v_1 + v_2|)(1 + |v_1 - v_2|)} e^{c(\Re\psi(v_1)T + \Re\psi(v_2)(T-t)} \mathrm{d}v_1 \mathrm{d}v_2\,.$$

Step 4. In the last step we consider the integral of $J(t)$ over $[0, T]$.

$$\begin{aligned}
&\int_0^T J(t)\mathrm{d}t \\
&\le C \int_{\mathbb{R}^2} \frac{(1 + |\Re\psi(v_1/2))| + |\Re\psi(v_2/2)|)e^{c\Re\psi(v_1)T}}{(1 + |v_1 + v_2|)(1 + |v_1 - v_2|)} \frac{1 - e^{c\Re\psi(v_2)T}}{\Re\psi(v_2)} \mathrm{d}v_1 \mathrm{d}v_2 \\
&\le C \int_{\mathbb{R}^2} \frac{(1 + |\Re\psi(v_1/2))| + |\Re\psi(v_2/2)|)e^{c\Re\psi(v_1)T}}{(1 + |v_1 + v_2|)(1 + |v_1 - v_2|)} \frac{1}{1 + |\Re\psi(v_2)|} \mathrm{d}v_1 \mathrm{d}v_2 \\
&\le C \int_{\mathbb{R}^2} \frac{(1 + |\Re\psi(v_1)|)e^{c\Re\psi(v_1)T}}{(1 + |v_1 + v_2|)(1 + |v_1 - v_2|)} \mathrm{d}v_1 \mathrm{d}v_2,
\end{aligned}$$

where the last inequality follows from (A.1).

From Lemma A.2 we then get

$$\int_0^T J(t)\mathrm{d}t \le C \int_{\mathbb{R}} \frac{(1 + |\Re\psi(v_1)|)(1 + \log(1 + |v_1|))e^{c\Re\psi(v_1)}}{1 + |v_1|} \mathrm{d}v_1,$$

and also

$$\int_0^T J(t)\mathrm{d}t \le C \int_{\mathbb{R}} \frac{(1 + \log(1 + |v_1|))e^{c\Re\psi(v_1)}}{1 + |v_1|} \mathrm{d}v_1,$$

for different constants c and C. Lemma now A.1 allows to conclude that this integral is finite, completing the proof of the theorem under the assumption H1.

Suppose now that one of the two alternative assumptions is satisfied. Then, from Lemma A.1, or, if $a > 0$, directly from the form of the characteristic function, we get

$$J(t) \leq C \int_{\mathbb{R}^2} \frac{(1+|v_1+v_2|^{\alpha/2})(1+|v_1-v_2|^{\alpha/2})}{(1+|v_1+v_2|)(1+|v_1-v_2|)} \times e^{-cT|v_1|^\alpha - c(T-t)|v_2|^\alpha} \mathrm{d}v_1 \mathrm{d}v_2,$$

where we set $\alpha = 2$ if $a > 0$. To finish the proof in this case, it is now sufficient to repeat the arguments from the beginning of step 3 onwards, taking $\psi(u) = -|u|^\alpha$. □

A similar result for the quadratic hedging strategy and options with regular pay-offs can be obtained along the same lines; the difference between the two strategies will become important in the case of irregular pay-offs.

Theorem 4.3 (Quadratic hedging, european-like options) *Let the pay-off function and the Lévy process satisfy the conditions* (2.6)–(2.8) *and* (4.2) *for some* $R \in \mathbb{R}$*, and assume that one of the three alternative conditions holds:*
- *ν satisfies the assumption H1 and $a = 0$;*
- *ν satisfies the assumption H2-α with $\alpha \in (0,2)$ and $a = 0$;*
- *$a > 0$.*

Let the hedging strategy be given by Proposition 2.3. Then

$$\lim_{h\downarrow 0} \frac{1}{h} E\left[\left(\int_0^T F_t^h \, dS_t\right)^2\right] = \frac{A}{2} E\left[\int_0^T S_t^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t(z) e^{2z} \nu(dz)\right) dt\right].$$

Proof As in the proof of Theorem 4.1, define I_i, $i = 1, \ldots, 4$ as in (4.4)–(4.5) but with μ, σ and γ now given by (2.21)–(2.23). We now need to prove that $\int_0^T I_i(t)dt < \infty$ for $i = 1, \ldots, 4$, and by the same arguments as in the proof of Theorem 4.1, we get that $I_i(t) \leq J(t)$ with

$$J(t) = C \int_{\mathbb{R}^2} \frac{(1+\sqrt{|\Re\psi(u_1)|})(1+\sqrt{|\Re\psi(u_2)|})|\Upsilon(u_1)\Upsilon(u_2)|}{(1+|u_1|^2)|(1+|u_2|^2)} \times e^{c(\Re\psi(u_1+u_2)t + \Re\psi(u_1)(T-t) + \Re\psi(u_2)(T-t))} \mathrm{d}u_1 \mathrm{d}u_2$$

for some constants $C > 0$ and $c > 0$. From Lemma A.4 we have that

$$|\Upsilon(u)| \leq C(1+\sqrt{|\psi(u)|}) \leq C(1+|u|),$$

which implies

$$J(t) \leq C \int_{\mathbb{R}^2} \frac{(1+\sqrt{|\Re\psi(u_1)|})(1+\sqrt{|\Re\psi(u_2)|})}{(1+|u_1|)|(1+|u_2|)} \times e^{c(\Re\psi(u_1+u_2)t + \Re\psi(u_1)(T-t) + \Re\psi(u_2)(T-t))} \mathrm{d}u_1 \mathrm{d}u_2.$$

This is exactly the same expression as in Theorem 4.1, and hence the claim holds. □

Next, we turn to options with irregular pay-offs. In this case the convergence rate of the discretization error to zero is not necessarily $r(h) = h$, but depends on the properties of the Lévy measure of X near zero. Therefore, we need to make a precise assumption about these properties. For the same reason (to compute the precise convergence rate and the constant rather than just an upper bound) it is necessary to fix the pay-off profile.

Theorem 4.4 (Delta hedging, digital options) *Let the pay-off function be given by $G(S_T) = 1_{S_T \geq K}$ and assume (4.1) for some $R > 0$. Let the hedging strategy be given by Proposition 2.2.*

1. *Assume that $a = 0$ and ν satisfies the assumption H4-α with $\alpha \in (1, 2)$. Then the hedging error satisfies*

$$\lim_{h \downarrow 0} \frac{1}{r(h)} E\left[\left(\int_0^T F_t^h dS_t\right)^2\right] = \frac{AD_\alpha}{2\pi(f_+ + f_-)^{1/\alpha}} p_T(\log K),$$

with $r(h) = h^{1-1/\alpha}$, where D_α is a constant depending only on α and given explicitly by

$$D_\alpha := \frac{1}{(\Gamma(-\alpha)\cos(\pi(2-\alpha)/2))^{1/\alpha}} \int_{\mathbb{R}} dv \frac{1 - e^{-|v|^\alpha} - |v|^\alpha e^{-|v|^\alpha}}{|v|^\alpha(1 - e^{-|v|^\alpha})}$$

and p_T is the density of X_T, which can be computed from the characteristic function via

$$p_T(\log K) = \frac{1}{2\pi} \int_{\mathbb{R}} dv e^{-iv \log K} e^{T\psi(v)}.$$

2. *Assume that $a > 0$. Then the hedging error satisfies*

$$\lim_{h \downarrow 0} \frac{1}{\sqrt{h}} E\left[\left(\int_0^T F_t^h dS_t\right)^2\right] = \frac{AD}{2\pi a} p_T(\log K),$$

with

$$D := \int_{\mathbb{R}} dv \frac{1 - e^{-v^2} - v^2 e^{-v^2}}{v^2(1 - e^{-v^2})}. \tag{4.8}$$

Proof We use the notation introduced in the proof of Theorem 4.1. The proof below covers both cases by setting $\alpha = 2$ in the case $a > 0$. As a preliminary remark, observe that by Lemma A.6, the risk-neutral characteristic exponent $\bar{\psi}$ also has the property (A.5) of Lemma A.1. This shows that condition (2.12) is satisfied and Proposition 2.2 holds.

Step 1. Let

$$e_i(u_1, u_2, t) := f_i(u_1, u_2)\bar{\phi}_{T-t}(-u_1 - iR)\bar{\phi}_{T-t}(-u_2 - iR)\phi_t(-u_1 - u_2 - 2iR)K^{iu_1 + iu_2 - 2R}.$$

Then, with a change of variables,

$$I_i(t) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e_i(u_1,u_2,t)du_1du_2$$
$$= \frac{h^{-1/\alpha}}{8\pi^2}\int_{\mathbb{R}^2} e_i\left(\frac{v_1+v_2h^{-1/\alpha}}{2},\frac{v_1-v_2h^{-1/\alpha}}{2},t\right)dv_1dv_2\,.$$

In this first step we would like to show that

$$\int_0^T h^{-1}(\bar\eta(t)-t)e_i\left(\frac{v_1+v_2h^{-1/\alpha}}{2},\frac{v_1-v_2h^{-1/\alpha}}{2},t\right)dt\,,$$

is bounded from above by a function which does not depend on h and is integrable with respect to v_1 and v_2. This will on one hand prove the assumption (3.1) (with $r(h) = h^{1-1/\alpha}$) and on the other hand will enable us to use the dominated convergence theorem for computing the limit in (3.2).

Using property (A.5), corollary A.5, and the estimate (4.6),

$$|e_i(u_1,u_2,t)| \le Ce^{-c\{t|u_1+u_2|^\alpha+(T-t)|u_1|^\alpha+(T-t)|u_2|^\alpha\}}(1+|u_1|^{\alpha/2})(1+|u_2|^{\alpha/2})$$

for some constants $c, C > 0$ which may change from line to line. By Lemma A.3, we then get:

$$\left|e_i\left(\frac{v_1+v_2}{2},\frac{v_1-v_2}{2},t\right)\right| \le Ce^{-c\{T|v_1|^\alpha+(T-t)|v_2|^\alpha\}}(1+|v_1|^\alpha+|v_2|^\alpha),$$

and finally, evaluating the time integral explicitly using the formula

$$\int_0^T(\bar\eta(t)-t)e^{a(T-t)}dt = \sum_{i=1}^n e^{a(T-T_i)}\int_0^h te^{at}dt = \frac{(ahe^{ah}-e^{ah}+1)(1-e^{aT})}{a^2(1-e^{ah})}, \tag{4.9}$$

we get

$$\int_0^T h^{-1}(\bar\eta(t)-t)\left|e_i\left(\frac{v_1+v_2h^{-1/\alpha}}{2},\frac{v_1-v_2h^{-1/\alpha}}{2},t\right)\right|dt$$
$$\le Ch^{-2}(h(1+|v_1|^\alpha)+|v_2|^\alpha)e^{-cT|v_1|^\alpha}\int_0^T(\bar\eta(t)-t)e^{-c(T-t)h^{-1}|v_2|^\alpha}dt$$
$$= C(1+|v_1|^\alpha)e^{-cT|v_1|^\alpha}\frac{-c|v_2|^\alpha e^{-c|v_2|^\alpha}-e^{-c|v_2|^\alpha}+1}{|v_2|^{2\alpha}}\frac{1-e^{-c|v_2|^\alpha h^{-1}}}{1-e^{-c|v_2|^\alpha}}h$$
$$+ Ce^{-cT|v_1|^\alpha}\frac{-c|v_2|^\alpha e^{-c|v_2|^\alpha}-e^{-c|v_2|^\alpha}+1}{|v_2|^\alpha}\frac{1-e^{-c|v_2|^\alpha h^{-1}}}{1-e^{-c|v_2|^\alpha}}$$
$$\le C(1+|v_1|^\alpha)e^{-cT|v_1|^\alpha}\frac{-c|v_2|^\alpha e^{-c|v_2|^\alpha}-e^{-c|v_2|^\alpha}+1}{|v_2|^\alpha(1-e^{-c|v_2|^\alpha})},$$

where the last inequality follows from the bound $1-e^{-x} \le x$, $x \ge 0$. Since the last expression is integrable with respect to v_1 and v_2 (it is bounded near zero and behaves like $\frac{1}{|v_2|^\alpha}$ at infinity), step 1 is completed.

Step 2. Let us now compute the renormalized limiting hedging error

$$\begin{aligned}\varepsilon_0 &:= \lim_{h\to 0} A h^{1/\alpha-1} E\int_0^T S_t^2(\bar\eta(t)-t)\left(\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right)dt\\ &= \lim_{h\to 0}\frac{A}{8\pi^2}\int_{\mathbb{R}^2} dv_1\, dv_2\\ &\quad\times\int_0^T h^{-1}(\bar\eta(t)-t)\{e_2+e_4\}\left(\frac{v_1+v_2h^{-1/\alpha}}{2},\frac{v_1-v_2h^{-1/\alpha}}{2},t\right)dt\,.\end{aligned}$$

By the dominated convergence theorem, whose application is justified by Step 1, we can compute the limit inside the integral with respect to v_1 and v_2.

$$\begin{aligned}&\lim_{h\to 0}\int_0^T h^{-1}(\bar\eta(t)-t)\{e_2+e_4\}\left(\frac{v_1+v_2h^{-1/\alpha}}{2},\frac{v_1-v_2h^{-1/\alpha}}{2},t\right)dt\\ &= e^{T\psi(-v_1-2iR)}K^{iv_1-2R}L_1L_2,\end{aligned}$$

with

$$\begin{aligned}L_1 &= \lim_{h\to 0} h\Bigg\{\psi(-v_1-2iR)-\psi\left(-\frac{v_1+v_2h^{-1/\alpha}}{2}-i(R+1)\right)\\ &\qquad -\psi\left(-\frac{v_1-v_2h^{-1/\alpha}}{2}-i(R+1)\right)+\psi(-2i)\Bigg\},\\ L_2 &= \lim_{h\to 0}\int_0^T dt(\bar\eta(t)-t)h^{-2}\\ &\qquad\times e^{(T-t)\{-\psi(-v_1-2iR)+\bar\psi(-\frac{v_1+v_2h^{-1/\alpha}}{2}-iR)+\bar\psi(-\frac{v_1-v_2h^{-1/\alpha}}{2}-iR)\}},\end{aligned}$$

provided that both limits exist. Now, a direct computation using Lemma A.6 and equations (A.2) of Lemma A.1 yields $L_1 = 2^{-\alpha}(c_+ + c_-)|v_2|^\alpha$, $v_2 \neq 0$, where the constants c_+ and c_- are defined in Lemma A.1. To compute L_2, we first observe that for all $v_2 \neq 0$

$$\begin{aligned}&\lim_{h\to 0}\Bigg\{-\psi(-v_1-2iR)+\bar\psi\left(-\frac{v_1+v_2h^{-1/\alpha}}{2}-iR\right)\\ &\qquad\qquad +\bar\psi\left(-\frac{v_1-v_2h^{-1/\alpha}}{2}-iR\right)\Bigg\} = -\infty,\\ &\lim_{h\to 0} h\Bigg\{-\psi(-v_1-2iR)+\bar\psi\left(-\frac{v_1+v_2h^{-1/\alpha}}{2}-iR\right)\\ &\qquad\qquad +\bar\psi\left(-\frac{v_1-v_2h^{-1/\alpha}}{2}-iR\right)\Bigg\} = -2^{-\alpha}(c_+ + c_-)|v_2|^\alpha \neq 0.\end{aligned}$$

Combined with the explicit formula (4.9), these two limits allow to conclude that

$$L_2 = \frac{\kappa(v_2)e^{\kappa(v_2)} - e^{\kappa(v_2)} + 1}{\kappa(v_2)^2(1-e^{\kappa(v_2)})}, \quad \kappa(v_2) = -2^{-\alpha}(c_+ + c_-)|v_2|^\alpha.$$

Finally, assembling L_1 and L_2 together and performing the integration with respect to v_1 and v_2, the proof is completed. □

The behavior of the quadratic hedging strategy for options with irregular pay-off is very different from that of delta hedging: the convergence rate improves rather than deteriorates when the Blumenthal-Getoor index α decreases, and in many cases the convergence takes place in the regular regime even for digital options.

Theorem 4.5 (Martingale quadratic hedging, digital options, regular regime) *Let the pay-off function and the Lévy process satisfy the conditions* (2.1)–(2.2) *and* (4.2) *for some $R \in \mathbb{R}$, and assume that one of the two alternative conditions holds:*

- $a = 0$ *and ν satisfies the assumptions H1 and H3-α_+ for some $\alpha_+ \in (0,1]$.*
- $a = 0$ *and ν satisfies the assumptions H2-α_- and H3-α_+ with $0 < \alpha_- \le \alpha_+ < \frac{3}{2}$.*

Let the hedging strategy be given by Proposition 2.3. Then

$$\lim_{h\downarrow 0} \frac{1}{h} E\left[\left(\int_0^T F_t^h\, dS_t\right)^2\right] = \frac{A}{2} E\left[\int_0^T S_t^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t(z) e^{2z} \nu(dz)\right) dt\right].$$

Proof First, from Lemmas A.1 and A.6 it follows that

$$\int_{\mathbb{R}} \frac{|\bar{\phi}_{T-t}(u-iR)|}{1+|u|} du < \infty, \quad \forall t < T,$$

which means that condition (2.3) is satisfied for all $t < T$ and therefore Proposition 2.3 holds.

From the assumption H3-α_+, for all $u \in \mathbb{R}$,

$$\begin{aligned} |\Upsilon(u)| &= \frac{1}{\bar{A}} \left| \int_{\mathbb{R}} \nu(dx)(e^{Rx-iux} - 1)(e^x - 1) \right| \\ &\le C + C \left| \int_{|x|\le 1} \nu(dx)(e^{-iux} - 1) e^{Rx}(e^x - 1) \right| \\ &\le C \int_{|x|\le 1} \nu(dx) |ux|^{\alpha_+ - 1} |e^{-iux} - 1|^{2-\alpha_+} e^{Rx} |e^x - 1| \le C(1 + |u|^{\alpha_+ - 1}). \end{aligned} \tag{4.10}$$

for some constant $C > 0$ which changes from line to line.

As in the proof of Theorem 4.1, define I_i, $i = 1, \ldots, 4$ as in (4.4)–(4.5) but with μ, σ and γ now given by (2.21)–(2.23). We now need to prove that

$\int_0^T I_i(t)dt < \infty$ for $i = 1, \ldots, 4$, and by the same arguments as in the proof of Theorem 4.1, we get that $I_i(t) \leq J(t)$ with

$$J(t) = C \int_{\mathbb{R}^2} \frac{(1 + \sqrt{|\Re\psi(u_1)|})(1 + \sqrt{|\Re\psi(u_2)|})|\Upsilon(u_1)\Upsilon(u_2)|}{(1 + |u_1|)(1 + |u_2|)} \times e^{c(\Re\psi(u_1+u_2)t + \Re\psi(u_1)(T-t) + \Re\psi(u_2)(T-t))} \mathrm{d}u_1 \mathrm{d}u_2$$

for some constants $C > 0$ and $c > 0$.

Using (4.10), this reduces to

$$J(t) \leq C \int_{\mathbb{R}^2} \frac{(1 + \sqrt{|\Re\psi(u_1)|})(1 + \sqrt{|\Re\psi(u_2)|})}{(1 + |u_1|^{(2-\alpha_+)\wedge 1})(1 + |u_2|^{(2-\alpha_+)\wedge 1})} \times e^{c(\Re\psi(u_1+u_2)t + \Re\psi(u_1)(T-t) + \Re\psi(u_2)(T-t))} \mathrm{d}u_1 \mathrm{d}u_2,$$

and the same arguments as in the proof of Theorem 4.1 lead to

$$\int_0^T J(t)\mathrm{d}t \leq C \int_{\mathbb{R}^2} \frac{(1 + |\Re\psi(v_1)|)e^{c\Re\psi(v_1)T}}{(1 + |v_1 + v_2|^{(2-\alpha_+)\wedge 1})(1 + |v_1 - v_2|^{(2-\alpha_+)\wedge 1})} \mathrm{d}v_1 \mathrm{d}v_2. \tag{4.11}$$

If $\alpha_+ \leq 1$, the above expression reduces to

$$\int_0^T J(t)\mathrm{d}t \leq C \int_{\mathbb{R}^2} \frac{(1 + |\Re\psi(v_1)|)e^{c\Re\psi(v_1)T}}{(1 + |v_1 + v_2|)(1 + |v_1 - v_2|)} \mathrm{d}v_1 \mathrm{d}v_2,$$

which is exactly the same as in Theorem 4.1, and so the proof is completed.

Suppose $\alpha_+ > 1$. By assumptions of the theorem this means that H2-α_- is satisfied with $\alpha_- > 0$. By Lemmas A.1 and A.2, the integral (4.11) then reduces to

$$\begin{aligned}\int_0^T J(t)\mathrm{d}t &\leq C \int_{\mathbb{R}} \frac{(1 + |\Re\psi(v_1)|)e^{c\Re\psi(v_1)T}}{1 + |v_1|^{2\alpha_+ - 3}} \mathrm{d}v_1 \\ &\leq C \int_{\mathbb{R}} \frac{(1 + |\Re\psi(v_1)|)e^{-cT|v_1|^{\alpha_-}}}{1 + |v_1|^{2\alpha_+ - 3}} \mathrm{d}v_1,\end{aligned}$$

which is clearly finite. □

Theorem 4.6 (Martingale quadratic hedging, digital options, irregular regime) *Let the pay-off function be given by $G(S_T) = 1_{S_T \geq K}$ and assume (4.2) for some $R > 0$. Let the hedging strategy be given by Proposition 2.3.*

1. *Let $a = 0$ and let ν satisfy the assumption H4-α with $\alpha \in \left(\frac{3}{2}, 2\right)$. Then the hedging error satisfies*

$$\lim_{h \downarrow 0} \frac{1}{r(h)} E\left[\left(\int_0^T F_t^h dS_t\right)^2\right] = \frac{1}{2\pi} A Q_\alpha \frac{\gamma_+ \gamma_-}{\bar{A}^2} (f_+ + f_-)^{\frac{3}{\alpha} - 2} p_T(\log K)$$

with $r(h) = h^{\frac{3}{\alpha}-1}$, where Q_α is a constant depending only on α and given by

$$Q_\alpha := \left(\Gamma(-\alpha)\cos\left(\frac{\pi(2-\alpha)}{2}\right)\right)^{\frac{3}{\alpha}-2}\int_{\mathbb{R}} dv \frac{1-e^{-|v|^\alpha}-|v|^\alpha e^{-|v|^\alpha}}{|v|^{4-\alpha}(1-e^{-|v|^\alpha})},$$

and the constants γ_+, γ_- are defined in equations (4.12)–(4.13) in terms of f_+, f_- and α.

2. *Let $a > 0$. Then the hedging error satisfies*

$$\lim_{h\downarrow 0}\frac{1}{\sqrt{h}} E\left[\left(\int_0^T F_t^h dS_t\right)^2\right] = \frac{AD}{2\pi a} p_T(\log K)\frac{a^4}{\bar{A}^2}$$

with D as in (4.8).

Remark 4.7 The limiting case when $a = 0$ and ν satisfies H4-$\frac{3}{2}$ is not covered by Theorems 4.5 and 4.6. While it is easy to show that the convergence rate in this case will be better than $r(h) = h^{1-\varepsilon}$ for every $\varepsilon > 0$, it may not necessarily be equal to $r(h) = h$ but include, for example, additional logarithmic factors.

Remark 4.8 When $a > 0$ the convergence rate is $r(h) = \sqrt{h}$ both for delta hedging and the quadratic hedging, but the corresponding constant differs by a factor $\frac{a^4}{\bar{A}^2}$, which is strictly smaller than one whenever the underlying Lévy process has jumps. Therefore, also in this case the quadratic hedging strategy is superior to delta hedging.

Proof As in the proof of Theorem 4.4, we establish the result in two steps: first, we find an upper bound and second, we will use the dominated convergence theorem to compute the limiting renormalized hedging error. If $a > 0$, we set $\alpha = 2$.

Step 1. Let

$$e_i(u_1, u_2, t) := f_i(u_1, u_2)\bar{\phi}_{T-t}(-u_1 - iR)\bar{\phi}_{T-t}(-u_2 - iR)\phi_t(-u_1 - u_2 - 2iR) \times K^{iu_1+iu_2-2R}\frac{\Upsilon(u_1)\Upsilon(u_2)}{(R-iu_1)(R-iu_2)}$$

Then, with a change of variables,

$$\frac{1}{r(h)}\int_0^T (\bar{\eta}(t) - t)I_i(t)dt = \frac{1}{2}\int_0^T h^{1-\frac{4}{\alpha}}(\bar{\eta}(t) - t)\int_{\mathbb{R}^2} e_i\left(\frac{v_1 + v_2 h^{-1/\alpha}}{2}, \frac{v_1 - v_2 h^{-1/\alpha}}{2}, t\right) dv_1 dv_2$$

In this first step we would like to show that

$$\int_0^T h^{1-\frac{4}{\alpha}}(\bar{\eta}(t) - t)e_i\left(\frac{v_1 + v_2 h^{-1/\alpha}}{2}, \frac{v_1 - v_2 h^{-1/\alpha}}{2}, t\right) dt$$

has an integrable bound.

First, we need to analyze the behavior of $\Upsilon(u)$ as $u \to \infty$. Suppose first that $a = 0$. Then

$$\lim_{u\to+\infty} \frac{\Upsilon(u)}{u^{\alpha-1}} = \lim_{u\to+\infty} \frac{1}{u^{\alpha-1}\bar{A}} \int_{\mathbb{R}} (e^{Rx-iux} - 1)(e^x - 1)\nu(dx)$$
$$= \lim_{u\to+\infty} \frac{1}{u^{\alpha-1}\bar{A}} \left\{ \int_{\mathbb{R}} (e^{-iux} - 1)x\nu(dx) \right.$$
$$\left. + \int_{\mathbb{R}} e^{-iux}\{e^{(1+R)x} - e^{Rx} - x\}\nu(dx) + \int_{\mathbb{R}} \{1 + x - e^x\}\nu(dx) \right\}.$$

Since the two terms in the last line are bounded and $\alpha > 1$,

$$\lim_{u\to+\infty} \frac{\Upsilon(u)}{u^{\alpha-1}} = \lim_{u\to+\infty} \frac{1}{u^{\alpha-1}\bar{A}} \int_{\mathbb{R}} (e^{-iux} - 1)x\nu(dx)$$
$$= \lim_{u\to+\infty} \frac{1}{u^{\alpha-1}\bar{A}} \left\{ \int_0^{\varepsilon} (e^{-iux} - 1)\frac{f(x)}{x^\alpha}dx - \int_0^{\varepsilon} (e^{iux} - 1)\frac{f(-x)}{x^\alpha}dx \right\},$$

where ε is chosen such that $|f(x)| \le N$ for all x with $|x| \le \varepsilon$ and some $N < \infty$. By a change of variables and dominated convergence we then get

$$\lim_{u\to+\infty} \frac{\Upsilon(u)}{u^{\alpha-1}}$$
$$= \lim_{u\to+\infty} \frac{1}{\bar{A}} \left\{ \int_0^{\varepsilon u} (e^{-ix} - 1)\frac{f(x/u)}{x^\alpha}dx - \int_0^{\varepsilon u} (e^{ix} - 1)\frac{f(-x/u)}{x^\alpha}dx \right\}$$
$$= \frac{f_+}{\bar{A}} \int_0^\infty \frac{(e^{-ix} - 1)}{x^\alpha}dx - \frac{f_-}{\bar{A}} \int_0^\infty \frac{(e^{ix} - 1)}{x^\alpha}dx.$$

Evaluating the integrals (see [20, lemma 14.11]) and treating the limit $u \to -\infty$ in a similar manner, we finally obtain

$$\lim_{u\to+\infty} \frac{\Upsilon(u)}{|u|^{\alpha-1}} = \frac{\Gamma(1-\alpha)}{\bar{A}}\{f_+e^{-i\pi(1-\alpha)/2} - f_-e^{i\pi(1-\alpha)/2}\} := \frac{\gamma_+}{\bar{A}}, \quad (4.12)$$
$$\lim_{u\to-\infty} \frac{\Upsilon(u)}{|u|^{\alpha-1}} = \frac{\Gamma(1-\alpha)}{\bar{A}}\{f_+e^{i\pi(1-\alpha)/2} - f_-e^{-i\pi(1-\alpha)/2}\} := \frac{\gamma_-}{\bar{A}}. \quad (4.13)$$

If $a > 0$, a similar computation which is omitted to save space yields

$$\lim_{u\to\infty} \frac{\Upsilon(u)}{u} = -i\frac{a^2}{\bar{A}}. \quad (4.14)$$

Using property (A.5), corollary A.5, estimate (4.6) and limits (4.12), (4.13) and (4.14),

$$|e_i(u_1, u_2, t)|$$
$$\le Ce^{-c\{t|u_1+u_2|^\alpha + (T-t)|u_1|^\alpha + (T-t)|u_2|^\alpha\}}(1 + |u_1|^{3\alpha/2-2})(1 + |u_2|^{3\alpha/2-2})$$

for some constants $c, C > 0$ which may change from line to line. By Lemma A.3, we then get:

$$\left|e_i\left(\frac{v_1+v_2}{2}, \frac{v_1-v_2}{2}, t\right)\right| \le Ce^{-c\{T|v_1|^\alpha+(T-t)|v_2|^\alpha\}}(1+|v_1|^{3\alpha-4}+|v_2|^{3\alpha-4}),$$

and finally, evaluating the time integral using the formula (4.9),

$$\begin{aligned}
&\int_0^T h^{1-\frac{4}{\alpha}}(\bar\eta(t)-t)\left|e_i\left(\frac{v_1+v_2h^{-1/\alpha}}{2}, \frac{v_1-v_2h^{-1/\alpha}}{2}, t\right)\right| dt \\
&\le C(h^{1-\frac{4}{\alpha}}(1+|v_1|^{3\alpha-4})+h^{-2}|v_2|^{3\alpha-4})e^{-cT|v_1|^\alpha} \\
&\qquad\qquad \times \int_0^T (\bar\eta(t)-t)e^{-c(T-t)h^{-1}|v_2|^\alpha}dt \\
&= C(1+|v_1|^{3\alpha-4})e^{-cT|v_1|^\alpha}\frac{-c|v_2|^\alpha e^{-c|v_2|^\alpha}-e^{-c|v_2|^\alpha}+1}{|v_2|^{2\alpha}}\frac{1-e^{-c|v_2|^\alpha h^{-1}}}{1-e^{-c|v_2|^\alpha}}h^{3-\frac{4}{\alpha}} \\
&\quad + Ce^{-cT|v_1|^\alpha}\frac{-c|v_2|^\alpha e^{-c|v_2|^\alpha}-e^{-c|v_2|^\alpha}+1}{|v_2|^{4-\alpha}}\frac{1-e^{-c|v_2|^\alpha h^{-1}}}{1-e^{-c|v_2|^\alpha}} \\
&\le C(1+|v_1|^{3\alpha-4})e^{-cT|v_1|^\alpha}\frac{-c|v_2|^\alpha e^{-c|v_2|^\alpha}-e^{-c|v_2|^\alpha}+1}{|v_2|^{4-\alpha}(1-e^{-c|v_2|^\alpha})},
\end{aligned}$$

where the last inequality follows from the bound $1-e^{-x} \le x^{3-\frac{4}{\alpha}}$, $x \ge 0$, which holds because $3-\frac{4}{\alpha} \in (\frac{1}{3}, 1)$. Since the last expression is integrable with respect to v_1 and v_2 (it behaves like $\frac{1}{|v_2|^{4-2\alpha}}$ near zero and like $\frac{1}{|v_2|^{4-\alpha}}$ at infinity), step 1 is completed.

Step 2. Similarly to the proof of Theorem 4.4, we compute

$$\begin{aligned}
&\lim_{h\to 0}\int_0^T h^{1-\frac{4}{\alpha}}(\bar\eta(t)-t)\{e_2+e_4\}\left(\frac{v_1+v_2h^{-1/\alpha}}{2}, \frac{v_1-v_2h^{-1/\alpha}}{2}, t\right)dt \\
&= e^{T\psi(-v_1-2iR)}K^{iv_1-2R}L_1L_2L_3,
\end{aligned}$$

where L_1 and L_2 are the same as in the proof of Theorem 4.4, and

$$L_3 = \lim_{h\to 0} h^{2-\frac{4}{\alpha}}\frac{\Upsilon\left(\frac{v_1+v_2h^{-1/\alpha}}{2}\right)\Upsilon\left(\frac{v_1-v_2h^{-1/\alpha}}{2}\right)}{\left(R-i\frac{v_1+v_2h^{-1/\alpha}}{2}\right)\left(R-i\frac{v_1-v_2h^{-1/\alpha}}{2}\right)} = \gamma_+\gamma_-\left(\frac{v_2}{2}\right)^{2\alpha-4}$$

if $a=0$ and $L_3 = \frac{a^4}{A^2}$ if $a>0$. Assembling the three factors together, the proof is completed. □

A Characteristic function estimates in exponential Lévy models and other useful results

Below we use the common notation introduced in the beginning of section 2.

Lemma A.1

1. *Let the Lévy measure ν satisfy the assumption H1 on page 16. Then (i) for every $t > 0$ there exist constants $C > 0$ and $c > 0$ such that*

$$|\phi_t(z)| \leq C|z|^{-c}, \quad z \in \mathbb{R}$$

and (ii) there exists a constant c such that

$$u \geq v \quad \text{implies} \quad \Re\psi(u) \leq \Re\psi(v) + c \quad \text{for all} \quad u, v > 0\,. \tag{A.1}$$

2. *Let ν satisfy the assumption H2-α with $\alpha \in (0, 2)$. Then there exist $c > 0$ and $C > 0$ such that*

$$|\phi_t(z)| \leq Ce^{-ct|z|^\alpha}, \quad \forall t > 0, \forall z.$$

3. *Let ν satisfy the assumption H4-α with $\alpha \in (1, 2)$ and let $a = 0$. Then the characteristic exponent ψ satisfies*

$$\lim_{u\to+\infty} \frac{\psi(u)}{|u|^\alpha} = -c_+ \quad \text{and} \quad \lim_{u\to-\infty} \frac{\psi(u)}{|u|^\alpha} = -c_-, \tag{A.2}$$

where

$$c_+ = -\Gamma(-\alpha)\{f_+e^{-i\pi\alpha/2} + f_-e^{i\pi\alpha/2}\} \tag{A.3}$$
$$c_- = -\Gamma(-\alpha)\{f_+e^{i\pi\alpha/2} + f_-e^{-i\pi\alpha/2}\} \tag{A.4}$$

and there exist constants $c_1, c_3 \in \mathbb{R}$ and $c_2, c_4 > 0$ such that for all $u \in \mathbb{R}$,

$$c_1 - c_2|u|^\alpha < \Re\psi(u) < c_3 - c_4|u|^\alpha. \tag{A.5}$$

If $a > 0$ then equations (A.2) *hold with $c_+ = c_- = \frac{a^2}{2}$ and inequality* (A.5) *holds with $\alpha = 2$.*

Proof

1. The property (i) is Lemma 28.5 in [20]; let us concentrate on property (ii). Since this property is linear in ψ and clearly satisfied by a Lévy process with zero Lévy measure, we can suppose without loss of generality that $a = 0$. Let $u \geq v$ and $u, v > 0$, then

$$\begin{aligned}\Re\psi(u) - \Re\psi(v) = &\left\{\int_{\mathbb{R}} (\cos(ux) - 1)\nu_0(dx) - \int_{\mathbb{R}} (\cos(vx) - 1)\nu_0(dx)\right\}\\ &+ \left\{\int_{\mathbb{R}} (\cos(ux) - 1)\frac{k(x)}{|x|}dx - \int_{\mathbb{R}} (\cos(vx) - 1)\frac{k(x)}{|x|}dx\right\}.\end{aligned} \tag{A.6}$$

A change of variables ($y = ux$ and $y = vx$) yields for the second term

$$\begin{aligned}&\int_{\mathbb{R}} (\cos(ux) - 1)\frac{k(x)}{|x|}dx - \int_{\mathbb{R}} (\cos(vx) - 1)\frac{k(x)}{|x|}dx\\ &\qquad = \int_{\mathbb{R}} \frac{(\cos(y) - 1)}{|y|}(k(y/u) - k(y/v))dy \leq 0.\end{aligned}$$

Thus, (A.1) is satisfied with $c = 2\int_{\mathbb{R}} \nu_0(dx)$.

2. This follows from the proof of Proposition 28.3 in [20].
3. Choose $\varepsilon > 0$ and $N < \infty$ such that $|f(x)| \leq N$ for all x with $|x| \leq \varepsilon$. Since $\alpha > 1$,

$$\begin{aligned}\lim_{u\to+\infty} \frac{\psi(u)}{u^\alpha} &= \lim_{u\to+\infty} \frac{1}{u^\alpha} \int_{|x|\leq\varepsilon} (e^{iux} - iux - 1)\nu(dx)\\ &= \lim_{u\to+\infty} \frac{1}{u^\alpha} \int_{|x|\leq\varepsilon} (e^{iux} - iux - 1)\frac{f(x)}{|x|^{1+\alpha}}dx.\end{aligned}$$

By a change of variables and dominated convergence we then get

$$\lim_{u\to+\infty}\frac{\psi(u)}{u^\alpha}=\lim_{u\to+\infty}\int_{|x|\le\varepsilon u}(e^{ix}-ix-1)\frac{f(x/u)}{|x|^{1+\alpha}}dx$$
$$=f_-\int_{-\infty}^{0}(e^{ix}-ix-1)\frac{dx}{|x|^{1+\alpha}}+f_+\int_0^{+\infty}(e^{ix}-ix-1)\frac{dx}{|x|^{1+\alpha}}.$$

These integrals are explicitly computed in [20, page 84], and the case $u\to-\infty$ can be treated in a similar manner. The estimates (A.5) follow directly from (A.2). For the case $a>0$ see [3, page 16].
□

Lemma A.2 *Let $\alpha>\frac{1}{2}$, $\alpha\neq 1$. Then there exists $C<\infty$ with*

$$\int_{\mathbb{R}}\frac{dv}{(1+|u+v|^\alpha)(1+|u-v|^\alpha)}\le C(1+|u|)^{1-2\alpha}.$$

In the case $\alpha=1$, there exits $C<\infty$ with

$$\int_{\mathbb{R}}\frac{dv}{(1+|u+v|)(1+|u-v|)}\le C\frac{1+\log(1+|u|)}{1+|u|}.$$

Proof In the case $\alpha\neq 1$, we have

$$\int_{\mathbb{R}}\frac{dv}{(1+|u+v|^\alpha)(1+|u-v|^\alpha)}=\int_0^\infty\frac{dv}{(1+||u|+v|^\alpha)(1+||u|-v|^\alpha)}$$
$$\le\frac{2}{1+|u|^\alpha}\int_0^{2|u|}\frac{du}{1+||u|-v|^\alpha}+2\int_{2|u|}^\infty\frac{dv}{(1+|v|^\alpha)(1+|\frac{v}{2}|^\alpha)}$$
$$\frac{C}{(1+|u|)^\alpha}\int_0^{|u|}\frac{dv}{(1+v)^\alpha}+C\int_{2|u|}^\infty\frac{dv}{(1+v)^{2\alpha}}\le\frac{C}{(1+|u|)^{2\alpha-1}},$$

where C is a constant which may change from inequality to inequality. In the case $\alpha=1$ the proof is done in a similar manner (the logarithmic factor appears in the first integral of the last line). □

Lemma A.3
1. *For any Lévy process X,*

$$\Re\psi(u)\le\frac{1}{4}\Re\psi(2u),\quad u\in\mathbb{R}. \tag{A.7}$$

2. *Assume that there exists a constant $C>0$ such that*

$$u\ge v\quad\Rightarrow\quad\Re\psi(u)\le\Re\psi(v)+C\,,\quad u,v>0\,. \tag{A.8}$$

Then

$$\Re\psi((u+v)/2)+\Re\psi((u-v)/2)\le(\Re\psi(u)+\Re\psi(v))/8+C/4\,,\qquad u,v\in\mathbb{R}\,, \tag{A.9}$$

$$\sqrt{|\Re\psi((u+v)/2)||\Re\psi((u-v)/2)|}\le 8(|\Re\psi(u/2)|+|\Re\psi(v/2)|)+2C\,,\qquad u,v\in\mathbb{R}\,, \tag{A.10}$$

Proof By the Lévy-Khintchine formula

$$\Re\psi(2u)=-4a^2\frac{u^2}{2}+\int_{\mathbb{R}}(\cos(2ux)-1)\nu(\mathrm{d}x)$$
$$=-4a^2\frac{u^2}{2}+2\int_{\mathbb{R}}(\cos(ux)-1)^2\nu(\mathrm{d}x)+4\int_{\mathbb{R}}(\cos(ux)-1)\nu(\mathrm{d}x)$$
$$\ge-4a^2\frac{u^2}{2}+4\int_{\mathbb{R}}(\cos(ux)-1)\nu(\mathrm{d}x)\,,$$

which proves (A.7). Combined with (A.8), this immediately yields

$$\Re\psi((u+v)/2) \le (\Re\psi(u) + \Re\psi(v))/8 + C/4\,, \quad u,v > 0\,, \tag{A.11}$$

and therefore, since $\Re\psi \le 0$, for all $u, v \in \mathbb{R}$,

$$\begin{aligned}&\Re\psi((u+v)/2) + \Re\psi((u-v)/2) = \Re\psi(|u+v|/2) + \Re\psi(|u-v|/2)\\ &= \Re\psi((|u|+|v|)/2)+\Re\psi((|u|-|v|)/2) \le \Re\psi((|u|+|v|)/2) \le (\Re\psi(|u|)+\Re\psi(|v|))/8+C/4\,.\end{aligned}$$

Finally, taking absolute values in the above inequality, we have

$$|\Re\psi((u+v)/2)| + |\Re\psi((u-v)/2)| \ge (|\Re\psi(|u|)| + |\Re\psi(|v|)|)/8 - C/4,$$

and after a change of variables,

$$|\Re\psi((u+v)/2)| + |\Re\psi((u-v)/2)| \le 8(|\Re\psi(u/2)| + |\Re\psi(v/2)|) + 2C,$$

from which (A.10) follows. □

Lemma A.4 *Let $R, R' \in \mathbb{R}$ with $R \le R'$ and assume*

$$\int_{|x|>1} e^{-xR}\nu(dx) < \infty, \quad \textit{and} \quad \int_{|x|>1} e^{-xR'}\nu(dx) < \infty.$$

Then there exists $C > 0$ such that for all $u, v \in \mathbb{C}$ with $\Im u \in [R, R']$, $\Im v \in [R, R']$ and $\Im u + \Im v \in [R, R']$,

$$|\psi(u+v) - \psi(u) - \psi(v)| \le C(1 + \sqrt{|\Re\psi(\Re u)|})(1 + \sqrt{|\Re\psi(\Re v)|}).$$

Proof From the Lévy-Khintchine formula,

$$\begin{aligned}|\psi(u+v) - \psi(u) - \psi(v)| &= \left|-a^2uv + \int_{\mathbb{R}}(e^{\mathrm{i}ux}-1)(e^{\mathrm{i}vx}-1)\nu(\mathrm{d}x)\right|\\ &\le c_1 + c_2a^2(1+|\Re u|)(1+(|\Re v|)\\ &+ \left(\int_{|x|\le 1}|e^{\mathrm{i}ux}-1|^2\nu(\mathrm{d}x)\right)^{\frac{1}{2}}\left(\int_{|x|\le 1}|e^{\mathrm{i}vx}-1|^2\nu(\mathrm{d}x)\right)^{\frac{1}{2}}.\end{aligned} \tag{A.12}$$

for some constants c_1 and c_2. Let $u = \alpha + i\beta$, then

$$\Re\psi(\Re u) = -\frac{a^2\alpha^2}{2} - \int_{\mathbb{R}}(\cos\alpha x - 1)\nu(dx). \tag{A.13}$$

Therefore,

$$\begin{aligned}&\int_{|x|\le 1}|e^{iux}-1|^2\nu(\mathrm{d}x)\\ &\quad= \int_{|x|\le 1}(e^{-\beta x}-1)^2\nu(\mathrm{d}x) + 2\int_{|x|\le 1}e^{-\beta x}(1-\cos(\alpha x))\nu(\mathrm{d}x) \le c_3 + 2e^{|\beta|}|\Re\psi(\alpha)|\end{aligned} \tag{A.14}$$

for some $c_3 < \infty$. Combining (A.12), (A.13) and (A.14), the proof is completed. □

Corollary A.5 *Let R and R' be as in Lemma A.4. Then there exist constants $C_1 \in \mathbb{R}$ and $C_2 > 0$ for all $u \in \mathbb{C}$ with $\Im u \in [R, R']$,*

$$\Re\psi(u) \le C_1 + C_2\Re\psi(\Re u).$$

Lemma A.6 *Let $R, R', \bar{R}, \bar{R}' \in \mathbb{R}$ with $R \le R'$ and $\bar{R} \le \bar{R}'$, such that*

$$\int_{|x|>1} (e^{-xR} + e^{-xR'})\nu(dx) < \infty, \quad \text{and} \qquad \int_{|x|>1} (e^{-x\bar{R}} + e^{-x\bar{R}'})\bar{\nu}(dx) < \infty.$$

Then there exists $C > 0$ such that for all $u, v \in \mathbb{C}$ with $\Re u = \Re v$, $\Im u \in [R, R']$ and $\Im v \in [\bar{R}, \bar{R}']$,

$$|\psi(u) - \bar{\psi}(v)| \le C(1 + \sqrt{|\Re\psi(\Re u)|}).$$

Proof Let $z = \Re u = \Re v$. The difference in question can be rewritten as

$$\begin{aligned}\psi(u) - \bar{\psi}(v) = \psi(z) - \bar{\psi}(z) \\ + \psi(u) - \psi(\Re u) - \psi(i\Im u) \\ + \bar{\psi}(\Re v) + \bar{\psi}(i\Im v) - \bar{\psi}(v) \\ + \psi(i\Im u) - \psi(i\Im v). \end{aligned} \tag{A.15}$$

Let us start with the first line. From Theorem 33.1 in [20], $\gamma - \bar{\gamma} - \int_{-1}^{1} x(\nu - \bar{\nu})(\mathrm{d}x) = a^2\eta$ for some $\eta \in \mathbb{R}$. This relation yields

$$|\psi(z) - \bar{\psi}(z)| = \left| a^2\eta z + \int_{\mathbb{R}} (e^{\mathrm{i}zx} - 1)(e^{\varphi(x)} - 1)\nu(\mathrm{d}x) \right|,$$

where $\varphi(x)$ is defined by $e^{\varphi(x)} = \nu(\mathrm{d}x)/\bar{\nu}(\mathrm{d}x)$. Equation (A.13) shows that when $a > 0$, the first term in the right-hand side satisfies the required bound; let us focus on the second term (the integral). In the following, C denotes a constant which may change from line to line.

$$\begin{aligned} &\left| \int_{\mathbb{R}} (e^{\mathrm{i}zx} - 1)(e^{\varphi(x)} - 1)\nu(\mathrm{d}x) \right| \le C + \left| \int_{|x|\le 1} (e^{\mathrm{i}zx} - 1)(e^{\varphi(x)} - 1)\nu(\mathrm{d}x) \right| \\ &\le C + \left| \int_{\{x:|x|\le 1\}\cap\{x:|\varphi(x)|\le 1\}} (e^{\mathrm{i}zx} - 1)(e^{\varphi(x)} - 1)\nu(\mathrm{d}x) \right| \\ &\le C + \left(\int_{\{x:|x|\le 1\}} |e^{\mathrm{i}zx} - 1|^2 \nu(\mathrm{d}x) \right)^{1/2} \left(\int_{\{x:|\varphi(x)|\le 1\}} (e^{\varphi(x)} - 1)^2 \nu(\mathrm{d}x) \right)^{1/2} \\ &\le C + C \left(\int_{\{x:|x|\le 1\}} |e^{\mathrm{i}zx} - 1|^2 \nu(\mathrm{d}x) \right)^{1/2} \le C(1 + \sqrt{|\Re\psi(z)|}), \end{aligned}$$

where the second and the fourth inequality follow from [20, Remark 33.3] and the last one follows from (A.14). Finally,

$$|\psi(z) - \bar{\psi}(z)| \le C(1 + \sqrt{|\Re\psi(z)|}). \tag{A.16}$$

Applying Lemma A.4 to the second and the third line in (A.15), and observing that the fourth line is bounded by a constant, we get

$$|\psi(u) - \bar{\psi}(v)| \le C \left(1 + \sqrt{|\Re\psi(z)|} + \sqrt{|\Re\bar{\psi}(z)|} \right).$$

Now, using (A.16) for a second time, the proof is completed. □

B Martingale representations for Fourier integrals

Lemma B.1 *Let the process F be defined by*

$$F_t = \int_{\mathbb{R}} f(u)\bar{\phi}_{T-t}(-u-iR)S_t^{R'-iu}du, \tag{B.1}$$

where f is a deterministic function satisfying

$$\int_{\mathbb{R}} |f(u)\bar{\phi}_{T-t}(-u-iR)|du < \infty, \quad \forall t < T. \tag{B.2}$$

Assume

$$\int_{|x|>1} e^{2R'x}\nu(dx) < \infty \quad \text{and} \quad \int_{|x|>1} e^{Rx}\bar{\nu}(dx) < \infty. \tag{B.3}$$

Then the representation (2.11) *holds for F with*

$$\mu_t = \int_{\mathbb{R}} f(u)\bar{\phi}_{T-t}(-u-iR)S_t^{R'-iu} \times (\psi(-u-iR') - \bar{\psi}(-u-iR))du, \tag{B.4}$$

$$\sigma_t = a\int_{\mathbb{R}} f(u)\bar{\phi}_{T-t}(-u-iR)(R'-iu)S_t^{R'-iu}du, \tag{B.5}$$

$$\gamma_t(z) = \int_{\mathbb{R}} f(u)\bar{\phi}_{T-t}(-u-iR)S_t^{R'-iu}(e^{(R'-iu)z}-1)du. \tag{B.6}$$

Proof Let $t < T$. Applying the Itô formula under the integral sign in (B.1), we find, under the condition (B.3),

$$\begin{aligned} &F_t - F_0 \\ &= \int_{\mathbb{R}} du f(u)\int_0^t \bar{\phi}_{T-s}(-u-iR)S_s^{R'-iu}(\psi(-u-iR') - \bar{\psi}(-u-iR))ds \\ &+ \int_{\mathbb{R}} du f(u)\int_0^t \bar{\phi}_{T-s}(-u-iR)(R'-iu)S_s^{R'-iu} a dW_s \\ &+ \int_{\mathbb{R}} du f(u)\int_0^t \bar{\phi}_{T-s}(-u-iR)S_{s-}^{R'-iu}\int_{\mathbb{R}}(e^{(R'-iu)z}-1)\tilde{J}_X(ds\times dz). \end{aligned} \tag{B.7}$$

To finish the proof, we apply a suitable Fubini-type theorem to each of the three terms. For the first term, we use the standard Fubini theorem (path by path), whose applicability condition is

$$\begin{aligned} &\int_{\mathbb{R}} du \int_0^t \left| f(u)\bar{\phi}_{T-s}(-u-iR)S_s^{R'-iu}(\psi(-u-iR') - \bar{\psi}(-u-iR))\right| ds \\ &\le C\sup_{s\le t} S_s^{R'} \int_{\mathbb{R}} du \int_0^t \left| f(u)e^{(T-s)\Re\bar{\psi}(-u-iR)}\right| \left(1 + \sqrt{|\Re\bar{\psi}(u)|}\right) ds \\ &\le C\sup_{s\le t} S_s^{R'} \int_{\mathbb{R}} du \int_0^t \left| f(u)e^{(T-s)c\Re\bar{\psi}(u)}\right| \left(1 + \sqrt{|\Re\bar{\psi}(u)|}\right) ds \\ &\le C\sup_{s\le t} S_s^{R'} \int_{\mathbb{R}} du \left| f(u)e^{(T-t)c\Re\bar{\psi}(u)}\right| < \infty \quad a.s., \end{aligned}$$

where we used Lemma A.6 to pass from line 1 to line 2 and Corollary A.5 from line 2 to line 3, and the constants $c > 0$ and $C > 0$ may change from line to line. Note that $\sup_{s\le t} S_s^{R'} < \infty$ a.s. because S is càdlàg.

Let us now assume that $\sigma > 0$ and study the second term in the right-hand side of (B.7), which can be written as

$$\int_{\mathbb{R}} \mu(du) \int_0^t H_s^u dW_s, \tag{B.8}$$

where

$$\mu(du) = |f(u)\bar{\phi}_{T-t}(-u-iR)|du,$$

is a finite positive measure on $\mathbb{R}$ and

$$H_s^u = \frac{af(u)\bar{\phi}_{T-s}(-u-iR)}{2\pi|f(u)\phi_{T-t}(-u-iR)|}(R'-iu)S_s^{R'-iu}.$$

By the Fubini theorem for stochastic integrals (see [19, page 208]), we can interchange the two integrals in (B.8) provided that

$$E\int_0^t \mu(du)|H_s^u|^2 ds < \infty. \tag{B.9}$$

From Corollary A.5 it follows that

$$\frac{|\bar{\phi}_{T-s}(-u-iR)|}{|\bar{\phi}_{T-t}(-u-iR)|} \leq C,$$

for all $s \leq t \leq T$ for some constant $C > 0$ which does not depend on s and t. To prove (B.9) it is then sufficient to check

$$E\int_0^t \int_{\mathbb{R}} |f(u)\bar{\phi}_{T-t}(-u-iR)||S_s^{2(R'-iu)}|^2(R'-iu)^2 dudt < \infty.$$

After evaluating the expectation explicitly using (B.3), the finiteness of this integral follows from

$$|\bar{\phi}_{T-t}(-u-iR)| \leq Ce^{-(T-t)\frac{\sigma^2u^2}{2}}. \tag{B.10}$$

Therefore, the second term on the right-hand side of (B.7) is equal to $\int_0^t \sigma_s dW_s$.

Let us now turn to the third term in the right-hand side of (B.7). Here we need to apply the Fubini theorem for stochastic integrals with respect to a compensated Poisson random measure [1, Theorem 5] and the applicability condition boils down to

$$E\int_0^t \int_{\mathbb{R}} |f(u)\bar{\phi}_{T-t}(-u-iR)||S_s^{2(R'-iu)}|^2 \int_{\mathbb{R}} |e^{(R'-iu)z}-1|^2\nu(dz)dudt < \infty. \tag{B.11}$$

If $\sigma > 0$, this is once again guaranteed by (B.10), and when $\sigma = 0$,

$$\int_{\mathbb{R}} |e^{(R'-iu)z}-1|^2\nu(dz) = \psi(-2iR') - 2\Re\psi(-u-iR').$$

Therefore, evaluating the expectation explicitly, and using Lemma A.6, the integrability condition (B.11) reduces to

$$\int_{\mathbb{R}} |f(u)\bar{\phi}_{T-t}(-u-iR)|(1+|\Re\bar{\psi}(u)|)du < \infty,$$

which holds by Corollary A.5 and assumption (B.2). □

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