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Risk-sensitive asset management with jump-diffusion price processes

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Risk-Sensitive Control

Control theory: Jacobson, Whittle, Bensoussan, Fleming,..

Asset Management: Bielecki-Pliska, Kuroda-Nagai, Peng-Nagai

Conventional control: $\max \mathbb{E}[F]$ for some performance function F.

Risk-sensitive control: maximize

$$-\frac{2}{\theta} \log \mathbb{E}\left[e^{-\frac{\theta}{2}F}\right] = \mathbb{E}[F] - \frac{\theta}{2}\mathrm{var}[F] + o(\theta).$$

Conventional control recovered as $\theta \to 0$.

In risk-sensitive asset management, F is the log-return, i.e. $F = \log V$ where V is portfolio value. Objective is then to maximize

$$-\frac{2}{\theta} \log \mathbb{E}\left[e^{-\frac{\theta}{2}\log V}\right] = -\frac{2}{\theta} \log \mathbb{E}\left[V^{-\theta/2}\right].$$

The optimization problem is then equivalent to maximizing power utility, but has an aspect of 'risk-return trade-off' à la Markowitz. As $\theta \to 0$ we revert to the growth-optimal portfolio.

I: JUMP-DIFFUSION PRICES WITH DIFFUSION FACTORS

The Risk-Sensitive Investment Problem

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be the underlying probability space.

Take a market with a money market asset S_0 with dynamics

$$\frac{dS_0(t)}{S_0(t)} = (a_0 + A_0' X(t))\, dt, \qquad S_0(0) = s_0 \tag{1}$$

and m risky assets following jump-diffusion SDEs

$$\begin{aligned} \frac{dS_i(t)}{S_i(t^-)} &= (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t) + \int_{\mathbf{Z}} \gamma_i(z) \bar{N}_{\mathbf{p}}(dt, dz), \\ & S_i(0) = s_i, \quad i = 1, \dots, m \end{aligned} \tag{2}$$

$X(t)$ is a n-dimensional vector of economic factors following

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \qquad X(0) = x \tag{3}$$

- $W(t)$ is a $\mathbb{R}^{m+n}$-valued $(\mathcal{F}_t)$-Brownian motion with components $W_k(t)$, $k = 1, \dots, (m+n)$.
- $\bar{N}_{\mathbf{p}}(dt, dz)$ is a Poisson random measure (see e.g. Ikeda and Watanabe defined as

$$\bar{N}_{\mathbf{p}}(dt, dz) = \begin{cases} N_{\mathbf{p}}(dt, dz) - \nu(dz)dt =: \tilde{N}_{\mathbf{p}}(dt, dz) & \text{if } z \in \mathbf{Z}_0 \\ N_{\mathbf{p}}(dt, dz) & \text{if } z \in \mathbf{Z} \backslash \mathbf{Z}_0 \end{cases}$$

- the jump intensity $\gamma(z)$ satisfies appropriate well-posedness conditions.
- assume that

$$\Sigma\Sigma' > 0 \tag{4}$$

The wealth, $V(t)$ of the investor in response to an investment strategy $h(t) \in \mathcal{H}$, follows the dynamics

$$\begin{aligned} \frac{dV(t)}{V(t^-)} &= \left(a_0 + A_0'X(t)\right)dt + h'(t)\left(\hat{a} + \hat{A}X(t)\right)dt + h'(t)\Sigma dW_t \\ &\quad + \int_{\mathbf{Z}} h'(t)\gamma(z)\bar{N}_{\mathbf{p}}(dt, dz) \end{aligned} \tag{5}$$

with initial endowment $V(0) = 0$, where $\hat{a} := a - a_0\mathbf{1}$, $\hat{A} := A - \mathbf{1}A_0'$ and $\mathbf{1} \in \mathbf{R}^m$ denotes the m-element unit column vector.

The objective is to maximize a function of the log-return of wealth

$$J(x,t,h;\theta) := -\frac{1}{\theta}\ln\mathbf{E}\left[e^{-\theta\ln V(t,x,h)}\right] = -\frac{1}{\theta}\ln\mathbf{E}\left[V^{-\theta}(t,x,h)\right] \tag{6}$$

By Itô,

$$e^{-\theta \ln V(t)} = v^{-\theta} \exp\left\{\theta \int_0^t g(X_s, h(s); \theta) ds\right\} \chi_t^h \tag{7}$$

where

$$\begin{aligned} g(x, h; \theta) &= \frac{1}{2}(\theta + 1) h'\Sigma\Sigma' h - a_0 - A_0' x - h'(\hat{a} + \hat{A}x) \\ &+ \int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left[(1 + h'\gamma(z))^{-\theta} - 1\right] + h'\gamma(z) 1_{\mathbf{Z}_0}(z)\right\} \nu(dz) \end{aligned} \tag{8}$$

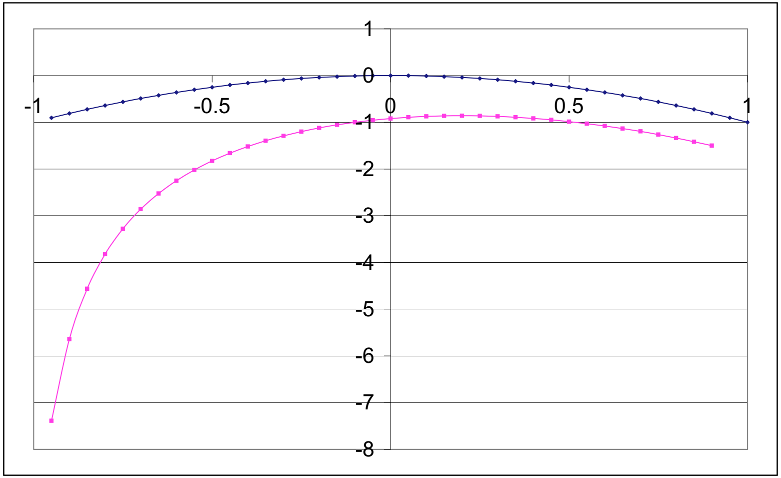
The Doléans exponential χ_t^h is given by

$$\begin{aligned}\chi_t^h \;:=\; \exp\Bigg\{ & -\theta\int_0^t h(s)'\Sigma dW_s - \frac{1}{2}\theta^2\int_0^t h(s)'\Sigma\Sigma' h(s)ds \\ & + \int_0^t\int_{\mathbf{Z}} \ln\left(1 - G(z,h(s);\theta)\right)\tilde{N}_{\mathbf{p}}(ds,dz) \\ & + \int_0^t\int_{\mathbf{Z}} \left\{\ln\left(1 - G(z,h(s);\theta)\right) + G(z,h(s);\theta)\right\}\nu(dz)ds \Bigg\},\end{aligned} \tag{9}$$

with

$$G(z,h;\theta) \;=\; 1 - \left(1 + h'\gamma(z)\right)^{-\theta} \tag{10}$$

Figure 1: Function $-x^2 - 1/(1+x)$



Solving the Stochastic Control Problem

The process involves

1. change of measure;
2. deriving the HJB PDE;
3. identifying a (unique) candidate optimal control;
4. proving a verification theorem;
5. proving existence of a $C^{1,2}$ solution to the HJB PDE.

Changes of measure for semimartingales: the Doléans-Dade theorem

The Girsanov theorem for Brownian motion shows that on Wiener space 'change of measure is change of drift' and gives an exponential formula for the Radon-Nikodým derivative. The analogous result for general semimartingales is known as the Doléans-Dade theorem. It is described in detail §II.8 of Protter's book. In particular, Theorem 37 of that section states the following. We are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Theorem. Let M be an $\mathcal{F}_t$-semimartingale with $M_0 = 0$. Then there exists a unique semimartingale Z, denoted $Z = \mathcal{E}(M)$, satisfying the equation

$$Z_t = 1 + \int_0^t Z_{s-} dM_t. \tag{11}$$

Z is given explicitly by

$$Z_t = e^{M_t - \frac{1}{2}[M,M]_t^c} \prod_{0<s\leq t} (1 + \Delta M_s) e^{-\Delta M_s}, \tag{12}$$

where the infinite product converges.

In (12), $[M,M]^c_t$ denotes the quadratic variation of the continuous martingale part M^c of M.

When M is a local martingale, Z is a positive local martingale and hence a supermartingale, so that $\mathbb{E}[Z_T] \le 1$. By standard arguments, it is a martingale on any finite time interval $[0,T]$ provided $\mathbb{E}[Z_T] = 1$. We may then define a measure $\mathbb{Q}$ on $\mathcal{F}_T$ by its Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(M)_T. \tag{13}$$

Theorem. Let M, N be local martingales. Define $Z = \mathcal{E}(M)$, assume $\mathbb{E}[Z_T] = 1$ and define $\mathbb{Q}$ by (13). Let A be a predictable process and define $X_t = N_t - A_t$. Then X is a $\mathbb{Q}$-local martingale iff A is the predictable compensator of $[M,N]$. Here, $[M,N]$ is the cross-variation process defined by

$$[M,N] = \frac{1}{4}([M+N, M+N] - [M-N, M-N]).$$

Proof

It is standard that X is a $\mathbb{Q}$-local martingale iff XZ is a $\mathbb{P}$-local martingale. By the Ito product formula

$$d(XZ) = X_{-}dZ + Z_{-}dN - Z_{-}dA + d[Z, N],$$

and from (11)

$$[Z, N] = [Z \cdot M, N] = Z \cdot [M, N].$$

Thus

$$d(XZ) = X_{-}dZ + Z_{-}dN + Z_{-}(d[M, N] - dA),$$

and XZ is a local martingale iff $[M, N] - A$ is a local martingale. □

Application to point processes

Let N_t be a Poisson process with constant rate λ and let a_t be a predictable integrable process. We take

$$M_t = \int_0^t a_s(dN_s - \lambda ds).$$

Then $\Delta M_t = a_t$ and from (12) we have

$$\begin{aligned} \log \mathcal{E}(M)_t &= M_t - \sum_{s \le t}(\Delta M_s + \log(1 + \Delta M_s)) \\ &= \int_0^t \log(1 + a_s)dN_s - \int_0^t a_s \lambda ds. \end{aligned}$$

If we now define μ_t by $\mu_t = \lambda(1 + a_s)$ then

$$M_t = \int_0^t \frac{\mu_s - \lambda}{\lambda}(dN_s - \lambda ds)$$

and ...

$$
\begin{aligned}
\mathcal{E}(M)_t &= \prod_{T_i \le t} \left(\frac{\mu_{T_i}}{\lambda} \right) e^{-\int_0^t (\mu_s - \lambda) ds} \\
&= \exp\left(\int_0^t \log\left(\frac{\mu_s}{\lambda} \right) dN_s - \int_0^t (\mu_s - \lambda) ds \right).
\end{aligned}
$$

where (T_i) are the jump times of N_t. Since $\Delta M_t = (\mu_t - \lambda)/\lambda$ and $\Delta N_t = 1$, the predictable compensator of $[M, N]$ is $A_t = \int_0^t (\mu_s - \lambda) ds$ and we conclude that *under measure* $\mathbb{Q}$ *defined by* (13), N_t *is a point process with rate* μ_t.

Back to our problem ...

The next step is the one introduced by Kuroda and Nagai. Let $\mathbb{P}_h^\theta$ be the measure on $(\Omega, \mathcal{F}_T)$ defined via the Radon-Nikodým derivative

$$\frac{d\mathbb{P}_h^\theta}{d\mathbb{P}} := \chi_T^h \tag{14}$$

For a change of measure to be possible, we must ensure that $G(z, h(s); \theta) < 1$, which is satisfied iff $h'(s)\gamma(z) > -1$ a.s. $d\nu$.

$$W_t^h = W_t + \theta \int_0^t \Sigma' h(s) ds$$

is a standard Brownian motion under the measure $\mathbb{P}_h^\theta$ and $X(t)$ satisfies the SDE:

$$dX(t) = (b + BX(t) - \theta\Lambda\Sigma' h(t))\, dt + \Lambda dW_t^h, \qquad t \in [0, T] \tag{15}$$

For the jump term, the change of measure is equivalent to an absolutely continuous change of the jump measure ν, so that for a set A with $\nu(A) < \infty$, the jump martingale $\tilde{N}^h_{\mathbf{p}}(t, A)$ under $\mathbb{P}^\theta_h$ is

$$\begin{aligned}\tilde{N}^h_{\mathbf{p}}(t, A) &= N_{\mathbf{p}}(t, A) - \int_0^t \int_A \{1 - G(s, X(s), z, h(s); \theta)\} \, \nu(dz)ds \\ &= N_{\mathbf{p}}(t, A) - \int_0^t \int_A \left\{(1 + h'\gamma(s, X(s), z))^{-\theta}\right\} \nu(dz)ds\end{aligned}$$

Remark: For the present diffusion-factor problem, we don't need to use this fact.

Introduce two auxiliary criterion functions under $\mathbb{P}_h^\theta$:

- the risk-sensitive control problem:

$$I(v, x; h; t, T; \theta) = -\frac{1}{\theta} \ln \mathbf{E}_{t,x}^{h,\theta} \left[\exp \left\{ \theta \int_t^T g(X_s, h(s); \theta) ds - \theta \ln v \right\} \right] \quad (16)$$

  where $\mathbf{E}_{t,x} [\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h^\theta$ and with initial conditions (t, x).

- the exponentially transformed criterion

$$\tilde{I}(v, x, h; t, T; \theta) := \mathbf{E}_{t,x}^{h,\theta} \left[\exp \left\{ \theta \int_t^T g(s, X_s, h(s); \theta) ds - \theta \ln v \right\} \right] \quad (17)$$

Note that the optimal control problem has become a diffusion problem.

The HJB PDE

The HJB PDE associated with the risk-sensitive control criterion (16) is

$$\frac{\partial \Phi}{\partial t}(t,x) + \sup_{h \in \mathcal{J}} L_t^h \Phi(t,x) = 0, \qquad (t,x) \in (0,T) \times \mathbb{R}^n \tag{18}$$

where

$$\begin{aligned} L_t^h \Phi(t,x) &= \left(b + Bx - \theta \Lambda \Sigma' h(s)\right)' D\Phi \\ &\quad + \frac{1}{2} \mathrm{tr}\left(\Lambda \Lambda' D^2 \Phi\right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi - g(x,h;\theta) \end{aligned} \tag{19}$$

and subject to terminal condition $\Phi(T,x) = \ln v$ This is a quasi-linear PDE with two sources of non-linearity:

- the $\sup_{h \in \mathcal{J}}$;
- the quadratic growth term $(D\Phi)' \Lambda \Lambda' D\Phi$;

We can address the second linearity by considering instead the semi-linear PDE associated with the exponentially-transformed problem (48):

$$\frac{\partial \tilde{\Phi}}{\partial t}(t,x) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2 \tilde{\Phi}(t,x)\right) + H(t,x,\tilde{\Phi},D\tilde{\Phi}) \;=\; 0 \tag{20}$$

subject to terminal condition $\tilde{\Phi}(T,x) = v^{-\theta}$ and where

$$H(s,x,r,p) \;=\; \inf_{h\in\mathcal{J}}\left\{(b + Bx - \theta\Lambda\Sigma' h(s))' p + \theta g(x,h;\theta) r\right\} \tag{21}$$

for $r \in \mathbb{R}$, $p \in \mathbb{R}^n$.

In particular $\tilde{\Phi}(t,x) = \exp\{-\theta\Phi(t,x)\}$.

Identifying a (Unique) Candidate Optimal Control

The supremum in (18) can be expressed as

$$\begin{aligned}
&\sup_{h\in\mathcal{J}} L_t^h\Phi \\
= \;& (b+Bx)'\,D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi + a_0 + A_0'x \\
& + \sup_{h\in\mathcal{J}}\left\{-\frac{1}{2}(\theta+1)\,h'\Sigma\Sigma' h - \theta h'\Sigma\Lambda' D\Phi + h'(\hat{a}+\hat{A}x)\right. \\
& \left. -\frac{1}{\theta}\int_{\mathbf{Z}}\left\{\left[(1+h'\gamma(z))^{-\theta}-1\right] + \theta h'\gamma(z)\mathit{1}_{\mathbf{Z}_0}(z)\right\}\nu(dz)\right\} \qquad (22)
\end{aligned}$$

- Under Assumption 4 the supremum is concave in h $\forall z \in \mathbb{Z}$ a.s. $d\nu$.
- The supremum is reached for a unique maximizer $\hat{h}(t,x,p)$.
- By measurable selection, $\hat{h}$ can be taken as a Borel measurable function on $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$.

Verification Theorem

Broadly speaking, the verification theorem states that if we have

- a $C^{1,2}([0,T] \times \mathbb{R}^n)$ bounded function ϕ which satisfies the HJB PDE (18) and its terminal condition;
- the stochastic differential equation

$$dX(t) = (b + BX(t) - \theta\Lambda\Sigma' h(t))\, dt + \Lambda dW_t^\theta$$

  defines a unique solution $X(s)$ for each given initial data $X(t) = x$; and,
- there exists a Borel-measurable minimizer $\tilde{h}^*(t, X_t)$ of $\tilde{h} \mapsto \tilde{L}^{\tilde{h}}\tilde{\phi}$ defined in (19);

then $\tilde{\Phi}$ is the value function and $\tilde{h}^*(t, X_t)$ is the optimal Markov control process.

... and similarly for $\tilde{\Phi}$ and the exponentially-transformed problem.

Existence of a $C^{1,2}$ Solution to the HJB PDE

To show that there exists a unique $C^{1,2}$ solution $\tilde{\Phi}$ to the HJB PDE (20) for the exponentially transformed problem, we follow similar arguments to those developed by Fleming and Rishel (Theorem 6.2 and Appendix E). Namely, we use an approximation in policy space alongside functional analysis-related results on linear parabolic partial differential equations.

The approximation in policy space algorithm was originally proposed by Bellman in the 1950s as a numerical method to compute the value function.

Our approach has two steps. First, we use the approximation in policy space algorithm to show existence of a classical solution in a bounded region. Next, we extend our argument to unbounded state space.

To derive this second result we follow a different argument than Fleming and Rishel which makes more use of the actual structure of the control problem.

Zero Beta Policy:

By reference to the definition of the function g in equation (42) (Slide 7), a *'zero beta' (0β) control policy* $\check{h}(t)$ is an admissible control policy for which the function g is independent of the state variable x (Fischer Black).

A zero beta policy exists as long as the coefficient matrix A has full rank.

Without loss of generality, in the following we will fix a 0β control $\check{h}$ as a constant function of time so that

$$g(x, \check{h}; \theta) = \check{g}$$

where $\check{g}$ is a constant.

Functional analysis notation: denote by

- $L^\eta(K)$ the space of η-th power integrable functions on $K \subset Q$;
- $\|\cdot\|_{\eta,K}$ the norm in $L^\eta(K)$;
- $\mathcal{L}^\eta(Q), 1 < \eta < \infty$ the space of all functions ψ such that for $\psi(t,x)$ and all its generalized partial derivatives $\frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x_i}, \frac{\partial^2 \psi}{\partial x_i x_j}, i,j = 1,\ldots,n$ are in $L^\eta(K)$;
- $\|\psi\|^{(2)}_{\eta,K}$ the Sobolev-type norm associated with $\mathcal{L}^\eta(Q), 1 < \eta < \infty$ and defined as

$$\|\psi\|^{(2)}_{\eta,K} := \|\psi\|_{\eta,K} + \left\|\frac{\partial \psi}{\partial t}\right\|_{\eta,K} + \sum_{i=1}^{n} \left\|\frac{\partial \psi}{\partial x_i}\right\|_{\eta,K} + \sum_{i,j=1}^{n} \left\|\frac{\partial^2 \psi}{\partial x_i x_j}\right\|_{\eta,K}$$

Note: (i) $\psi' = \partial\psi/\partial x_i$ in L^η if $\int \psi' \beta \, dx = -\int \psi \frac{\partial \beta}{\partial x_i} dx \quad \forall \beta \in C_0^1$.

(ii) If $\psi_n \to \psi$ in L^η and $\{||\psi_n'||_{L^\eta}\}$ is bounded, then $\psi_n' \to \psi'$ weakly in L^η (meaning $\int \psi_n' \chi dx \to \int \psi' \chi dx \quad \forall \chi \in L^r, \ \eta^{-1} + r^{-1} = 1$)

Step 1: Approximation in policy space - bounded space

Consider the following auxiliary problem: fix $R > 0$ and let $\mathcal{B}_R$ be the open n-dimensional ball of radius $R > 0$ centered at 0 defined as $\mathcal{B}_R := \{x \in \mathbb{R}^n : |x| < R\}$.

We construct an investment portfolio by solving the optimal risk-sensitive asset allocation problem as long as $X(t) \in \mathcal{B}_R$ for $R > 0$. Then, as soon as $X(t) \notin \mathcal{B}_R$, we switch all of the wealth into the 0β policy $\check{h}$ from the exit time t until the end of the investment horizon at time T.

The HJB PDE for this auxiliary problem can be expressed as

$$\frac{\partial\tilde{\Phi}}{\partial t}+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t)D^2\tilde{\Phi}\right)+H(t,x,\tilde{\Phi},D\tilde{\Phi})=0$$
$$\forall(t,x)\in Q_R:=(0,T)\times\mathcal{B}_R$$

subject to boundary conditions

$$\tilde{\Phi}(t,x) \;=\; \Psi(t,x)$$
$$\forall(t,x)\in\partial^*Q_R:=((0,T)\times\partial\mathcal{B}_R)\cup(\{T\}\times\mathcal{B}_R)$$

and where

- $\Psi(T,x)=e^{-\theta\ln v}\ \forall x\in\mathcal{B}_R$;
- $\Psi(t,x):=\psi(t):=e^{\theta\check{g}(T-t)}\ \forall(t,x)\in(0,T)\times\partial\mathcal{B}_R$ and where $\check{h}$ is a fixed arbitrary 0β policy. ψ is obviously of class $C^{1,2}(\overline{Q_R})$ and the Sobolev-type norm

$$\|\Psi\|^{(2)}_{\eta,\partial^*Q_R}=\|\psi\|^{(2)}_{\eta,Q_R} \tag{23}$$

is finite.

Define a sequence of functions $\tilde{\Phi}^1$, $\tilde{\Phi}^2$,... $\tilde{\Phi}^k$,... on $\overline{Q_R} = [0,T] \times \overline{\mathcal{B}_R}$ and of bounded measurable feedback control laws h^0, h^1,... h^k,... where h^0 is an arbitrary control (for example, the 0β control). Assuming h^k is defined, $\tilde{\Phi}^{k+1}$ solves the boundary value problem:

$$\begin{aligned}
&\frac{\partial \tilde{\Phi}^{k+1}}{\partial t} + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t)D^2\tilde{\Phi}^{k+1}\right) \\
&+f(t,x,h^k)'D\tilde{\Phi}^{k+1} + \theta g(t,x,h^k)\tilde{\Phi}^{k+1} = 0
\end{aligned} \tag{24}$$

subject to boundary conditions

$$\tilde{\Phi}(t,x) \;=\; \Psi(t,x)\,\forall(t,x) \in \partial^* Q_R := ((0,T) \times \partial\mathcal{B}_R) \cup (\{T\} \times \mathcal{B}_R)$$

Based on standard results on parabolic Partial Differential Equations (Appendix E in Fleming and Rishel, Chapter IV in Ladyzhenskaya, Solonnikov and Uralceva, the boundary value problem (24) admits a unique solution in $\mathcal{L}^\eta(Q_R)$.

Moreover, for almost all $(t,x) \in Q_R$, $k = 1, 2, \ldots$, we define h^{k+1} by the prescription

$$h^{k+1} = \text{Argmin}_{h \in \mathcal{J}} \left\{ f(t,x,h)' D\tilde{\Phi}^{k+1} + \theta g(t,x,h)\tilde{\Phi}^{k+1} \right\} \tag{25}$$

so that

$$\begin{aligned} & f(t,x,h^{k+1})' D\tilde{\Phi}^{k+1} + \theta g(t,x,h^{k+1})\tilde{\Phi}^{k+1} \\ = \; & \inf_{h \in \mathcal{J}} \left\{ f(t,x,h)' D\tilde{\Phi}^{k+1} + \theta g(t,x,h)\tilde{\Phi}^{k+1} \right\} \\ = \; & H(t,x,\tilde{\Phi}^{k+1}, D\tilde{\Phi}^{k+1}) \end{aligned} \tag{26}$$

Observe that the sequence $\left(\tilde{\Phi}^k\right)_{k \in \mathbb{N}}$ is globally bounded:

- bounded from below by 0 (by Feynman-Kac).
- bounded from above (optimality principle and 'zero beta' (0β) control policy)

These bounds do not depend on the radius R and are therefore valid over the entire space $(0,T) \times \mathbb{R}^n$.

Step 2: Convergence Inside the Cylinder $(0, T) \times \mathcal{B}_R$

It can be shown using a control argument that the sequence $\left\{\tilde{\Phi}^k\right\}_{k\in\mathbb{N}}$ is non increasing and as a result converges to a limit $\tilde{\Phi}$ as $k \to \infty$. Since the Sobolev-type norm $\|\tilde{\Phi}^{k+1}\|^{(2)}_{\eta,Q_R}$ is bounded for $1 < \eta < \infty$, we can show that the Hölder-type norm $|\tilde{\Phi}^k|^{1+\mu}_{Q_R}$ is also bounded by apply the following estimate given by equation (E.9) in Appendix E of Fleming and Rishel

$$|\tilde{\Phi}^k|^{1+\mu}_{Q_R} \leq M_R \|\tilde{\Phi}^k\|^{(2)}_{\eta,Q_R} \tag{27}$$

for some constant M_R (depending on R) and where

$$\begin{aligned} \mu &= 1 - \frac{n+2}{\eta} \\ |\tilde{\Phi}^k|^{1+\mu}_{Q_R} &= |\tilde{\Phi}^k|^{\mu}_{Q_R} + \sum_{i=1}^{n} |\tilde{\Phi}^k_{x_i}|^{\mu}_{Q_R} \end{aligned}$$

$$|\tilde{\Phi}^k|^{\mu}_{Q_R} = \sup_{(t,x)\in Q_R} |\tilde{\Phi}^k(t,x)| + \sup_{\substack{(x,y)\in\overline{G} \\ 0\le t\le T}} \frac{|\tilde{\Phi}^k(t,x) - \tilde{\Phi}^k(t,y)|}{|x-y|^{\mu}}$$

$$+ \sup_{\substack{x\in\overline{G} \\ 0\le s,t\le T}} \frac{|\tilde{\Phi}^k(s,x) - \tilde{\Phi}^k(t,x)|}{|s-t|^{\mu/2}}$$

As $k \to \infty$,

- $D\tilde{\Phi}^k$ converges to $D\tilde{\Phi}$ uniformly in $L^{\eta}(Q_R)$;
- $D^2\tilde{\Phi}^k$ converges to $D^2\tilde{\Phi}$ weakly in $L^{\eta}(Q_R)$; and
- $\frac{\partial \tilde{\Phi}^k}{\partial t}$ converges to $\frac{\partial \tilde{\Phi}}{\partial t}$ weakly in $L^{\eta}(Q_R)$.

We can then prove that $\tilde{\Phi} \in C^{1,2}(Q_R)$.

Step 3: Convergence from the Cylinder $[0,T) \times \mathcal{B}_R$ to the State Space $[0,T) \times \mathbb{R}^n$

Let $\{R_i\}_{i\in\mathbb{N}} > 0$ be a non decreasing sequence with $\lim_{i\to\infty} R_i \to \infty$ and let $\{\tau_i\}_{i\in\mathbb{N}}$ be the sequence of stopping times defined as

$$\tau_i := \inf\{t : X(t) \notin \mathcal{B}_{R_i}\} \wedge T$$

Note that $\{\tau_i\}_{i\in\mathbb{N}}$ is non decreasing and $\lim_{i\to\infty} \tau_i = T$.

Denote by $\tilde{\Phi}^{(i)}$ the limit of the sequence $\left(\tilde{\Phi}^k\right)_{k\in\mathbb{N}}$ on $(0,T) \times \mathcal{B}_{R_i}$, i.e.

$$\tilde{\Phi}^{(i)}(t,x) = \lim_{k\to\infty} \tilde{\Phi}^k(t,x) \qquad \forall (t,x) \in (0,T) \times \mathcal{B}_{R_i} \tag{28}$$

The sequence $(\tilde{\Phi}^{(i)})_{i\in\mathbb{N}}$ is bounded and non increasing: it converges to a limit $\tilde{\Phi}$. This limit satisfies the boundary condition. We now apply Ascoli's theorem to show that $\tilde{\Phi}$ is $C^{1,2}$ and satisfies the HJB PDE. These statements are local properties so we can restrict ourselves to a finite ball Q_R.

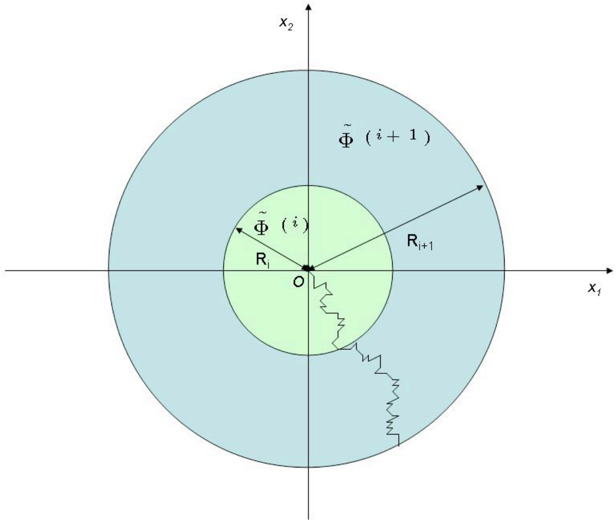
Using the following estimate given by equation (E8) in Appendix E of Fleming and Rishel, we deduce that

$$\|\tilde{\Phi}^{(i)}\|_{\eta,Q_R}^{(2)} \leq M\|\Psi\|_{\eta,\partial^* Q_R}^{(2)} \tag{29}$$

for some constant M.

Combining (29) with assumption (23) implies that $\|\tilde{\Phi}^{(i)}\|_{\eta,Q_R}^{(2)}$ is bounded for $\eta > 1$. Critically, the bound M does not depend on k. Moreover, by Step 2 $\tilde{\Phi}^{(i)}$ and $D\tilde{\Phi}^{(i)}$ are uniformly bounded on any compact subset of $\overline{Q_0}$. By equation (29) we know that $\|\tilde{\Phi}\|_{\eta,Q_R}^{(2)}$ is bounded for any bounded set $Q_R \subset Q_0$.

Figure 2: Convergence of the Sequence $\left\{\tilde{\Phi}^{(i)}\right\}_{i\in\mathbb{N}}$



On Q_R, $\tilde{\Phi}^{(i)}$ also satisfies the Hölder estimate

$$|\tilde{\Phi}^{(i)}|_{Q_R}^{1+\mu} \leq M_1 \|\tilde{\Phi}^{(i)}\|_{\eta, Q_R}^{(2)}$$

for some constant M_1 depending on Q_R and η.

We find, that $\frac{\partial \tilde{\Phi}^{(i)}}{\partial t}$ and $\frac{\partial^2 \tilde{\Phi}^{(i)}}{\partial x_i x_j}$ also satisfy a uniform Hölder condition on any compact subset of Q.

By Ascoli's theorem, we can find a subsequence $\left(\tilde{\Phi}^l\right)_{l\in\mathbb{N}}$ of $\left(\tilde{\Phi}^{(i)}\right)_{i\in\mathbb{N}}$ such that $\left(\tilde{\Phi}^l\right)_{l\in\mathbb{N}}$, $\left(\frac{\partial \tilde{\Phi}}{\partial t}^l\right)_{l\in\mathbb{N}}$, $\left(D\tilde{\Phi}^l\right)_{l\in\mathbb{N}}$ and $\left(D^2\tilde{\Phi}^l\right)_{l\in\mathbb{N}}$ tends to respective limits $\tilde{\Phi}$, $\frac{\partial \tilde{\Phi}}{\partial t}$ $D\tilde{\Phi}$ and $D^2\tilde{\Phi}$ uniformly on each compact subset of $[0,T]\times\mathbb{R}^n$.

Finally, the function $\tilde{\Phi}$ is the desired solution of equation (20) with terminal condition $\tilde{\Phi}(T,x) = e^{-\theta \ln v}$.

This completes the proof.

II: THE FULLY NONLINEAR CASE

Factor Dynamics

The factor process $X(t) \in \mathbb{R}^n$ is allowed to have a full jump-diffusion dynamics, satisfying the SDE

$$dX(t) = b\left(t, X(t^-)\right) dt + \Lambda(t, X(t))dW(t) + \int_{\mathbf{Z}} \xi\left(t, X(t^-), z\right) \bar{N}_{\mathbf{p}}(dt, dz), \qquad X(0) = x \tag{30}$$

The standing assumptions are as follows.

(i) The function b defined as $b : [0, T] \times \mathbb{R}^n(t, x) \mapsto b(t, x) \in \mathbb{R}^n$ is bounded and Lipshitz continuous

$$|b(t, y) - b(s, x)| \leq K_b\left(|t - s| + |y - x|\right) \tag{H0}$$

for some constant $K_b > 0$.

(ii) the function Λ defined as $\Lambda : [0, T] \times \mathbb{R}^n(t, x) \mapsto \Lambda(t, x) \in \mathbb{R}^{n \times N}$ is bounded and Lipschitz continuous, i.e.

$$|\Lambda(t, y) - \Lambda(s, x)| \leq K_\Lambda\left(|t - s| + |y - x|\right) \tag{H1}$$

for some constant $K_\Lambda > 0$.

(iii) There exists $\eta_\Lambda > 0$ such that

$$\nu' \Lambda \Lambda'(t,x) \nu \geq \eta_\Lambda |\nu|^2 \tag{H2}$$

for all $\nu \in \mathbb{R}^n$

(iv) the first order derivatives of b and Λ are bounded, i.e. there exists $K_b' > 0$ and $K_\Lambda' > 0$ such that

$$|b_t| + |b_x| \leq K_b' \tag{H3}$$

$$|\Lambda_t| + |\Lambda_x| \leq K_\Lambda' \tag{H4}$$

(v) the function ξ defined as $\xi : [0,T] \times \mathbb{R}^n \times \mathbf{Z}(t,x,z) \mapsto \xi(t,x,z) \in \mathbb{R}$ is bounded and Lipshitz continuous, i.e.

$$|\xi(t,y,z) - \xi(s,x,z)| \leq K_\xi \left(|t-s| + |y-x|\right) \tag{H5}$$

for some constant $K_\xi > 0$. Moreover,the vector valued function $\xi(t,x,z)$ satisfy:

$$\int_{\mathbf{Z}} |\xi(t,x,z)| \nu(dz) < \infty, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n \tag{H7}$$

The minimal condition on ξ under which the factor equation (30) is well posed is

$$\int_{\mathbf{Z}_0} |\xi(t, x, z)|^2 \nu(dz) < \infty,$$

see Definition II.4.1 in Ikeda and Watanabe. However, here it is essential to impose the stronger condition (H7) in order to obtain the connection between the HJB partial integro-differential equation (PIDE) and a related PDE, when interpreted in the viscosity sense.

Asset Market Dynamics

Let S_0 denote the wealth invested in the money market account with dynamics given by the equation:

$$\frac{dS_0(t)}{S_0(t)} = a_0\left(t, X(t)\right) dt, \qquad S_0(0) = s_0 \tag{31}$$

where the function a_0 defined as $a_0 : [0,T] \times \mathbb{R}^n (t,x) \mapsto a_0(t,x) \in \mathbb{R}$ is bounded, of class $C^{1,1}\left([0,T] \times \mathbb{R}^n\right)$ and is Lipshitz continuous

$$|a_0(t,y) - a_0(s,x)| \leq K_0 \left(|t-s| + |y-x|\right) \tag{H8}$$

for some constant $K_0 > 0$. Moreover, the first order derivatives of a_0 are bounded, i.e. there exists $K_0' > 0$ such that

$$\left|\frac{\partial a_0}{\partial t}\right| + |Da_0| \leq K_0' \tag{H9}$$

Let $S_i(t)$ denote the price at time t of the ith security, with $i = 1, \ldots, m$. The dynamics of risky security i can be expressed as:

$$\frac{dS_i(t)}{S_i(t^-)} = \left[a\left(t, X(t^-)\right)\right]_i dt + \sum_{k=1}^{N} \sigma_{ik}(t, X(t))dW_k(t) + \int_{\mathbf{Z}} \gamma_i(t,z)\bar{N}_{\mathbf{p}}(dt,dz),$$
$$S_i(0) = s_i, \quad i = 1, \ldots, m \tag{32}$$

where the coefficients a, σ, γ satisfy similar conditions as the coefficients of the state process $X(t)$.

We also require

$$|\Lambda\Sigma'(t,y) - \Lambda\Sigma'(s,x)| \leq K_{\Lambda\Sigma}\left(|t-s| + |y-x|\right) \tag{H17}$$

for some constant $K_{\Lambda\Sigma} > 0$

Assumptions on function γ

Define

$$\mathbf{S} := \operatorname{supp}(\nu) \in \mathcal{B}_{\mathbf{Z}}$$

and

$$\tilde{\mathbf{S}} := \operatorname{supp}(\nu \circ \gamma^{-1}) \in \mathcal{B}(\mathbb{R}^m)$$

where supp$(\cdot)$ denotes the support of the measure support. Let $\prod_{i=1}^{m}[\gamma_i^{min}, \gamma_i^{max}]$ be the smallest closed hypercube containing $\tilde{\mathbf{S}}$, then we assume that $\gamma(t,z) \in \mathbb{R}^m$ satisfies

$$-1 \le \gamma_i^{min} \le \gamma_i(z) \le \gamma_i^{max} < +\infty, \qquad i = 1, \ldots, m$$

and

$$\gamma_i^{min} < 0 < \gamma_i^{max}, \qquad i = 1, \ldots, m$$

for $i = 1, \ldots, m$.

Define the set $\mathcal{J}_0$ as

$$\mathcal{J}_0 = \left\{ h \in \mathbb{R}^m : -1 - h'\psi < 0 \quad \forall \psi \in \tilde{\mathbf{S}} \right\} \tag{33}$$

For a given $z \in \mathbf{S}$, the equation $h'\gamma(t,x,z) = -1$ describes a hyperplane in $\mathbb{R}^m$. Under the above assumptions, $\mathcal{J}_0$ is a convex subset of $\mathbb{R}^m$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

Portfolio Dynamics

We assume that the systematic (factor-driven) and idiosyncratic (asset-driven) jump risks are uncorrelated, i.e $\forall(t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbf{Z}$, $\gamma(t, z)\xi'(t, x, z) = 0$.

This assumption implies that there are no simultaneous jumps in the factor process and any asset price process. This is restrictive, but appears to be essential in the argument below.

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security and factor processes up to time t.

Definition An $\mathbb{R}^m$-valued control process $h(t)$ is in class $\mathcal{H}_0$ if the following conditions are satisfied:

- $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0, t]) \otimes \mathcal{G}_t\}_{t \geq 0}$ and is càdlàg;
- $P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1$;
- $h(t) \in \mathcal{J}_0 \quad \forall t$ a.s.

Note: A control process $h(t)$ satisfying these conditions is bounded.

By the budget equation, the proportion invested in the money market account is equal to $h_0(t) = 1 - \sum_{i=1}^{m} h_i(t)$. This implies that the wealth, $V(t)$ of the investor in response to an investment strategy $h(t) \in \mathcal{H}$, follows the dynamics

$$\begin{aligned} \frac{dV(t)}{V(t^-)} &= a_0\left(t, X(t)\right) dt + h'(t)\left[a(t, X(t)) - a_0(t, X(t))\mathbf{1}\right] dt \\ &\quad + h'(t)\Sigma(t, X(t))dW_t + \int_{\mathbf{Z}} h'(t)\gamma(t, z)\bar{N}_{\mathbf{p}}(dt, dz) \end{aligned}$$

where $\mathbf{1} \in \mathbf{R}^m$ denotes the m-element unit column vector and with $V(0) = v$. Defining $\hat{a} := a - a_0\mathbf{1}$ and $\hat{A} := A - \mathbf{1}A_0'$, we can express the portfolio dynamics as

$$\begin{aligned} \frac{dV(t)}{V(t^-)} &= \left(a_0\left(t, X(t)\right)\right) dt + h'(t)\left(\hat{a} + \hat{A}X(t)\right) dt + h'(t)\Sigma(t, X(t))dW_t \\ &\quad + \int_{\mathbf{Z}} h'(t)\gamma(t, z)\bar{N}_{\mathbf{p}}(dt, dz) \end{aligned}$$

Investment Constraints

We consider $r \in \mathbb{N}$ fixed investment constraints expressed in the form

$$\Upsilon' h(t) \leq \upsilon \tag{34}$$

where $\Upsilon \in \mathbb{R}^m \times \mathbb{R}^r$ is a matrix and $\upsilon \in \mathbb{R}^r$ is a column vector. For the constrained control problem to be sensible, we need Υ and υ to satisfy the following assumptions:

assumptions: (i) The system

$$\Upsilon' y \leq \upsilon$$

for the variable $y \in \mathbb{R}^m$ admits at least two solutions.

(ii) The rank of the matrix Υ is equal to $\min(r, n)$.

We define the feasible region $\mathcal{J}$ as

$$\mathcal{J} := \{h \in \mathcal{J}_0 : \Upsilon' h \leq \upsilon\}$$

and the constrained class of investment processes, $\mathcal{H}$, as

$$\mathcal{H} := \{h(t) \in \mathcal{H}_0 : h(t) \in \mathcal{J} \ \forall t \in [0, T], \text{a.s.}\} \tag{35}$$

The feasible region $\mathcal{J}$ is a a convex subset of $\mathbb{R}^r$ and as a result of Assumption , $\mathcal{J}$ has at least one interior point.

Definition A control process $h(t)$ is in class $\mathcal{A}$ if the following conditions are satisfied:

- $h \in \mathcal{H}$;
- $\mathbf{E}\chi_T^h = 1$ where χ_t^h is the Doléans exponential defined for $t \in [0, T]$ by

$$\begin{aligned} \chi_t^h \;:=\; & \exp\left\{ -\theta \int_0^t h(s)'\Sigma(s, X(s))dW_s - \frac{1}{2}\theta^2 \int_0^t h(s)'\Sigma\Sigma'(s, X(s))h(s)ds \right. \\ & + \int_0^t \int_{\mathbf{Z}} \ln\left(1 - G(s, z, h(s); \theta)\right) \tilde{N}_{\mathbf{p}}(ds, dz) \\ & \left. + \int_0^t \int_{\mathbf{Z}} \left\{ \ln\left(1 - G(s, z, h(s); \theta)\right) + G(s, z, h(s); \theta) \right\} \nu(dz)ds \right\}, \end{aligned} \tag{36}$$

and

$$G(t, z, h; \theta) \;=\; 1 - \left(1 + h'\gamma(t, z)\right)^{-\theta} \tag{37}$$

Problem Setup

We assume that the objective of the investor is to maximize the risk adjusted growth of his/her portfolio of assets over a finite time horizon. This implies finding $h^*(t) \in \mathcal{H}$ to maximize the control criterion

$$J(t, x, h; \theta) := -\frac{1}{\theta} \ln \mathbf{E} \left[e^{-\theta \ln V(t,x,h)} \right] \tag{38}$$

We define the value function Φ corresponding to the maximization of the auxiliary criterion function $J(v, x; h; t, T)$, i.e.

$$\Phi(t, x) = \sup_{h \in \mathcal{A}} J(v, x; h; t, T) \tag{39}$$

By Itô, the log of the portfolio value in response to a strategy h is

$$\begin{aligned}\ln V(t) &= \ln v + \int_0^t a_0\left(s, X(s)\right) + h(s)'\hat{a}\left(s, X(s)\right) ds \\ &\quad -\frac{1}{2}\int_0^t h(s)'\Sigma\Sigma'(s, X(s))h(s)ds + \int_0^t h(s)'\Sigma(s, X(s))dW(s) \\ &\quad + \int_0^t \int_{\mathbf{Z}_0} \left\{\ln\left(1 + h(s)'\gamma(s,z)\right) - h(s)'\gamma(s,z)\right\} \nu(dz)ds \\ &\quad + \int_0^t \int_{\mathbf{Z}} \ln\left(1 + h(s)'\gamma(s,z)\right) \bar{N}_{\mathbf{p}}(ds, dz) \end{aligned} \tag{40}$$

Hence,

$$e^{-\theta \ln V(t)} = v^{-\theta} \exp\left\{\theta \int_0^t g(s, X_s, h(s); \theta)ds\right\} \chi_t^h \tag{41}$$

where

$$\begin{aligned} g(t,x,h;\theta) &= \frac{1}{2}\left(\theta+1\right) h'\Sigma\Sigma'(t,x)h - a_0(t,x) - h'\hat{a}(t,x) \\ &\quad + \int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left[\left(1 + h'\gamma(t,z)\right)^{-\theta} - 1\right] + h'\gamma(t,z)\mathit{1}_{\mathbf{Z}_0}(z)\right\} \nu(dz) \end{aligned} \tag{42}$$

and the Doléans exponential χ_t^h is given by (36).

With the above conditions, for a given fixed h, the functional g is bounded and Lipschitz continuous in the state variable x.

Change of Measure For $h \in \mathcal{A}$ and $\theta > 0$ let $\mathbb{P}_h^\theta$ be the measure on $(\Omega, \mathcal{F}_T)$ defined via the Radon-Nikodým derivative

$$\frac{d\mathbb{P}_h^\theta}{d\mathbb{P}} = \chi_T^h \tag{43}$$

For $h \in \mathcal{A}$,

$$W_t^h = W_t + \theta \int_0^t \Sigma(s, X(s))' h(s) ds$$

is a standard Brownian motion under the measure $\mathbb{P}_h^\theta$ and we have

$$\begin{aligned}
\int_0^t \int_{\mathbf{Z}} \tilde{N}_{\mathbf{p}}^h(ds, dz) &= \int_0^t \int_{\mathbf{Z}} N_{\mathbf{p}}(ds, dz) - \int_0^t \int_{\mathbf{Z}} \{1 - G(s, X(s), z, h(s); \theta)\} \nu(dz) ds \\
&= \int_0^t \int_{\mathbf{Z}} N_{\mathbf{p}}(ds, dz) - \int_0^t \int_{\mathbf{Z}} \left\{ (1 + h'\gamma(s, X(s), z))^{-\theta} \right\} \nu(dz) ds
\end{aligned}$$

As a result, $X(s)$, $0 \leq s \leq t$ satisfies the SDE:

$$
\begin{aligned}
dX(s) &= f(s, X(s), h(s); \theta)ds + \Lambda(s, X(s))dW_s^\theta \\
&\quad + \int_{\mathbf{Z}} \xi\left(s, X(s^-), z\right) \tilde{N}_{\mathbf{p}}^\theta(ds, dz)
\end{aligned} \tag{44}
$$

where

$$
f(t, x, h; \theta) := b(t, x) - \theta \Lambda \Sigma(t, x)' h(s) + \int_{\mathbf{Z}} \xi(t, x, z) \left[(1 + h'\gamma(t, z))^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \tag{45}
$$

and b is the $\mathbb{P}$-measure drift of the factor process, see (30).

The drift function f is Lipschitz continuous with coefficient $K_f = K_b + \theta K_{\Lambda\Sigma} + K_\xi K_0$ where $K_0 > 0$ is a constant.

Moreover the generator $\mathcal{L}$ of the state process $X(t)$ is defined as

$$
\begin{aligned}
\mathcal{L}\tilde{\Phi}(t, x) &:= f(t, x, h; \theta)' D\tilde{\Phi} + \frac{1}{2} \mathrm{tr}\left(\Lambda\Lambda'(t, X) D^2 \tilde{\Phi} \right) \\
&\quad + \int_{\mathbf{Z}} \left\{ \tilde{\Phi}\left(x + \xi(t, x, z)\right) - \tilde{\Phi}(x) - \xi(t, x, z)' D\tilde{\Phi} \right\} \nu(dz) ds
\end{aligned} \tag{46}
$$

We will now introduce the following two auxiliary criterion functions under the measure $\mathbb{P}_h^\theta$:

- the auxiliary function directly associated with the risk-sensitive control problem:

$$I(v, x; h; t, T; \theta) = -\frac{1}{\theta} \ln \mathbf{E}_{t,x}^{h,\theta} \left[\exp \left\{ \theta \int_t^T g(s, X_s, h(s); \theta) ds - \theta \ln v \right\} \right] \tag{47}$$

  where $\mathbf{E}_{t,x}^{h,\theta}[\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h^\theta$ and with initial conditions (t, x).

- the exponentially transformed criterion

$$\tilde{I}(v, x, h; t, T; \theta) := \mathbf{E}_{t,x}^{h,\theta} \left[\exp \left\{ \theta \int_t^T g(s, X_s, h(s); \theta) ds - \theta \ln v \right\} \right] \tag{48}$$

  which we will find convenient to use in the derivation of some of the results.

The criterion $\tilde{I}$ defined in (48) as is akin to a discounted payoff of 1 at terminal time T discounted at a stochastic controlled rate of $\theta g(\cdot)$, minus an initial investment equal to $\theta \ln v$.

The Risk-Sensitive Control Problems under $\mathbb{P}_h^\theta$

We will show that the value function Φ defined in (39) satisfies the HJB PIDE

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in J} L_t^h \Phi(t, X(t)) = 0 \tag{49}$$

where J is defined in (33), and

$$\begin{aligned} L_t^h \Phi(t,x) = \;& f(t,x,h;\theta)' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t,x) D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda'(t,x) D\Phi - g(t,x,h;\theta) \\ & + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta}\left(e^{-\theta[\Phi(t,x+\xi(t,x,z))-\Phi(t,x)]} - 1 \right) - \xi(t,x,z)' D\Phi \right\} \nu(dz) \end{aligned} \tag{50}$$

and subject to terminal condition

$$\Phi(T,x) = \ln v, \qquad x \in \mathbb{R}^n \tag{51}$$

Similarly, let $\tilde{\Phi}$ be the value function for the auxiliary criterion function $\tilde{I}(v, x; h; t, T)$. Then $\tilde{\Phi}$ is defined as

$$\tilde{\Phi}(t, x) = \inf_{h \in \mathcal{A}} \tilde{I}(v, x; h; t, T) \tag{52}$$

The corresponding HJB PIDE is

$$\begin{aligned} & \frac{\partial \tilde{\Phi}}{\partial t}(t, x) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t, x) D^2 \tilde{\Phi}(t, x)\right) + H(t, x, \tilde{\Phi}, D\tilde{\Phi}) \\ & + \int_{\mathbf{Z}} \left\{ \tilde{\Phi}\left(t, x + \xi(t, x, z)\right) - \tilde{\Phi}(t, x) - \xi(t, x, z)' D\tilde{\Phi}(t, x) \right\} \nu(dz) \\ = \;& 0 \end{aligned} \tag{53}$$

subject to terminal condition

$$\tilde{\Phi}(T, x) = v^{-\theta} \tag{54}$$

where, for $r \in \mathbb{R}$, $p \in \mathbb{R}^n$,

$$H(s, x, r, p) = \inf_{h \in \mathcal{U}} \left\{ f(s, x, h; \theta)' p + \theta g(s, x, h; \theta) r. \right\} \tag{55}$$

The function H is Lipschitz and satisfies the linear growth condition

$$|H(s,x,r,p)| \leq C\,(1+|p|)\,, \quad \forall (s,x) \in Q_0$$

The value functions Φ and $\tilde{\Phi}$ are related through a strictly monotone continuous transformation

$$\tilde{\Phi}(t,x) = \exp\left\{-\theta\Phi(t,x)\right\} \tag{56}$$

Thus an admissible (optimal) strategy for the exponentially transformed problem is also admissible (optimal) for the risk-sensitive problem.

Proposition The supremum in (49) admits a unique maximizer $\hat{h}(t,x,p)$ for $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, which is an interior point of the set $\mathcal{J}$.

This is proved by showing that the maximization is equivalent to a convex programming problem, and the result then follows from Luenberger [57].

Properties of the Value Function

Proposition The exponentially transformed value function $\tilde{\Phi}$ is positive and bounded, i.e. there exists $M > 0$ such that

$$0 \leq \tilde{\Phi}(t,x) \leq M \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n$$

Proof By definition,

$$\tilde{\Phi}(t,x) \;=\; \inf_{h \in \mathcal{A}(\mathcal{T})} \mathbf{E}^{h,\theta}_{t,x} \left[\exp \left\{ \theta \int_t^T g(s, X_s, h(s); \theta) ds - \theta \ln v \right\} \right] \geq 0$$

Moreover, fix $h = 0$. By the Dynamic Programming Principle

$$\tilde{\Phi}(t,x) \;\leq\; e^{\theta \left[\int_t^T g(X(s),0;\theta) ds - \ln v \right]} = e^{\theta \left[\int_t^T a_0(s,X(s) ds - \ln v \right]}$$

Because a_0 is bounded, there exists $\check{a}_0$ such that $|a_0(t,x)| \leq \check{a}_0$ and hence

$$e^{\theta \left[\int_t^T a_0(s,X(s) ds - \ln v \right]} \;\leq\; e^{\theta (\check{a}_0 (T-t) - \ln v)} =: M$$

which concludes the proof.

Proposition The value function $\tilde{\Phi}$ is Lipschitz continuous in the state variable x.

1 Proving Smoothness: An Overview

The ultimate objective is to prove that the value functions Φ and $\tilde{\Phi}$ are the unique classical ($C^{1,2}$) solutions of the corresponding HJB equations. The argument involves 7 steps:

I. $\tilde{\Phi}$ is a continuous viscosity solution (VS-PIDE) of (53) **-**

First, change notation and rewrite the HJB PIDE as

$$-\frac{\partial \tilde{\Phi}}{\partial t}(t,x) + F(t,x,\tilde{\Phi},D\tilde{\Phi},D^2\tilde{\Phi},\mathcal{I}[t,x,\tilde{\Phi}]) \;=\; 0 \tag{57}$$

subject to terminal condition $\tilde{\Phi}(t,x) = v^{-\theta}$ where

$$\begin{aligned} F(t,x,\tilde{\Phi},D\tilde{\Phi},D^2\tilde{\Phi},\mathcal{I}[t,x,\tilde{\Phi}]) \;&=\; H_v(t,x,\tilde{\Phi},D\tilde{\Phi}) - \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t,x)D^2\tilde{\Phi}(t,x)\right) \\ &\quad -\mathcal{I}[t,x,\tilde{\phi}] \end{aligned}$$

$$\mathcal{I}[t,x,\tilde{\Phi}] \;:=\; \int_{\mathbf{Z}} \left\{\tilde{\Phi}\left(t,x+\xi(t,x,z)\right) - \tilde{\Phi}(t,x) - \xi(t,x,z)'D\tilde{\Phi}(t,x)\mathit{1}_{\mathbf{Z}_0}\right\}\nu(dz) \tag{58}$$

$$
\begin{aligned}
H_v(s,x,r,p) &= -H(s,x,r,p) \\
&= \sup_{h\in\mathcal{U}} \{-f_v(s,x,h;\theta)'p - \theta g(s,x,h;\theta)r\}
\end{aligned}
$$

for $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ and where

$$
\begin{aligned}
f_v(t,x,h;\theta) &:= f(t,x,h;\theta) - \int_{\mathbf{Z}\setminus\mathbf{Z}_\delta} \xi(t,x,z)\nu(dz) \\
&= b(t,x) - \theta\Lambda\Sigma(t,x)'h(s) + \int_{\mathbf{Z}} \xi(t,x,z)\left[\left(1+h'\gamma(t,z)\right)^{-\theta} - 1\right]\nu(dz)
\end{aligned}
\tag{59}
$$

Now use methods similar to those of Touzi [55] to show that $\tilde{\Phi}$ is a (discontinuous) (**VS-PIDE**) of (57). But we already know $\tilde{\Phi}$ is continuous.

II. From PIDE to PDE. Change notation and rewrite the HJB PIDE as the parabolic PDE *à la* Pham [58]:

$$
\frac{\partial\tilde{\Phi}}{\partial t}(t,x) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t,x)D^2\tilde{\Phi}(t,x)\right) + H_a(t,x,\tilde{\Phi},D\tilde{\Phi}) + d_a^{\tilde{\Phi}}(t,x) = 0 \tag{60}
$$

subject to terminal condition $\tilde{\Phi}(T,x) = v^{-\theta}$ and with

$$H_a(s,x,r,p) = \inf_{h\in\mathcal{U}} \{f_a(x,h)'p + \theta g(x,h;\theta)r\} \tag{61}$$

for $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ and where

$$\begin{aligned} f_a(x,h) &:= f(x,h) - \int_{\mathbf{Z}} \xi(t,x,z)\nu(dz) \\ &= b(t,x) - \theta\Lambda\Sigma(t,x)'h(s) + \int_{\mathbf{Z}} \xi(t,x,z) \left[(1 + h'\gamma(t,z))^{-\theta} - 1_{\mathbf{Z}_0}(z) - 1 \right] \nu(dz) \end{aligned} \tag{62}$$

and

$$d_a^{\tilde{\Phi}}(t,x) = \int_{\mathbf{Z}} \left\{ \tilde{\Phi}\left(t, x + \xi(t,x,z)\right) - \tilde{\Phi}(t,x) \right\} \nu(dz) \tag{63}$$

III. Viscosity solution to the PDE (60) **-** consider a viscosity solution (**VS-PDE**) $\check{\phi}$ of the semi-linear PDE (61) (always interpreted as an equation for 'unknown' $\check{\phi}$ with the last term prespecified, with $\tilde{\Phi}$ defined as in A.)

$$\frac{\partial \check{\phi}}{\partial t}(t,x) + \frac{1}{2}\text{tr}\left(\Lambda\Lambda'(t,x)D^2\check{\phi}(t,x)\right) + H_a(t,x,\check{\phi},D\check{\phi}) + d_a^{\tilde{\Phi}}(t,x) \ = \ 0 \quad (64)$$

Then $\tilde{\Phi}$ *is a viscosity solution of the PDE* (60) - this is essentially due to the fact that by choosing $\tilde{\Phi}$, PIDE (57) and PDE (60) are in essence the same equation. Hence, if $\tilde{\Phi}$ solves one of them, then it solves both.

IV. Uniqueness of the viscosity solution to the PDE (60) **-** If a function u solves the PDE (60) it does not mean that u also solves the PIDE (57) because the term d_a in the PDE (60) depends on $\tilde{\Phi}$ regardless of the choice of u. Thus, if we were to show the existence of a classical solution u to PDE (60), we would not be sure that this solution is the value function $\tilde{\Phi}$ unless we can show that PDE (60) admits a unique solution. This only requires applying a "classical" comparison result for viscosity solutions (see Theorem 8.2 in Crandall, Ishii and Lions [59]) provided appropriate conditions on f_a and d_a are satisfied.

V.Existence of a Classical Solution to the HJB PDE (60). We use the argument in Appendix E of Fleming and Rishel [29] to show the existence of a classical solution to the PDE (60).

VI. Any classical solution is a viscosity solution. Observe that a classical solution is also a viscosity solution[1] Hence, the classical solution to the PDE (60) is also the unique viscosity solution of both (60) and (57). This shows $\tilde{\Phi}$ is $C^{1,2}$ and satisfies (53) in the classical sense.

VII. Verification Theorem. We prove as in the diffusion factor model that the classical solutions $\tilde{\Phi}$ and Φ do solve the original control problems.

[1] Broadly speaking the argument is that if the solution of the PDE is smooth, then we can use it as a test function in the definition of viscosity solutions. If we do this, we will recover the classical maximum principle and therefore prove that the solution of the PDE is a classical solution.

Additional references

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