# Numerical Methods for Hamilton Jacobi Bellman Equations in Finance
# Lecture 1
# Introduction

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# Overview of Course

- Lecture 1: Examples of HJB Equations in Finance
- Lecture 2: Viscosity Solutions and Basic Properties of a Numerical Scheme
  - Monotonicity, Stability and Consistency
- Lecture 3: Examples
  - Passport options, Pension Investment
- Lecture 4: Guaranteed Minimum Withdrawal Benefits
- Lecture 5: Gas Storage

# Outline For Lecture 1

# Overview

Many problems in finance can be viewed as some form of optimal stochastic control (Survey article, see (Pham, 2005)).

$\hookrightarrow$ These problems often involve modelling realistic market effects, and/or complex decision making.

- Continuous time mean-variance portfolio optimization
- Optimal trade execution, hedging with liquidity effects (i.e. price impact) and transaction costs
- Investments in endowments and pension plans
- Optimal operation of gas storage facilities
- Variable annuity products with market guarantees (GMWB, GMDB)

These problems give rise to non-linear Hamilton Jacobi Bellman (HJB) Partial Differential Equations (PDEs), or HJB Partial Integro Differential Equations (PIDEs).

# Uncertain Volatility

Uncertain Volatility model proposed in (Avellenada *et al* (1995), Lyons (1995)).

Suppose that an asset price $S$ follows the risk neutral process

$$\begin{aligned} dS &= rS\, dt + \sigma S\, dZ \\ & r = \text{ interest rate} \\ & \sigma = \text{ volatility} \\ & dZ = \text{ increment of a Weiner process} \end{aligned}$$

where volatility is stochastic, but we only know that

$$\sigma \in [\sigma_{min}, \sigma_{max}]$$

What is the *fair* price to charge for a contingent claim?

# Uncertain Volatility cont'd

Let $V(S, \tau = T - t)$ be the value of a contingent claim, then

$$\text{Short Position:} \qquad V_\tau = \sup_{Q \in \hat{Q}} \left\{ \frac{Q^2 S^2}{2} V_{SS} + S V_S - rV \right\}$$

$$\text{Long Position:} \qquad V_\tau = \inf_{Q \in \hat{Q}} \left\{ \frac{Q^2 S^2}{2} V_{SS} + S V_S - rV \right\}$$

$$\hat{Q} = [\sigma_{min}, \sigma_{max}]$$

$[\sigma_{min}, \sigma_{max}]$ is the admissible control set

$Q$ is the control for this HJB equation

# Worst Case Hedge

- Suppose we are short the option
- Consider a strategy where we value the option (solve the HJB equation)
- We delta hedge, using delta values from our HJB solution
- No matter what happens, as long as

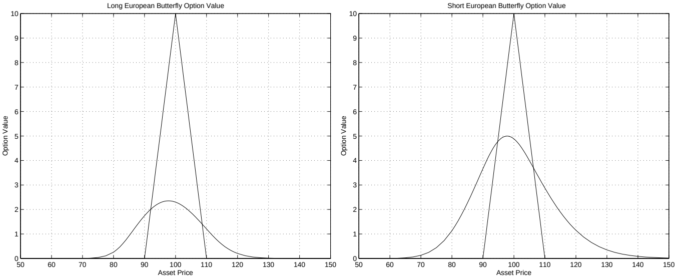
$$\sigma_{\text{min}} \leq \sigma \leq \sigma_{\text{max}},$$

  we will always end up with a non-negative balance in the hedging portfolio
- So the HJB equation solution is an upper bound to the cost of constructing a self-financing hedge
- This is the least upper bound, i.e. there exists at least one asset path, volatility scenario where our hedging portfolio is worth exactly zero at expiry

# Bid-Ask Prices

If buyers sellers priced options based on worst case hedging (from their own perspectives)

- Long/short prices would correspond to bid-ask prices
- Similar HJB PDE for other cases
  - Uncertain dividends
  - Hedging with transaction costs
  - Different interest rates for lending/borrowing



# Stock Borrowing Fees, Unequal Borrowing Lending Rates

Assume that risky asset $S$ follows GBM (under the risk neutral measure)

$$dS = rS\ dt + \sigma S\ dZ$$

Contingent claims are hedged using delta hedging, usual Black Scholes assumptions with the exceptions

- Hedger can borrow at rate $r_b$.
- Hedger can receive $r_l < r_b$ for cash on deposit.
- Stock borrowing fee $r_f$ charged for shorting stock.
- Effectively, the hedger receives interest at rate $r_l - r_f$, on the proceeds of a short sale.

# HJB Equation: Short Position on Contingent Claim

Let the no-arbitrage value of a contingent claim be $V = V(S, \tau = T - t)$.

$$
\begin{aligned}
V_\tau &= \sup_{Q \in \hat{Q}} \left\{ \frac{\sigma^2 S^2}{2} V_{SS} + q_3 q_1 (S V_S - V) \right. \\
&\quad \left. + (1 - q_3)[(r_l - r_f) S V_S - q_2 V] \right\} \\
&\quad Q = (q_1, q_2, q_3) \\
&\quad \hat{Q} = (\{r_l, r_b\}, \{r_l, r_b\}, \{0, 1\}) \qquad (1)
\end{aligned}
$$

For Long position on claim, replace sup by inf in (1).

# Continuous Time Mean Variance Asset Allocation

Suppose an investor saves for retirement by contributing to a pension account at a rate $\pi$ per year.

She can divide her wealth $W$ in the pension account into

- A fraction $p$ invested in a risky asset $X$ which follows

$$\begin{aligned} dX &= (r + \xi\sigma)X\ dt + \sigma X\ dZ \\ \xi &= \text{ the market price of risk} \end{aligned}$$

- A fraction $(1 - p)$ in a riskless asset $B$ which follows

$$\frac{dB}{dt} = rB$$

The process followed by $W = B + X$ is

$$dW = (r + p\xi\sigma)W\ dt + \pi\ dt + p\sigma W\ dZ.$$

# Optimal Strategy

Define

$$\begin{aligned} p(W,t) &= \text{dynamic fraction invested in the risky asset} \\ W_T &= \text{terminal wealth} \end{aligned}$$

Let

$$\begin{aligned} E_{p(\cdot)}^{t=0}[W_T] &= \text{Expected gain under strategy } p(\cdot) \\ \text{Var}_{p(\cdot)}^{t=0}[W_T] &= \text{Variance under strategy } p(\cdot) \end{aligned}$$

So that

$$\text{Var}_{p(\cdot)}^{t=0}[W_T] = E_{p(\cdot)}^{t=0}[(W_T)^2] - \left(E_{p(\cdot)}^{t=0}[W_T]\right)^2$$

# Minimum Variance

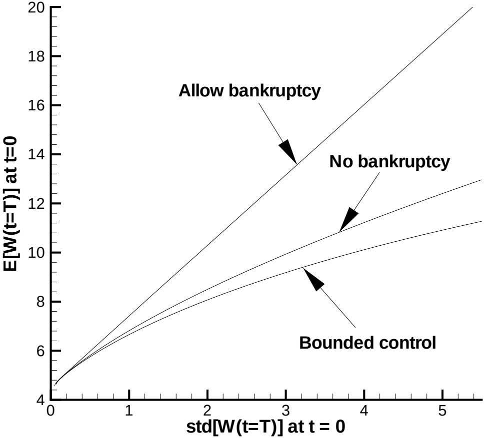
The objective is to determine the strategy $p(\cdot)$ such that

$$\min Var^{t=0}_{p(\cdot)}[W_T] = E^{t=0}_{p(\cdot)}[(W_T)^2] - d^2$$
$$\text{subject to } \begin{cases} E^{t=0}_{p(\cdot)}[W_T] = d \\ p(\cdot) \in Z \end{cases}$$
$$Z = \text{ set of admissible controls}$$

Given an expected return $d = E^{t=0}_{p(\cdot)}[W_t]$, strategy $p(\cdot)$ produces the smallest possible variance.

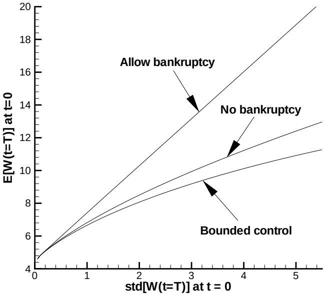
Varying the parameter $d$ traces out a curve in the expected value - standard deviation plane.

Figure: Typical Efficient Frontiers



# Efficient Frontier

- Each point on the frontier is optimal in the sense that no other strategy gives smaller risk for given expected gain.
- Any rational investor will choose points on this curve.
- Different investors will choose different points depending on her risk preferences.



# Eliminate Constraint

Original problem is convex optimization, use Lagrange multiplier $\gamma$ to eliminate constraint.

$$\max_{\gamma} \min_{p(\cdot)\in Z} E_{p(\cdot)}^{t=0}\left[ (W_T)^2 - d^2 - \gamma(E_{p(\cdot)}^{t=0}[W_T] - d) \right] . \qquad (2)$$

Suppose somehow we know the $\gamma$ which solves (2), for fixed $d$.

Then the optimal strategy $p^*(\cdot)$ which solves (2) is given by (for fixed $\gamma$)

$$\min_{p(\cdot)\in Z} E_{p(\cdot)}^{t=0}[(W_T - \frac{\gamma}{2})^2] . \qquad (3)$$

Note $d$ has disappeared from (3).

# Construction of Efficient Frontier

We can alternatively regard $\gamma$ as a parameter, and determine the optimal strategy $p^*(\cdot)$ which solves

$$\min_{p(\cdot)\in Z} E^{t=0}_{p(\cdot)}[(W_T - \frac{\gamma}{2})^2] \ .$$

Once $p^*(\cdot)$ is known, we can easily determine $E^{t=0}_{p^*(\cdot)}[W_T]$, $E^{t=0}_{p^*(\cdot)}[(W_T)^2]$.

Varying $\gamma$ traces out the efficient frontier.

Let $V(W,\tau) = E^{t=T-\tau}[(W_T - \gamma/2)^2]$, then $p^*(\cdot)$ is determined from solution of HJB equation

$$V_\tau = \sup_{p\in z}\left\{[(p\mu + (1-p)r)W + \pi]V_W + (p\sigma)^2 W^2 V_{WW}\right\}$$

$$V(W,\tau=0) = (W - \frac{\gamma}{2})^2$$

# Guaranteed Minimum Withdrawal Benefit (GMWB)

Popular variable annuity product sold in Canada/US.

- Designed for holders of defined contribution pension plans.
- Investor hands over lump sum to an insurance company, insurance company invests sum in risky assets.
- The investor can withdraw up to a contract amount each year, up to total amount of lump sum, regardless of actual amount in investment account.
- At end of contract, investor gets amount remaining in the investment account (net of withdrawals).
- The investor can participate in market gains, but still has a guaranteed cash flow, in the case of market losses.
- This insulates pensioners from losses in the early years of retirement.

# Some more details

- Denote the amount in the risky investment by $W$.
- The investor also has a virtual guarantee account $A$.
- The investor can choose to withdraw up to the specified contract *rate* $G_r$ without penalty.
- Usually, a penalty ($\kappa > 0$) is charged for any withdrawal above rate $G_r$ (instantaneous withdrawal allowed).

The risk neutral process followed by $W$ is then (including withdrawals $dA$).

$$\begin{aligned} dW &= (r-\alpha)W dt + \sigma W dZ + dA, \quad \text{if } W > 0 \\ dW &= 0, \quad \text{if } W = 0 \\ & \alpha = \text{ fee paid for guarantee } \; ; \quad A = \text{ guarantee account} \end{aligned}$$

Let the value of the guarantee be $V = V(W, A, \tau = T - t)$. Define

$$\mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r-\alpha) W V_W - rV.$$

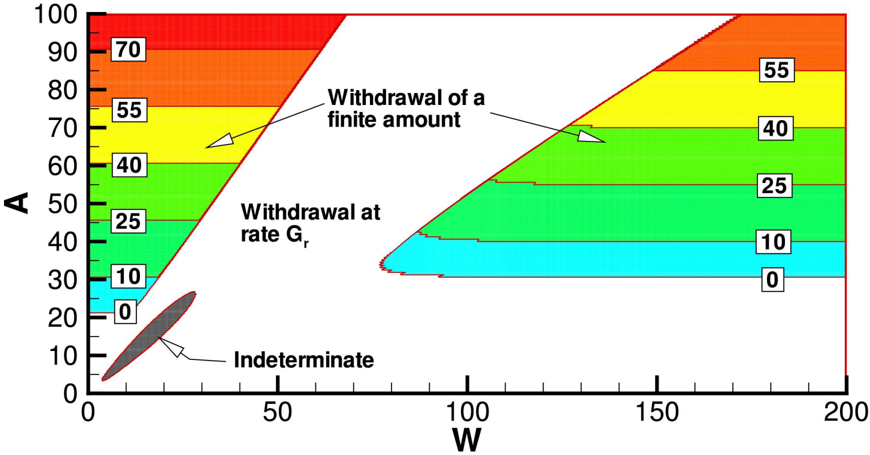
# Impulse Control (HJB Variational Inequality)

Optimal strategy: withdraw at finite rate ($\leq G_r$) or instantaneously withdraw a finite amount (infinite rate).

$$\min\left\{ V_\tau - \mathcal{L}V - \max_{\hat{\gamma}\in[0,G_r]} \left(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\right), \right.$$
$$\left. V - \sup_{\gamma\in(0,A]} \left[ V(\max(W-\gamma,0), A-\gamma, \tau) + (1-\kappa)\gamma - c \right] \right\}$$
$$= 0$$

$c$ is a small fixed cost which ensures that the Impulse Control problem is well-posed.

Figure: Optimal withdrawal strategy: GMWB



# GMWB: Singular Control Formulation

This problem can also be posed as a singular control

$$\min\left[V_\tau - \mathcal{L}V - G\max(\mathcal{F}V, 0), \kappa - \mathcal{F}V\right] = 0$$

$$\mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r-\alpha)WV_W - rV$$

$$\mathcal{F}V = 1 - V_W - V_A$$

This singular control formulation can be solved numerically using a penalty method (Dai, Kwok, Zong: Mathematical Finance (2008)).

The penalty method can viewed as a particular type of control, hence the methods described in these lectures can be used.

# HJB Equations: A General Form

Many of the HJB equations we have seen can be written as

$$
\begin{aligned}
\mathcal{L}^Q V &\equiv a(x,\tau,Q)V_{xx} + b(x,\tau,Q)V_x - c(x,\tau,Q)V \\
& Q = \text{ control} \\
& c(x,\tau,Q) \geq 0
\end{aligned}
$$

$$
\begin{aligned}
V_\tau &= \sup_{Q \in \hat{Q}} \left\{ \mathcal{L}^Q V + d(x,\tau,Q) \right\} \\
& \hat{Q} = \text{ set of admissible controls} \quad .
\end{aligned}
$$

$\hookrightarrow$ Standard approach: determine optimal $Q(t)$ by differentiating $\{\mathcal{L}^Q V + d(Q)\}$ w.r.t. $Q$ and setting to zero.

From a computational perspective: this is a bad idea. We will see why later!

# Viscosity Solution

In general, the HJB equation may not have smooth solutions.

What does it mean to solve a differential equation when the *solution* is not differentiable?

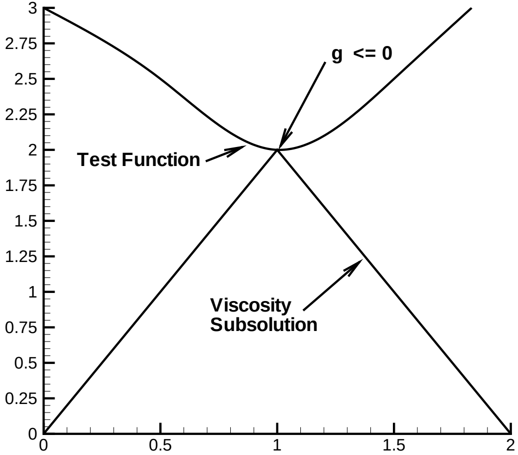
We can write our general HJB equation in the form

$$\begin{aligned} g(V_{xx}, V_x, V_\tau, V, x, \tau) &= V_\tau - \sup_{Q \in \hat{Q}} \left\{ \mathcal{L}^Q V + d(x, \tau, Q) \right\} \\ &= 0 \end{aligned}$$

Suppose we have a $C^{2,1}$ test function $\phi$ such that $\phi \geq V$, and $\phi$ touches $V$ at a single point $(x_0, \tau_0)$.
For simplicity, assume that $V$ is continuous. Otherwise, $\phi$ should touch the upper semi-continuous envelope of $V$.

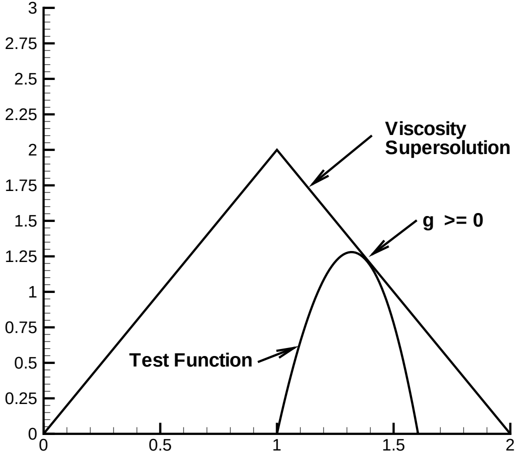
# Subsolution

Figure: If, for any point $(x_0, \tau_0)$, for any test function $\phi \geq V$, where $\phi$ touches $V$ at the single point $(x_0, \tau_0)$, $g(\phi_{xx}, \phi_x, \phi_\tau, \phi, x_0, \tau_0) \leq 0$, then $V$ is a viscosity subsolution.



# Supersolution

Figure: If, for any point $(x_0, \tau_0)$, for any test function $\phi \leq V$, where $\phi$ touches $V$ at the single point $(x_0, \tau_0)$, $g(\phi_{xx}, \phi_x, \phi_\tau, \phi, x_0, \tau_0) \geq 0$, then $V$ is a viscosity supersolution.



# Viscosity Solution

Any solution which is both a subsolution and a supersolution is a *viscosity solution*

Note that we never evaluate $g(V_{xx}, V_x, V_\tau, ...)$ but only $g(\phi_{xx}, \phi_x, \phi_\tau, ...)$, $\rightarrow$ no problems with non-differentiable $V$.

Numerical issues:

- We want to ensure that our numerical scheme converges to the viscosity solution
- For examples of cases where seemingly reasonable discretizations converge to non-viscosity solutions, see *Pooley, Forsyth, Vetzal*, IMA J. Num. Anal. (2003)
- Sufficient conditions known which ensure that a numerical scheme converges to the viscosity solution (Barles, Souganidis (1991))

# Technical Point I

## Remark

*There may also be some points where a smooth $C^{2,1}$ test function cannot touch the solution from either above or below. As a pathological example, consider the function*

$$f(x) = \begin{cases} \sqrt{x} & x \geq 0, \\ -\sqrt{-x} & x < 0. \end{cases} \tag{4}$$

*This function cannot be touched at the origin from below (or above) by any smooth function with bounded derivatives. Note that the definition of a viscosity solution only specifies what happens when the test function touches the viscosity solution at a single point (from either above or below). The definition is silent about cases where this cannot happen.*

# Technical Point II

From now on, we make the following Assumption

## Assumption (Strong Comparison)

*We assume that any HJB PDE we will examine, along with appropriate boundary conditions, satisfies the strong comparison property, which then implies that there exists a unique, continuous viscosity solution to the HJB equation.*

This has been proven for many cases, but not precisely all the problems we will look at.

# Summary

- Many problems in finance can be formulated in terms of optimal stochastic control $\rightarrow$ nonlinear HJB equation/variational inequality.
- Solutions are in general non-differentiable $\rightarrow$ viscosity solution.
- There are precise rules to follow which will guarantee convergence of a numerical scheme to the viscosity solution
- It is a bad idea (numerically) to analytically determine the optimal control (i.e. by differentiating and setting equal to zero), and then to discretize the PDE
- A better approach: first discretize and then optimize the discrete equations.

# Numerical Methods for Hamilton Jacobi Bellman Equations in Finance
# Lecture 2
# Sufficient Conditions for Convergence to the Viscosity Solution

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# Outline For Lecture 2

# A cautionary tale

Recall the uncertain volatility model

$$V_\tau = \max_{q \in \hat{Q}} \left\{ \frac{q^2 S^2}{2} V_{SS} + S V_S - rV \right\}$$

$$\hat{Q} = [\sigma_{\min}, \sigma_{\max}]$$

Suppose we solve this equation using a standard finite difference method with Crank-Nicolson timestepping.

| Parameter | Value |
| --- | --- |
| $S_{init}$ | 100 |
| $\sigma_{\max}$ | .25 |
| $\sigma_{\min}$ | .15 |
| $r$ | .10 |
| $T$ | .25 |
| Payoff | Butterfly |
| Position | Short |

Table: Data used for the uncertain volatility model.

# Convergence Study

- Solve this problem on a sequence of grids/timesteps
- At each refinement level we add new fine grid nodes and double timesteps

| Nodes | Timesteps | Option value |
|---|---|---|
| 61 | 50 | 2.13890 |
| 121 | 100 | 2.12377 |
| 241 | 200 | 2.08726 |
| 481 | 400 | 2.05960 |

Table: Value of the uncertain volatility claim at $S = 100$, Crank-Nicolson timestepping.

Appears to converge to about $2.05

# Crank-Nicolson Solution: Base Initial Grid

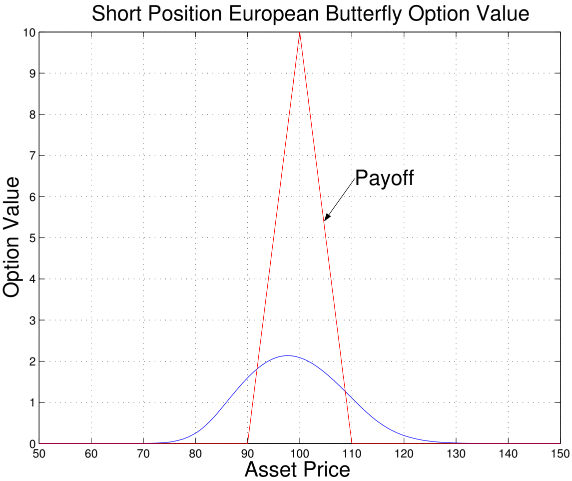



Figure: Option value, uncertain volatility model, Crank-Nicolson timestepping, base initial grid.

# Add Nodes to Base Grid

Convergence was a bit slow, let's add more nodes near the strike on the base grid, and repeat the convergence study.

| Nodes | Timesteps | Option value |
|---|---|---|
| 65 | 50 | 1.06626 |
| 129 | 100 | 0.97531 |
| 257 | 200 | 0.91380 |
| 513 | 400 | 0.87161 |

Table: Convergence data, uncertain volatility, Crank-Nicolson timestepping, $S = 100$, different initial grid.

Appears to converge to about \$0.85, compare with \$2.05 before!

# Crank-Nicolson Solution: Initial grid with more nodes near the strike

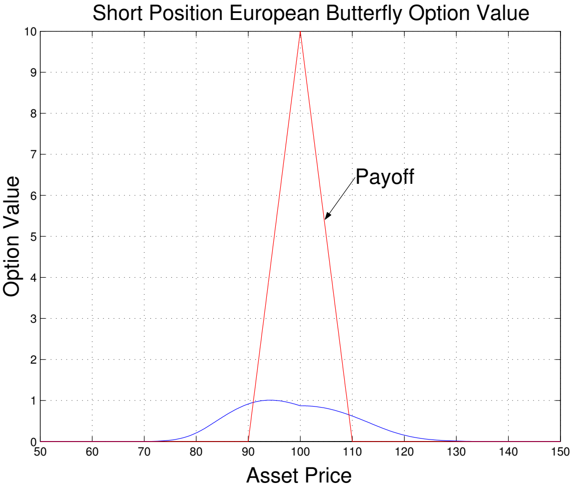



Figure: Option value, uncertain volatility model, Crank-Nicolson timestepping, more nodes near the strike.

# What Happened?

For a linear PDE

- Crank-Nicolson is unconditionally stable, and converges to the exact solution, regardless of the initial grid/timesteps
- For this nonlinear HJB equation
- It appears that we can converge to different solutions depending on our starting grid!

$\hookrightarrow$ We need to be careful to converge to the correct, i.e. viscosity solution

# Definitions

Define a solution domain $\Omega = \Omega_{in} \cup \Omega_{bnd}$.

$$\begin{aligned} \Omega_{in} &= \text{Interior points} \\ \Omega_{bnd} &= \text{Boundary points} \end{aligned}$$

Let

$$\begin{aligned} \mathbf{x} = (x, \tau) \quad &; \quad V = \text{ solution} \\ D^2 V = V_{xx} \quad &; \quad DV = (V_x, V_\tau) \end{aligned}$$

We can write the general form for the HJB equation in a way that includes boundary conditions

$$\begin{aligned} g(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}) &= g_{in}(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}) \\ &\quad \mathbf{x} \in \Omega_{in} \\ g(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}) &= g_{bnd}(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}) \\ &\quad \mathbf{x} \in \Omega_{bnd} \end{aligned}$$

# Definitions cont'd

where

$$
\begin{aligned}
g_{in}(D^2V, DV, V, \mathbf{x}) &= V_\tau - \sup_{Q \in \hat{Q}} \left\{ \mathcal{L}^Q V + d(x, \tau, Q) \right\} \\
&= 0 \\
g_{bnd}(D^2V, DV, V, \mathbf{x}) &= \text{specified boundary condition}
\end{aligned}
$$

Suppose we define a grid $\{x_0, x_1, ..., x_i, ...\}$ and a set of timesteps $\{\tau^0, \tau^1, ..., \tau^n, ...\}$.

Let the discretization parameter $h$ be given by $(C_1, C_2 = const.)$

$$
\begin{aligned}
\max_n(\tau^{n+1} - \tau^n) &= C_1 h \\
\max_i(x_{i+1} - x_i) &= C_2 h
\end{aligned}
$$

# Discretization: General Form

Let $V_i^n$ be the approximate value of the solution, i.e. $V_i^n \simeq V(x_i, \tau^n) = V(\mathbf{x}_i^n)$.

Then we can write a general discretization of the HJB equation $g(D^2V, DV, V, \mathbf{x})$ at node $\mathbf{x}_i^{n+1} = (x_i, \tau^{n+1})$

$$
\begin{aligned}
& G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n, V_{i+1}^n, V_{i-1}^n, \ldots) \\
&= G_i^{n+1}(h, V_i^{n+1}, \left\{V_m^p\right\}_{\substack{p\neq n+1 \\ \text{or } m\neq i}}) \\
&= 0
\end{aligned}
$$

$\left\{V_m^p\right\}_{\substack{p\neq n+1 \\ \text{or } m\neq i}}$ refers to the discrete solution values at nodes neighbouring (in space and time) node $(x_i, \tau^{n+1})$.

# Example: Linear Heat Equation

Discretize

$$\begin{aligned} V_t &= V_{xx} \\ & V(x=0,t) = V(x=1,t) = 1 \end{aligned}$$

using a mesh $x_i = i\Delta x, i = 0, .., i_{max},\ t^n = n\Delta t,\ V(x_i, t^n) \simeq V_i^n$.

$$\begin{aligned} \frac{V_i^{n+1} - V_i^n}{\Delta t} &= \frac{V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}}{(\Delta x)^2} \\ & \Delta t = h \ \ ; \ \ \Delta x = h \end{aligned}$$

$$\begin{aligned} G_{i=0}^{n+1} &= V_{i=0}^{n+1} - 1 \\ G_i^{n+1} &= \frac{V_i^{n+1} - V_i^n}{h} - \left( \frac{V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}}{h^2} \right); \quad i \neq 0, i_{\max} \\ G_{i=i_{\max}}^{n+1} &= V_{i=i_{\max}}^{n+1} - 1 \end{aligned}$$

# Sufficient Conditions: Convergence

We use the results from (Barles, Sougandinis, 1991).

## Theorem (Convergence)

*Any numerical scheme which is consistent, $l_\infty$ stable, and monotone, converges to the viscosity solution.*

**Stability**

- Usual requirement (discrete solution bounded in $l_\infty$ as mesh, timestep $\rightarrow 0$)
- Can prove using maximum analysis
- Usually, only fully implicit timestepping is unconditionally $l_\infty$ stable (but not always, e.g. jump diffusions).

# Monotonicity

## Definition (Monotonicity)

The discretization

$$G_i^{n+1}\left(h, V_i^{n+1}, \left\{V_m^p\right\}_{\substack{p\neq n+1 \\ \text{or } m\neq i}}\right)$$

is monotone if

$$G_i^{n+1}\left(h, V_i^{n+1}, \left\{X_m^p\right\}_{\substack{p\neq n+1 \\ \text{or } m\neq i}}\right) \leq G_i^{n+1}\left(h, V_i^{n+1}, \left\{Y_m^p\right\}_{\substack{p\neq n+1 \\ \text{or } m\neq i}}\right)$$
$$\forall\; X_m^p \geq Y_m^p, \forall m, p$$

# Monotonicity cont'd

- A discretization which is a positive coefficient method is monotone (usually)
- For 1-d problems, multi-dimensional problems with diffusion only in one direction, forward/backward/central differencing generate a positive coefficient scheme
- Often results in low order schemes (first order)
- Has nice financial interpretation: enforces discrete no-arbitrage inequalities. i.e. if payoff from contingent claim A is greater than claim B (on the same underlying) then the value of Claim A must always be greater than Claim B at all earlier times.
- More later

# Monotonicity: Heat equation

Recall that

$$G_i^{n+1} = \frac{V_i^{n+1} - V_i^n}{h} - \left( \frac{V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}}{h^2} \right); \quad i \neq 0, i_{\max}$$

We have to show that $G_i^{n+1}$ is a nonincreasing function of $(V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n)$.

First step, consider

$$\begin{aligned} & G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1} + \epsilon, V_{i-1}^{n+1}, V_i^n) - G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n) \\ &= -\left( \frac{V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1} + \epsilon}{h^2} \right) + \left( \frac{V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}}{h^2} \right) \\ &= -\frac{\epsilon}{h^2} < 0 \quad ; \quad \epsilon > 0 \end{aligned}$$

Similar result if we perturb $V_i^n, V_{i-1}^{n+1}$

# Consistency

Consistency in the viscosity sense defined a technical way. I include this for completeness.

## Definition (Semi-continuous envelopes)

If $C$ is a *closed set*, and $f(x)$ is a function defined on $C$, then the upper semi-continuous envelope $f^*(x)$ and the lower semi-continuous envelope $f_*(x)$, $x \in C$, is defined by

$$f^*(x) = \limsup_{\substack{y \longrightarrow x \\ y \in C}} f(y) \quad ; \quad f_*(x) = \liminf_{\substack{y \longrightarrow x \\ y \in C}} f(y)$$

## Remark

*Note that $C$ is closed here, which means that we consider the neighbours of $x$ and $x$ itself.*

# Consistency

The scheme $G_i^{n+1}(h, V_i^{n+1}, \{X_m^p\}_{\substack{p\neq n+1\\ m\neq i}})$ is consistent if, for all $\hat{\mathbf{x}} = (\hat{x}, \hat{\tau})$ in the computational domain, and for all smooth, bounded test functions $\phi(\mathbf{x})$, we have

$$\begin{aligned}
&\limsup_{\substack{h\to 0\\ \mathbf{x}_i^{n+1}\to\hat{\mathbf{x}}\\ \xi\to 0}} G_i^{n+1}\left(h, \phi(\mathbf{x}_i^{n+1}) + \xi, \left\{\phi(\mathbf{x}_m^p) + \xi\right\}_{\substack{p\neq n+1\\ \text{or } m\neq i}}\right) \\
&\qquad\qquad \leq g^*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big), \\
&\liminf_{\substack{h\to 0\\ \mathbf{x}_i^{n+1}\to\hat{\mathbf{x}}\\ \xi\to 0}} G_i^{n+1}\left(h, \phi(\mathbf{x}_i^{n+1}) + \xi, \left\{\phi(\mathbf{x}_m^p) + \xi\right\}_{\substack{p\neq n+1\\ \text{or } m\neq i}}\right) \\
&\qquad\qquad \geq g_*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big),
\end{aligned} \tag{1}$$

# Why do we ever need this complex definition of consistency?

At points near the boundary, the viscosity definition of consistency is more relaxed than the classical definition of consistency.

- Sometimes, a scheme (i.e. a semi-Lagrangian method) may never be consistent (in the classical sense) near the boundary.
- But the scheme is consistent in the viscosity sense!
- Consistency in the viscosity sense can be very useful in such situations.
- In fact, if you do anything reasonable in a complex situation, you usually find that everything is fine in the viscosity sense.
- The whole idea of viscosity solutions allows you to do what you were going to do anyway, except now you know it is OK.

# Discretization

Recall that we can write our HJB equation as

$$\begin{aligned} g(V_{xx}, V_x, V_\tau, V, x, \tau) &= V_\tau - \sup_{Q \in \hat{Q}} \left\{ \mathcal{L}^Q V + d(x, \tau, Q) \right\} \\ &= 0 \end{aligned} \tag{2}$$

where

$$\mathcal{L}^Q V \equiv a(x, \tau, Q) V_{xx} + b(x, \tau, Q) V_x - c(x, \tau, Q) V$$

Discretize (2)

- Fully implicit timestepping
- Forward/backward/central differencing

# Fully Implicit Timestepping

Define a grid $[x_0, x_1, .., x_{i_{\max}}]$, and let $V^n = [V_0^n, V_1^n, ..., V_{i_{\max}}^n]'$. Let $\mathcal{L}_h^Q$ be the discrete form of the operator $\mathcal{L}^Q$, so that

$$
\begin{aligned}
(\mathcal{L}_h^Q V^{n+1})_i &= \alpha_i^{n+1}(Q) V_{i-1}^{n+1} + \beta_i^{n+1}(Q) V_{i+1}^{n+1} \\
&\quad -(\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)) V_i^{n+1}.
\end{aligned}
$$

The discrete form of the HJB equation is then

$$
\frac{V_i^{n+1} - V_i^n}{\Delta \tau} = \sup_{Q^{n+1} \in \hat{Q}} \left\{ (\mathcal{L}_h^{Q^{n+1}} V^{n+1})_i + d_i^{n+1} \right\} .
$$

# General Form

We can write the discretized PDE in our general form

$$
\begin{aligned}
& G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n) \\
&= \frac{V_i^{n+1} - V_i^n}{\Delta\tau} \\
&+ \inf_{Q \in \hat{Q}} \Big\{ (\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)) V_i^{n+1} \\
&\qquad - \alpha_i^{n+1}(Q) V_{i-1}^{n+1} - \beta_i^{n+1}(Q) V_{i+1}^{n+1} - d_i^{n+1}(Q) \Big\} \\
&= 0 \qquad (3)
\end{aligned}
$$

Assume $c_i^{n+1} \geq 0$, and that the following condition holds (using central/forward/backward differencing)

Condition (Positive Coefficient)

$$\alpha_i^{n+1}(Q) \geq 0, \quad \beta_i^{n+1}(Q) \geq 0, \quad c_i^{n+1}(Q) \geq 0 \quad ; \quad \forall Q \in \hat{Q} \quad . \tag{4}$$

It is now easy to show that this discretization is monotone.

We have to show that $G_i^{n+1}(\cdot)$ is a nonincreasing function of the neighbour nodes $V_m^p, (m \neq i$ or $p \neq n+1)$.

# Discretized equations

Recall that the discretized equations are

$$
\begin{aligned}
&G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n) \\
&= \frac{V_i^{n+1} - V_i^n}{\Delta\tau} + \inf_{Q \in \hat{Q}} \Big\{ (\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)) V_i^{n+1} \\
&\qquad - \alpha_i^{n+1}(Q) V_{i-1}^{n+1} - \beta_i^{n+1}(Q) V_{i+1}^{n+1} - d_i^{n+1}(Q) \Big\}
\end{aligned}
$$

# Monotonicity

Consider the case where we perturb $V_{i+1}^{n+1}$ by $\epsilon > 0$, we need to show

$$
\begin{aligned}
&G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1} + \epsilon, V_{i-1}^{n+1}, V_i^n) \\
&-G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n) \\
&\leq 0
\end{aligned}
$$

Given two functions $S(z)$, $T(z)$, $z \in D$, then a useful fact is that

$$
\inf_{x \in D} S(x) - \inf_{y \in D} T(y) \quad \leq \quad \sup_{z \in D} \Big( S(z) - T(z) \Big)
$$

$$G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1} + \epsilon, V_{i-1}^{n+1}, V_i^n) - G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n)$$

$$= \inf_{Q \in \hat{Q}} \Big\{ (\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)) V_i^{n+1}$$

$$-\alpha_i^{n+1}(Q) V_{i-1}^{n+1} - \beta_i^{n+1}(Q) V_{i+1}^{n+1} - \beta_i^{n+1}(Q)\epsilon - d_i^{n+1}(Q) \Big\}$$

$$- \inf_{Q^* \in \hat{Q}} \Big\{ (\alpha_i^{n+1}(Q^*) + \beta_i^{n+1}(Q^*) + c_i^{n+1}(Q^*)) V_i^{n+1}$$

$$-\alpha_i^{n+1}(Q^*) V_{i-1}^{n+1} - \beta_i^{n+1}(Q^*) V_{i+1}^{n+1} - d_i^{n+1}(Q^*) \Big\}$$

$$\leq \sup_{Q \in \hat{Q}} \Big\{ -\beta_i^{n+1}(Q)\epsilon \Big\} = -\epsilon \inf_{Q \in \hat{Q}} \Big\{ \beta_i^{n+1}(Q) \Big\} \leq 0$$

which follows from $\beta_i^{n+1}(Q) \geq 0$ and we have used $(\inf S - \inf T) \leq \sup(S - T)$.

Similarly, we can show that a positive perturbation of any of $\{V_{i-1}^{n+1}, V_i^n\}$ causes a decrease in the value of $G_i^{n+1}(\cdot)$.

We can now state the following result.

## Proposition (Monotonicity)

*If the positive coefficient condition (4) is satisfied, then (3) is a monotone discretization, as defined in Definition 1.*

**Note:** In order to ensure a positive coefficient discretization

- our choice of central/forward/backward differencing will depend, in general, on the control $Q$.
- more about this later

# What about Consistency? How do we show this?

According to the definition of lim sup, there exist sequences $h_k, i_k, n_k, \xi_k$, such that

$$h_k \to 0, \xi_k \to 0, \mathbf{x}_{i_k}^{n_k+1} \to \hat{\mathbf{x}} \text{ as } k \to \infty$$

and

$$\lim_{k_0 \to \infty} \sup_{k > k_0} G_{i_k}^{n_k+1}\left(h_k, \phi(\mathbf{x}_{i_k}^{n_k+1}) + \xi_k, \left\{\phi(\mathbf{x}_m^p) + \xi_k\right\}_{\substack{p \neq n_k+1 \\ \text{or } m \neq i_k}}\right)$$
$$= \limsup_{\substack{h \to 0 \\ \mathbf{x}_i^{n+1} \to \hat{\mathbf{x}} \\ \xi \to 0}} G_i^{n+1}\left(h, \phi(\mathbf{x}_i^{n+1}) + \xi, \left\{\phi(\mathbf{x}_m^p) + \xi\right\}_{\substack{p \neq n+1 \\ \text{or } m \neq i}}\right)$$

At interior points, it is usually easy to show (using Taylor series) that (recall that $\phi$ is infinitely differentiable)

$$G_{i_k}^{n_k+1}\left(h_k, \phi(\mathbf{x}_{i_k}^{n_k+1}) + \xi_k, \left\{\phi(\mathbf{x}_m^p) + \xi_k\right\}_{\substack{p\neq n_k+1 \\ \text{or } m\neq i_k}}\right)$$
$$= g_{in}\left(D^2\phi(\mathbf{x}_{i_k}^{n_k+1}), D\phi(\mathbf{x}_{i_k}^{n_k+1}), \phi(\mathbf{x}_{i_k}^{n_k+1}), \mathbf{x}_{i_k}^{n_k+1}\right) + O(h_k) + O(\xi_k)$$

Since usually $g_{in}(\cdot)$ is continuous, then at interior points $g^* = g_* = g_{in}$, and we are done.

**Note:** Usually, the complex definition of consistency boils down to a classic definition of consistency, which is easily proved using Taylor series.

# Matrix Form of the Discrete Equations

Let the value of the optimal control at node $i$, timestep $n$ be $Q_i^n$, and define

$$Q^n = [Q_0^n, Q_i^n, .., Q_{i_{max}}^n]$$

Define the matrix $A(Q^n)$, so that

$$[A(Q^n) V^n]_i \quad = \quad (\mathcal{L}_h^{Q^n} V^{n+1})_i$$

and the vector $D(Q^{n+1})^{n+1} = [d_0^n, d_1^n, ..., d_{i_{max}}^n]$ so that fully implicit timestepping can be written as

$$V^{n+1} \quad = \quad V^n + \Delta\tau \sup_{Q^{n+1} \in \hat{Q}} \left\{ A^{n+1}(Q^{n+1}) V^{n+1} + D^{n+1}(Q^{n+1}) \right\}$$

# Matrix Form of the Discrete Equations II

If $\hat{Q}$ is compact, and the objective function is continuous, then we can write the discretized equations as

$$[I - \Delta\tau A^{n+1}(Q^{n+1})]V^{n+1} = V^n + \Delta\tau D^{n+1}(Q^{n+1})$$
$$\text{where } Q_i^{n+1} \in \underset{Q_i^{n+1} \in \hat{Q}}{\arg\max} \left\{ \left[ A^{n+1}(Q^{n+1}) V^{n+1} + D^{n+1}(Q^{n+1}) \right]_i \right\}$$

From the positive coefficient condition, $[I - \Delta\tau A^{n+1}(Q^{n+1})]$ is an $\mathcal{M}$ matrix, hence

$$[I - \Delta\tau A^{n+1}(Q^{n+1})]^{-1} \geq 0$$

# Nonlinear Algebraic Equations

$$V^{n+1} = V^n + \Delta\tau \sup_{Q^{n+1} \in \hat{Q}} \left\{ A^{n+1}(Q^{n+1}) V^{n+1} + D^{n+1}(Q^{n+1}) \right\} \quad (5)$$

- Note that this is a set of nonlinear algebraic equations
- But, from $\mathcal{M}$ matrix property, the solution exists and is unique

**Arbitrage Inequality**

Suppose we have two different initial vectors $W^n, U^n$, so that

$$W^{n+1} = W^n + \Delta\tau \sup_{Q^{n+1} \in \hat{Q}} \left\{ A^{n+1}(Q^{n+1}) W^{n+1} + D^{n+1}(Q^{n+1}) \right\} \quad (6)$$

$$U^{n+1} = U^n + \Delta\tau \sup_{Q^{n+1} \in \hat{Q}} \left\{ A^{n+1}(Q^{n+1}) U^{n+1} + D^{n+1}(Q^{n+1}) \right\} \quad (7)$$

# Arbitrage Inequality

### Proposition (Discrete Comparison)

*If a positive coefficient discretization is used to discretize the HJB PDE, then if $W^n$, $U^n$ are discrete solutions to (6)-(7), with $W^n \geq U^n$, then $W^{n+1} \geq U^{n+1}$.*

### Proof.

Manipulate (6)-(7), use the $\mathcal{M}$ matrix property of $[I - \Delta\tau A^{n+1}(Q^{n+1})]$. □

### Remark (Monotonicity)

*We can see here that monotonicity $\rightarrow$ arbitrage inequality, i.e. inequality of payoffs translates into inequality of value at all earlier times. Note that this holds regardless of timestep or mesh size.*

# Back to Our Uncertain Volatility Equation

Recall that, with C-N timestepping, using different initial grids, we appeared to converge to two different values

- Grid1 $\rightarrow$ \$2.05
- Grid2 $\rightarrow$ \$0.85

What went wrong here? C-N timestepping does not satisfy two of the conditions we need

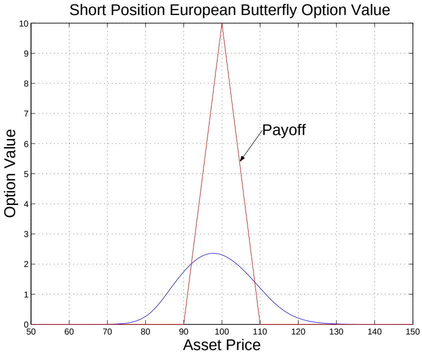
- C-N is not monotone (for timesteps $>$ explicit timestep size)
- C-N is not $l_\infty$ stable.

However, fully implicit timestepping, positive coefficient discretization is

- Unconditionally monotone, $l_\infty$ stable
- Consistent

# Fully Implicit Timestepping

| Nodes | Time steps | Option value |
|---|---|---|
| 61 | 50 | 2.3211 |
| 121 | 100 | 2.3107 |
| 241 | 200 | 2.3045 |
| 481 | 400 | 2.3012 |




We now know for sure that \$2.30 is the correct solution!

# Summary: Lecture II

- Convergence to the viscosity solution ensured if discretization is Consistent, $l_\infty$ stable and monotone.
- Consistency: most of the time, this is just classical consistency, applied to infinitely differentiable test functions
  - $\rightarrow$ Consistency in the viscosity sense is very forgiving, when it comes to points near the boundaries
- $l_\infty$ Stability: standard maximum analysis can be used to prove this
- Monotonicity: most interesting condition: preserves discrete arbitrage inequalities
  - $\rightarrow$ Positive coefficient discretization guarantees this property
- Seemingly reasonable discretizations can converge to wrong values if these conditions are not satisfied!

# Numerical Methods for Hamilton Jacobi Bellman Equations in Finance
# Lecture 3
# Examples: Passport Options, Asset Allocation in Pension Plans

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# Outline For Lecture 3

Example: Passport options

Example: Pension Plan

Discretization

Policy Iteration

Central Differencing as Much as Possible

Numerical Results

Discretized Control

Summary

# Example I: Passport Options

- Option on a trading account
- Holder is allowed to go long/short underlying asset $S$ at any time during $[0, T]$
- Assume $S$ follows Geometric Brownian Motion
- Holder can hold $q$ shares at any time, where $|q| < C$
- Let $W$ be the accumulated gain in the portfolio
- At maturity, holder's position is

$$\max(W, 0)$$

# Passport Options

Option Value $V(S, W, t) = Su(x = W/S, t)$

$$
\begin{aligned}
u_\tau &= -\gamma u + \sup_{|q|\le 1}\Big[((r-\gamma-r_c)q - (r-\gamma-r_t)x)u_x \\
&\quad + \frac{\sigma^2}{2}(x-q)^2 u_{xx}\Big] ,
\end{aligned}
$$

$$
\begin{aligned}
x &= W/S \; ; \quad \tau = T - t \; ; \quad r = \text{ risk-free rate} \; , \\
\gamma &= \text{dividend rate} \; ; \quad r_c = \text{cost of carry} \; , \\
r_t &= \text{ interest rate on trading account}
\end{aligned}
$$

# Asset Allocation: Pension Plan

- Cairns *et al* , JEDC, 2006 *"Stochastic Lifestyling: Optimal Dynamic Asset Allocation for Defined Contribution Pension Plans"*
- Holder of defined contribution pension plan has annual salary $Y(t)$ (stochastic), contributes $\pi \; Y$ per year to pension plan
- Invests a fraction $p$ of accumulated wealth $W$ in risky asset $S$, $(1-p)$ in riskfree asset
- Desires to maximize expected utility of wealth/income ratio at retirement $T$

$$\text{Power Law Utility} \quad = \quad U(x, T) = \frac{x^{\gamma}}{\gamma} \quad ; \quad \gamma < 0$$

$$x = \frac{W}{Y} \quad ; \quad \text{Wealth/Income ratio}$$

# Pension Plan II

Bond process:

$$dB = rB\ dt$$
$$B = \text{Bond price} \; ; \quad r = \text{risk free rate}$$

Underlying risky asset process:

$$dS = (r + \xi_1 \sigma_1) S\ dt + \sigma_1 S\ dZ_1 \ ,$$
$$S = \text{risky asset} \; ; \qquad \xi_1 = \text{market price of risk}$$
$$\sigma_1 = \text{volatility}; \qquad dZ_1 = \text{increment Wiener process}$$

Wealth process:

$$dW = (r + p\xi_1 \sigma_1) W\ dt + p\sigma_1 W\ dZ_1 + \pi Y dt \ .$$
$$W = \text{accumulated wealth}; \quad \pi Y = \text{contribution rate},$$
$$p = \text{fraction invested in S}; \quad (1 - p) = \text{fraction in bond}$$

# Stochastic Salary

Assume that yearly salary is also stochastic:

$$
\begin{aligned}
dY &= (r+\mu_Y)Y\ dt + \sigma_{Y_0} Y\ dZ_0 + \sigma_{Y_1} Y\ dZ_1\ , \\
& \quad Y = \text{ yearly salary;} \quad r+\mu_Y = \text{ salary drift rate,} \\
& \quad \sigma_{Y_1} = \text{ random salary component} \\
& \qquad\qquad \text{correlated with market.} \\
& \quad \sigma_{Y_0} = \text{ random salary component uncorrelated with market}
\end{aligned}
$$

$Z_0, Z_1$ are uncorrelated, i.e.

$$dZ_0\ dZ_1 = 0\ .$$

# Maximize expected utility $V(x, \tau)$

Studies show: we are happy if our wealth $W$ at retirement is large compared to our pre-retirement yearly salary $Y$.

Let $x = W/Y$, and maximize expected terminal utility by determining $V(x, \tau)$ such that

$$
\begin{aligned}
V(x,\tau) &= \sup_{p \in \hat{P}} E[U(x(T))|x(T-\tau) = x] \\
&\quad \tau = T - t \\
&\quad p(t) = \text{fraction of } W \text{ in risky asset} \\
&\quad \hat{P} = \text{set of admissible allocation strategies} \\
&\quad U(x) = U(W/Y) = \text{utility} \\
&\quad T = \text{retirement date}
\end{aligned}
$$

# HJB Equation: Pension plan

$$x = \frac{\text{Wealth}}{\text{Yearly income}}$$

Standard dynamic programming arguments:

$$V_\tau = \sup_{p \in \hat{P}} \left\{ \mu_X^p V_x + \frac{1}{2} (\sigma_X^p)^2 V_{xx} \right\} \quad ; \quad x \in [0, \infty) \ ,$$

where

$$\begin{aligned} V(x, \tau = 0) &= \gamma^{-1} x^\gamma \quad ; \quad \gamma < 0 \quad , \\ & \mu_X^p = \pi + x(-\mu_Y + p\sigma_1(\xi_1 - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2) \quad , \\ & (\sigma_X^p)^2 = x^2(\sigma_{Y_0}^2 + (p\sigma_1 - \sigma_{Y_1})^2) \ . \end{aligned}$$

# No-bankruptcy

Note that the salary process is always non-negative.

In order to ensure that $W \geq 0$, which implies that $x \geq 0$

$$V_\tau \quad = \quad \pi V_x \quad ; \quad x \rightarrow 0$$

As we shall see, this means that $(px) \rightarrow 0$ as $x \rightarrow 0$.

- The *amount* invested in the risky asset tends to zero as the (wealth-income ratio $\rightarrow 0$).
- But, the ratio of wealth invested in $S$ becomes infinite.

# General Form (Passport and Pension Allocation)

Both of these PDEs can be be written in the general form

$$\mathcal{L}^Q V \equiv a(x,\tau,Q)V_{xx} + b(x,\tau,Q)V_x - c(x,\tau,Q)V$$
$$Q = \text{ control}$$

$$V_\tau = \sup_{Q \in \hat{Q}} \left\{ \mathcal{L}^Q V + d(x,\tau,Q) \right\}$$
$$\hat{Q} = \text{ set of admissible controls} \quad .$$

Textbook approach: determine optimal $Q(t)$ by differentiating $\{\mathcal{L}^Q V + d(Q)\}$ w.r.t. $Q$ and setting to zero.

Typical method used to get analytic solutions.

# Standard Approach

Apply this idea (differentiating w.r.t. $p$ and setting to zero) to Pension Plan example

$$
\begin{aligned}
V_\tau &= (\pi + \delta x) V_x + \frac{\sigma_{Y0}^2 x^2}{2} V_{xx} - \frac{(\xi_1 - \sigma_{Y1})^2}{2} \left( \frac{V_x^2}{V_{xx}} \right) \\
&\delta = -\mu_Y + \sigma_{Y0}^2 + \sigma_{Y1} \xi_1
\end{aligned} \quad (1)
$$

Now, we have taken a nice HJB equation, and made a mess of it!

It will be very difficult to construct a monotone, consistent, stable method for (1).

Moral of story:

- First discretize
- Then, optimize

Find max/min of discrete equations not analytic approximations to discrete equations.

# Discretization

Fully implicit discretization: Let $(\mathcal{L}_h^Q V^n)_i$ denote the discrete form of the differential operator $\mathcal{L}^Q V$ at node $(x_i, \tau^n)$. Use central, forward, backward differencing to obtain

$$\begin{aligned}(\mathcal{L}_h^Q V^{n+1})_i &= \alpha(Q)_i^{n+1} V_{i-1}^{n+1} + \beta(Q)_i^{n+1} V_{i+1}^{n+1} \\ &\quad -(\alpha(Q)_i^{n+1} + \beta(Q)_i^{n+1} + c(Q)_i^{n+1}) V_i^{n+1}\end{aligned}$$

The discrete form of the HJB equation is then

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \sup_{Q^{n+1} \in \hat{Q}} \left\{ (\mathcal{L}_h^{Q^{n+1}} V^{n+1})_i + d_i^{n+1} \right\} .$$

# Central, Forward, Backward

Recall our positive coefficient condition (which ensures monotonicity)

$$\alpha(Q)_i^{n+1} \geq 0 \quad ; \quad \beta(Q)_i^{n+1} \geq 0 \quad ; \quad c(Q)_i^{n+1} \geq 0 \tag{2}$$

In order to ensure this condition, we have to be careful how we discretize the term

$$b(x, \tau, Q) V_x$$

in the PDE.

A simple-minded idea is to use either forward or backward differencing depending on the sign of $b(x, \tau, Q)$.

However, this is only first order correct, and is not always necessary.

We would like to use central differencing as much as possible, and yet still have a positive coefficient scheme.

# Convergence

## Lemma (Stability)

*The fully implicit, positive coefficient scheme is unconditionally stable.*

### Proof.

Follows from a straightforward maximum analysis. □

## Lemma (Consistency)

*The fully implicit positive coefficient scheme is consistent.*

### Proof.

Taylor series applied to smooth test functions. □

## Lemma (Monotonicity)

*The fully implicit positive coefficient scheme is monotone .*

### Proof.

Follows from lecture 2. □

# Nonlinear discretized equations

Recall the matrix form of the discretization of $\mathcal{L}^Q$ :

$$(\mathcal{L}_h^Q V^n)_i = [A^n(Q^n)V^n]_i$$

so that the discretized equations are

$$\left[I - \Delta\tau A^{n+1}(Q^{n+1})^{n+1}\right] V^{n+1} = V^n + \Delta\tau D^{n+1}(Q^{n+1})^{n+1}$$

where

$$Q_i^{n+1} = \underset{Q_i^{n+1} \in \hat{Q}}{\arg\max} \left\{ \left[A^{n+1}(Q^{n+1})V^{n+1} + D^{n+1}(Q^{n+1})\right]_i \right\}$$

# Policy Iteration

We use the following iterative scheme to solve the nonlinear equations.

Let $(V^{n+1})^0 = V^n$

Let $\hat{V}^k = (V^{n+1})^k$

For $k = 0, 1, 2, \ldots$ until convergence

Solve

$$\left[I - \Delta\tau A^{n+1}(Q^k)\right] \hat{V}^{k+1} = V^n + \Delta\tau D^{n+1}(Q^k)$$

$$Q_i^k = \underset{Q_i^k \in \hat{Q}}{\arg\max} \left\{ F_i(Q_i^k) \right\}$$

$$F_i(Q_i^k) = \left[A^{n+1}(Q^k)\hat{V}^k + D^{n+1}(Q^k)\right]_i$$

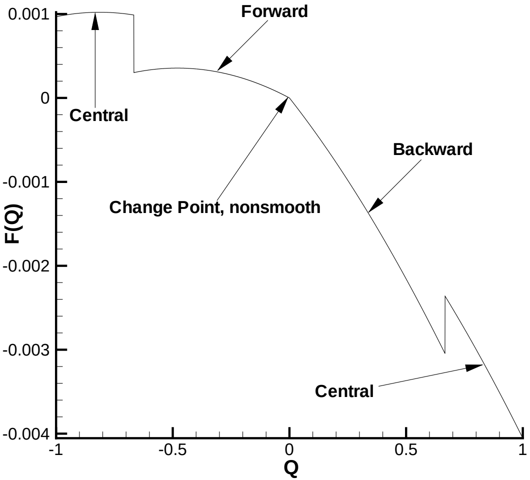
EndFor

# Central Differencing as Much as Possible

Given a control $Q_i^n$ at node $i$, the following algorithm is used to determine the differencing method

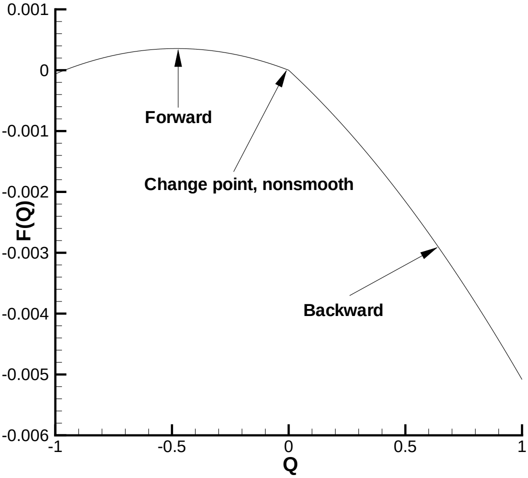
```
diff = central
IF α_i < 0 or β_i < 0 THEN
    diff = backward
    IF α_i < 0 or β_i < 0 THEN
        diff = forward
    ENDIF
ENDIF
```

# Local Objective Function, Passport Option, Central Differencing as much as possible




Local objective function is a discontinuous function of the control.

# Local Objective Function, Passport Option, Forward/Backward only




Local objective function is a continuous function of the control.

# Policy Iteration

In view of the fact that the local objective function can be a discontinuous function of the control

- At points of discontinuity, we want the limiting value of the objective function which produces the supremum
- At points of discontinuity, we can specify the supremum by specifying the control and the differencing method which gives this limiting value (abuse of notation: $\arg\sup(\cdot)$)
- A subtle point which may cause non-convergence if ignored

$$\left[ I - \Delta\tau A^{n+1}(Q^k) \right] \hat{V}^{k+1} = V^n + \Delta\tau D^{n+1}(Q^k)$$

$$Q_i^k \in \arg\sup_{Q_i^k \in \hat{Q}} \left\{ \left[ A^{n+1}(Q^k)\hat{V}^k + D^{n+1}(Q^k) \right]_i \right\} \quad (3)$$

# Convergence of Policy Iteration

Manipulation of algorithm (3) gives

$$\left[I - \Delta\tau A^{n+1}(Q^k)\right](\hat{V}^{k+1} - \hat{V}^k)$$
$$= \Delta\tau\left[\left(A^{n+1}(Q^k)\hat{V}^k + D^{n+1}(Q^k)\right) - \left(A^{n+1}(Q^{k-1})\hat{V}^k + D^{n+1}(Q^{k-1})\right)\right]$$

- Positive coefficient discretization $\rightarrow$ LHS is an $\mathcal{M}$ matrix
- RHS is non-negative (Why?)
- Easy to show that $\hat{V}^{k+1}$ is bounded independent of $k$
- Iterates form a bounded non-decreasing sequence
- If $\hat{V}^{k+1} = \hat{V}^k$, residual is zero

## Theorem (Convergence of Policy Iteration)

*If a positive coefficient discretization is used, then the Policy iteration algorithm converges to the unique solution of the nonlinear discretized equations for any initial iterate* $\hat{V}^0$*.*

**Note:** We do not require that the local objective function be a continuous function of the control.

- Using central differencing as much as possible should be more accurate.
- But we have no guarantee on how many iterations required to solve the nonlinear equations.
- We have no guarantee on the order of convergence as the mesh is refined
  - Forward/backward differencing only, can be *at most* first order
  - Central weighting as much as possible *may* be second order
- Is central differencing as much as possible useful in practice?

# Pension Allocation Example

Table: Data from Cairns et al (2006)

| | | | |
|---|---|---|---|
| $\mu_y$ | 0.0 | $\xi_1$ | 0.2 |
| $\sigma_1$ | 0.2 | $\sigma_{Y1}$ | 0.05 |
| $\sigma_{Y0}$ | 0.05 | $\pi$ | 0.1 |
| $T$ | 20 years | $\gamma$ | -5 |

Convergence study:

- Sequence of tests, on each grid refinement, add new node between each coarse grid node
- Divide timestep by four (not required for stability, but fully implicit method is only first order)
- If we are getting benefit from central weighting as much as possible, then ratio of changes $\rightarrow$ 4

# Computational Issues

Original domain $x \in [0, \infty)$.
$\hookrightarrow$ Computational domain $[0, 80]$.
$\hookrightarrow$ Dirichlet boundary condition at $x = x_{\text{max}}$.

Utility function undefined at $x = 0$.

$$\begin{aligned} V(x, \tau = 0) &= \gamma^{-1}(max(x, \varepsilon))^{\gamma} \\ &\quad \varepsilon = 10^{-3} \;\; ; \;\; \gamma < 0 \end{aligned}$$

Relative convergence tolerance for Policy Iteration.
$\hookrightarrow 10^{-7}$.

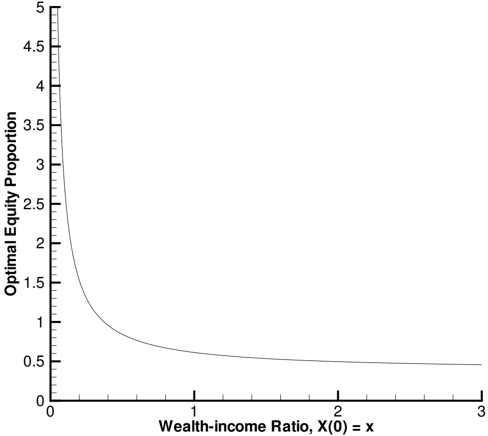
Modifying these parameters made no change in the solution to 7 digits.

# Convergence Study: Pension Allocation Example

| Nodes | Timesteps | Nonlinear iterations | CPU Time (Sec) | Utility | Ratio |
|---|---|---|---|---|---|
| Central Differencing as much as possible, $x = 0$ | | | | | |
| 87 | 160 | 331 | 0.04 | $-4.06482 \times 10^{-3}$ | |
| 173 | 640 | 1280 | 0.36 | $-3.65131 \times 10^{-3}$ | |
| 345 | 2560 | 5120 | 2.75 | $-3.58063 \times 10^{-3}$ | 5.851 |
| 689 | 10240 | 20480 | 21.31 | $-3.56354 \times 10^{-3}$ | 4.134 |
| 1377 | 40960 | 81920 | 168.07 | $-3.55922 \times 10^{-3}$ | 3.961 |
| Forward/backward differencing only, $x = 0$ | | | | | |
| 87 | 160 | 399 | 0.03 | $-6.73472 \times 10^{-3}$ | |
| 173 | 640 | 1296 | 0.22 | $-4.68055 \times 10^{-3}$ | |
| 345 | 2560 | 5135 | 1.68 | $-4.04828 \times 10^{-3}$ | 3.249 |
| 689 | 10240 | 20480 | 13.06 | $-3.79150 \times 10^{-3}$ | 2.462 |
| 1377 | 40960 | 81920 | 103.09 | $-3.67543 \times 10^{-3}$ | 2.213 |

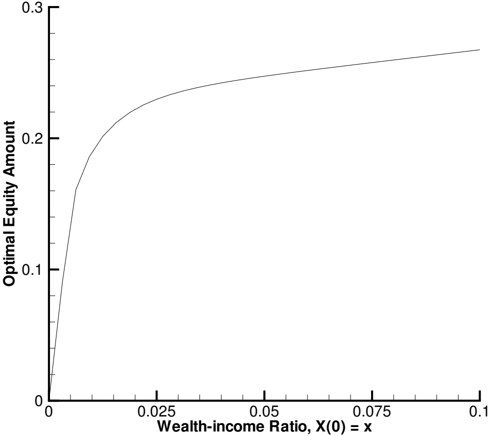
Similar results for Passport Options

# Pension Allocation: Optimal Equity fraction $p$




- Years to retirement $T = 20$
- fraction of salary per year invested 0.10
- $p \to \infty$ as wealth/income ratio $\to 0$

# Pension Allocation: Optimal Equity Amount ($px$)




- $p \to \infty$, $px \to 0$, when $x \to 0$

Recall $x =$ wealth in units of yearly income

# How do we solve the local optimization problem?

Recall that we need to determine (at each node)

$$Q_i^k \in \arg\sup_{Q_i^k \in \hat{Q}} \left\{ \left[ A^{n+1}(Q^k)\hat{V}^k + D^{n+1}(Q^k) \right]_i \right\}$$

For Passport options, Pension Allocation problem, we can form an analytic expression for the local objective function, as a function of the control and differencing method

- Determine the subintervals of control values where central, forward, backward give a positive coefficient discretization (central has priority)
- On each subinterval, objective function is smooth, use standard methods to find maximum.
- Global maximum found by comparing the maxima on each subinterval

But, this approach may not always be possible, if the objective function is complex

# Discretization of the Control

Suppose that $q \in \hat{Q}$, where $\hat{Q} = [q_{\min}, q_{\max}]$, $q_{\min}, q_{\max}$ bounded.

Discretize control, i.e. replace $\hat{Q}$ by $\hat{Q}'$

$$\hat{Q}' = [q_0, q_1, q_2, ..., q_k]$$

Let $max_i(q_{i+1} - q_i) = h$, then we have the following

## Proposition

*If the coefficients of the HJB equation are continuous, bounded functions of the control, then the discretized control problem*

$$V_\tau = \sup_{Q \in \hat{Q}'} \left\{ \mathcal{L}^Q V + d(S, \tau, Q) \right\} \tag{4}$$

*is consistent with the original control problem ($\hat{Q} = [q_{\min}, q_{\max}]$), as $h \to 0$.*

# Discretized Control II

So, if the local objective function is too complex to analytically determine the maximum, we can

- Discretize the control with parameter $h$
- Solve the local optimization problem by linear search
- Let $h \to 0$ as the mesh, timesteps $\to 0$
- This will converge to the viscosity solution

Advantage:
$\hookrightarrow$ Trivial implementation, can be applied to PDE with complex coefficients.

Disadvantage:
$\hookrightarrow$ Increase of computational complexity, effectively we increase the dimensionality of the problem by one.

# Convergence Study: Discretized Control, Pension Allocation

| x-Nodes | p-Nodes | Timesteps | CPU (Sec) | Utility | Ratio |
|---|---|---|---|---|---|
| | | $x = 0$ | | | |
| 173 | 113 | 640 | 1.9 | $-3.65307 \times 10^{-3}$ | |
| 345 | 225 | 2560 | 29.4 | $-3.58083 \times 10^{-3}$ | |
| 689 | 449 | 10240 | 457 | $-3.56358 \times 10^{-3}$ | 4.19 |
| 1377 | 897 | 40960 | 7240 | $-3.55923 \times 10^{-3}$ | 3.97 |
| | | $x = 1.0$ | | | |
| 173 | 113 | 640 | 1.9 | $-4.31662 \times 10^{-4}$ | |
| 345 | 225 | 2560 | 29.4 | $-4.26845 \times 10^{-4}$ | |
| 689 | 449 | 10240 | 457 | $-4.25619 \times 10^{-4}$ | 3.93 |
| 1377 | 897 | 40960 | 7240 | $-4.25306 \times 10^{-4}$ | 3.92 |

Note: we replace admissible set $p \in [0, \infty]$ by $p \in [0, 200]$.

On finest grid, solution same to 5 digits compared to continuous control.

# Lecture 3: Summary

- Discretize first, then maximize (NOT the other way around)
  - Maximize the discrete equations directly $\rightarrow$ ensures monotonicity, constraints on control easily handled
- Positive Coefficient Discretization
  - Guarantees convergence to viscosity solution
  - Guarantees convergence of policy iteration
- Central weighting as much as possible
  - Not usually done
  - Higher convergence rate at little cost
- HJB PDEs with complex coefficients
  - Discretize control, solve optimization problem by linear search
  - Guaranteed to converge, easy to implement
  - But more expensive

# Numerical Methods for Hamilton Jacobi Bellman Equations in Finance
# Lecture 4
# Guaranteed Minimum Withdrawal Benefits (GMWB)

Peter Forsyth

University of Waterloo
Cheriton School of Computer Science

January 18-20

# Outline For Lecture 4

# Motivation

Variable annuity products: sold by insurance companies to retail investors.

These products are guarantees on investments in pension plans.

From a paper we wrote in 2002 (segregated funds are a Canadian version of Variable Annuities)

> *"If one adopts the no-arbitrage perspective...in many cases these contracts appear to be significantly underpriced, in the sense that the current deferred fees being charged are insufficient to establish a dynamic hedge for providing the guarantee. This is particularly true for cases where the underlying asset has relatively high volatility. This finding might raise concerns at institutions writing such contracts."* Windcliff, Forsyth, LeRoux, Vetzal, North American Actuarial J., 6 (2002) 107-125

# What Happened?

As described in a Globe and Mail article (Report on Business, December 2, 2008, "Manulife, in red, raises new equity,"), one of the large Canadian insurance companies, Manulife, posted a large mark-to-market writedown to account for losses associated with these segregated fund guarantees. From the Globe and Mail Streetwise Blog, November 7, 2008

> *"Concerns that the market selloff will translate into massive future losses at Canada's largest insurer sent Manulife shares reeling last month. Those concerns were a result of Manulife's strategy of not fully hedging products such as annuities and segregated funds, which promise investors income no matter what markets do."*

# Why did this happen?

These products contain embedded options which allow the investors many opportunities to optimize the value of the guarantee.

Pricing of these products requires solution of an optimal stochastic control problem (an HJB PDE).

- This was beyond the technical abilities of most insurance companies
- Insurance companies used simplistic models which underestimated the risk involved.
- These models showed that there was no need to hedge these products, (Quote from actuary: ) *"Over any ten year period, the market never goes down."*
- Insurance company executives were able to boast of large (apparent) profits, which then triggered rich bonus payments to traders and executives.

# Retirement Risk Zone

Consider an investor with a retirement account, which is invested in the stock market
Over the long run (before retirement), it does not matter if

- the market first drops by 10% per year over several years and then goes up by 20% per year for several years; or
- the market first goes up by 20% per year and then drops by 10% per year

$$(.9)(.9)...(1.2)(1.2)... = (1.2)(1.2)...(.9)(.9)...$$

# The Retirement Risk Zone II

This is not the case once the investor retires, and begins to make withdrawals from the retirement account

The outcomes will be very different in the cases:

- in the first few years after retirement, the market has losses, and the account is further depleted by withdrawals, followed by some years of good market returns; compared to
- a few years of good market returns, after retirement (including withdrawals), followed by some years of losses

Losses in the early years of retirement can be devastating in the long run! Early bad returns can cause complete depletion of the account.

# A Typical GMWB Example

Investor pays \$100 to an insurance company, which is invested in a risky asset.

Denote amount in risky asset sub-account by $W = 100$.

The investor also has a virtual guarantee account $A = 100$.

Suppose that the contract runs for 10 years, and the guaranteed withdrawal rate is \$10 per year.

# A Typical GMWB Example II

At the end of each year, the investor can choose to withdraw up to \$10 from the account. If $\$\gamma \in [0, 10]$ is withdrawn, then

$$\begin{aligned} W_{new} &= \max(W_{old} - \gamma, 0) \quad ; \quad \textit{Actual investment} \\ A_{new} &= A_{old} - \gamma \quad ; \quad \textit{Virtual account} \end{aligned}$$

This continues for 10 years. At the end of 10 years, the investor can withdraw anything left, i.e. $\max(W_{new}, A_{new})$.

Note: the investor can continue to withdraw cash as long as $A > 0$, *even if* $W = 0$ (recall that $W$ is invested in a risky asset).

# Example: Order of Random Returns

Good Returns at Start

| Time | Return (%) | Balance ($) | Withdrawal ($) |
|---|---|---|---|
| 1 | 41.65 | 141.65 | 10 |
| 2 | 31.12 | 172.62 | 10 |
| 3 | 20.15 | 195.39 | 10 |
| 4 | -30.25 | 129.31 | 10 |
| 5 | 18.05 | 140.85 | 10 |
| 6 | 16.82 | 152.86 | 10 |
| 7 | 20.12 | 171.60 | 10 |
| 8 | 7.44 | 173.62 | 10 |
| 9 | -40.90 | 96.70 | 10 |
| 10 | -7.5 | 80.20 | 10 |
| Total Withdrawal Amount ($) | | | 170.20 |
| Ten year balance if no withdrawal ($) | | | 151.37 |

# Same Random Returns: Different Order

No GMWB: poor returns at start

| Time | Return (%) | Balance ($) | Withdrawal ($) |
|---|---|---|---|
| 1 | -30.25 | 69.75 | 10 |
| 2 | -40.90 | 35.31 | 10 |
| 3 | 16.82 | 29.57 | 10 |
| 4 | 7.44 | 21.03 | 10 |
| 5 | 41.65 | 15.62 | 10 |
| 6 | 20.12 | 6.75 | 6.75 |
| 7 | 31.12 | 0 | 0 |
| 8 | 18.05 | 0 | 0 |
| 9 | 20.15 | 0 | 0 |
| 10 | -7.5 | 0 | 0 |
| Total Withdrawal Amount ($) | | | 56.75 |
| Ten year balance if no withdrawal ($) | | | 151.37 |

# Unlucky Order of Returns: With GMWB

GMWB Protection

| Time | Return (%) | Balance ($) | Withdrawal ($) |
|---|---|---|---|
| 1 | -30.25 | 69.75 | 10 |
| 2 | -40.90 | 35.31 | 10 |
| 3 | 16.82 | 29.57 | 10 |
| 4 | 7.44 | 21.03 | 10 |
| 5 | 41.65 | 15.62 | 10 |
| 6 | 20.12 | 6.75 | 10 |
| 7 | 31.12 | 0 | 10 |
| 8 | 18.05 | 0 | 10 |
| 9 | 20.15 | 0 | 10 |
| 10 | -7.5 | 0 | 10 |
| Total Withdrawal Amount ($) | | | 100 |
| Ten year balance if no withdrawal ($) | | | 151.37 |

# Why is this useful?

The investor can participate in market gains, but still has a guaranteed cash flow, in the case of market losses.

This insulates pensioners from losses in the early years of retirement.

This protection is paid for by deducting a yearly fee $\alpha$ from the amount in the risky account $W$ each year.

The simple form of GMWB described has many variants in practice: Guaranteed Lifetime Withdrawal Benefit (GLWB), ratchet increase of virtual account $A$ if no withdrawals, etc.

We will keep things simple here, and look at the basic GMWB.

Most variable annuities sold in North America have some type of market guarantee.

# Some More Details

The investor can choose to withdraw up to the specified contract *rate* $G_r$ without penalty.

Usually, a penalty ($\kappa > 0$) is charged for withdrawals above $G_r$.

Let $\hat{\gamma}$ be the rate of withdrawal selected by the holder.

Then, the rate of actual cash received by the holder of the GMWB is

$$\hat{f}(\hat{\gamma}) = \begin{cases} \hat{\gamma} & \text{if } 0 \leq \hat{\gamma} \leq G_r, \\ \hat{\gamma} - \kappa(\hat{\gamma} - G_r) & \text{if } \hat{\gamma} > G_r. \end{cases}$$

# Stochastic Process

Let $S$ denote the value of the risky asset, we assume that the risk neutral process followed by $S$ is

$$
\begin{aligned}
dS &= rSdt + \sigma S dZ \\
&\quad r = \text{ risk free rate; } \quad \sigma = \text{ volatility} \\
&\quad dZ = \phi\sqrt{dt} \;\; ; \;\; \phi \sim \mathcal{N}(0,1)
\end{aligned}
$$

The risk neutral process followed by $W$ is then (including withdrawals $dA$).

$$
\begin{aligned}
dW &= (r-\alpha)Wdt + \sigma W dZ + dA, \quad \text{if } W > 0 \\
dW &= 0, \quad \text{if } W = 0 \\
&\quad \alpha = \text{ fee paid for guarantee } \; ; \; A = \text{ guarantee account}
\end{aligned}
$$

# No-arbitrage Value

Let $V(W, A, \tau)$ ($\tau = T - t$, $T$ is contract expiry) be the no-arbitrage value of the GMWB contract (i.e. the cost of hedging).

At contract expiry ($\tau = 0$) we have (payoff = withdrawal)

$$V(W, A, \tau = 0) \quad = \quad \max(W, A(1 - \kappa))$$

It turns out that it is optimal to withdraw at a rate $\hat{\gamma}$

- $\hat{\gamma} \in [0, G_r]$, or
- $\hat{\gamma} = \infty$ (instantaneously withdraw a finite amount)

# Impulse Control

Let

$$\mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r-\alpha)WV_W - rV.$$

Since we have the option of withdrawing at a finite rate at each point in $(W, A, \tau)$, Ito's Lemma and no-arbitrage arguments give

$$V_\tau - \mathcal{L}V - \max_{\hat{\gamma}\in[0,G_r]}\left(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\right) \geq 0$$

Note that $\hat{\gamma}$ is a finite withdrawal *rate*. Withdrawals only allowed if $A > 0$.

# Impulse Control II

We also have the option of withdrawing a finite amount instantaneously (withdrawing at an infinite rate) at each point in $(W, A, \tau)$

$$V(W, A, \tau) - \sup_{\gamma \in (0, A]} \left[ V(\max(W - \gamma, 0), A - \gamma, \tau) + (1 - \kappa)\gamma - c \right] \geq 0 \ .$$

where $\gamma$ is a finite withdrawal *amount*.

$c > 0$ is a fixed cost (which can be very small). This is required to make the Impulse Control problem well-posed.

Note that this equation specifies that any amount in the remaining guarantee account can be withdrawn instantaneously (i.e. $\gamma \in (0, A]$) with a penalty.

# HJB Variational Inequality

Since it must be optimal to either withdraw at a finite rate or withdraw a finite amount at each point, then this can all be written compactly as a Hamilton Jacobi Bellman Variational Inequality

$$\min\Big\{ V_\tau - \mathcal{L}V - \max_{\hat{\gamma}\in[0,G_r]}\big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big),$$

$$V - \sup_{\gamma\in(0,A]}\big[V(\max(W-\gamma,0), A-\gamma, \tau) + (1-\kappa)\gamma - c\big]\Big\}$$

$$= 0$$

# Previous Work

- Milevsky, Salisbury, (2006, Insurance: Mathematics and Economics), pose GMWB pricing problem as a singular control.
- Dai, Kwok, Zong, (Mathematical Finance, 2008), solve singular control formulation using a penalty method.
- Zakamouline (Mathematical Methods Operations Research, 2005) argues that in general one can pose singular control problems as impulse control with negligible difference (infinitesimal fixed cost)
    - Claims that impulse control is more general
- Chen, Forsyth (Numerische Mathematik, 2008), solve impulse control formulation (this lecture)

# Alternative Approach: Discrete Withdrawal Times

Rather than attempt to solve the HJB Impulse Control problem directly, let's replace this problem by a *discrete withdrawal* problem

- Assume that the holder can only withdraw at discrete *withdrawal* times $\tau_1, ..., \tau_N$, with $\tau_{i+1} - \tau_i = \Delta t_w$
- Use dynamic programming idea, work backwards from $t = T(\tau = 0)$, so that $V(W, A, 0) = \max(W, A(1-\kappa))$
- During the interval from $\tau = 0$ to $\tau = \tau_1$ (the first withdrawal time going backwards) we solve

$$V_\tau - \mathcal{L}V = 0 \;\; ; \;\; \mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r-\alpha)WV_W - rV .$$

# Optimum Strategy: Discrete Withdrawals

At $\tau_1$, we assume that the holder withdraws the optimum *amount* $\gamma$

$$V(W, A, \tau_1^+) = \max_{\gamma \in [0,A]} \left[ V\big(\max(W-\gamma, 0), A-\gamma, \tau_1\big) + f(\gamma) \right],$$

where now the cash flow term is

$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \le \gamma \le G, \\ \gamma - \kappa(\gamma - G) - c & \text{if } \gamma > G. \end{cases}$$

$$G = G_r \Delta t_w$$

# Discrete Withdrawals

Then, from $\tau_1^+$ to $\tau_2$, we solve

$$V_\tau - \mathcal{L}V = 0 \quad ; \quad \text{No A dependence in } \mathcal{L}V$$

Then, we determine the optimum withdrawal at $\tau_2^+$, and so on, back down to $\tau = T(t = 0)$ today.

This would appear to be a reasonable approximation to reality.

In fact, most real contracts allow only discrete withdrawals.

# Discrete Withdrawal: A Numerical Scheme

Define nodes in the $W$ direction $[W_0, W_1, \ldots, W_{i_{\max}}]$, and in the A direction $[A_0, A_1, \ldots, A_{j_{\max}}]$.

Let $V_{i,j}^n \simeq V(W_i, A_j, \tau^n)$. $[V^n]_{i,j} = V_{i,j}^n$.

Let $(\mathcal{L}_h V)_{i,j}^n$ be a discrete form of the operator $\mathcal{L}V$.

Away from withdrawal times, we solve

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta\tau} = (\mathcal{L}_h V)_{i,j}^{n+1}$$

# A Numerical Scheme II

At withdrawal time $\tau_n$, we then solve the local optimization problem at each node

$$V_{i,j}^{n+} = \max_{\gamma_{i,j}^n \in [0, A_j]} \left[ \mathcal{I}_{i,j}(\gamma_{i,j}^n) V^n + f\left(\gamma_{i,j}^n\right) \right],$$

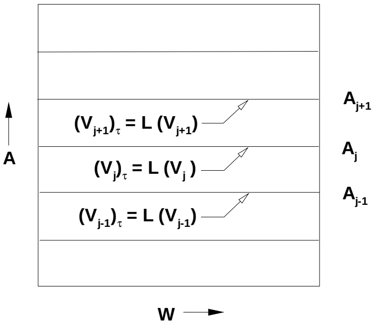
where $\mathcal{I}$ is an interpolation operator

$$\begin{aligned} \mathcal{I}_{i,j}(\gamma) V^n &= V^n(\max(W_i - \gamma, 0), A_j - \gamma) \\ &\quad + \text{ interpolation error} \end{aligned}$$

We use a linear interpolant of $V_{i,j}^n$ to determine the optimum withdrawal at each node $\gamma_{i,j}^n$.

# Numerical Scheme III

- Away from withdrawal times, we solve a decoupled set of 1-d PDEs.
- At withdrawal times, we solve a set of decoupled optimization problems.




Vast majority of CPU time spent solving the local optimization problem at each node:

$$V_{i,j}^{n+} = \max_{\gamma_{i,j}^n \in [0,A_j]} \left[ \mathcal{I}_{i,j}(\gamma_{i,j}^n) V^n + f\left(\gamma_{i,j}^n\right)\right],$$

- This is *embarrassingly parallel*, but requires access to global data.

# Obvious Question

If we let $\Delta\tau_w \rightarrow 0$, does this discrete withdrawal approximation converge to the solution of the Impulse Control HJB equation?

If we allow discrete withdrawals every timestep, then our numerical method is

$$V_{i,j}^{n+1} - \max_{\gamma_{i,j}^n \in [0, A_j]} \left[ \mathcal{I}_{i,j}(\gamma_{i,j}^n) V^n + f\left(\gamma_{i,j}^n\right) \right] - \Delta\tau \left( \mathcal{L}_h V \right)_{i,j}^{n+1} = 0 \ .$$

where the cash flow term $f\left(\gamma_{i,j}^n\right)$ is

$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \le \gamma \le G, \\ \gamma - \kappa(\gamma - G) - c & \text{if } \gamma > G. \end{cases}$$

$$G = G_r \Delta\tau$$

and $\mathcal{I}$ is a linear interpolation operator

# Does it Converge?

We want to show that this scheme converges as $\Delta\tau$, $\Delta A$, $\Delta W$ $\rightarrow 0$ to the viscosity solution of

$$\min\Big\{ V_\tau - \mathcal{L}V - \max_{\hat{\gamma}\in[0,G_r]} \big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big),$$
$$V - \sup_{\gamma\in(0,A]} \big[V(\max(W-\gamma,0), A-\gamma, \tau) + (1-\kappa)\gamma - c\big]\Big\}$$
$$= 0$$

This seems intuitively obvious, but there are some subtle points.

# Monotonicity, Stability and Consistency

## Lemma (Monotonicity and Stability)

*Provided $(\mathcal{L}_h V^{n+1})$ is discretized using a positive coefficient method and linear interpolation is used when solving the local optimization problem at each node, then the scheme is unconditionally $l_\infty$ stable and monotone.*

## Proof.

Straightforward □

## Lemma (Consistency)

*Provided the discrete operator $(\mathcal{L}_h V^{n+1})$ is consistent in the classical sense, and linear interpolation is used to solve the local optimization problem at each node, then the numerical scheme is consistent as defined in (Barles, Souganidis (1991)).*

## Proof.

Not so straightforward ($\liminf, \limsup$ form needed for boundary conditions) □

# Convergence

### Theorem (Strong Comparison Result)

*The GMWB Impulse Control problem satisfies the Strong Comparison Result, i.e. there is a unique, continuous viscosity solution to the Impulse Control Problem. (Seydel, 2008)*

### Theorem (Convergence to the Viscosity Solution)

*The discrete withdrawal numerical method, with withdrawal interval* $\Delta t_w \to 0$ *converges to the unique viscosity solution of the Impulse Control problem.*

### Proof.

This scheme is consistent, stable, and monotone, hence converges to the viscosity solution (Barles, Souganidis (1991)). □

# One scheme for all problems

So, we now have a single scheme which

- Can be used to price GMWB contracts with finite withdrawal intervals (the usual case in real contracts, i.e. withdrawals only allowed once or twice a year)
- We can also price GMWB contracts in the limit as the withdrawal interval $\rightarrow 0$
- Convergence to the Impulse Control problem guaranteed
- No need for different method for these two cases!
- Scheme is simple and intuitive to implement $\rightarrow$ might be actually used by practitioners.

# Local Optimization

Recall that at each node, at each timestep, we have to solve

$$V_{i,j}^{n+1} - \max_{\gamma_{i,j}^n \in [0, A_j]} \left[ V_{\hat{i},\hat{j}}^n + f\left(\gamma_{i,j}^n\right) \right] - \Delta\tau \left(\mathcal{L}_h V\right)_{i,j}^{n+1} = 0 \ .$$

where $V_{\hat{i},\hat{j}}^n$ is a linear interpolation of $V^n\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n\big)$.

Obvious method: use one-d optimization method at each node

But, these methods are not guaranteed to get the global max of the objective function.

We have seen some problems where only a local max was found, $\rightarrow$ convergence to the wrong solution of the PDE.

# Local Optimization

In order to maximize

$$\max_{\gamma_{i,j}^n \in [0,A_j]} \left[ V_{\hat{i},\hat{j}}^n + f\left(\gamma_{i,j}^n\right) \right] - \Delta\tau \left(\mathcal{L}_h V\right)_{i,j}^{n+1}$$

We discretize the control values $\gamma_{i,j}^n$ in $[0, A_j]$, and find the maximum value by evaluating the objective function at all the discrete control values.

Provided the discretization step in $[0, A_j]$, $\rightarrow 0$ as $\Delta\tau \rightarrow 0$, then this is a consistent (hence convergent) method.

# Examples

Recall that the investor pays no extra up-front fee for the guarantee (only the initial premium $w_0$).

The insurance company deducts an annual fee $\alpha$ from the balance in the sub-account $W$.

Problem: let $V(\alpha, W, A, \tau)$ be the value of the GMWB contract, for given yearly guarantee fee $\alpha$.

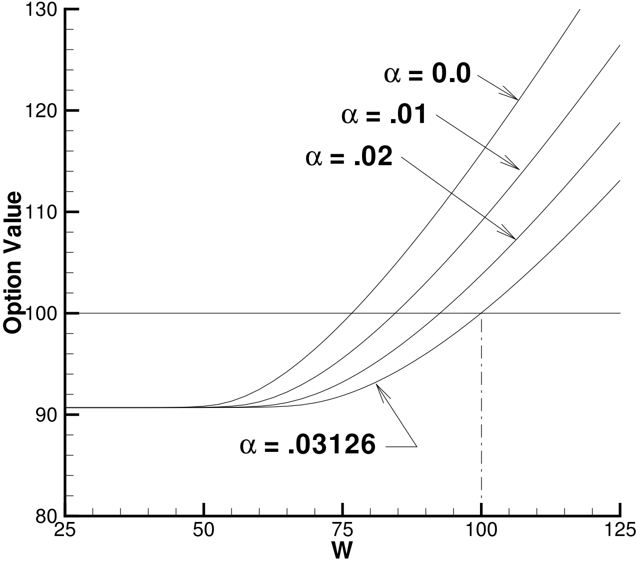
Assume that the investor pays an initial premium $w_0$ at $t = 0$ ($\tau = T$).

Find the no-arbitrage fee $\alpha$ such that $V(\alpha, w_0, w_0, T) = w_0$ (we do this by a Newton iteration).

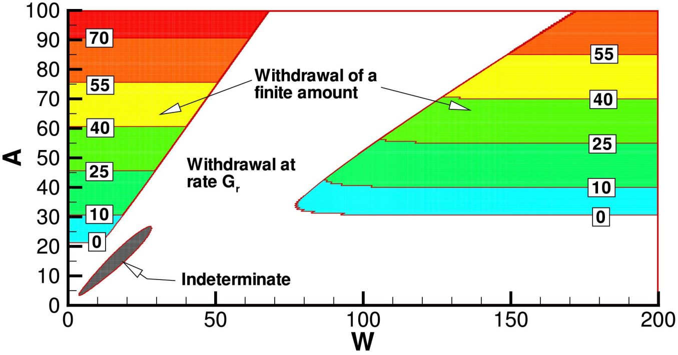
# Data

| Parameter | Value |
|---|---|
| Expiry time $T$ | 10.0 years |
| Interest rate $r$ | .05 |
| Maximum withdrawal rate $G_r$ | 10/year |
| Withdrawal penalty $\kappa$ | .10 |
| Volatility $\sigma$ | .30 |
| Initial Lump-sum premium $w_0$ | 100 |
| Initial guarantee account balance | 100 |
| Initial sub-account value | 100 |
| Continuous Withdrawal | Yes |

# The No-arbitrage Fee ($t = 0$, $A = 100$)



# Optimal Withdrawal Strategy




$t = 0$, fair fee charged for $w_0 = 100$. Indeterminate region: appears to converge to optimal withdrawal rate $\hat{\gamma} = 0$ ?

# Indeterminate Region?

In the continuous withdrawal region, we have

$$V_\tau = \mathcal{L}V + \max_{\hat{\gamma}\in[0,G_r]} \left[\hat{\gamma}(1 - V_W - V_A)\right] \quad (1)$$

In the *indeterminate* region, we observe that (mesh, timestep $\to 0$)

$$\Big[1 - V_W(W_i, A_j, \Delta\tau) - V_A(W_i, A_j, \Delta\tau)\Big]_{i,j} \to 0^-$$

If $(1 - V_W - V_A) \to 0$, then any withdrawal rate in $[0, G_r]$ is optimal.

Control may not be unique (but value is unique).

# No-arbitrage Fee

- $\sigma = .15 \rightarrow \alpha = .007$ (70 bps)
- $\sigma = .20 \rightarrow \alpha = .014$ (140 bps)
- $\sigma = .30 \rightarrow \alpha = .031$ (310 bps)
- Current volatility of $S\&P \simeq .25$
- Typical fees charged: $\alpha = .005$ (50 bps) too low for current market conditions.
- Insurance companies seem to be charging fees based on marketing considerations, not hedging costs.
- Fee should be even higher if other (typical) contract options considered

# Other Issues

Can easily use the same method if we assume underlying process is a jump diffusion (Chen, Forsyth, SIAM J. Scientific Computing (2007)).

Effect of discrete withdrawals, volatility, non-optimal withdrawals, etc. (Chen, Vetzal, Forsyth, Insurance: Mathematics and Economics (2008)).

A penalty method for singular control formulation of a GMWB (Dai et al, Mathematical Finance (2008)), (Huang, Forsyth, Working paper (2009)).

Impulse control for a Guaranteed Minimum Death Benefit (Belanger, Forsyth, Labahn, Applied Mathematical Finance (forthcoming)).

# Summary

- We have developed a single scheme which can be used to price GMWB contracts with finite withdrawal intervals, and the limiting case of infinitesimal withdrawal intervals
- In the case of infinitesimal withdrawal intervals, we have proven convergence to the viscosity solution of the Impulse Control problem
- For an infinitesimal fixed cost, solutions agree with a singular control formulation
- Insurance companies seem to be charging fees which are too low to cover hedging costs. Another subprime problem?

# Numerical Methods for Hamilton Jacobi Bellman Equations in Finance
# Lecture 5
# Gas Storage

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January 18-20

# Outline For Lecture 5

Motivation

HJB Equation

Semi-Lagrangian Discretization

Convergence

Numerical Examples

Summary: Gas Storage

Summary: Lectures

Open Problems

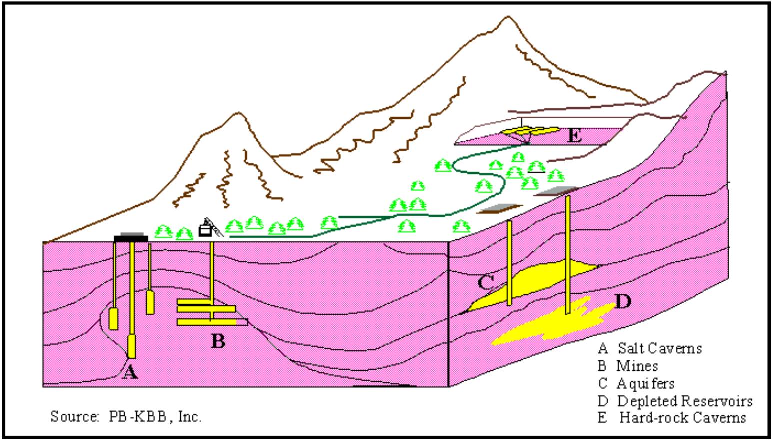
# Introduction

Natural gas prices show seasonality effects (higher in winter, lower in summer).

- Natural gas storage facilities are constructed to provide a cushion of available gas
- Gas is released during periods of high demand
- Gas is stored in seasons of low demand
- Underground caverns are used for long term storage
- Objective: determine the no-arbitrage value of leasing a storage facility for a fixed term
- A by-product of this valuation is the optimal operating strategy (i.e. when to inject, produce or do nothing)
- This injection strategy is the control in this case

# An Underground Storage Facility

**Figure 1. Types of Underground Natural Gas Storage Facilities**



# Variable Definitions

$$
\begin{aligned}
P &= \text{Current Spot Price of Gas} \\
I &= \text{Current amount of working gas inventory} \\
& \quad I \in [0, I_{\max}] \\
c &= \text{Rate of gas production from storage} \\
& \quad c \in C(I) = [c_{\min}(I), c_{\max}(I)] \\
& \quad c > 0 \rightarrow \text{ production} \\
& \quad c < 0 \rightarrow \text{ injection}
\end{aligned}
$$

Gas inventory satisfies

$$
\begin{aligned}
\frac{dI}{dt} &= -(c + a(I, c)) \\
& \quad a(I, c) = \text{ production/injection losses}
\end{aligned}
$$

# More Definitions

Max/min production rates are nonlinear functions of inventory $I$, e.g.

$$c_{\max}(I) \;=\; k_1\sqrt{I} \;\; ; \;\; k_1 = \text{ const.} \quad \text{(ideal gas law)}$$

Revenue obtained from selling gas

$$\begin{aligned}\text{Revenue} \;&=\; (c - b(c))\,P \\ & b(c) = \text{ gas loss during transportation} \\ & \qquad \text{Revenue } > 0 \rightarrow \text{ gas released and sold} \\ & \qquad \text{Revenue } < 0 \rightarrow \text{ gas purchased and stored}\end{aligned}$$

Let $V(P, I, \tau = T - t)$ be the value of leasing the facility for $T$ years.

# Risk neutral stochastic process for spot price

$$
\begin{aligned}
dP &= \alpha(K(t) - P)dt + \sigma P dZ \\
K(t) &= K_0 + \beta_{SA} \sin(4\pi(t - t_{SA})),
\end{aligned}
$$

where

- $\alpha > 0$ is the mean-reverting rate,
- $K(t) \geq 0$ is the long-term equilibrium price that incorporates seasonality,
- $\sigma$ is the volatility,
- $dZ$ is an increment of the standard Gauss-Wiener process,
- $K_0 \geq 0$ is the equilibrium price without seasonality effect,
- $\beta_{SA}$ is the semiannual seasonality parameter,
- $t_{SA}$ is the seasonality centering parameters, representing the time of semiannual peak of equilibrium price in summer and winter.

# Is this a good model for gas prices?

Actually, this simple model does not fit gas prices very well.

A better model

- Regime switching
- Stochastic process switches between a number of regimes
- E.g. a low demand regime where the gas price is mean reverting towards a low price
- A high demand regime where the gas price is mean reverting to a high price, with a high volatility
- A simple two regime model can fit gas forward curves reasonably well with a small number of parameters (Chen and Forsyth, Quantitative Finance (forthcoming))
- However, basic numerical techniques can be illustrated with simple stochastic model
- These methods can be easily generalized to regime switching, other jump processes, etc.

# HJB PDE

The value of leasing the facility is the risk neutral discounted cash flows from buying/selling gas in $[0, T]$

$$\begin{aligned} V(P, I, \tau) &= \sup_{c(s)\in C(I(s))} E^{\mathcal{Q}}\bigg[\int_t^T e^{-r(s-t)}[c(s) - b(c(s))]P(s)\, ds \\ &\qquad + e^{-r(T-t)} V(P(T), I(T), T)\bigg] \end{aligned}$$

Following the usual steps we get the HJB PDE for $V(P, I, \tau)$

$$\begin{aligned} V_\tau &= \mathcal{L}V + \max_{c\in C(I)} \Big\{(c - b(c))P - (c + a(c))V_I\Big\} \\ \mathcal{L}V &\equiv \frac{1}{2}(\sigma P)^2 V_{PP} + \alpha(K(t) - P)V_P - rV \end{aligned}$$

# Value at $t = T$

We use the following terminal value

$$V(P, I, \tau = 0) \;\; = \;\; -2P \max(I^* - I, 0)$$

The financial meaning behind this value at $t = T$ is as follows

- The gas storage is leased to you with an initial inventory of gas $I^*$
- You must return the facility to the owner with $I \geq I^*$, otherwise severe penalties are charged (double prevailing spot price)
- Familiar concept, this is what happens when you return a rental car.

# Semi-Lagrangian form

Let

$$\frac{DV}{D\tau} = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial I}(c + a(c))$$

be the Lagrangian derivative along the trajectory

$$\frac{dI}{d\tau} = c + a(c)\ .$$

We can then write the HJB PDE as

$$\min_{c \in C(I)} \left\{ \frac{DV}{D\tau} - (c - a(c))P - \mathcal{L}V \right\} = 0.$$

# Discretization

Define a set of nodes

$$[P_1, P_2, ..., P_{imax}] \;\; ; \;\; [I_1, I_2, ..., I_{jmax}]$$

and a set of discrete times $[\tau^0, \tau^1, ..., \tau^N]$.

Let $V_{i,j}^n$ be a discrete approximation to $V(P_i, I_j, \tau^n)$, with

$$[V^n]_{i,j} = V_{i,j}^n$$

Recall that

$$\mathcal{L}V \equiv \frac{1}{2}(\sigma P)^2 V_{PP} + \alpha(K(t) - P)V_P - rV$$

# Discretization: Operator $\mathcal{L}$

Then, we discretize $\mathcal{L}$ in the $P$ direction using forward, backward, central differencing:

$$(\mathcal{L}_h V^n)_{i,j} = \alpha_i^n V_{i-1,j}^n + \beta_i^n V_{i+1,j}^n - (\alpha_i^n + \beta_i^n + r) V_{i,j}^n,$$

As usual, we require that the positive coefficient condition holds

$$\alpha_i^n \geq 0 \quad ; \quad \beta_i^n \geq 0 \quad i = 1, \ldots, i_{\text{max}} \; ; \; j = 1, \ldots, j_{\text{max}} \; ; \; n = 1, \ldots, N.$$

# Semi-Lagrangian Discretization

Recall the Lagrangian trajectory equation

$$\frac{dI}{d\tau} = c + a(c) \ . \qquad (1)$$

Let $c_{i,j}^{n+1}$ be the optimal control at node $(P_i, I_j, \tau^{n+1})$.

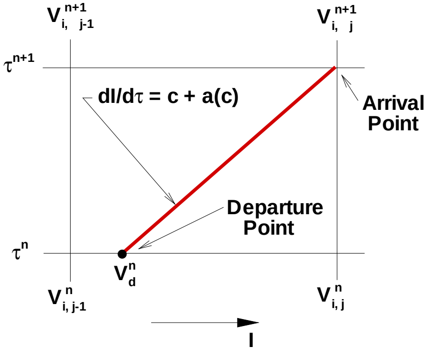
Using a first order method, we integrate (1) backwards from $\tau^{n+1}$ to $\tau^n$, starting from node $(P_i, I_j)$.

$$\begin{aligned} I^n_{j(i,n+1)} &= I_j - \Delta\tau^n(c_{i,j}^{n+1} + a(c_{i,j}^{n+1})) \\ & \Delta\tau^n = \tau^{n+1} - \tau^n \end{aligned}$$

The Lagrangian derivative is approximated (to first order) by

$$\left(\frac{DV}{D\tau}\right)_{i,j}^{n+1} \simeq \frac{V_{i,j}^{n+1} - V(P_i, I^n_{j(i,n+1)}, \tau^n)}{\Delta\tau^n}$$

# Semi-Lagrangian Derivative




$$\left(\frac{DV}{D\tau}\right)_{ij}^{n+1} \simeq \frac{V_{i,j}^{n+1} - V_d^n}{\Delta\tau}$$

$$V_d^n = V(P_i, I_d, \tau^n)$$
$$I_d = I_j - \Delta\tau^n(c_{i,j}^{n+1} + a(c_{i,j}^{n+1}))$$

$$I_d = I_{j(i,n+1)}^n$$

# Interpolation

Denote the approximate value of $V(P_i, I_{j(i,n+1)}, \tau^n)$ by

$$V(P_i, I_{j(i,n+1)}, \tau^n) \simeq V^n_{i,j(i,n+1)}$$

Since $I_{j(i,n+1)}$ will not in general coincide with a grid point $I_j$, we use an interpolation operator $\Phi^{n+1}$ to determine $V^n_{i,j(i,n+1)}$,

$$[\Phi V^n]_{i,j} = V^n_{i,j(i,n+1)} + \text{ interpolation error}$$

So that the final form for the Lagrangian derivative is

$$\left(\frac{DV}{D\tau}\right)^{n+1}_{i,j} \simeq \frac{V^{n+1}_{i,j} - [\Phi^{n+1} V^n]_{i,j}}{\Delta\tau^n}$$

# Admissible Controls

Note that

$$c_{i,j}^{n+1} \in C(I_j)$$

However, for any finite size timestep, it may be the case that

$$I_{j(i,n+1)}^{n} \quad = \quad I_j - \Delta\tau^n (c_{i,j}^{n+1} + a(c_{i,j}^{n+1}))$$

is outside the boundaries $[0, I_{\max}]$. This could happen near the boundaries.

Let

$$C_{i,j}^{n+1} \subseteq C(I_j)$$

denote the set of values of $c_{i,j}^{n+1}$ such that the resulting $\mathcal{I}_{j(i,n+1)}^{n} \in [0, I_{\max}]$

# Final Discretization

The final form for the discretization is then

$$V_{i,j}^{n+1} = \max_{c_{i,j}^{n+1} \in C_{i,j}^{n+1}} \left\{ [\Phi^{n+1} V^n]_{i,j} + \Delta\tau^n \big( c_{i,j}^{n+1} - a\big(c_{i,j}^{n+1}\big)\big) P_i \right\} + \Delta\tau^n (\mathcal{L}_h V)_{i,j}^{n+1}, \quad (2)$$

Define:

$$\Gamma_{i,j}^{n} = \max_{c_{i,j}^{n+1} \in C_{i,j}^{n+1}} \left\{ [\Phi^{n+1} V^n]_{i,j} + \Delta\tau^n \big( c_{i,j}^{n+1} - a\big(c_{i,j}^{n+1}\big)\big) P_i \right\} \quad (3)$$

so that (2) becomes

$$V_{i,j}^{n+1} = \Gamma_{i,j}^{n} + \Delta\tau^n (\mathcal{L}_h V)_{i,j}^{n+1} \quad (4)$$

This means that the discretization can be broken down into an interpolation/optimization step (3) and a time advance step (4).

# Viscosity Solution

## Assumption (Strong Comparison)

*We make the assumption that the HJB PDE (with the boundary conditions) satisfies the strong comparison property, i.e. a unique, continuous viscosity solution exists.*

## Remark

*The main problem is that the PDE is degenerate in the $I$ direction (i.e. no diffusion in the $I$ direction). However, the characteristics are outgoing (or zero) on these boundaries, independent of the control. Hence, no boundary data is required at these nodes.*
*The gas storage PDE almost meets almost all assumptions required to prove strong comparison in (Barles, Rouy: Comm. Partial Differential Equations (1998)), but not quite.*

# Convergence of the Scheme

## Lemma (Stability)

*Provided that a positive coefficient method is used to discretize the operator $\mathcal{L}V$, and linear interpolation is used for $\Phi^{n+1}$, then the scheme (2) is unconditionally $l_\infty$ stable.*

## Proof.

Straightforward maximum analysis. □

## Lemma (Monotonicity)

*Provided that a positive coefficient method is used to discretize the operator $\mathcal{L}V$, and linear interpolation is used for $\Phi^{n+1}$, then the scheme (2) is unconditionally monotone.*

## Proof.

Positive coefficient $\longrightarrow$ monotonicity. □

## Lemma (Consistency)

*The discretization scheme (2) is consistent (in the viscosity sense) with the HJB PDE and boundary conditions.*

### Proof.

The main problem is near the boundaries, since the numerical admissible set $c_{i,j}^{n+1} \in C_{i,j}^{n+1} \subseteq C(I_j)$. does not agree with the actual admissible set at that node.
However, the relaxed form of viscosity consistency comes to our rescue here, and everything works. □

## Theorem (Convergence)

*Provided all the conditions required for the above Lemmas are satisfied, then scheme (2) converges to the viscosity solution.*

### Proof.

Consistent, stable, monotone $\rightarrow$ convergence. □

# Computational Details

From (Thompson, Rasmussen, Davison (2008)), we know that

- Exact optimal controls are of the bang-bang type
- i.e. the optimal strategy is one of:
  - Produce at rate $c_{\max}(I)$
  - Inject at rate $c_{\min}(I)$
  - Do nothing

Solution of local optimization problem at each node

- *bang-bang* method $\rightarrow$ search only for optimal controls within the finite set of possible controls in the exact solution
- *no bang-bang* method $\rightarrow$ solve the discrete optimization problem

Both methods must converge to the bang-bang control solution

# Computational Details II

Assume

$$\frac{\Delta \tau_{max}}{C_1} = \frac{\Delta P_{max}}{C_2} = \frac{\Delta I_{max}}{C_3} = h$$

Since the maximum of a piecewise linear interpolant is at the nodes, *no bang-bang* examines a fixed number of nodes independent of $h$.

- Both bang-bang and no bang-bang methods have the same complexity as $h \to 0$.

Why bother with *no bang-bang*? We will see later.

Note that the time advance step

$$V_{i,j}^{n+1} = \Gamma_{i,j}^{n} + \Delta \tau^{n} (\mathcal{L}_h V)_{i,j}^{n+1}$$

results in a set of decoupled $1 - d$ problems.

Hence the complexity of the semi-Lagrangian method is the same as an explicit method, yet is unconditionally stable.

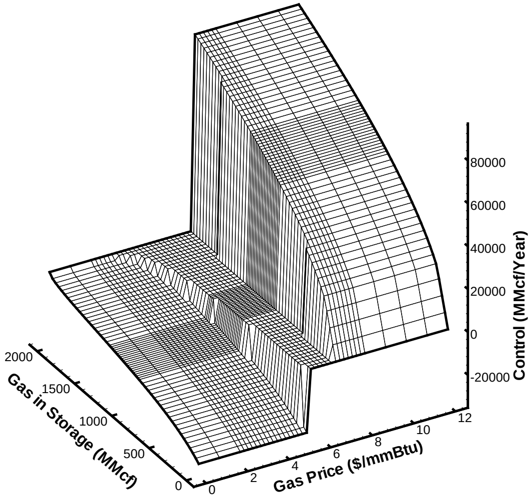
# Numerical Example

Case 1: No seasonality

$$dP = 2.38(6 - P)dt + 0.59PdZ.$$

| Parameter | Value |
|---|---|
| $r$ | 0.1 |
| T | 3 years |
| $I_{\text{max}}$ | 2000 MMcf |

# Control Surface: $t = 0$



# Case 2: Seasonality

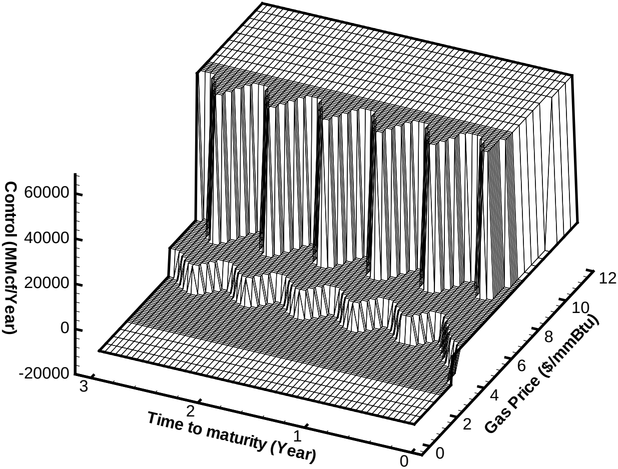



Figure: Control Surface: fixed $I = I_{\max}/2$

# Adding Jumps

Assume:

$$dP = [\alpha(K(t) - P) - \lambda\kappa P]dt + \sigma P dZ + (\eta - 1)P dq$$

where

- $dq$ is the independent Poisson process = $\begin{cases} 0 & \text{with probability } 1 - \lambda dt, \\ 1 & \text{with probability } \lambda dt, \end{cases}$
- $\lambda$ is the jump intensity
- When $dq = 1$, price jumps from $P$ to $P\eta$.
- We assume that $\eta$ follows a probability density function $g(\eta)$, which is log-normal.
- $\kappa$ is $E[\eta - 1]$, where $E[\,\cdot\,]$ is the expectation operator.

# HJB PIDE

$$
\begin{aligned}
V_\tau = & \frac{1}{2}\sigma^2 P^2 V_{PP} + [\alpha(K(t) - P) - \lambda\kappa P]V_P \\
& + \max_{c \in C(I)} \{(c - a(c))P - (c + a(c))V_I\} \\
& - rV + \left(\lambda \int_0^\infty V(P\eta)g(\eta)d\eta - \lambda V\right) .
\end{aligned} \tag{5}
$$

- No control in the integral term
- Use Semi-Lagrangian method, and PIDE techniques in (d'Halluin, Forsyth, Vetzal: IMA J. Numerical Analysis (2005)).

# Bang-Bang vs. No Bang-Bang

| $P$ grid nodes | $I$ grid nodes | No. of timesteps | Bang-bang method | | No-bang-bang method | |
|---|---|---|---|---|---|---|
| | | | Value | Ratio | Value | Ratio |
| 79 | 61 | 500 | 7995143 | n.a. | 8070698 | n.a. |
| 157 | 121 | 1000 | 7962386 | n.a. | 7999775 | n.a. |
| 313 | 241 | 2000 | 7951062 | 2.89 | 7971737 | 2.53 |
| 625 | 481 | 4000 | 7951032 | 377 | 7961554 | 2.75 |
| 1249 | 961 | 8000 | 7951976 | -0.03 | 7957509 | 2.52 |

Table: Value of a natural gas storage facility at $P = 6$ \$/mmBtu and $I = 1000$ MMcf, $t = 0$. Mean-reverting plus jumps. Ratio is the ratio of successive changes as the mesh is refined.

- No bang-bang $\rightarrow$ optimize discrete equations $\rightarrow$ smoother convergence.

# Regime Switching

- State of system characterized by a finite number of regimes
- Each regime has its own mean-reverting drift, volatility
- Poisson switching process between regimes
- Reasonable fit to market data
- Control surface now a function of what regime we are in
- (Chen, Forsyth: Quantitative Finance (forthcoming))

# Control Surface: fixed $I = I_{max}/2$

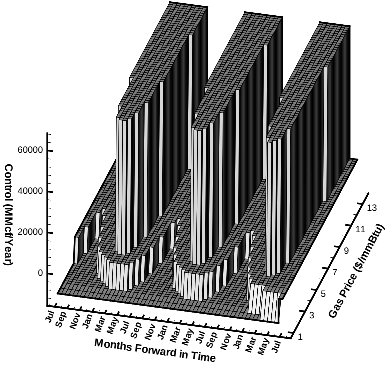



Figure: Control Surface: regime 0

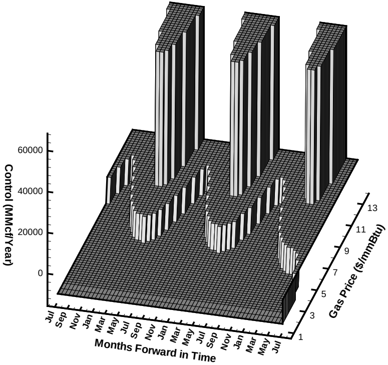



Figure: Control Surface: regime 1

- Regime 0: low mean reversion level .
- Regime 1: high mean reversion level.

# Summary: Gas Storage

- Semi-Lagrangian method, unconditionally stable, same work/timestep as an explicit method
- Guaranteed to converge to the viscosity solution
- Easily generalized to other stochastic processes: jumps, regime switching
- Solve discrete optimization problem $\rightarrow$ smoother convergence than using knowledge of exact solution (bang-bang controls)

# Summary of Lectures

- Seemingly reasonable discretizations may not converge to the viscosity solution
- Must ensure that discretization is stable, consistent, monotone $\rightarrow$ positive coefficient discretization
- Fully implicit timestepping, use central differencing as much as possible (but still positive coefficient)
  $\hookrightarrow$ Policy iteration guaranteed to converge
  $\hookrightarrow$ Better accuracy at small additional cost.
- Combined bounded control, impulse control, easily handled
- Singular control $\rightarrow$ penalty method
- For problems where control appears in first order terms $\rightarrow$ semi-Lagrangian method is effective

# Open Problems

- Impulse control, local optimization problem
  $\hookrightarrow$ $1-d$ optimization unreliable
  $\hookrightarrow$ Discretize control, use linear search $\rightarrow$ expensive
  $\hookrightarrow$ Better method?
- For multi-factor stochastic process, if non-zero correlation, this generates a cross derivative term in the PDE
  $\hookrightarrow$ How do we generate a positive coefficient discretization?
  $\hookrightarrow$ Current methods not completely satisfactory (equally spaced grids)
- Jump processes: control in jump integral term?
  $\hookrightarrow$ Possible solution: discretize control, use piecewise constant timestepping $\rightarrow$ expensive