# Sample-Path Large Deviations in Credit Risk

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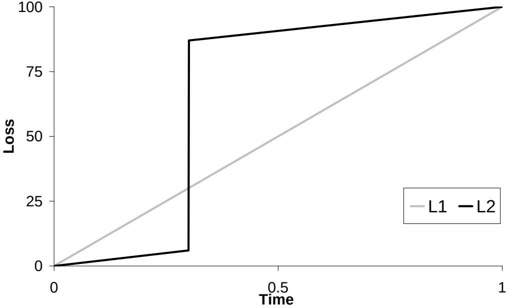
# Outline

# Motivation

- Goal is to characterize loss distribution in large portfolio
- Current techniques focus on single point in time
- Path dependent measures capture more of characteristics
- Consider events: $\{\exists t : L(t) > \zeta(t)\}$ or $\{\forall t : L(t) < \xi(t)\}$



# Model and Notation

- Model loss in portfolio consisting of $n$ obligors
- Companies identically and independently distributed
- Separately model the default time $\tau$ and loss given default $U$
- Assume $\tau$ and $U$ are independent

# Model and Notation (2)

- Loss process given by

$$L_n(t) = \sum_{i=1}^{n} U_i Z_i(t)$$

$$Z_i(t) = \mathbb{I}_{\{\tau_i \leq t\}}$$

Where $U_i \sim U$ and $\tau_i \sim \tau$

- Consider the loss process on time grid $\{1, 2, \ldots, N\}$.
- The distribution of the default times given by

$$p_i := \mathbb{P}(\tau = i)$$

$$F_i := \sum_{j=1}^{i} p_i$$

# Large Deviation Principle

- Let $(\mathcal{X}, d)$ be a metric space
- Let $\{\mu_n\}$ be a sequence of measures on Borel sets of $\mathcal{X}$.
- Study behavior of $\{\mu_n\}$ as $n \to \infty$.
- Large Deviation Principle states exponential upper and lower bounds

**Definition (Rate Function)**

*A Rate Function is a lower semicontinuous mapping $I : \mathcal{X} \to [0, \infty]$. This means that for all $\alpha \in [0, \infty)$ the set $\{x |\ I(x) \leq \alpha\}$ is a closed subset of $\mathcal{X}$.*

# Large Deviation Principle(2)

**Definition (Large Deviation Principle)**

*We say that $\{\mu_n\}$ satisfies the Large Deviation Principle (LDP) with rate function $I(\cdot)$ if*

(i) *(Upper bound) for all closed $F \subseteq \mathcal{X}$*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(F) \leq -\inf_{x\in F} I(x)$$

(ii) *(Lower bound) for all open $G \subseteq \mathcal{X}$*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(G) \geq -\inf_{x\in G} I(x)$$

# Large Deviation Principle(3)

**Definition (Large Deviation Principle (continued))**

*We say that a family of random variables $X = \{X_n\}$, with values in $\mathcal{X}$, satisfies a large deviation principle with rate function $I_X(\cdot)$ iff the laws $\{\mu_n^X\}$ satisfy a large deviation principle with rate function $I_X(\cdot)$.*

**Definition (Fenchel-Legendre Transform)**

*Let $X$ be a random variable. The Fenchel-Legendre Transform is given by*

$$\Lambda_X^\star(x) = \sup_\theta \left(\theta x - \Lambda_X(\theta)\right)$$

*where $\Lambda_X$ is the logarithmic moment generating function of $X$*

$$\Lambda_X(\theta) = \log\left(\mathbb{E}e^{\theta X}\right)$$

# Cramér's Theorem

**Theorem (Cramér)**

*Let $\{X_i\}$ be i.i.d. sequence of random variables and let $\mu_n$ be the law of the average $S_n = \sum_{i=1}^{n} X_i/n$. Then $\{\mu_n\}$ satisfies an LDP with rate function $\Lambda_X^\star(\cdot)$.*

**Example (Loss Process)**

For any $T > 0$, the average loss process $L_n(T)/n$ satisfies a Large Deviation Principle, where the rate function is given by the Legendre-Fenchel transform of the variable $U\,Z(T) = U\mathbb{I}_{\{\tau \le T\}}$, so

$$I(x) = \Lambda_{UZ(T)}^\star(x)$$

# Additional Notation

- Finite time grid $T_N = \{t_1 < t_2 < \cdots < t_N\}$, or for simplicity $T_N = \{1, 2, \ldots, N\}$
- Space of all nonnegative and nondecreasing functions on $T_N$:

$$\mathcal{S} = \left\{ f : T_N \to \mathbb{R}^+ | \; 0 \leq f_i \leq f_{i+1}, \text{ for } i < N \right\}$$

- Topology on induced by supremum norm $||f|| = \max_i |f_i|$
- Space of all probability measures on $T_N$

$$\Phi = \left\{ \varphi \in \mathbb{R}^N | \; \sum_{i=1}^{N} \varphi_i = 1, \; \varphi_i \geq 0, \; i \leq N \right\}$$

# Sample-Path Large Deviation Principle

**Theorem**

*Let $\Lambda_U(\theta) < \infty$ for all $\theta$. Then the path of the average loss process $L_n(\cdot)/n$, on the points $\{1, 2, \ldots, N\}$, satisfies a Large Deviation Principle with rate function $I_{U,p}$. Here, for $x \in \mathcal{S}$, $I_{U,p}$ is given by*

$$I_{U,p} = \inf_{\varphi \in \Phi} \sum_{i=1}^{N} \varphi_i \left( \log\left(\frac{\varphi_i}{p_i}\right) + \Lambda_U^\star\left(\frac{\Delta x_i}{\varphi_i}\right)\right)$$

*with $\Delta x_i = x_i - x_{i-1}$ and $x_0 = 0$.*

Remarks:

- Decompose influence of default times and losses given default
- Optimizing $\varphi$ can be interpreted as most like loss distribution, given path of $L_n(\cdot)/n$ is close to $x$

# Example 1

## Example

Let the loss amount $U$ have finite support on $[0, u]$. Then $\Lambda_U(\theta) < \infty$ for all $\theta$ as

$$\Lambda_U(\theta) = \log\left(\mathbb{E}e^{\theta U}\right) \leq \theta u < \infty$$

So the loss process $L_n(\cdot)/n$ satisfies the sample-path LDP.

In practice loss amounts are finite, thus any realistic model for the loss distribution satisfies the sample path LDP.

# Example 2

## Example

Assume loss amount $U$ is measured in units $u > 0$, e.g. $u, 2u, \ldots$.
Assume that $U$ has Poisson-like distribution with parameter $\lambda$, such that for $i = 1, 2, \ldots$

$$\mathbb{P}\left(U = (i+1)u\right) = e^{-\lambda} \frac{\lambda^i}{i!}$$

Then then $\Lambda_U$ is given by

$$\Lambda_U(\theta) = \theta u + \lambda \left(e^{\theta u} - 1\right)$$

which is finite for all $\theta$, showing that the sample-path LDP is satisfied for a distribution with infinite support.

# Remarks and Extensions

- Sample-path LDP is valid for wide range of distributions
- Assumptions not realistic, e.g. independent and identical distributions
- In practice defaults clearly not independent
- Different types of obligors can be distinguished
- Finite grid might be too restrictive

# Dependent Defaults

- Relax assumption that obligors are independent
- Use so-called (factor) copula approach
- Conditional on a factor $Y$, the default times and loss amounts are independent
- Apply theorem conditional on realization of $Y$, yielding conditional decay rate $r_y$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{n}L_n(\cdot) \in A \middle| Y = y\right) = r_y$$

- When $Y$ has finite outcomes, say in $\mathcal{Y}$ , the unconditional decay rate $r$ is given as $r = \max\{r_y | y \in \mathcal{Y}\}$

# Different Types

- Relax assumption that obligors are identically distributed
- Assume that there are $m$ different classes, default ratings for example
- Each class makes up fraction $a_i$ of portfolio
- Split loss process $L_n$ into $m$ sub-loss processes, and condition on realizations, which gives rate function

$$I_{U,p,m}(x) = \inf_{\varphi \in \Phi^m} \inf_{v \in V_x} \sum_{j=1}^{m} \sum_{i=1}^{N} a_i \varphi_i^j \left( \log\left(\frac{\varphi_i^j}{p_i^j}\right) + \Lambda_U^\star\left(\frac{v_i^j}{a_i \varphi_i^j}\right)\right)$$

$$V_x = \left\{ v \in \mathbb{R}_+^{m \times N} \middle| \sum_{j=1}^{m} v_i^j = \Delta x_i \text{ for all } i \leq N \right\}$$

$$\Phi^m = \Phi \times \ldots \times \Phi, \ (m \text{ times})$$

# Extend Finite Grid

- Extend current grid $\{1, 2, \ldots, N\}$ to $\mathbb{N}$
- Expected rate function $I_{U,p,\infty}$:

$$I_{U,p,\infty}(x) = \inf_{\varphi \in \Phi_\infty} \sum_{i=1}^{\infty} \varphi_i \left( \log\left(\frac{\varphi_i}{p_i}\right) + \Lambda_U^\star\left(\frac{\Delta x_i}{\varphi_i}\right)\right)$$

- Extend from grid $\{1, 2, \ldots, N\}$ to interval $[0, N]$
- Expected rate function $I_{U,p,[0,N]}$:

$$I_{U,p,[0,N]}(x) := \inf_{\varphi \in \mathcal{M}} \int_0^N \varphi(t) \left( \log\left(\frac{\varphi(t)}{p(t)}\right) + \Lambda_U^\star\left(\frac{x'(t)}{\varphi(t)}\right)\right) \mathrm{d}t$$

# Exact Asymptotic Results

- The sample-path LDP provides bounds for the exponential decay rate
- It does not provide exact expression for $\mathbb{P}\left(\frac{1}{n}L_n(\cdot) \in A\right)$
- For certain events it is possible to obtain exact expression, resulting in expressions like

$$\mathbb{P}\left(\frac{1}{n}L_n(\cdot) \in A\right) = \frac{C\, e^{-I_L\, n}}{\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right)$$

For certain constants $C$ and $I_L$

# Bahadur-Rao Theorem

- The exact asymptotic results depend on Bahadur-Rao theorem

**Theorem (Bahadur-Rao)**

*Let $X_i$ be an i.i.d. real valued sequence of random variables. Then we have*

$$
\begin{aligned}
\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq q\right) &= \frac{e^{-n\Lambda_X^\star(q)} C_{X,q}}{\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right) \\
C_{X,q} &= \frac{1}{\sigma\sqrt{2\pi\Lambda_X''(\sigma)}} \\
\Lambda_X'(\sigma) &= q
\end{aligned}
$$

# Crossing a Barrier

- Work on the infinite time grid $T = \mathbb{N}$
- Consider the event that at some point in time $t$ the loss is above some threshold $\zeta(t)$

$$\left\{ \exists t \in T \,\middle|\, \frac{1}{n} L_n(t) > \zeta(t) \right\}$$

- Need that $\zeta(t) > \mathbb{E}\left[U\right] \mathbb{P}\left(\tau \leq t\right)$
- Determine loss path quantiles

# Crossing a Barrier(2)

**Theorem**

*Assume that there exists unique $t^\star \in T$ such that*

$$I_{UZ}(t^\star) = \min_{t\in T} I_{UZ}(t),$$

*and assume that*

$$\liminf_{t\to\infty} \frac{I_{UZ}(t)}{\log t} > 0,$$

*where* $I_{UZ}(t) = \sup_\theta \left\{\theta\zeta(t) - \Lambda_{UZ(t)}(\theta)\right\} = \Lambda^\star_{UZ(t)}(\zeta(t))$. *Then*

$$\mathbb{P}\left(\exists t \in T \ s.t. \ \frac{1}{n} L_n(t) > \zeta(t)\right) = \frac{e^{-nI_{UZ}(t^\star)}C^\star}{\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right)$$

*Where $\sigma^\star$ is such that $\Lambda'_{UZ(t^\star)}(\sigma^\star) = \zeta(t^\star)$. The constant $C^\star$ follows from the Bahadur-Rao theorem, with $C^\star = C_{UZ(t^\star),\zeta(t^\star)}$.*

# Remarks

- Same type of decay rate as in Bahadur-Rao theorem
- Clearly it holds that

$$\mathbb{P}\left(\exists t \in T \ s.t. \ \frac{1}{n}L_n(t) > \zeta(t)\right) \geq \sup_{t\in T} \mathbb{P}\left(\frac{1}{n}L_n(t) > \zeta(t)\right).$$

- The theorem shows that this bound is tight
- The maximizing $t^\star$ dominates the contributions. So given the extreme event occurs, it will, with overwhelming probability, happen at time $t^\star$
- Relaxing the uniqueness requirements yields similar decay rate, but we lack a clean expression for the proportionality constant
- The second assumption makes sure that we can ignore the 'upper tail'

# Sufficient Conditions

**Lemma**

*The condition*

$$\liminf_{t\to\infty} \frac{I_{UZ}(t)}{\log t} > 0$$

*is satisfied, when*

$$\begin{aligned} \Lambda_U^\star(x)/x &\to \infty \\ \liminf_t \zeta(t)/\log t &> 0 \end{aligned}$$

Remarks

- Condition only depends on distribution of losses, and not of default times
- First condition holds quite general

# Sufficient Conditions(2)

- Second condition follows from

$$\begin{aligned}
\Lambda_{UZ(t)}(\theta) &= \log \mathbb{P}(\tau \le t)\, \mathbb{E}\left[e^{\theta U}\right] + \mathbb{P}(\tau > t) \\
&\le \log \mathbb{E}\left[e^{\theta U}\right] \\
I_{UZ}(t) &= \Lambda^{\star}_{UZ(t^{\star})}(\zeta(t)) \\
&\ge \Lambda^{\star}_{U}(\zeta(t)) = \sup_{\theta}\left(\theta\zeta(t) - \log \mathbb{E}\left[e^{\theta U}\right]\right) \\
\liminf_{t\to\infty} \frac{\Lambda^{\star}_{U}(\zeta(t))}{\log t} &= \liminf_{t\to\infty} \frac{\Lambda^{\star}_{U}(\zeta(t))}{\zeta(t)} \frac{\zeta(t)}{\log t} > 0
\end{aligned}$$

# Large Increments of Loss Process

- Look at increments of the average loss process $\frac{1}{n}(L_n(t) - L_n(s))$, for $s < t$, exceeding a threshold $\xi(s,t)$
- Need that $\xi(s,t) > \mathbb{E}[U](\mathbb{P}(\tau \le t) - \mathbb{P}(\tau \le s))$

Assumptions

- There is a unique $s^\star < t^\star \in T$ such that

$$I_{UZ}(s^\star, t^\star) = \min_{s<t} I_{UZ}(s,t),$$

- Write $I_{UZ}(s,t) = \sup_\theta \left(\theta\xi(s,t) - \Lambda_{U(Z(t)-Z(s))}(\theta)\right) = \Lambda^\star_{U(Z(t)-Z(s))}(\xi(s,t))$. and let

$$\inf_{s\in T} \liminf_{t\to\infty} \frac{I_{UZ}(s,t)}{\log t} > 0,$$

# Large Increments of Loss Process(2)

**Theorem**

*Under these assumptions*

$$\mathbb{P}\left(\exists s < t : \; \frac{1}{n}(L_n(t) - L_n(s)) > \xi(s,t)\right)$$
$$= \frac{e^{-nI_{UZ}(s^\star,t^\star)}C^\star}{\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right),$$

*where $\sigma^\star$ is such that $\Lambda'_{U(Z(t^\star)-Z(s^\star))}(\sigma^\star) = \xi(s^\star,t^\star)$. The constant $C^\star$ follows from the Bahadur-Rao theorem, with $C^\star = C_{U(Z(t^\star)-Z(s^\star)),\ \xi(s^\star,t^\star)}$.*

# Remarks

- Result is very similar to result for crossing a barrier
- Conditions look quite restrictive and difficult to check
- However, the following is sufficient

$$\liminf_{t\to\infty} \frac{\xi(s,t)}{\log t} > 0$$

For the latter assumption

# Conclusions

- Established sample-path LDP for the average loss process $L_n(t)/n$
- Shown how results can be extended
- Future research to formally prove the extensions
- Established the exact asymptotic behavior of the probability of ever crossing a barrier
- Established the exact asymptotic behavior of probability that loss increments cross a certain barrier