Discrete hedging in models with jumps

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Lunteren, January 18, 2010

Discrete hedging

- In option pricing models, the hedging strategy is usually computed as a function of stock price (greek) or in feedback form, which means that it varies continuously, and often has infinite variation.
- Continuous rebalancing is unfeasible: in practice, the strategy F_t is replaced with a discrete strategy, leading to a discretization error.
- The simplest choice is $F_t^n := F_{h[t/h]}$, $h = T/n$.
- This discretization error has only been studied in the case of continuous processes.
- Two main approaches: weak convergence (CLT for hedging error) and L^2 convergence

Discrete hedging: the complete market case

- Bertsimas, Kogan and Lo '98 introduced an *asymptotic approach* allowing to study discrete hedging in continuous time.

Suppose

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

and we want to hedge a European option with payoff $H(S_T)$ using delta-hedging $F_t = \frac{\partial C}{\partial S}$.

CLT for hedging error

The discrete hedging error is defined by

$$\varepsilon_T^n = H(S_T) - \int_0^T F_t^n dS_t$$

Then $\varepsilon_T^n \to 0$ but the renormalized error $\frac{1}{\sqrt{h}}\varepsilon_T^n$ converges to

$$\sqrt{\frac{1}{2}} \int_0^T \frac{\partial^2 C}{\partial S^2} \sigma(t, S_t)^2 dW_t^*,$$

where W^* is a Brownian motion independent of W.

- Hedging error decays as $\sqrt{h}$.
- It is orthogonal to the stock price.
- The amplitude is determined by the gamma $\frac{\partial^2 C}{\partial S^2}$

Approximating hedging portfolios

Hayashi and Mykland '05 interpreted the discrete hedging error as the error of approximating the "ideal" hedging portfolio $\int_0^T F_t dS_t$ with a feasible hedging portfolio $\int_0^T F_t^n dS_t$

- This makes sense in incomplete markets

Suppose F and S are Itô process:
$dF_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t$ and $dS_t = \mu_t dt + \sigma_t dW_t$. Then

$$\frac{1}{\sqrt{h}} \varepsilon_t^n \Rightarrow \sqrt{\frac{1}{2}} \int_0^t \tilde{\sigma}_s \sigma_s dW_s^*, \qquad \left(\tilde{\sigma}_t = \frac{\partial^2 C}{\partial S^2} \sigma(t, S_t) \right)$$

where $\quad \varepsilon_t^n := \int_0^t (F_t - F_t^n) dS_t.$

- Weak convergence of processes in the Skorokhod topology on the space $\mathbb{D}$ of càdlàg functions

L^2 hedging error for continuous processes

- Result by Zhang (1999): for call/put options, the L^2 hedging error converges to the expected square of the weak limit.

$$\lim_{n\to\infty} \frac{1}{h} E[(\varepsilon_T^n)^2] = \frac{1}{2} E\left[\int_0^T \left(\frac{\partial^2 C}{\partial S^2}\right)^2 \sigma(s, S_s)^4 ds\right].$$

- The constant may be improved by an intelligent choice of rebalancing dates (Brodén and Wiktorsson '08) but the convergence rate cannot be improved.
- See also related results by Gobet and Temam (01) and Geiss (02), (06), (07).

Hedging in incomplete markets

- Incomplete market: exact replication impossible.
- Hedging is now an approximation problem.
- Industry practice: sensitivities to risk factors

$$\text{Delta} = \frac{\partial C(t, S_t)}{\partial S}: \quad \text{infinitesimal moves, hedge with stock}$$

$$\text{Gamma} = \frac{\partial^2 C(t, S_t)}{\partial S^2}: \quad \text{bigger moves; hedge with liquid options}$$

- Quadratic hedging: control the residual error

$$\min_F E\left(c + \int_0^T F_t dS_t - Y\right)^2$$

All these strategies require a continuously rebalanced portfolio.

Discretization error in presence of jumps

Our idea: study the discretization error

$$\varepsilon_t^n := \int_0^t (F_{t-} - F_{t-}^n) dS_t$$

in presence of jumps in the underlying and the hedging strategy.

- Approximation error of the Lévy-driven Euler scheme: Jacod and Protter (98), Jacod (04)
- Related results in the approximation of quadratic variation by realized volatility

$$X_T^2 = X_0^2 + 2\int_0^T X_{t-} dX_t + [X,X]_T$$

- Limit theorems for the approximation error of quadratic variation: Jacod (08).

Model setup: Lévy-Itô processes

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|z|\leq 1} \gamma_s(z)\tilde{J}(ds \times dz) + \int_0^t \int_{|z|>1} \gamma_s(z) J(ds \times dz).$$

- J: Poisson random measure with intensity $dt \times \nu$
- μ and σ are càdlàg $(\mathcal{F}_t)$-adapted
- γ: $\Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that $(\omega, z) \mapsto \gamma_t(z)$ is

$\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$-measurable $\forall t$ and $t \to \gamma_t(z)$ is càglàd $\forall \omega, z$;

$$\gamma_t(z)^2 \leq A_t \rho(z), \qquad \int_{|z|\leq 1} \rho(z)\nu(dz) < \infty$$

with ρ positive deterministic and A càglàd $(\mathcal{F}_t)$-adapted.

Model setup

- The stock price S is a Lévy-Itô process with coefficients μ, σ, γ;
- The continuous-time strategy F is a Lévy-Itô process with coefficients $\tilde{\mu}, \tilde{\sigma}, \tilde{\gamma}$.
- The agent uses the discrete strategy $F_t^n := F_{h[t/h]}$ instead of the continuous strategy F_t.

Weak convergence: the normalizing sequence

The normalizing factor need not be equal to $1/\sqrt{h}$.

Suppose F and S move only by finite-intensity jumps. If there is only one jump between t_i and t_{i+1},

$$\int_{t_i}^{t_{i+1}} F_{t-} dS_t = \int_{t_i}^{t_{i+1}} F^n_{t-} dS_t$$

Therefore $P[\varepsilon^n_t \neq 0] = O(1/n)$ and

$$\frac{1}{h^\alpha}\varepsilon^n_t \to 0$$

in probability $\forall \alpha > 0$.

More generally, if S and F are Lévy-Itô processes without diffusion parts,

$$\frac{1}{\sqrt{h}}\varepsilon^n_t \to 0$$

in probability uniformly on t.

Weak convergence

The discretization error satisfies

$$\frac{1}{\sqrt{h}}\varepsilon^n_t \to \sqrt{\frac{1}{2}}\int_0^t \sigma_s\tilde{\sigma}_s dW^*_s + \sum_{i:T_i\leq t} \Delta F_{T_i}\sqrt{\zeta_i}\xi_i\sigma_{T_i} + \sum_{i:T_i\leq t} \Delta S_{T_i}\sqrt{1-\zeta_i}\xi'_i\tilde{\sigma}_{T_i-}.$$

W^* is a standard BM independent from W and J,
$(\xi_k)_{k\geq 1}$ and $(\xi'_k)_{k\geq 1}$ are two sequences of independent $N(0,1)$,
$(\zeta_k)_{k\geq 1}$ is sequence of independent $U([0,1])$
$(T_i)_{i\geq 1}$ are the jump times of J enumerated in any order.

Remarks on convergence

- The hedging error $\frac{1}{\sqrt{h}}\varepsilon_t^n$ converges weakly in finite-dimensional laws but not in Skorohod topology.
- The discretized error process $\frac{1}{\sqrt{h}}\varepsilon_{h[t/h]}^n$ converges in Skorohod topology to the same limit.

Idea of the proof

Main tool: if (X^n) and (Y^n) are two sequences of processes such that

$$\sup_t |X^n_t - Y^n_t| \to 0 \quad \text{in probability}$$

and $X^n \to X$ weakly then $Y^n \to X$ weakly.

Idea of the proof

Step 1 Remove the big jumps

Step 2 Remove the small jumps

Step 3 Now we can write

$$S_t = S_0 + S_t^d + S_t^c + S_t^j$$

$$S_t^d = \int_0^t \left(\mu_s + \int \gamma_s(z)\nu(dz) \right) ds$$

$$S_t^c = \int_0^t \sigma_s dW_s$$

$$S_t^j = \int_0^t \int \gamma_s(z) J(ds \times dz)$$

and $F_t = F_0 + F_t^d + F_t^c + F_t^j$.

Idea of the proof

The leading terms in the hedging error are

$$\frac{1}{\sqrt{h}}\int (F^c_t - F^{c,n}_t)dS^c_t \to \sqrt{\frac{1}{2}}\int_0^t \sigma_s\tilde{\sigma}_s dW^*_s$$

$$\frac{1}{\sqrt{h}}\int (F^j_t - F^{j,n}_t)dS^c_t = \sum_i \Delta F_{T_i}\frac{1}{\sqrt{h}}\int_{T_i}^{r(T_i)} \sigma_s dW_s$$

$$\to \sum_{i:T_i\leq t} \Delta F_{T_i}\sqrt{\zeta_i}\xi_i\sigma_{T_i}$$

$$\frac{1}{\sqrt{h}}\int (F^c_t - F^{c,n}_t)dS^j_t = \sum_i \Delta S_{T_i}\frac{1}{\sqrt{h}}\int_{l(T_i)}^{T_i} \tilde{\sigma}_s dW_s$$

$$\to \sum_{i:T_i\leq t} \Delta S_{T_i}\sqrt{1-\zeta_i}\xi'_i\tilde{\sigma}_{T_i-}.$$

Discretization error in presence of jumps

- In finance it is more common to measure risk by an L^2 criterion, therefore in this work we want to study the rate of convergence of $E[(\varepsilon^h_T)^2]$ to zero.
- Surprising result: Even in the most simple cases, the L^2 error does not converge to the expected square of the weak limit if there are jumps *both* in S and in F.

L^2 convergence: example

Suppose

$$F_t = S_t = N_t,$$

with N_t a Poisson process with intensity λ. Then

$$P\left[\int_0^T (N_{t-} - N_{h[t/h]})dN_t \neq 0\right] = O(h)$$

and therefore $h^{-\alpha}\varepsilon^h_T \to 0$ in probability for all $\alpha > 0$. However

$$\lim_{n\to\infty} E\left[\left(\frac{1}{\sqrt{h}}\int_0^T (N_{t-} - N_{h[t/h]})dN_t\right)^2\right] = \frac{\lambda^2 T}{2}.$$

Model setup: the stock price process

- Let $S_t = e^{X_t}$, where X is a Lévy process with characteristic triple (a, ν, b), such that $E[S_t^2] < \infty$ and denote

$$A := a^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz), \quad \phi_t(u) = E[e^{iuX_t}]$$

- There exists an equivalent martingale measure Q under which X is again a Lévy process with triple $(a, \bar{\nu}, \bar{b})$ and we denote

$$\bar{A} := a^2 + \int_{\mathbb{R}} (e^z - 1)^2 \bar{\nu}(dz), \quad \bar{\phi}_t(u) = E^Q[e^{iuX_t}]$$

Model setup: the strategy

- Assume the continuous-time hedging strategy F is a Lévy-Itô process

$$F_t = F_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u + \int_0^t \int_{\mathbb{R}} \gamma_{u-}(z)\tilde{J}(du \times dz),$$

  where $\tilde{J}$ is the compensated Poisson random measure of jumps of X.
- The rebalancing dates are equally spaced: $T_i = hi$ and we denote $l(t) = \sup\{T_i, T_i < t\}$ and $r(t) = \inf\{T_i, T_i \geq t\}$
- The agent uses the discrete-time strategy $F_{l(t)}$ instead of the continuous-time strategy F_t.

The general limit theorem

Choose a function $\rho(h)$ with $\lim_{h\downarrow 0}\rho(h) = 0$ and assume that

$$\frac{h}{\rho(h)} E\left[\int_0^T S_t^2 (r(t)-t)\left(\mu_t^2 + \int_{\mathbb{R}} \gamma_t^2(z)\nu(dz)\right) dt\right] \xrightarrow{h\to 0} 0.$$

Then

$$\lim_{h\downarrow 0} \frac{1}{\rho(h)} E\left[\left(\varepsilon_T^h\right)^2\right]$$
$$= \lim_{h\downarrow 0} \frac{A}{\rho(h)} E\left[\int_0^T S_t^2 (r(t)-t)\left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z) e^{2z}\nu(dz)\right) dt\right]$$

whenever the limit in the right-hand side exists.

The regular regime

Let the assumption of the theorem be satisfied and suppose

$$E\left[\int_0^T S_t^2\left(\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right)dt\right]<\infty$$

Then it is easy to see that

$$\lim_{h\downarrow 0}\frac{1}{h}E\left[(\varepsilon_T^n)^2\right]=\frac{A}{2}E\left[\int_0^T S_t^2\left(\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right)dt\right].$$

Therefore the best possible convergence rate in this setting, obtained for regular strategies, is $\rho(h)=h$. However, worse rates may arise in the presence of irregular pay-offs.

Idea of the proof

- For simplicity, suppose that S is a P-martingale.
- Define an auxiliary probability measure P^2 by

$$\frac{dP^2}{dP}|_{\mathcal{F}_t} = \frac{e^{2X_t}}{e^{t\psi(-2i)}}, \quad \psi(u) = \log E e^{iuX_1}.$$

- Under P^2, the proces $W_t^{(2)} = W_t - 2at$ is a standard Brownian motion and

$$\tilde{J}^{(2)}(dt \times dz) = \tilde{J}(dt \times dz) - dt \times (e^{2z} - 1)\nu(dz)$$

is a compensated Poisson random measure.

Idea of the proof

The hedging error satisfies

$$\frac{1}{\rho(h)}E\left[\left(\varepsilon^h_T\right)^2\right] = \frac{1}{\rho(h)}E\left[\left(\int_0^T (F_{t-} - F_{l(t)})dS_t\right)^2\right]$$

$$= \frac{A}{\rho(h)}E\left[\int_0^T (F_{t-} - F_{l(t)})^2 S_t^2 dt\right]$$

$$= \frac{A}{\rho(h)}\int_0^T e^{t\psi(-2i)}E^{P^2}[(F_{t-} - F_{l(t)})^2]dt$$

$$\approx \frac{A}{\rho(h)}\int_0^T dt\, e^{t\psi(-2i)}E^{P^2}\left[\int_{l(t)}^t \sigma_s dW_s^{(2)} + \int_{l(t)}^t \int \gamma_{s-}(z)\tilde{J}^{(2)}(ds\,dz)\right]^2$$

$$= \frac{A}{\rho(h)}\int_0^T dt\, e^{t\psi(-2i)}E^{P^2}\left[\int_{l(t)}^t \left(\sigma_s^2 + \int_{\mathbb{R}} \gamma_s^2(z)e^{2z}\nu(dz)\right) ds\right]$$

Idea of the proof

Using integration by parts and switching back to the probability P,

$$\frac{A}{\rho(h)}\int_0^T e^{t\psi(-2i)}E^{P^2}\left[\int_{l(t)}^t\left(\sigma_s^2+\int_{\mathbb{R}}\gamma_s^2(z)e^{2z}\nu(dz)\right)ds\right]dt$$

$$=\frac{A}{\rho(h)}\int_0^T dt\, E^{P^2}\left[\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right]\int_t^{r(t)}e^{s\psi(-2i)}ds$$

$$=\frac{A(1+O(h))}{\rho(h)}E\left[\int_0^T S_t^2(r(t)-t)\left(\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right)dt\right]$$

Fourier transform option pricing

We consider a European option with pay-off $G(S_T)$ at time T and denote by g its log-payoff function: $G(e^x) \equiv g(x)$. Suppose that there exists $R \in \mathbb{R}$ such that

$$g(x)e^{-Rx} \quad \text{has finite variation on } \mathbb{R},$$

$$g(x)e^{-Rx} \in L^1(\mathbb{R}),$$

$$E^Q[e^{RX_t}] < \infty \quad \text{and} \quad \int_{\mathbb{R}} \frac{|\bar{\phi}_{T-t}(u - iR)|}{1 + |u|} du < \infty.$$

Then

$$C(t, S_t) := E^Q[G(S_T)|\mathcal{F}_t] = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR)\bar{\phi}_{T-t}(-u - iR)S_t^{R-iu} du,$$

where

$$\hat{g}(u) := \int_{\mathbb{R}} e^{iux} g(x) dx$$

The strategies

- The delta hedging strategy is given by

$$F_t = \frac{\partial C(t,S_t)}{\partial S} = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)(R-iu)S_t^{R-iu-1}du.$$

- The quadratic hedging strategy minimizes

$$E^Q\left[\left(G(S_T) - C(0,S_0) - \int_0^T F_t dS_t\right)^2\right].$$

and is given by

$$F_t = \frac{d\langle C,S\rangle_t^Q}{d\langle S,S\rangle_t^Q} = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(u+iR)\bar{\phi}_{T-t}(-u-iR)S_t^{R-iu-1}\Upsilon(u)du$$

where $\quad \Upsilon(u) = \dfrac{\bar{\psi}(-u-i(R+1)) - \bar{\psi}(-u-iR) - \bar{\psi}(-i)}{\bar{\psi}(-2i) - 2\bar{\psi}(-i)}.$

European options

- For all parametric models found in the literature, both for delta hedging and the quadratic hedging the convergence takes place in the regular regime:

$$\lim_{h\downarrow 0} \frac{1}{h} E\left[\left(\varepsilon^h_T\right)^2\right] = \frac{A}{2} E\left[\int_0^T S_t^2\left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z) e^{2z} \nu(dz)\right) dt\right].$$

- The limit can be evaluated via Fourier transform as a 3-dimensional integral.

Digital options: delta hedging

- Assume a non-zero diffusion component or a stable-like behavior of small jumps: the Lévy measure ν has a density satisfying

$$\nu(x) = \frac{f(x)}{|x|^{1+\alpha}}, \quad \lim_{x \to 0+} f(x) = f_+, \quad \lim_{x \to 0-} f(x) = f_-$$

  for some constants $f_- > 0$ and $f_+ > 0$.
- Let the pay-off function be given by $G(S_T) = 1_{S_T \geq K}$.
- If $\alpha \in (1, 2]$, for delta hedging the discretization error satisfies

$$\lim_{h \downarrow 0} \frac{1}{\rho(h)} E\left[\left(\varepsilon_T^h\right)^2\right] = AD p_T(\log K),$$

  with $\rho(h) = h^{1-1/\alpha}$, where D depends only on α, f_+ and f_-.

Digital options: quadratic hedging

- Same assumptions as for delta hedging.
- If $\alpha \in \left(0, \frac{3}{2}\right)$, for quadratic hedging the convergence takes place in the *regular regime*:

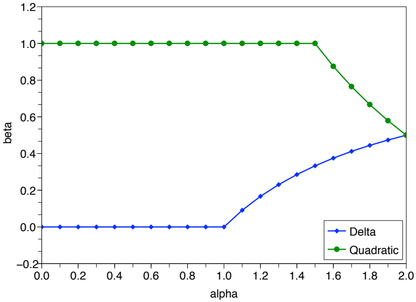
$$\lim_{h\downarrow 0} \frac{1}{h} E\left[\left(\varepsilon_T^h\right)^2\right] = \frac{A}{2} E\left[\int_0^T S_t^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t(z) e^{2z} \nu(dz)\right) dt\right].$$

- If $\alpha \in \left(\frac{3}{2}, 2\right]$, the discretization error satisfies

$$\lim_{h\downarrow 0} \frac{1}{\rho(h)} E\left[\left(\varepsilon_T^h\right)^2\right] = \frac{AQ}{\bar{A}^2} p_T(\log K)$$

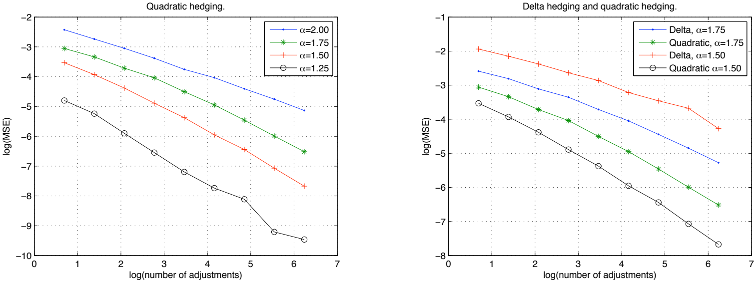
with $\rho(h) = h^{\frac{3}{\alpha}-1}$, where Q depends only on α, f_+ and f_-.

Comparison of delta and quadratic hedging



Convergence rate of the expected squared discretization error to zero as function of the stability index α for a digital option. The rate is given by $\rho(h) = h^{\beta}$, where β is plotted on the graph.

Numerical illustration



Convergence of the discretization error to zero for hedging a digital option in the CGMY model. Left: quadratic hedging. Right: delta hedging vs. quadratic hedging.

Concluding remarks

- Combined with recent results of C. Geiss and E. Laukkarinen (talk by C. Geiss at SPA'09), our findings allow to exhibit the non-equidistant rebalancing strategy allowing to recover the optimal rate $\frac{1}{n}$ in the irregular case.
- For pure-jump processes, the convergence rate may be improved beyond $\frac{1}{n}$ by taking suitable random rebalancing dates (work in progress).