

# GALOIS REPRESENTATIONS FOR EVEN GENERAL SPECIAL ORTHOGONAL GROUPS

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ABSTRACT. We prove the existence of  $\mathrm{GSpin}_{2n}$ -valued Galois representations corresponding to cohomological cuspidal automorphic representations of certain quasi-split forms of  $\mathrm{GSO}_{2n}$  under the local hypotheses that there is a Steinberg component and that the archimedean parameters are regular for the standard representation. This is based on the cohomology of Shimura varieties of abelian type, of type  $D^{\mathbb{H}}$ , arising from forms of  $\mathrm{GSO}_{2n}$ . As an application, under similar hypotheses, we compute automorphic multiplicities, prove meromorphic continuation of (half) spin  $L$ -functions, and improve on the construction of  $\mathrm{SO}_{2n}$ -valued Galois representations by removing the outer automorphism ambiguity.

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## INTRODUCTION

Inspired by conjectures of Langlands and Clozel's work [Clo90] for the group  $G = \mathrm{GL}_n$ , Buzzard–Gee [BG14, Conj. 5.16] formulate the following version of the Langlands correspondence (in one direction) for an arbitrary connected reductive group  $G$  over a number field  $F$ . Let  $\mathbb{A}_F$  denote the ring of adèles over  $F$ . Write  $\widehat{G}$  (resp.  ${}^L G$ ) for the Langlands dual group (resp.  $L$ -group) of  $G$  over  $\overline{\mathbb{Q}}_\ell$ . When  $g \in {}^L G(\overline{\mathbb{Q}}_\ell)$ , let  $g_{\mathrm{ss}}$  denote its semisimple part.

**Conjecture 1.** *Let  $\ell$  be a prime number and fix an isomorphism  $\iota: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ . Let  $\pi$  be a cuspidal  $L$ -algebraic automorphic representation of  $G(\mathbb{A}_F)$ . Then there exists a Galois representation*

$$\rho_\pi = \rho_{\pi, \iota}: \mathrm{Gal}(\overline{F}/F) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell),$$

*such that for all but finitely many primes  $\mathfrak{q}$  of  $F$  (excluding  $\mathfrak{q}|\ell$  and those such that  $\pi_{\mathfrak{q}}$  are ramified), the  $\widehat{G}$ -conjugacy class of  $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}} \in {}^L G(\overline{\mathbb{Q}}_\ell)$  is the Satake parameter of  $\pi_{\mathfrak{q}}$  via  $\iota$ .*

The conjecture of Buzzard–Gee is more precise (and does not assume cuspidality). They describe the image of each complex conjugation element and  $\ell$ -adic Hodge-theoretic properties of  $\rho_\pi$ . Moreover they predict [BG14, Conj. 5.17] that the compatibility holds at every  $\mathfrak{q}$  coprime to  $\ell$  such that  $\pi_{\mathfrak{q}}$  is unramified. In fact  $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})$ , instead of its semisimple part, appears in their conjecture. While  $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})$  is expected to be always semisimple, this seems to be a problem of different nature and out of reach. Thus we state the conjecture with  $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}}$ .

For most recent results on Conjecture 1 for  $\mathrm{GL}_n$  (in the regular case), we refer to [Sch15, HLTT16] and the references therein. Arthur's endoscopic classification [Art13] (see [Mok15, KMSW] for unitary groups)<sup>1</sup> provides a crucial input for constructing Galois representations as in the conjecture for symplectic, special orthogonal, and unitary groups by reducing the question to the case of general linear groups. When the group is  $\mathrm{SO}_{2n}$ , however, such an approach proves only a weaker local-global compatibility up to outer automorphisms (see (SO-i) in Theorem 6.3 below), falling short of proving Conjecture 1 (even under local hypotheses); we will return to this point as an application of our main theorem.

Our goal is to prove Conjecture 1 for a quasi-split form  $G^*$  of  $\mathrm{GSO}_{2n}$  over a totally real field under certain local hypotheses, as a sequel to our work [KS16] where we proved the conjecture for  $\mathrm{GSp}_{2n}$  under similar local hypotheses. The group  $\mathrm{GSO}_{2n}$  is closely related to the classical group  $\mathrm{SO}_{2n}$ , just like  $\mathrm{GSp}_{2n}$  is to  $\mathrm{Sp}_{2n}$ , but the similitude groups may well be regarded as non-classical groups. An important reason is that the Langlands dual groups of  $\mathrm{GSO}_{2n}$  and  $\mathrm{GSp}_{2n}$ , namely the general spin groups  $\mathrm{GSpin}_{2n}$  and  $\mathrm{GSpin}_{2n+1}$ , do not admit standard *embeddings* (into general linear groups of proportional rank). This makes the problem both nontrivial and interesting.

To be more precise, we set up some notation. Let  $F$  be a totally real number field, and  $n \in \mathbb{Z}_{\geq 3}$ . Let  $\mathrm{GSO}_{2n}$  denote the connected split reductive group over  $F$  which is the identity component of the orthogonal similitude group  $\mathrm{GO}_{2n}$ . (See §2 below for an explicit definition.) Our setup depends on the parity of  $n$ :

- (*n* **even**)  $E = F$ , and  $G^* = \mathrm{GSO}_{2n}$  (the split form over  $F$ ),
- (*n* **odd**)  $E$  is a totally imaginary quadratic extension of  $F$ , and  $G^*$  is a non-split quasi-split form of  $\mathrm{GSO}_{2n}$  relative to  $E/F$  (explicitly given as (8.4)).

We write  $\mathrm{GSO}_{2n}^{E/F}$  for the  $F$ -group  $G^*$  in either case. The setup is naturally designed so that there are Shimura varieties for (an inner twist of)  $\mathrm{Res}_{F/\mathbb{Q}} G^*$ . In particular  $G^*(F_y)$  has discrete series at every infinite place  $y$  of  $F$ . (Indeed  $G^*(F_y)$  has no discrete series if we swap the parity of  $n$  above.) There is a short exact sequence of  $F$ -groups

$$1 \rightarrow \mathrm{SO}_{2n}^{E/F} \longrightarrow \mathrm{GSO}_{2n}^{E/F} \xrightarrow{\mathrm{sim}} \mathbb{G}_m \rightarrow 1,$$

<sup>1</sup>The endoscopic classification is conditional in the following sense. At this time, the postponed articles [A25], [A26] and [A27] in the bibliography of [Art13] have not appeared. The proof of the weighted fundamental lemma for non-split groups has not become available yet either.

where  $\mathrm{SO}_{2n}^{E/F}$  is a quasi-split form of  $\mathrm{SO}_{2n}$ , defined similarly as  $\mathrm{GSO}_{2n}^{E/F}$ , and  $\mathrm{sim}$  denotes the similitude character. It is convenient to use the version of  $L$ -group relative to  $E/F$ , with coefficients in either  $\mathbb{C}$  or  $\overline{\mathbb{Q}}_\ell$ :

$${}^L G^* = \widehat{G}^* \rtimes \mathrm{Gal}(E/F) = \mathrm{GSpin}_{2n} \rtimes \mathrm{Gal}(E/F),$$

where the nontrivial element of  $\mathrm{Gal}(E/F)$  acts non-trivially on  $\mathrm{GSpin}_{2n}$ . (This identifies  ${}^L G^*$  with  $\mathrm{GPin}_{2n}$  if  $[E : F] = 2$ .) An important feature of the (general) spin groups  $\mathrm{GSpin}_m$  ( $m \in \mathbb{Z}_{\geq 2}$ ) is their spin representation  $\mathrm{spin}_m : \mathrm{GSpin}_m \rightarrow \mathrm{GL}_{2\lfloor m/2 \rfloor}$ . In case  $m$  is even, this representation is reducible and splits up into a direct sum  $\mathrm{spin}_m = \mathrm{spin}_m^+ \oplus \mathrm{spin}_m^-$  of two irreducible representations of dimension  $2^{\lfloor m/2 \rfloor - 1}$ . These representations  $\mathrm{spin}_m^\pm$  are called the *half spin* representations. Two other important representations are the *standard representation* and the *spinor norm* (see Lemma 3.1 for  $\mathrm{pr}^\circ$ )

$$\mathrm{std} : \mathrm{GSpin}_m \xrightarrow{\mathrm{pr}^\circ} \mathrm{SO}_m \rightarrow \mathrm{GL}_m, \quad \text{and} \quad \mathcal{N} : \mathrm{GSpin}_m \rightarrow \mathrm{GL}_1.$$

If  $m$  is odd,  $\mathrm{spin}$  is faithful. In the even case  $m = 2n$ , none of the representations  $\mathrm{spin}^+$ ,  $\mathrm{spin}^-$ ,  $\mathrm{std}$ , or  $\mathcal{N}$  is faithful, but  $\mathrm{spin}$  is faithful.

Let  $\pi$  be a cuspidal automorphic representation of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ . Consider the following hypotheses on  $\pi$ , where  $|\mathrm{sim}|$  denotes the composite  $\mathrm{GSO}_{2n}^{E/F}(F \otimes_{\mathbb{Q}} \mathbb{R}) \xrightarrow{\mathrm{sim}} (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \xrightarrow{|\cdot|} \mathbb{R}_{>0}^\times$ :

- (St) There is a finite  $F$ -place  $\mathfrak{q}_{\mathrm{St}}$  such that  $\pi_{\mathfrak{q}_{\mathrm{St}}}$  is the Steinberg representation of  $G^*(F_{\mathfrak{q}_{\mathrm{St}}})$  twisted by a character.
- (L-coh)  $\pi_\infty |\mathrm{sim}|^{n(n-1)/4}$  is  $\xi$ -cohomological for an irreducible algebraic representation  $\xi = \otimes_{y:F \rightarrow \mathbb{C}} \xi_y$  of the group  $(\mathrm{Res}_{F/\mathbb{Q}} G^*) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \prod_{y:F \rightarrow \mathbb{C}} (G^* \otimes_{F,y} \mathbb{C})$ .
- (std-reg) The highest weight of  $\xi$ , viewed as a cocharacter of  $\widehat{G}^* = \mathrm{GSpin}_{2n}(\mathbb{C})$ , is mapped to a regular cocharacter of  $\mathrm{GL}_{2n}$  under the standard representation  $\mathrm{GSpin}_{2n} \rightarrow \mathrm{GL}_{2n}$ .

In (L-coh), ‘ $\xi$ -cohomological’ means that the tensor product with  $\xi$  has nonvanishing relative Lie algebra cohomology in some degree (§1 below). Condition (L-coh) implies that  $\pi$  is L-algebraic. The other two conditions should be superfluous as they do not appear in Conjecture 1. Condition (St) plays an essential role in our argument, and would take significant new ideas and effort to get rid of. We assume (std-reg) for the reason that certain results for regular-algebraic self-dual cuspidal automorphic representations of  $\mathrm{GL}_N$ ,  $N > 2$ , are missing in the non-regular case. However we need less than (std-reg) for our argument to work. The necessary input for us to proceed without (std-reg) is formulated as Hypothesis 6.10, which we expect to be quite nontrivial but within reach nonetheless. Thus we assume either (std-reg) or Hypothesis 6.10 in the main theorem, hoping that (std-reg) will be removed as soon as the hypothesis is verified.

Let  $S_{\mathrm{bad}} = S_{\mathrm{bad}}(\pi)$  denote the finite set of rational primes  $p$  such that either  $p = 2$ ,  $p$  ramifies in  $F$ , or  $\pi_{\mathfrak{q}}$  ramifies at a place  $\mathfrak{q}$  of  $F$  above  $p$ . The following theorem assigns an  $\ell$ -adic Galois representation to  $\pi$  for each prime number  $\ell$  and each isomorphism  $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ .

**Theorem A.** *Assume that  $\pi$  satisfies conditions (St) and (L-coh). If (std-reg) does not hold for  $\pi$ , further assume Hypothesis 6.10 (for an  $\mathrm{SO}_{2n}(\mathbb{A}_F)$ -subrepresentation of  $\pi$ ). Then there exists, up to  $\widehat{G}$ -conjugation, a unique semisimple Galois representation attached to  $\pi$  and  $\iota$*

$$\rho_\pi = \rho_{\pi,\iota} : \mathrm{Gal}(\overline{F}/F) \rightarrow {}^L G^*$$

such that the following hold.

- (A1) For every prime  $\mathfrak{q}$  of  $F$  not above  $S_{\mathrm{bad}} \cup \{\ell\}$ ,  $\rho_\pi(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}}$  is  $\widehat{G}^*$ -conjugate to  $\iota \phi_{\pi_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}})$ , where  $\phi_{\pi_{\mathfrak{q}}}$  is the unramified Langlands parameter of  $\pi_{\mathfrak{q}}$ .
- (A2) The composition

$$\mathrm{Gal}(\overline{F}/F) \xrightarrow{\rho_\pi} {}^L G^* \xrightarrow{\mathrm{pr}^\circ} \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{Gal}(E/F)$$

corresponds to a cuspidal automorphic  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ -subrepresentation  $\pi^{\flat}$  contained in  $\pi$  in that  $\mathrm{pr}^{\circ}(\rho_{\pi}(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}})$  is  $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_{\ell})$ -conjugate to the Satake parameter of  $\pi_{\mathfrak{q}}^{\flat}$  via  $\iota$  at every  $\mathfrak{q}$  not above  $S_{\mathrm{bad}} \cup \{\ell\}$ . Further, the composition

$$\mathrm{Gal}(\overline{F}/F) \xrightarrow{\rho_{\pi}} {}^L G^* \xrightarrow{\mathcal{N}} \mathrm{GL}_1(\overline{\mathbb{Q}}_{\ell})$$

corresponds to the central character of  $\pi$  via class field theory and  $\iota$ .

(A3) For every  $\mathfrak{q}|\ell$ , the representation  $\rho_{\pi, \mathfrak{q}}$  is de Rham (in the sense that  $r \circ \rho_{\pi, \mathfrak{q}}$  is de Rham for all representations  $r$  of  $\widehat{G}^*$ ). Moreover

(a) The Hodge–Tate cocharacter of  $\rho_{\pi, \mathfrak{q}}$  is explicitly determined by  $\xi$ . More precisely, for all  $y: F \rightarrow \mathbb{C}$  such that  $\iota y$  induces  $\mathfrak{q}$ , we have

$$\mu_{\mathrm{HT}}(\rho_{\pi, \mathfrak{q}}, \iota y) = \iota \mu_{\mathrm{Hodge}}(\xi_y) + \frac{n(n-1)}{4} \mathrm{sim}.$$

(We still write  $\mathrm{sim}$  to mean the cocharacter of  $\mathrm{GSpin}_{2n}$  dual to  $\mathrm{sim}: G^* \rightarrow \mathbb{G}_m$ . See §1 below for the Hodge–Tate and Hodge cocharacters  $\mu_{\mathrm{HT}}$  and  $\mu_{\mathrm{Hodge}}$ .<sup>2</sup>)

(b) If  $\pi_{\mathfrak{q}}$  has nonzero invariants under a hyperspecial (resp. Iwahori) subgroup of  $G^*(F_{\mathfrak{q}})$  then either  $\rho_{\pi, \mathfrak{q}}$  or a quadratic character twist is crystalline (resp. semistable).

(c) If  $\ell \notin S_{\mathrm{bad}}$  then  $\rho_{\pi, \mathfrak{q}}$  is crystalline.

(A4) For every  $v|\infty$ ,  $\rho_{\pi, v}$  is odd (see §1 and Remark 12.6 below).

(A5) The Zariski closure of the image of  $\rho_{\pi}(\mathrm{Gal}(\overline{F}/E))$  in  $\mathrm{PSO}_{2n}$  maps onto one of the following four subgroups of  $\mathrm{PSO}_{2n}$ :

(a)  $\mathrm{PSO}_{2n}$ ,

(b)  $\mathrm{PSO}_{2n-1}$  (as a reducible subgroup),

(c) the image of a principal  $\mathrm{SL}_2$  in  $\mathrm{PSO}_{2n}$ , or

(d) (only when  $n = 4$ )  $G_2$  (embedded in  $\mathrm{SO}_7 \subset \mathrm{PSO}_8$ ) or  $\mathrm{SO}_7$  (as an irreducible subgroup via the projective spin representation).

(A6) If  $\rho': \mathrm{Gal}(\overline{F}/F) \rightarrow {}^L G^*$  is another semisimple Galois representation such that, for almost all finite  $F$ -places  $\mathfrak{q}$  where  $\rho'$  and  $\rho_{\pi}$  are unramified, the semisimple parts  $\rho'(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}}$  and  $\rho_{\pi}(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}}$  are conjugate, then  $\rho$  and  $\rho'$  are conjugate.

As explained below Conjecture 1, the existence of Galois representations

$$(0.1) \quad \rho_{\pi^{\flat}}: \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_{\ell}) \rtimes \mathrm{Gal}(E/F)$$

in a weaker form is known for any cuspidal automorphic representation  $\pi^{\flat}$  of  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$  satisfying  $(\mathrm{coh}^{\circ})$ ,  $(\mathrm{St}^{\circ})$ , and  $(\mathrm{std}\text{-}\mathrm{reg}^{\circ})$  (these conditions are variants for  $\mathrm{SO}_{2n}^{E/F}$ , see Section 6), and possibly a larger class of representations though we have not worked it out. The main ingredients are Arthur’s transfer [Art13, Thm. 1.5.2] from  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$  to  $\mathrm{GL}_{2n}(\mathbb{A}_F)$ , and collective results on the Langlands correspondence for  $\mathrm{GL}_{2n}(\mathbb{A}_F)$  in the self-dual case. Statements (SO-i)–(SO-v) of Theorem 6.3 in the text summarize what we know about  $\rho_{\pi^{\flat}}$ . A main drawback of Theorem 6.3 is that the conjugacy class of each  $\rho_{\pi^{\flat}}(\mathrm{Frob}_{\mathfrak{q}})_{\mathrm{ss}}$  is determined only up to  $\mathrm{O}_{2n}$ -conjugacy, rather than  $\mathrm{SO}_{2n}$ -conjugacy.

Using Theorem A we can upgrade Theorem 6.3 and remove this “outer” ambiguity (coming from the outer automorphism) as long as  $\pi^{\flat}$  can be extended to a cohomological representation  $\pi$  of  $\mathrm{GSO}_{2n}^{E/F}$ . If  $\pi$  is  $\xi$ -cohomological then  $\xi$  must satisfy condition (cent) of §9, so a necessary condition for such a cohomological extension to exist is the following condition (which is void for  $F = \mathbb{Q}$ ):

(cent<sup>o</sup>) the central character  $\{\pm 1\} = \mu_2(F_y) \rightarrow \mathbb{C}^{\times}$  of  $\pi_y$  at  $y \in \mathcal{V}_{\infty}$  is independent of  $y$ .

**Theorem B.** *Let  $\pi^{\flat}$  be a cuspidal automorphic representation of  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$  satisfying (cent<sup>o</sup>), (coh<sup>o</sup>), (St<sup>o</sup>), and (std-reg<sup>o</sup>). Then Conjecture 1 holds (for every  $\ell$  and  $\iota$ ). The associated Galois representation  $\rho_{\pi^{\flat}}$  is characterized uniquely up to  $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_{\ell})$ -conjugation.*

<sup>2</sup>More precisely, the Hodge cocharacter is a half-integral cocharacter, but the sum on the right hand side makes it integral. The two cocharacters are well-defined only up to conjugacy, but the formula makes sense because  $\mathrm{sim}$  is a central cocharacter.

See Theorem 13.1 below for a precise and stronger statement. The crux of the argument lies in showing that  $\pi^b$  extends to an automorphic representation  $\pi$  of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$  satisfying conditions of Theorem A. As Theorem A has no outer ambiguity, this yields Theorem B.

Theorem B offers a new perspective on the local Langlands correspondence for quasi-split forms of  $\mathrm{SO}_{2n}$  over  $p$ -adic fields. By localizing the theorem at finite places, we get a candidate for the correspondence, not just up to  $\mathrm{O}_{2n}$ -conjugacy as in [Art13]. More precisely, let  $H$  denote a quasi-split form of  $\mathrm{SO}_{2n}$  over a  $p$ -adic field  $k$ , assumed to be split if  $n$  is even. Then we can find  $E/F$  as above (depending on the parity of  $n$ ) and a prime  $\mathfrak{q}$  of  $F$  such that  $F_{\mathfrak{q}} \simeq k$  and  $\mathrm{SO}_{2n, \mathfrak{q}}^{E/F} \simeq H$ . If  $\sigma$  is an irreducible discrete series representation of  $H(k)$  then a candidate for the  $L$ -parameter for  $\sigma$  is described by the following procedures.

- (1) Find  $\pi^b$  satisfying  $(\mathrm{cent}^\circ)$ ,  $(\mathrm{coh}^\circ)$ ,  $(\mathrm{St}^\circ)$ , and  $(\mathrm{std}\text{-}\mathrm{reg}^\circ)$  such that  $\pi_{\mathfrak{q}}^b \simeq \sigma$ .
- (2) Obtain  $\rho_{\pi^b}$  from Theorem B (which relies on Theorem A).
- (3) Take  $\mathrm{WD}(\rho_{\pi^b}|_{\Gamma_{F_{\mathfrak{q}}}})$ , which can be viewed as an  $L$ -parameter for  $H(k)$ .

The globalization in (1) is possible by a standard trace formula argument proving the limit multiplicity formula. See §1 below for the definition of WD. The  $L$ -parameter resulting from the above is in the  $\mathrm{O}_{2n}$ -orbit of the  $L$ -parameter in [Art13] by Theorem 6.3 (SO-i), but could a priori depend on various choices. It is an interesting problem to relate the global construction here to the purely local constructions by Kaletha [Kal19, Kal] and Fargues–Scholze [FS]. In fact all this can be mimicked for  $\mathrm{GSO}_{2n}$  in place of  $\mathrm{SO}_{2n}$ , using Theorem A rather than Theorem B, so a similar question may be asked in the  $\mathrm{GSO}_{2n}$ -case.

As another application of Theorem A, we compute the automorphic multiplicities  $m(\pi)$  for certain automorphic representations  $\pi$  of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ .

**Theorem C.** *Let  $\pi$  be a cuspidal automorphic representation of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$  satisfying  $(L\text{-coh})$ ,  $(\mathrm{St})$  and  $(\mathrm{std}\text{-}\mathrm{reg})$ . Then we have  $m(\pi) = 1$ .*

In fact we also prove that  $m(\pi^b) = 1$  for cuspidal automorphic representations  $\pi^b$  of  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$  such that  $(\mathrm{coh}^\circ)$ ,  $(\mathrm{St}^\circ)$  and  $(\mathrm{std}\text{-}\mathrm{reg}^\circ)$  hold, and this serves as a prerequisite. Arthur’s multiplicity formula [Art13] determines the multiplicity of  $\pi^b$  up to an outer automorphism orbit, but notice that we compute the honest multiplicity. To refine Arthur’s formula, we utilize potential automorphy results [BLGGT14] combined with an  $L$ -function argument. The point is to rule out the case where  $\mathrm{std}\rho_\pi$  is reducible but the transfer of  $\pi$  to  $\mathrm{GL}_{2n}$  is cuspidal (see Proposition 14.1). To compute  $m(\pi)$  for  $\mathrm{GSO}_{2n}^{E/F}$  we rely on Theorem A and a result of Bin Xu [Xu18] to show that  $m(\pi) = m(\pi^b)$  for  $\pi^b \subset \pi$  a well-chosen  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ -subrepresentation.

Our final application is meromorphic continuation of the (half) spin- $L$  functions. Let  $\pi$  be a cuspidal automorphic representation of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$  unramified away from a finite set of places  $S$ . To make uniform statements, define a set

$$\mathfrak{e} := \begin{cases} \{+, -\}, & \text{if } n \text{ is even (thus } E = F), \\ \{\emptyset\}, & \text{if } n \text{ is odd (thus } [E : F] = 2), \end{cases}$$

with the understanding that  $\mathrm{spin}^\emptyset = \mathrm{spin}$ . The partial (half-)spin  $L$ -function for  $\pi$  away from  $S$  is by definition

$$(0.2) \quad L^S(s, \pi, \mathrm{spin}^\epsilon) := \prod_{\mathfrak{p} \notin S} \frac{1}{\det(1 - q_{\mathfrak{p}}^{-s} \mathrm{spin}^\epsilon(\phi_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{p}})))}, \quad \epsilon \in \mathfrak{e},$$

where  $q_{\mathfrak{p}} := \#(\mathcal{O}_F/\mathfrak{p})$  and  $\phi_{\pi_{\mathfrak{p}}}$  is the unramified  $L$ -parameter of  $\pi_{\mathfrak{p}}$ . Consider the following hypothesis for  $L$ -parameters  $\phi_{\pi_y}$  at infinite places  $y$ .

**(spin-reg)**  $\mathrm{spin}^\epsilon(\phi_{\pi_y})$  is regular for every infinite place  $y$  of  $F$  and every  $\epsilon \in \mathfrak{e}$ .

Then (spin-reg) implies (std-reg) when  $n \geq 3$ . This hypothesis ensures that  $\mathrm{spin}^\epsilon(\rho_\pi)$  has distinct Hodge–Tate weights. Our construction and Theorem A allow us to apply the potential automorphy theorem of Barnet-Lamb–Gee–Geraghty–Taylor [BLGGT14] to the weakly compatible system of  $\mathrm{spin}^\epsilon(\rho_\pi)$  (as  $\ell$  and  $\iota$  vary). Thereby we obtain the following.

**Theorem D.** *Assume  $n \geq 3$ . Let  $\pi$  be a cuspidal automorphic representation of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$  satisfying (L-coh), (St) and (spin-reg). Then there exists a finite totally real extension  $F'/F$  (which can be chosen to be disjoint from any prescribed finite extension of  $F$  in  $\overline{F}$ ) such that  $\mathrm{spin}^\epsilon \circ \rho_\pi|_{\mathrm{Gal}(\overline{F}/F')}$  is automorphic for each  $\epsilon \in \mathfrak{e}$ . More precisely, there exists a cuspidal automorphic representation  $\Pi^\epsilon$  of  $\mathrm{GL}_{2^n/|\mathfrak{e}|}(\mathbb{A}_{F'})$  such that*

- *for each finite place  $\mathfrak{q}'$  of  $F'$  not above  $S_{\mathrm{bad}}$ , the representation  $\iota^{-1}\mathrm{spin}^\epsilon \circ \rho_\pi|_{W_{F',\mathfrak{q}'}}$  is unramified and its Frobenius semisimplification is the Langlands parameter for  $\Pi_{\mathfrak{w}}^\epsilon$ ,*
- *at each infinite place  $y'$  of  $F'$  above a place  $y$  of  $F$ , we have  $\phi_{\Pi_{y'}^\epsilon}|_{W_{\mathbb{C}}} \simeq \mathrm{spin}^\epsilon \circ \phi_{\pi_y}|_{W_{\mathbb{C}}}$ .*

*In particular the partial spin  $L$ -function  $L^S(s, \pi, \mathrm{spin}^\epsilon)$  admits a meromorphic continuation and is holomorphic and nonzero in an explicit right half plane (e.g., in the region  $\Re(s) \geq 1$  if  $\pi$  has unitary central character).*

We now give a sketch of the argument for Theorem A. For simplicity, we restrict the discussion here to  $n$  even to put ourselves in the *split* case, and assume  $F = \mathbb{Q}$  to simplify notation. We also ignore all character twists in the following sketch and keep the isomorphism  $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$  implicit. (See the main text for correct twists.)

The basic idea is to construct  $\rho_\pi$  and prove its expected properties by understanding what should be  $\mathrm{spin}^+ \circ \rho_\pi$ ,  $\mathrm{spin}^- \circ \rho_\pi$ ,  $\mathrm{std} \circ \rho_\pi$ , and  $\mathcal{N} \circ \rho_\pi$ . One already has access to  $\mathrm{std} \circ \rho_\pi$  via Arthur's endoscopic classification and known instances of the global Langlands correspondence. The seemingly innocuous  $\mathcal{N} \circ \rho_\pi$  is not so trivial but can be understood from the first two. We realize  $\mathrm{spin}^+ \circ \rho_\pi$  and  $\mathrm{spin}^- \circ \rho_\pi$  in the cohomology of suitable Shimura varieties; this is the port of embarkation. In fact  $\rho_\pi$  would not be recovered from  $\mathrm{spin}^+ \circ \rho_\pi$ ,  $\mathrm{spin}^- \circ \rho_\pi$ ,  $\mathrm{std} \circ \rho_\pi$ , and  $\mathcal{N} \circ \rho_\pi$  in general due to essential group-theoretic difficulties (e.g.,  $\mathrm{GSpin}_{2n}$  is not acceptable in the sense of [Lar94, Lar96]), but condition (St) mitigates the matter. Another important role of (St) is to remove complexity associated with endoscopy.

Our Shimura varieties are associated with an inner twist  $G/\mathbb{Q}$  of the split group  $\mathrm{GSO}_{2n}$  (unique up to isomorphism) which splits at all primes  $p \neq p_{\mathrm{St}}$ , and whose derived subgroup is isomorphic to the quaternionic orthogonal group  $\mathrm{SO}^*(2n)$  over  $\mathbb{R}$  (which is not isomorphic to  $\mathrm{SO}(a, b)$  for any signature  $a + b = 2n$ ). Concretely  $G(\mathbb{R})$  is isomorphic to the group  $\mathrm{GSO}_{2n}^J(\mathbb{R})$  in §8 below.

The group  $G$  admits two abelian-type Shimura data  $(G, X^\epsilon)$  with  $\epsilon \in \{+, -\}$ , corresponding to the two edges of the “fork” in the Dynkin diagram of type  $D_n$  (see Section 9). These two Shimura data are not isomorphic. (The analogous Shimura data *are* isomorphic via an outer automorphism when  $n$  is odd; see Lemma (ii) below. Even then, we distinguish the two data as the outer automorphism changes isomorphism classes of representations.)

Let  $\pi$  be as in Theorem A. Using a trace formula argument, we transfer  $\pi$  to a  $\xi$ -cohomological cuspidal automorphic representation  $\pi^\natural$  of  $G(\mathbb{A})$  with isomorphic unramified local components as  $\pi$  such that  $\pi^\natural$  is Steinberg at a finite prime. Let  $\rho_\pi^{\mathrm{Sh}, \epsilon}$  be the  $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -representation on the  $\pi^{\natural, \infty}$ -isotypical part of the (semisimplified) compact support cohomology of the  $\ell$ -adic local system  $\mathcal{L}_\xi/\mathrm{Sh}(G, X^\epsilon)$  attached to  $\xi$ . Conjecturally the two representations  $\rho_\pi^{\mathrm{Sh}, \epsilon}$  should realize  $\mathrm{spin}^\epsilon \circ \rho_\pi$  up to semi-simplification (and up to a twist and a multiplicity that we ignore), in the non-endoscopic case. In particular, if  $\phi_{\pi_p} : W_{\mathbb{Q}_p} \rightarrow \mathrm{GSpin}_{2n}(\mathbb{C})$  is the unramified  $L$ -parameter of  $\pi_p$  at a prime  $p \neq \ell$  where  $\pi_p$  is unramified, then  $\rho_\pi^{\mathrm{Sh}, \epsilon}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$  ought to be unramified and satisfy

$$(0.3) \quad \mathrm{Tr}(\mathrm{Frob}_p^j, \rho_\pi^{\mathrm{Sh}, \epsilon}) = \mathrm{Tr} \mathrm{spin}^\epsilon(\phi_{\pi_p}(\mathrm{Frob}_p)^j) \in \overline{\mathbb{Q}}_\ell, \quad j \gg 1.$$

Employing Kisin's results on the Langlands–Rapoport conjecture [Kis17] and the Langlands–Kottwitz method for Shimura varieties of abelian type in the upcoming work of Kisin–Shin–Zhu [KSZ], we prove (0.3) for almost all  $p$ .

Let  $\pi^b \subset \pi$  be an irreducible cuspidal automorphic  $\mathrm{SO}_{2n}(\mathbb{A})$ -subrepresentation. From the aforementioned weaker version of Conjecture 1 for  $\mathrm{SO}_{2n}$ , we construct (see Theorem 6.3 below)

$$\rho_{\pi^b} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell).$$

such that

$$(0.4) \quad \rho_{\pi^b}(\text{Frob}_p)_{\text{ss}} \overset{\text{out}}{\sim} \text{pr}^\circ(\phi_{\pi_p}(\text{Frob}_p)) \in \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell),$$

for all primes  $p \neq \ell$  where  $\pi^b$  is unramified. Here  $\overset{\text{out}}{\sim}$  indicates  $\text{O}_{2n}(\overline{\mathbb{Q}}_\ell)$ -conjugacy, and  $\text{pr}^\circ : \text{GSpin}_{2n} \rightarrow \text{SO}_{2n}$  is the natural surjection.

We expect  $\rho_\pi$  to lift  $\rho_{\pi^b}$  (up to outer automorphism) and to sit inside  $\rho^{\text{Sh}} := \rho_\pi^{\text{Sh},+} \oplus \rho_\pi^{\text{Sh},-}$  as illustrated below. By  $\text{spin}$  we mean the unique projective representation of  $\text{SO}_{2n}$  that the projectivization of  $\text{spin}$  factors through.

$$(0.5) \quad \begin{array}{ccccc} & & \rho_\pi^{\text{Sh}} & & \\ & \curvearrowright & & \curvearrowleft & \\ \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \overset{\rho_\pi}{\dashrightarrow} & \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) & \xrightarrow{\text{spin}} & \text{GL}_{2n}(\overline{\mathbb{Q}}_\ell) \\ & & \text{pr}^\circ \downarrow & & \downarrow \\ & \curvearrowright & \rho_{\pi^b} & & \\ & \dashrightarrow & \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell) & \xrightarrow{\text{spin}} & \text{PGL}_{2n}(\overline{\mathbb{Q}}_\ell). \end{array}$$

We deduce from (0.3) and (0.4) that the outer diagram commutes, after a conjugation if necessary. In fact this is not straightforward because two  $\text{PGL}_{2n}$ -valued Galois representations need not be conjugate even if they map each  $\text{Frob}_p$  into the same conjugacy class for almost all  $p$ . We get around the difficulty by using a classification of reductive subgroups of  $\text{SO}_{2n}$  containing a regular unipotent element by Saxl–Seitz [SS97]. This is applicable since (St) tells us that the image of  $\rho_{\pi^b}$  contains a regular unipotent element. As a consequence, the Zariski closure of the image of  $\rho_{\pi^b}$  is connected mod center. If it is connected, we have the commutativity of (0.5) after a conjugation, and it follows that there exists  $\rho_\pi$  completing the diagram. If the Zariski closure is connected only mod center, then we need a variant of (0.5) as explained in §10. A similar group-theoretic consideration shows that  $\rho_\pi$  is characterized up to isomorphism by the images of Frobenius elements at almost all primes, cf. (A6).

Having constructed  $\rho_\pi$ , we verify that  $\rho_\pi$  enjoys the expected properties. Let us focus here on (A1). By construction,

$$\text{spin}(\rho_\pi(\text{Frob}_p)_{\text{ss}}) \sim \text{spin}(\phi_{\pi_p}(\text{Frob}_p)), \quad \text{for almost all } p.$$

The key point is to refine this, or break the symmetry, by showing the same relation with  $\text{spin}^+$  and  $\text{spin}^-$  in place of  $\text{spin}$  (cf. proof of Proposition 10.5 below) with the help of (0.3). Roughly speaking, we are in a situation

$$\rho^{\text{Sh},+} \oplus \rho^{\text{Sh},-} \simeq \text{spin}^+ \rho_\pi \oplus \text{spin}^- \rho_\pi$$

and want to match the  $+$  and  $-$  parts. The problem is easy enough if  $\text{spin}^+ \rho_\pi \simeq \text{spin}^- \rho_\pi$  as there is little to distinguish. If  $\text{spin}^+ \rho_\pi \not\simeq \text{spin}^- \rho_\pi$  then the idea is that the  $+$  and  $-$  parts do not overlap at sufficiently many places (by a Chebotarev type argument) to match the  $+$  and  $-$  parts unambiguously. If  $\text{spin}^+ \rho_\pi$  and  $\text{spin}^- \rho_\pi$  are irreducible, it is quite doable to promote this idea to a robust argument. In general, e.g., when the image of  $\rho_{\pi^b}$  is Zariski dense in a principal  $\text{PGL}_2$  mod center,  $\text{spin}^+ \rho_\pi$  and  $\text{spin}^- \rho_\pi$  are highly reducible. We deal with the intricacy by brute force via explicit group-theoretic computations (Case 3 in the proof of Proposition 10.3). This finishes the sketch of proof for Theorem A.

**Structure of the paper.** The paper splits roughly into four parts consisting of Sections 1–8 (preparation), Sections 9–12 (the core argument), Sections 13–15 (applications), and the appendices. Let us go over these parts in more detail. In Sections 1–5 we define (variants of) orthogonal groups and spin groups along with subgroups containing regular unipotent elements and the outer automorphism. We define the spin groups and their spin representations through root data as well as Clifford algebras by fixing the underlying quadratic spaces, and clarify the relationship between them. The former is natural in the context of Langlands correspondence whereas the latter has the advantage that various maps are determined and diagrams commute on the nose and not just up to conjugation. In Section 6 we construct Galois representations for certain cuspidal automorphic representations of quasi-split even orthogonal groups. This relies

on Arthur's book [Art13] and the known construction of automorphic Galois representations, but a few extra steps are taken to get the information that we need later on. In particular we study what happens to the Steinberg representation under Arthur's transfer from  $\mathrm{SO}_{2n}^{E/F}$  to  $\mathrm{GL}_{2n}$  (this relies on Appendix B). In Section 7 we list a number of basic results on comparing representations of  $\mathrm{SO}_{2n}^{E/F}$  with those of  $\mathrm{GSO}_{2n}^{E/F}$ . Section 8 discusses properties of the real points of  $\mathrm{GSO}_{2n}^{E/F}$  and introduces certain global inner forms  $G$  of  $\mathrm{GSO}_{2n}^{E/F}$ . The core argument starts in Section 9, where we take the cohomology of Shimura varieties associated with two Shimura data  $(G, X^\pm)$  to find two Galois representations  $\rho_\pi^{\mathrm{Sh}, \pm}$  attached to  $\pi$  as in the main theorem. In Section 10 we construct a  $\mathrm{GSpin}_{2n}$ -valued Galois representation  $\rho_\pi$  of  $\mathrm{Gal}(\overline{F}/E)$  from  $\rho_\pi^{\mathrm{Sh}, \pm}$  and  $\rho_{\pi^b}$ . This representation is not quite the one of Theorem A: The image of Frobenius under  $\rho_\pi$  is controlled only outside an unspecified finite set of primes, and moreover  $\rho_\pi$  should be extended to a representation of  $\mathrm{Gal}(\overline{F}/F)$ . The two problems are resolved in Sections 11 and 12 respectively. We emphasize that neither of these arguments is formal, the first one relies on Bin Xu's work [Xu18] and the second on a subtle global argument. The proof of Theorem A is also completed in Section 12. Sections 13–15 present applications of our main theorem to the construction of Galois representations for  $\mathrm{SO}_{2n}^{E/F}$ , automorphic multiplicity, and meromorphic continuation of (half)-spin  $L$ -functions.

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## 1. NOTATION AND PRELIMINARIES

We fix the following notation.

- $n \geq 3$  is an integer.<sup>3</sup>
- When  $X$  is a square matrix,  $\mathcal{E}\mathcal{V}(X)$  denotes the multi-set of eigenvalues of  $X$ .
- When  $A$  is a multi-set with elements in a ring  $R$  with  $r \in R$ , then  $r \cdot A := \{ra \mid a \in A\}$ . For  $n \in \mathbb{Z}_{>0}$ , write  $A^{\oplus n}$  for the multi-set consisting of  $a \in A$  whose multiplicity in  $A^n$  is  $n$  times that in  $A$ .
- $F$  is a number field. (In the main text,  $F$  is a totally real field with a distinguished embedding into  $\mathbb{C}$ .)
- $\mathcal{O}_F$  is the ring of integers of  $F$ .
- $\mathbb{A}_F$  is the ring of adèles of  $F$ ,  $\mathbb{A}_F := (F \otimes \mathbb{R}) \times (F \otimes \widehat{\mathbb{Z}})$ .
- If  $S$  is a finite set of  $F$ -places, then  $\mathbb{A}_F^S \subset \mathbb{A}_F$  is the ring of adèles with trivial components at the places in  $S$ , and  $F_S := \prod_{v \in S} F_v$ ;  $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$ .
- If  $\mathfrak{q}$  is a finite  $F$ -place, we write  $q_{\mathfrak{q}}$  for the cardinality of the residue field of  $\mathfrak{q}$ .
- If  $S$  is a set of prime numbers we write  $S^F$  for the set of  $F$ -places above  $S$ .
- If  $p$  is a prime number, then  $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ .
- $\ell$  is a prime number (typically different from  $p$ ).
- $\overline{\mathbb{Q}}_\ell$  is a fixed algebraic closure of  $\mathbb{Q}_\ell$ , and  $\iota: \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$  is an isomorphism.
- For each prime number  $p$  we fix the positive root  $p^{1/2} \in \mathbb{R}_{>0} \subset \mathbb{C}$ . From  $\iota$  we then obtain a choice for  $p^{1/2} \in \overline{\mathbb{Q}}_\ell$ . If  $q$  is a power of  $p$ , we obtain similarly a preferred choice  $q^{1/2}$  in  $\overline{\mathbb{Q}}_\ell$  and in  $\mathbb{C}$ .
- $\Gamma = \Gamma_F := \mathrm{Gal}(\overline{F}/F)$  is the absolute Galois group of  $F$ .
- For a finite extension  $E$  of  $F$  in  $\overline{F}$ , write  $\Gamma_E := \mathrm{Gal}(\overline{F}/E)$  and  $\Gamma_{E/F} := \mathrm{Gal}(E/F)$ .
- $\Gamma_v = \Gamma_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$  is (one of) the local Galois group(s) of  $F$  at the place  $v$ ,  $W_{F_v} \subset \Gamma_v$  is the corresponding Weil group.
- For each  $F$ -place  $v$ , choose an embedding  $\iota_v: \overline{F} \hookrightarrow \overline{F}_v$ , which induces  $\Gamma_v \hookrightarrow \Gamma$  that is canonical up to conjugation.
- $\mathcal{V}_\infty := \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$  is the set of infinite places of  $F$ .

<sup>3</sup>We should mention that if  $n \leq 3$ , there are exceptional isomorphisms of  $\mathrm{GSO}_{2n}$  (and its outer forms) to other simpler groups; for instance for  $n = 3$  the Shimura varieties that we obtain are (closely related to) Shimura varieties for unitary similitude groups, in particular more general results are already known.



- $c_y \in \Gamma$  is the complex conjugation (well-defined as a conjugacy class) induced by any embedding  $\overline{F} \hookrightarrow \mathbb{C}$  extending  $y \in \mathcal{V}_\infty$ .
- If  $S$  is a finite set of  $F$ -places, write  $\Gamma_{F,S}$  for the Galois group  $\text{Gal}(F(S)/F)$  where  $F(S) \subset \overline{F}$  is the maximal extension of  $F$  that is unramified away from  $S$ . If  $S$  is a set of rational places we write  $\Gamma_{F,S} := \Gamma_{F,S^F}$ .
- $\text{Frob}_\mathfrak{q}$  at a finite prime  $\mathfrak{q}$  of  $F$  means the *geometric* Frobenius element in the quotient of  $\Gamma_\mathfrak{q}$  by the inertia subgroup, or the image thereof in  $\Gamma_{F,S}$ . (The image in  $\Gamma_{F,S}$  depends on the choice of  $\iota_\mathfrak{q}$  but its conjugacy class is independent of the choice.)
- When  $G$  is a connected reductive group over  $F$ , write  $\widehat{G}$  and  ${}^L G = \widehat{G} \rtimes \Gamma_F$  for the Langlands dual group and the  $L$ -group, respectively (with coefficients in  $\mathbb{C}$  or  $\overline{\mathbb{Q}}_\ell$ , depending on the context). If  $G$  splits over a finite extension  $E/F$  in  $\overline{F}$  then  $\widehat{G} \rtimes \Gamma_{E/F}$  denotes the  $L$ -group with respect to  $E/F$ . (Namely such a semi-direct product is always understood with the  $L$ -action of  $\Gamma_{E/F}$  on  $\widehat{G}$ .) Often we use  ${}^L G$  to mean  $\widehat{G} \rtimes \Gamma_{E/F}$ .<sup>4</sup>
- When  $H$  is a reductive group over  $\overline{\mathbb{Q}}_\ell$ , we also use  $H$  to mean the topological group  $H(\overline{\mathbb{Q}}_\ell)$  by abuse of notation. This should be clear from the context and not leading to confusion.
- When  $F$  is a  $p$ -adic field and  $G$  is the set of  $F$ -points of a reductive group over  $F$ , we write  $\text{St}_G$  for the Steinberg representation of  $G$  (defined in [BW00, X.4.6] for instance). Moreover, we write  $\mathbf{1}_G$  for the trivial representation of  $G$ . In certain cases, when  $G$  is clear, we write  $\text{St} = \text{St}_G$  or  $\mathbf{1} = \mathbf{1}_G$ . We also write sometimes  $\text{St}_n$  for  $\text{St}_{\text{GL}_n(F)}$  (in case  $F$  is clear from the context).
- If  $G$  is an algebraic group over a field, we write  $Z(G)$  for its center (as a group scheme over the same field).

Fix  $G$  and  $E/F$  as above. We introduce some notions on the Galois side. By an ( $\ell$ -adic) *Galois representation* of  $\Gamma_F$  (with values in  $\widehat{G} \rtimes \Gamma_{E/F}$ ), we mean a continuous homomorphism (for a fixed  $E/F$  as above)

$$\rho : \Gamma_F \rightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$$

which factors through  $\Gamma_{F,S}$  for some finite set  $S$  and commutes with the obvious projections onto  $\Gamma_{E/F}$ . Similarly we define a Galois representation with the source  $\Gamma_\mathfrak{q}$  and/or with values in  ${}^L G(\overline{\mathbb{Q}}_\ell)$ . Two Galois representations are considered isomorphic if they are conjugate by an element of  $\widehat{G}(\overline{\mathbb{Q}}_\ell)$ . We say that  $\rho$  as above is (*totally*) *odd* if for every real place  $y$  of  $F$ , the following holds: writing  $\text{Ad}$  for the adjoint action of  ${}^L G$  on  $\text{Lie } G(\overline{\mathbb{Q}}_\ell)$ , the image of  $c_y$  under the composite

$$\Gamma_y \hookrightarrow \Gamma \xrightarrow{\rho} {}^L G(\overline{\mathbb{Q}}_\ell) \xrightarrow{\text{Ad}} \text{GL}(\text{Lie } G(\overline{\mathbb{Q}}_\ell))$$

has trace equal to the rank of the derived subgroup of  $\widehat{G}$ . (Compare with [Gro].)

An  ${}^L G$ -valued *Weil–Deligne representation* is a pair  $(r, N)$  consisting of a morphism

$$r : W_{F_\mathfrak{q}} \rightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$$

which has open kernel on the inertia subgroup and commutes with the canonical projections onto  $\Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$ , and a nilpotent operator  $N \in \text{Lie } \widehat{G}(\overline{\mathbb{Q}}_\ell)$  such that  $\text{Ad}_r(w)N = |w|N$ , where  $|\cdot| : W_{F_\mathfrak{q}} \rightarrow \|\mathfrak{q}\|^\mathbb{Z}$  is the homomorphism sending a geometric Frobenius element to  $\|\mathfrak{q}\|$ ; here  $\|\mathfrak{q}\| \in \mathbb{Z}_{>0}$  denotes the norm of  $\mathfrak{q}$ . The Frobenius-semisimplification  $(r^{\text{ss}}, N)$  is obtained by replacing  $r$  with its semisimplification. We say  $(r, N)$  is Frobenius-semisimple if  $r = r^{\text{ss}}$ .

Let  $\rho : \Gamma_F \rightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$  be a Galois representation. Write  $\mathfrak{p}$  for the prime of  $E$  induced by  $\iota_\mathfrak{q} : \overline{F} \hookrightarrow \overline{F}_\mathfrak{q}$ . Then the restriction (via  $\iota_\mathfrak{q}$ )

$$\rho|_{\Gamma_\mathfrak{q}} : \Gamma_{F_\mathfrak{q}} \rightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E_\mathfrak{p}/F_\mathfrak{q}}$$

gives rise to an  ${}^L G$ -valued Weil–Deligne representation  $\text{WD}(\rho|_{\Gamma_{F_\mathfrak{q}}}) = (r, N)$  (well-defined up to  $\widehat{G}(\overline{\mathbb{Q}}_\ell)$ -conjugacy). The construction follows from the case of  $G = \text{GL}_n$  by the Tannakian

<sup>4</sup>This is harmless for us as the inflation map induces a bijection of isomorphism classes of  ${}^L G$ -valued Galois representations when  $\Gamma_{E/F}$  is replaced with  $\Gamma_F$  in the semi-direct product.

formalism via algebraic representations of  $\widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E_p/F_q}$ . (The case  $\mathfrak{q}|\ell$  is more subtle than  $\mathfrak{q} \nmid \ell$ . In the former case, a detailed explanation is given in the proof of [KS16, Lem. 3.2], where  $\widehat{G}$  is denoted by  $H$ . In *loc. cit.*  $\Gamma_{E_p/F_q}$  is trivial but the same argument extends.) When  $\mathfrak{q} \nmid \ell$ , one can alternatively appeal to Grothendieck's  $\ell$ -adic monodromy theorem to construct  $\text{WD}(\rho|_{\Gamma_{F_q}})$ .

A local  $L$ -parameter  $\phi : W_{F_q} \times \text{SU}(2) \rightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E_p/F_q}$  is associated with a Frobenius-semisimple  ${}^L G$ -valued Weil–Deligne representation  $(r, N)$  given by the following recipe:

$$r(w) = \phi \left( w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right), \quad \text{and} \quad N = \phi \left( 1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right).$$

This induces a bijection on the sets of equivalence classes of such objects [GR10, Prop. 2.2]. In practice (where only equivalence classes matter), we will use them interchangeably.

We introduce some further notation and conventions in representation theory. If  $\pi$  is a representation on a complex vector space then we set  $\iota\pi := \pi \otimes_{\mathbb{C}, \iota} \overline{\mathbb{Q}}_\ell$ . Similarly if  $\phi$  is a local  $L$ -parameter of a connected reductive group  $G$  so that  $\phi$  maps into  ${}^L G(\mathbb{C})$  then  $\iota\phi$  is the parameter with values in  ${}^L G(\overline{\mathbb{Q}}_\ell)$  obtained from  $\phi$  via  $\iota$ . If  $G$  is a locally profinite group equipped with a Haar measure, then we write  $\mathcal{H}(G)$  for the *Hecke algebra* of locally constant, complex valued functions with compact support. We write  $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(G)$  for the same algebra, but now consisting of  $\overline{\mathbb{Q}}_\ell$ -valued functions. We normalize every parabolic induction by the half power of the modulus character as in [BZ77, 1.8], so that it preserves unitarity.

Let  $G$  be a real reductive group,  $K$  a maximal compact subgroup of  $G(\mathbb{R})$ , and  $\tilde{K} := K \cdot Z(G)(\mathbb{R})$ . Let  $\xi$  be an irreducible algebraic representation of  $G$  over  $\mathbb{C}$ . An irreducible admissible representation of  $G(\mathbb{R})$  is said to be  $\xi$ -*cohomological* if  $H^i(\text{Lie } G(\mathbb{C}), \tilde{K}, \pi \otimes_{\mathbb{C}} \xi) \neq 0$  for some  $i \geq 0$ . If this is the case, we associate a Hodge cocharacter (well-defined up to  $\tilde{G}$ -conjugacy) as in [KS16, Def 1.14]:

$$\mu_{\text{Hodge}}(\xi) : \mathbb{G}_m \rightarrow \widehat{G}.$$

Let  $L$  be a finite extension of  $\mathbb{Q}_\ell$ . Let  $H$  be a connected reductive group over  $\overline{\mathbb{Q}}_\ell$ , and  $\rho : \text{Gal}(\overline{L}/L) \rightarrow H(\overline{\mathbb{Q}}_\ell)$  a continuous morphism. If  $\rho$  is Hodge–Tate with respect to each  $i : L \hookrightarrow \overline{\mathbb{Q}}_\ell$ , we define a Hodge–Tate cocharacter over  $\overline{\mathbb{Q}}_\ell$  (well-defined up to  $H$ -conjugacy) as in [KS16, Def 1.10]:

$$\mu_{\text{HT}}(\rho, i) : \mathbb{G}_m \rightarrow H.$$

We recall the following lemma that can be easily deduced from the Chebotarev density theorem, as it will be needed in §10. The *density* of a set  $S$  consisting of primes of  $F$  is defined to be the limit  $d(S) = \lim_{n \rightarrow \infty} a_n(S)/a_n(F)$ , where  $a_n(F)$  is the number of primes  $\mathfrak{q}$  with bounded norm  $\|\mathfrak{q}\| < n$  and  $a_n(S)$  is the number of  $\mathfrak{q} \in S$  with  $\|\mathfrak{q}\| < n$  [Ser97, Sect. I.2.2]. Depending on  $S$ , the limit  $d(S)$  may or may not exist — in the former case, we say  $S$  has density  $d(S)$ , and otherwise we leave the density undefined.

**Lemma 1.1.** *Let  $S$  be a finite set of  $F$ -places, where  $F$  is now any number field. Let  $G/\overline{\mathbb{Q}}_\ell$  be a linear algebraic group and let  $r : \Gamma_S \rightarrow G(\overline{\mathbb{Q}}_\ell)$  be a Galois representation. Write  $H$  for the Zariski closure of the image of  $r$ . Let  $X \subset G$  be a strict, closed subvariety that is invariant by  $G$  conjugation and such that  $\dim(X) < \dim(H)$ . Then the set of  $F$ -places  $\mathfrak{q} \notin S$  with  $r(\text{Frob}_\mathfrak{q}) \in X$  has density 0.*

*Proof.* Replacing  $G$  by  $H$ , we may assume that  $r$  has Zariski dense image in  $G$  and  $H = G$ . As  $\pi_0 G$  is finite, we may, after passing to an open subgroup of  $\Gamma$ , assume that  $G$  is connected. Let  $\mu$  be the Haar measure on  $\Gamma$  with total volume 1. Then  $Y = r^{-1}(X)$  is a measurable subset (it is closed, hence measurable), with boundary of measure 0, and  $Y$  is stable under  $\Gamma$ -conjugation. By the Chebotarev density theorem the set of places  $\mathfrak{q} \notin S$  such that  $\text{Frob}_\mathfrak{q} \in Y$  has measure equal to  $\mu(Y)$  (See, e.g., [Ser97, I-8 Cor. 2b]). If  $\mu(Y) > 0$ , then  $Y$  contains a translate of an open subgroup  $U$  of  $\Gamma$ . Thus the image of  $\Gamma$  lies in  $[\Gamma : U]$ -translates of  $X$ , contradicting the assumption that  $G$  is connected.  $\square$

2. ROOT DATA OF  $\mathrm{GSO}_{2n}$  AND  $\mathrm{GSpin}_{2n}$ 

Let  $\mathrm{GO}_{2n}/\mathbb{Q}$  be the algebraic group such that for all  $\mathbb{Q}$ -algebras  $R$  we have

$$\mathrm{GO}_{2n}(R) = \left\{ g \in \mathrm{GL}_{2n}(R) \mid \exists \mathrm{sim}(g) \in R^\times : g^t \cdot \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix} \cdot g = \mathrm{sim}(g) \cdot \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix} \right\}.$$

(in the above formula  $1_n$  is the  $n \times n$  identity matrix.) The group  $\mathrm{GO}_{2n}$  is disconnected; its neutral component  $\mathrm{GSO}_{2n} \subset \mathrm{GO}_{2n}$  is defined by the condition  $\det(g) = \mathrm{sim}(g)^n$ . The groups  $\mathrm{GO}_{2n}$ ,  $\mathrm{GSO}_{2n}$  are split and defined by a quadratic form of signature  $(n, n)$ . An element  $t$  of the diagonal torus  $\mathrm{T}_{\mathrm{GSO}} \subset \mathrm{GSO}_{2n}$  is of the form

$$t = \mathrm{diag}(t_i)_{i=1}^{2n} = \mathrm{diag}(t_1, t_2, \dots, t_n, t_0 t_1^{-1}, t_0 t_2^{-1}, \dots, t_0 t_n^{-1}), \quad t_0 := \mathrm{sim}(t)$$

hence  $\mathrm{T}_{\mathrm{GSO}} \simeq \mathbb{G}_m^{n+1}$  by sending  $t$  to  $(t_0, t_1, \dots, t_n)$ . We identify  $X^*(\mathrm{T}_{\mathrm{GSO}}) = \bigoplus_{i=0}^n \mathbb{Z} \cdot e_i$  and  $X_*(\mathrm{T}_{\mathrm{GSO}}) = \bigoplus_{i=0}^n \mathbb{Z} \cdot e_i^*$  accordingly. We let  $\mathrm{B}_{\mathrm{GSO}}$  be the Borel subgroup of  $\mathrm{GSO}_{2n}$  of matrices of the form

$$(2.1) \quad g = \begin{pmatrix} A & AB \\ 0 & cA^{t,-1} \end{pmatrix}, \quad A \in B_{\mathrm{GL}_n}, \quad B \in M_n, \quad B^t = -B \quad \text{and} \quad c = \mathrm{sim}(g),$$

where  $B_{\mathrm{GL}_n} \subset \mathrm{GL}_n$  is the upper triangular Borel subgroup. (To see that this is indeed a Borel, notice that any block matrix  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  with  $C = 0$  is of the above form if and only if  $g \in \mathrm{GSO}_{2n}$ , and moreover the displayed group is solvable of dimension  $n^2 + 1$ ).

We realize the split forms of even (special) orthogonal groups in  $\mathrm{GO}_{2n}/\mathbb{Q}$ . Namely we write  $\mathrm{O}_{2n}$  (resp.  $\mathrm{SO}_{2n}$ ) for the subgroup of  $\mathrm{GO}_{2n}$  (resp.  $\mathrm{GSO}_{2n}$ ) where  $\mathrm{sim}$  is trivial.

**Lemma 2.1.** *The root datum of  $\mathrm{GSO}_{2n}$  with respect to  $\mathrm{B}_{\mathrm{GSO}}$  is described as follows.*

- (i) *The set of roots (resp. coroots) consists of  $\pm(e_i - e_j)$  and  $\pm(e_i + e_j - e_0)$  (resp.  $\pm(e_i^* - e_j^*)$  and  $\pm(e_i^* + e_j^*)$ ) with  $1 \leq i < j \leq n$ .*
- (ii) *The positive roots are  $\{e_i + e_j - e_0\}_{1 \leq i < j \leq n} \cup \{e_i - e_j\}_{1 \leq i < j \leq n}$  and the positive coroots  $\{e_i^* \pm e_j^*\}_{1 \leq i < j \leq n}$ .*
- (iii) *The simple roots are  $\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n$ , and  $\alpha_n = e_{n-1} + e_n - e_0$ .*
- (iv) *The simple coroots  $\Delta^\vee$  are  $\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \dots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*$ , and  $\alpha_n^\vee = e_{n-1}^* + e_n^*$ .*

*Remark 2.2.* The root datum of  $\mathrm{SO}_{2n}$  is described similarly. Putting  $\mathrm{T}_{\mathrm{SO}} := \mathrm{T}_{\mathrm{GSO}} \cap \mathrm{SO}_{2n}$  and  $\mathrm{B}_{\mathrm{SO}} := \mathrm{B}_{\mathrm{GSO}} \cap \mathrm{SO}_{2n}$ , we have  $\mathrm{T}_{\mathrm{SO}} = \{t \in \mathrm{T}_{\mathrm{GSO}} : t_0 = 1\}$  as well as  $X^*(\mathrm{T}_{\mathrm{SO}}) = \bigoplus_{i=1}^n e_i \cdot \mathbb{Z}$  and  $X_*(\mathrm{T}_{\mathrm{SO}}) = \bigoplus_{i=1}^n e_i^* \cdot \mathbb{Z}$ . To describe (positive or simple) roots and coroots, we only need to formally set  $e_0 = 0$  in the lemma above.

*Proof.* The standard computation for  $\mathrm{SO}_{2n}$  as in [FH91, 18.1] can be easily adapted to  $\mathrm{GSO}_{2n}$ .  $\square$

We define the following element<sup>5</sup>

$$(2.2) \quad \vartheta^\circ := - \begin{pmatrix} 1_{n-1} & & & \\ & 0 & & 1 \\ & & 1_{n-1} & \\ & & & 0 \end{pmatrix} \in \mathrm{O}_{2n}.$$

Since  $\det(\vartheta^\circ) = -1$  we have  $\vartheta^\circ \notin \mathrm{SO}_{2n}$ . We write  $\theta^\circ \in \mathrm{Aut}(\mathrm{GSO}_{2n})$  for the automorphism given by  $\vartheta^\circ$ -conjugation.

**Lemma 2.3.** *The automorphism  $\theta^\circ$  stabilizes  $\mathrm{B}_{\mathrm{GSO}}$  and  $\mathrm{T}_{\mathrm{GSO}}$ , and acts on  $\mathrm{T}_{\mathrm{GSO}}$  by*

$$(t_0, t_1, \dots, t_n) \mapsto (t_0, t_1, \dots, t_{n-1}, t_0 t_n^{-1}).$$

*Furthermore  $\theta^\circ(\alpha_i) = \alpha_i$  for  $i < n - 2$ ,  $\theta^\circ(\alpha_{n-1}) = \alpha_n$ , and  $\theta^\circ(\alpha_n) = \alpha_{n-1}$ .*

<sup>5</sup>We put the minus sign for  $\vartheta^\circ$  to make it compatible more easily with  $\vartheta \in \mathrm{GSpin}_{2n}$  to be introduced later.

*Proof.* By a direct computation,  $\theta^\circ(\mathrm{T}_{\mathrm{GSO}}) = \mathrm{T}_{\mathrm{GSO}}$  and  $\theta^\circ(\mathrm{B}_{\mathrm{GSO}}) = \mathrm{B}_{\mathrm{GSO}}$ . Since  $\theta^\circ$  only switches  $t_n$  and  $t_{2n} = t_0 t_n^{-1}$ , its action on  $\mathrm{T}_{\mathrm{GSO}}$  is explicitly described as in the lemma. Thus  $\theta^\circ(e_i) = e_i$  for  $1 \leq i \leq n-1$  and  $\theta^\circ(e_n) = e_0 - e_n$ , from which the last assertion follows.  $\square$

We define  $\mathrm{GSpin}_{2n}$  to be the Langlands dual group  $\widehat{\mathrm{GSO}}_{2n}$  over  $\mathbb{C}$  (or later over  $\overline{\mathbb{Q}}_\ell$  via  $\iota: \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ ). That is,  $\mathrm{GSpin}_{2n}$  is the connected reductive group over  $\mathbb{C}$ , equipped with a Borel subgroup  $B_{\mathrm{GSpin}}$  and a maximal torus  $\mathrm{T}_{\mathrm{GSpin}}$ , whose based root datum is dual to the one of  $\mathrm{GSO}_{2n}$  that we described above. In particular

$$X_*(\mathrm{T}_{\mathrm{GSpin}}) = X^*(\mathrm{T}_{\mathrm{GSO}}) \quad \text{and} \quad X^*(\mathrm{T}_{\mathrm{GSpin}}) = X_*(\mathrm{T}_{\mathrm{GSO}}).$$

Via the identification  $X^*(\mathrm{T}_{\mathrm{GSO}}) = \mathbb{Z}^{n+1}$ , we represent elements  $s \in \mathrm{T}_{\mathrm{GSpin}}$  as  $(s_0, s_1, \dots, s_n)$ . In Section 3 we will also define an explicit model of  $\mathrm{GSpin}_{2n}$  over  $\mathbb{Q}$  using Clifford algebras.

**Lemma 2.4.** *There is a unique  $\theta \in \mathrm{Aut}(\mathrm{GSpin}_{2n})$  that fixes  $\mathrm{T}_{\mathrm{GSpin}}, B_{\mathrm{GSpin}}$ , switches  $\alpha_{n-1}^\vee$  and  $\alpha_n^\vee$ , leaves the other  $\alpha_i^\vee$  invariant, and induces the trivial automorphism of the cocenter of  $\mathrm{GSpin}_{2n}$ . We have  $\theta^2 = 1$  and on the torus  $\mathrm{T}_{\mathrm{GSpin}}$  the involution  $\theta$  is given by*

$$(2.3) \quad (s_0, s_1, \dots, s_n) \mapsto (s_0 s_n, s_1, \dots, s_{n-1}, s_n^{-1}).$$

*Proof.* We have  $\theta(e_i^* - e_{i+1}^*) = e_i^* - e_{i+1}^*$  ( $1 \leq i < n$ ) and  $\theta(e_{n-1}^* - e_n^*) = e_{n-1}^* + e_n^*$ . Thus

$$(2.4) \quad \theta(e_i^*) = e_i^* \quad (1 \leq i < n) \quad \text{and} \quad \theta(e_n^*) = -e_n^*.$$

The center of  $\mathrm{GSO}_{2n}$  is the image of  $\mathbb{G}_m \ni z \mapsto (z^2, z, \dots, z) \in \mathrm{T}_{\mathrm{GSO}}$ . The dual map is

$$(2.5) \quad \mathrm{T}_{\mathrm{GSpin}} \rightarrow \mathbb{G}_m, \quad (s_0, s_1, \dots, s_n) \mapsto s_0^2 s_1 \cdots s_n.$$

Thus  $\theta(2e_0^* + e_1^* + \cdots + e_n^*) = 2e_0^* + e_1^* + \cdots + e_n^*$ , so  $\theta(2e_0^*) - e_n^* = 2e_0^* + e_n^*$  and  $\theta(e_0^*) = e_0^* + e_n^*$ .  $\square$

**Lemma 2.5.** *We have  $Z(\mathrm{GSpin}_{2n}) = \{(s_0, \dots, s_n) : s_1 = s_2 = \cdots = s_n \in \{\pm 1\}\}$ , which is isomorphic to  $\mathbb{G}_m \times \{\pm 1\}$  via  $(s_0, \dots, s_n) \mapsto (s_0, s_1)$ . In the latter coordinate,  $\theta(s_0, s_1) = (s_0 s_1, s_1)$ .*

*Proof.* Let  $s \in \mathrm{T}_{\mathrm{GSpin}}$ . Then  $s \in Z(\mathrm{GSpin}_{2n})$  if and only if  $\alpha^\vee(t) = 1$  for all  $\alpha^\vee \in \Delta^\vee$ . From Lemma 2.1(iii) we obtain  $s_i/s_{i+1} = 1$  ( $i \leq n-1$ ), and  $s_{n-1}s_n = 1$ . Hence  $s \in Z(\mathrm{GSpin}_{2n})$  if and only if  $s_1 = \cdots = s_n \in \{\pm 1\}$ . By (2.3) we get  $\theta(s_0, s_1) = (s_0 s_1, s_1)$ .  $\square$

The Weyl group of  $\mathrm{GSO}_{2n}$  (and  $\mathrm{GSpin}_{2n}$ ) is equal to  $\{\pm 1\}^{n'} \rtimes \mathfrak{S}_n$ , where  $\{\pm 1\}^{n'}$  is the group of  $a \in \{\pm 1\}^n$  such that  $\prod_1^n a(i) = 1$ . The action of  $\mathrm{W}_{\mathrm{GSO}}$  on  $\mathrm{T}_{\mathrm{GSO}}$  is determined by

$$(2.6) \quad \begin{cases} \sigma \cdot (t_0, t_1, \dots, t_n) = (t_0, t_{\sigma^{-1}1}, \dots, t_{\sigma^{-1}n}) & \sigma \in \mathfrak{S}_n \\ a \cdot (t_0, t_1, \dots, t_n) = (t_0, t_0 t_1^{-1}, t_0 t_2^{-1}, t_3, \dots, t_n) & a = (-1, -1, 1, \dots, 1) \in \{\pm 1\}^{n'}. \end{cases}$$

We define, for  $\varepsilon \in \{\pm 1\}$  the following cocharacter

$$(2.7) \quad \mu_\varepsilon := \begin{cases} (1, 1, \dots, 1, 1) & \text{if } \varepsilon = (-1)^n \\ (1, 1, \dots, 1, 0) & \text{if } \varepsilon = (-1)^{n+1} \end{cases} \in \mathbb{Z}^{n+1} = X_*(\mathrm{T}_{\mathrm{GSO}}) = X^*(\mathrm{T}_{\mathrm{GSpin}}).$$

Then  $\mu_\varepsilon$  is a minuscule cocharacter of  $\mathrm{GSO}_{2n}$  with  $\langle \alpha_i, \mu_\varepsilon \rangle = 1$  if and only if  $i = n$  (for  $\varepsilon = (-1)^n$ ) and  $i = n-1$  (for  $\varepsilon = (-1)^{n+1}$ ).

**Definition 2.6.** For  $\varepsilon \in \{+, -\}$ , define the *half spin representation*  $\mathrm{spin}^\varepsilon = \mathrm{spin}_{2n}^\varepsilon$  to be the irreducible representation of  $\mathrm{GSpin}_{2n}$  whose highest weight is equal to  $\mu_\varepsilon$  in  $X^*(\mathrm{T}_{\mathrm{GSpin}})$ . By the *spin representation* of  $\mathrm{GSpin}_{2n}$  we mean  $\mathrm{spin} := \mathrm{spin}^+ \oplus \mathrm{spin}^-$ .

These representations will be realized explicitly via Clifford algebras. Our sign convention is natural in that  $\mathrm{spin}^+$  (resp.  $\mathrm{spin}^-$ ) accounts for even (resp. odd) degree elements. See (4.2) and Lemma 4.1 below.

Under the action of the Weyl group the minuscule  $\mu_\varepsilon$  has  $2^{n-1}$  translates. Thus the half spin representations have dimension  $2^{n-1}$ , and more precisely the weights of  $\text{spin}_{2n}^{(-1)^n}$  are

$$(2.8) \quad \text{T}_{\text{GSpin}} \ni (s_0, s_1, \dots, s_n) \mapsto \left( s_0 \prod_{i \in U} s_i \right)_{U \subset \{1, 2, \dots, n\}, 2 | \#U} \in \mathbb{Z}^{n+1} = X_*(\text{T}_{\text{GSO}}) = X^*(\text{T}_{\text{GSpin}})$$

and  $\text{spin}_{2n}^{(-1)^{n+1}}$  has similar weights, except that the cardinality of  $U$  is now required to be odd. By computing the  $\theta$ -action on weights, we verify that (see Lemma 4.4 for an explicit intertwiner)

$$\text{spin}^+ \circ \theta \simeq \text{spin}^- \quad \text{and} \quad \text{spin}^- \circ \theta \simeq \text{spin}^+.$$

**Lemma 2.7.** *The kernel  $Z^\varepsilon$  of  $\text{spin}^\varepsilon$  is central in  $\text{GSpin}_{2n}$  and finite of order 2. The non-trivial element  $z_\varepsilon$  of  $Z^\varepsilon$  equals  $(\varepsilon, -1) \in \mathbb{G}_m \times \{\pm 1\}$ . The spin representation of  $\text{GSpin}_{2n}$  is faithful.*

*Proof.* Since  $\text{GSpin}_{2n}$  is simple modulo the center, the kernel  $Z^\varepsilon \subset \text{GSpin}_{2n}$  must be central. The central character is the restriction of  $\mu_\varepsilon: \text{T}_{\text{GSpin}} \rightarrow \mathbb{G}_m$  to the center  $Z(\text{GSpin}_{2n}) \subset \text{T}_{\text{GSpin}}$ . Let  $s = (s_0, s_1, \dots, s_n) = (a, b) \in Z(\text{GSpin}_{2n}) \subset \text{T}_{\text{GSpin}}$ . Then (see proof of Lemma 2.5)

$$(2.9) \quad \mu_\varepsilon(s) = \begin{cases} s_0 s_1 \cdots s_n = ab^n & \text{if } \varepsilon = (-1)^n \\ s_0 s_1 \cdots s_{n-1} = ab^{n-1} & \text{if } \varepsilon = (-1)^{n+1}. \end{cases}$$

The first assertion follows by considering the 4 different cases where  $n$  even or odd and  $\varepsilon = \pm 1$ . For the second point, it suffices to observe that  $Z^+ \cap Z^- = \{1\}$ .  $\square$

### 3. CLIFFORD ALGEBRAS AND CLIFFORD GROUPS

We recall how  $\text{GSpin}_{2n}$  is realized using the Clifford algebra and define a number of fundamental maps such as  $i_{\text{std}}: \text{GSpin}_{2n-1} \hookrightarrow \text{GSpin}_{2n}$  and the projection  $\text{GSpin}_{2n} \twoheadrightarrow \text{GSO}_{2n}$ . We also give a concrete definition of outer automorphisms of  $\text{GSpin}_{2n}$  and  $\text{GSO}_{2n}$  that are compatible with each other via the projection map. Our main reference is [Bas74], which introduces Clifford algebras over arbitrary commutative rings (with unity). Other useful references are [Bou07, §9] and [FH91, §20].

Let  $V$  be a quadratic space over  $\mathbb{Q}$  with quadratic form  $Q$ , giving rise to the groups  $\text{O}(V)$ ,  $\text{GO}(V)$ ,  $\text{SO}(V)$  and  $\text{GSO}(V)$ . The *Clifford algebra*  $C(V)$  is a universal map  $V \rightarrow C(V)$  which is initial in the category of  $\mathbb{Q}$ -linear maps  $f: V \rightarrow A$  into associative  $\mathbb{Q}$ -algebras  $A$  with unity  $1_A$  such that  $f(v)^2 = Q(v) \cdot 1_A$  for all  $v \in V$ . (See [Bas74, (2.3)] or [Bou07, §9.1].)

We define  $\langle x, y \rangle := Q(x+y) - Q(x) - Q(y)$  for  $x, y \in V$ , and similarly  $\langle x, y \rangle = (x+y)^2 - x^2 - y^2$  for  $x, y \in C(V)$ . In particular  $\langle x, y \rangle$  measures if  $x$  and  $y$  anti-commute in  $C(V)$ :

$$(3.1) \quad \langle x, y \rangle = (x+y)^2 - x^2 - y^2 = xy + yx \in C(V).$$

The map  $V \rightarrow C(V)$  induces a map  $V \rightarrow C(V)^{\text{opp}}$  (sending each  $v \in V$  to the same element), where  $C(V)^{\text{opp}}$  is the opposite algebra. The latter factors through a unique  $\mathbb{Q}$ -algebra map  $\beta: C(V) \rightarrow C(V)^{\text{opp}}$ . It is readily checked that  $\beta^2$  is the identity on  $C(V)$ . By the universal property  $\beta$  is the unique involution of  $C(V)$  that is the identity on  $V$ .

The universal property also yields a surjection from the tensor algebra

$$\bigoplus_{d \in \mathbb{Z}_{\geq 0}} V^{\otimes d} \rightarrow C(V).$$

Define  $C^+ = C(V)^+$  (resp.  $C^- = C(V)^-$ ) to be the image of  $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} V^{\otimes 2d}$  (resp.  $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} V^{\otimes 2d+1}$ ) so that  $C(V) = C(V)^+ \oplus C(V)^-$ . In fact the discussion of Clifford algebras so far works when  $V$  is replaced with a quadratic space on a module over an arbitrary commutative ring, in a way compatible with base change: in particular if  $R$  is a (commutative)  $\mathbb{Q}$ -algebra then  $C(V \otimes_{\mathbb{Q}} R) = C(V) \otimes_{\mathbb{Q}} R$  [Bou07, §9.1, Prop 2]. By scalars in  $C(V \otimes_{\mathbb{Q}} R)$  we mean  $R$  times the multiplicative unity. We keep using  $\beta$  to denote the main involution of  $C(V \otimes_{\mathbb{Q}} R)$ .

The *Clifford group*  $\text{GPin}(V)$  is the  $\mathbb{Q}$ -group such that for every  $\mathbb{Q}$ -algebra  $R$ ,

$$\text{GPin}(V)(R) = \{x \in C(V \otimes_{\mathbb{Q}} R)^\times : x(V \otimes_{\mathbb{Q}} R)x^{-1} = V \otimes_{\mathbb{Q}} R, x \text{ is homogeneous}\},$$

where homogeneity of  $x$  means that  $x \in C(V \otimes_{\mathbb{Q}} R)^{\varepsilon}$  for some sign  $\varepsilon$ . The *special Clifford group*  $\mathrm{GSpin}(V)$  is defined similarly with  $C^+$  in place of  $C$ . The embedding of invertible scalars in  $C(V \otimes_{\mathbb{Q}} R)$  induces a central embedding

$$(3.2) \quad \mathbb{G}_m \rightarrow \mathrm{GSpin}(V).$$

Since  $x\beta(x) \in R$  for  $x \in C(V \otimes_{\mathbb{Q}} R)$  by [Bas74, Prop 3.2.1 (a)], we have the *spinor norm* morphism

$$\mathcal{N}: \mathrm{GPin}(V) \rightarrow \mathbb{G}_m, \quad x \mapsto x\beta(x)$$

over  $\mathbb{Q}$ . (The involution in *loc. cit.* differs from our  $\beta$  by  $C(-1_P)$  in their notation, so our  $\mathcal{N}$  does not coincide with their  $N$ , but  $\mathcal{N}$  and  $N$  have the same kernel.) Evidently, composing  $\mathcal{N}$  with (3.2) yields the squaring map.

Define  $\mathrm{Spin}(V)$  by the following exact sequence of algebraic groups:

$$1 \rightarrow \mathrm{Spin}(V) \rightarrow \mathrm{GSpin}(V) \xrightarrow{\mathcal{N}} \mathbb{G}_m \rightarrow 1.$$

**Lemma 3.1.** *The following are true.*

- (i) The map  $\mathrm{pr}^{\circ} = \mathrm{pr}_V^{\circ}: \mathrm{GPin}(V) \rightarrow \mathrm{O}(V)$ ,  $x \mapsto (v \mapsto xv x^{-1})$  is surjective for  $n$  even, and  $\mathrm{pr}^{\circ}: \mathrm{GPin}(V) \rightarrow \mathrm{SO}(V)$  is surjective when  $n$  is odd.
- (ii) We have  $\ker(\mathrm{pr}^{\circ}) = \mathbb{G}_m$  via (3.2).
- (iii)  $\mathrm{pr}: \mathrm{GPin}(V) \rightarrow \mathrm{GO}(V)$ ,  $x \mapsto (v \mapsto xv\beta(x))$  is a surjection, and  $\mathrm{sim} \circ \mathrm{pr} = \mathcal{N}^2$ .
- (iv) The map  $\mathrm{pr}$  factors as  $\mathrm{GPin}(V) \xrightarrow{(\mathrm{pr}^{\circ}, \mathcal{N})} \mathrm{O}(V) \times \mathrm{GL}_1 \xrightarrow{\mathrm{mult}} \mathrm{GO}(V)$ , where the latter is the multiplication map. The map  $(\mathrm{pr}^{\circ}, \mathcal{N})$  has kernel  $\{\pm 1\}$  (scalars in  $C(V)$ ) and image  $\mathrm{O}(V) \times \mathrm{GL}_1$  (resp.  $\mathrm{SO}(V) \times \mathrm{GL}_1$ ) for  $n$  even (resp. odd).
- (v) The multiplication map  $\mathrm{Spin}(V) \times \mathbb{G}_m \rightarrow \mathrm{GSpin}(V)$  is a surjection with kernel  $\{\pm(1, 1)\}$ , where  $\{\pm 1\} \hookrightarrow \mathrm{Spin}(V)$  via (3.2).

*Proof.* (i) The surjectivity can be checked on field-valued points. This is proved in [Bou07, §9.5, Thm. 4].

(ii) As  $V \subset C(V)$  generates the Clifford algebra, the identity  $xvx^{-1} = v$  implies  $xyx^{-1} = y$  for all  $y \in C(V)$ , and the analogue holds for  $C(V \otimes_{\mathbb{Q}} R)$  for  $\mathbb{Q}$ -algebras  $R$ . Thus  $\ker(\mathrm{pr}^{\circ})(R)$  consists of invertible elements in the center of  $C(V \otimes_{\mathbb{Q}} R)$ . Let  $W \subset V$  be an isotropic subspace. Then  $C(V \otimes_{\mathbb{Q}} R) \simeq \mathrm{End}(\wedge(W \otimes_{\mathbb{Q}} R))$  as super  $R$ -algebras by [Bas74, (2.4) Thm.], so the center of  $C(V \otimes_{\mathbb{Q}} R)$  is  $R$ , implying that  $\ker(\mathrm{pr}^{\circ}) = \mathbb{G}_m$ .

(iii) We observe that  $\mathrm{pr}(x)$  preserves  $V$ : as  $x(V \otimes_{\mathbb{Q}} R)x^{-1} = V \otimes_{\mathbb{Q}} R$  and  $x\beta(x) \in R^{\times}$  imply that  $x(V \otimes_{\mathbb{Q}} R)\beta(x) = V \otimes_{\mathbb{Q}} R$ . Moreover  $\mathrm{pr}(x) \in \mathrm{GO}(V)$  as

$$(3.3) \quad Q(xv\beta(x)) = xv\beta(x)xv\beta(x) = \mathcal{N}(x)^2 Q(v).$$

Moreover  $\mathrm{pr}$  and  $\mathrm{pr}^{\circ}$  coincide on  $\mathrm{Pin}(V)$ , so  $(\mathrm{S})\mathrm{O}(V)$  is in the image of  $\mathrm{pr}$ . On the other hand,  $\mathcal{N}$  is seen to be surjective by considering scalar elements, telling us that the image of  $\mathrm{pr}$  also contains  $\mathbb{G}_m$  (scalar matrices in  $\mathrm{GO}(V)$ ). Since  $\mathbb{G}_m$  and  $(\mathrm{S})\mathrm{O}(V)$  generate  $\mathrm{G}(\mathrm{S})\mathrm{O}(V)$ , the surjectivity of  $\mathrm{pr}$  follows. The equality  $\mathrm{sim} \circ \mathrm{pr} = \mathcal{N}^2$  follows from (3.3).

(iv) The first part follows from  $\mathrm{pr}(x)(v) = xv\beta(x) = xv x^{-1}x\beta(x) = \mathrm{pr}^{\circ}(x)(v)\mathcal{N}(x)$  when  $x \in \mathrm{GPin}(V)$  and  $v \in V$ . The second part is easily seen from (i) and (ii).

(v) This readily follows from the preceding points.  $\square$

If  $V$  is odd dimensional then  $\mathrm{SO}(V) \times \{\pm 1\} = \mathrm{O}(V)$ , and the group  $\mathrm{GO}(V)$  is connected. For convenience we define  $\mathrm{GSO}(V) := \mathrm{GO}(V)$  in this case. If  $\dim(V)$  is even, then  $\mathrm{O}(V)$  (resp.  $\mathrm{GO}(V)$ ) has two connected components but does not admit a direct product decomposition into  $\mathrm{O}(V)$  (resp.  $\mathrm{GSO}(V)$ ) and  $\{\pm 1\}$ .

Assume that we have an orthogonal sum decomposition  $\varphi: W_1 \oplus W_2 \xrightarrow{\sim} V$  of non-degenerate quadratic spaces over  $\mathbb{Q}$ . As super algebras we have ([Bas74, (2.3)] or [Bou07, §9.3, Cor. 3, Cor. 4])

$$C_{\varphi}: C(W_1) \widehat{\otimes} C(W_2) \xrightarrow{\sim} C(V), \quad w_1 \widehat{\otimes} w_2 \mapsto w_1 w_2.$$

By definition, the algebra given by  $\widehat{\otimes}$  on the left side has underlying vector space  $C(W_1) \otimes C(W_2)$  and product

$$(a \widehat{\otimes} b) \cdot (c \widehat{\otimes} d) := (-1)^{k_b \cdot k_c} ac \widehat{\otimes} bd,$$

if  $a, c \in C(W_1)$ ,  $b, d \in C(W_2)$  are homogeneous elements of degree  $k_a, k_b, k_c, k_d \in \mathbb{Z}/2\mathbb{Z}$ . The sign is there to make  $C_\varphi$  compatible with products since  $bc = (-1)^{k_b k_c} cb$  in  $C(V)$ .

In fact  $C_\varphi$  intertwines the involution  $\beta$  on  $C(V)$  with the involution

$$\beta': C(W_1) \widehat{\otimes} C(W_2) \rightarrow C(W_1) \widehat{\otimes} C(W_2), \quad \beta'(a \widehat{\otimes} b) = (-1)^{k_a k_b} \beta_1(a) \widehat{\otimes} \beta_2(b),$$

for homogeneous elements  $a \in C(W_1)$ ,  $b \in C(W_2)$  of degree  $k_a, k_b \in \mathbb{Z}/2\mathbb{Z}$ , where  $\beta_1, \beta_2$  are the involutions of  $C(W_1)$  and  $C(W_2)$  (see below (3.1)). To verify that  $\beta$  is compatible with  $\beta'$ , observe that  $\beta$  on  $C(V)$  restricts to  $\beta_1, \beta_2$  via the obvious inclusions  $C(W_1) \hookrightarrow C(V)$  and  $C(W_2) \hookrightarrow C(V)$  induced by  $W_1 \subset V$  and  $W_2 \subset V$  (since  $\beta$  acts as the identity on both  $W_1$  and  $W_2$ ), and use the property that  $\beta_1, \beta_2$ , and  $\beta$  are preserving degrees. It follows that

$$\beta(ab) = \beta(b)\beta(a) = (-1)^{k_a k_b} \beta(a)\beta(b) = (-1)^{k_a k_b} \beta_1(a)\beta_2(b).$$

**Lemma 3.2.** *The mapping  $C_\varphi$  induces a morphism  $\mathrm{GSpin}(W_1) \times \mathrm{GSpin}(W_2) \rightarrow \mathrm{GSpin}(V)$ .*

*Proof.* We check that the image of  $C_\varphi$  is in  $\mathrm{GSpin}(V)$ . Let  $g \in \mathrm{GSpin}(W_1)$ ,  $h \in \mathrm{GSpin}(W_2)$ . Note that  $C_\varphi(g \widehat{\otimes} h) = gh \in C^+(V)$ . Let  $w_1 + w_2 \in V$  with  $w_i \in W_i$ ,  $i = 1, 2$ . To verify that  $gh \in \mathrm{GSpin}(V)$ , since homogeneous elements of even degree commute with each other if they are perpendicular, we see that

$$gh(w_1 + w_2)h^{-1}g^{-1} = gw_1g^{-1} + hw_2h^{-1} \in V.$$

□

Write

$$\begin{aligned} G(\mathrm{Pin}(W_1) \times \mathrm{Pin}(W_2)) &:= \{(g, h) \in \mathrm{GPin}(W_1) \times \mathrm{GPin}(W_2) \mid \mathcal{N}(g) = \mathcal{N}(h)\}, \\ G(\mathrm{O}(W_1) \times \mathrm{O}(W_2)) &:= \{(g, h) \in \mathrm{GO}(W_1) \times \mathrm{GO}(W_2) \mid \mathrm{sim}(g) = \mathrm{sim}(h)\}. \end{aligned}$$

**Lemma 3.3.** *The diagram*

$$\begin{array}{ccc} \mathrm{GSpin}(W_1) \times \mathrm{GSpin}(W_2) & \longrightarrow & \mathrm{GSpin}(V) \\ \mathrm{pr}_{W_1}^\circ \times \mathrm{pr}_{W_2}^\circ \downarrow & & \downarrow \mathrm{pr}_V^\circ \\ \mathrm{SO}(W_1) \times \mathrm{SO}(W_2) & \xrightarrow{i_{W_1, W_2}} & \mathrm{SO}(V) \end{array}$$

*commutes, where  $i_{W_1, W_2}$  is the block diagonal embedding.*

*Proof.* Immediate from the computation in the proof of the preceding lemma. □

In later chapters we will carry out explicit computations. It will then be convenient to work with fixed bases and quadratic forms. For this reason we now fix quadratic forms on the vector spaces  $V_{2n} = \mathbb{C}^{2n}$  and  $V_{2n-1} = \mathbb{C}^{2n-1}$ . We take the following quadratic forms:

$$(3.4) \quad \begin{aligned} Q_{2n}: & x_1 x_{n+1} + x_2 x_{n+2} + \dots + x_n x_{2n} \text{ on } \mathbb{C}^{2n} \\ Q_{2n-1}: & y_1 y_{n+1} + \dots + y_{n-2} y_{2n-2} + y_{2n-1}^2 \text{ on } \mathbb{C}^{2n-1}. \end{aligned}$$

Using them, we write  $\mathrm{SO}_m = \mathrm{SO}(V_m)$ ,  $\mathrm{GSO}_m = \mathrm{GSO}(V_m)$ , and likewise for  $\mathrm{O}_m, \mathrm{GO}_m$ , for  $m = 2n$  and  $m = 2n - 1$ . This is identical to the convention of §2 for  $m$  even. Similarly we write  $\mathrm{pr}_{2n-1}^\circ = \mathrm{pr}_{V_{2n-1}}^\circ$  and  $\mathrm{pr}_{2n}^\circ = \mathrm{pr}_{V_{2n}}^\circ$ .

Now we claim that  $\mathrm{GSpin}(V_{2n})$  is isomorphic to  $\mathrm{GSpin}_{2n}$  of §8 that is, the Clifford algebra definition is compatible with the root-theoretic definition as the Langlands dual of  $\mathrm{GSO}_{2n}$ . (An analogous argument shows that  $\mathrm{GSpin}_{2n-1}$  is dual to  $\mathrm{GSp}_{2n-2}$ .) As this is a routine exercise, we only sketch the argument. First,  $\mathrm{pr}^\circ$  restricts to a connected double covering  $\mathrm{Spin}(V_m) \rightarrow \mathrm{SO}(V_m)$  ([FH91, Prop. 20.38]), which must then be the unique (up to isomorphism) simply connected covering. This determines the root datum of  $\mathrm{Spin}(V_m)$ . From this, we compute the root datum of  $\mathrm{GSpin}(V_m)$  via the central isogeny  $\mathrm{Spin}(V_m) \times \mathbb{G}_m \rightarrow \mathrm{GSpin}(V_m)$  of Lemma

3.1. Finally when  $m = 2n$ , we deduce that the outcome is dual to the root datum of  $\mathrm{GSO}_{2n}$  in Lemma 2.1. Therefore  $\mathrm{GSpin}(V_{2n})$  is isomorphic to  $\mathrm{GSpin}_{2n}$  of §8. Henceforth we identify

$$(3.5) \quad \mathrm{GSpin}(V_{2n}) = \mathrm{GSpin}_{2n}.$$

In fact we may and will choose  $B_{\mathrm{GSpin}}$  and  $T_{\mathrm{GSpin}}$  to be the preimages of  $B_{\mathrm{SO}}$  and  $T_{\mathrm{SO}}$  via  $\mathrm{pr}^\circ : \mathrm{GSpin}_{2n} \rightarrow \mathrm{SO}_{2n}$ . subgroups of  $\mathrm{GSpin}(V_{2n})$ . We fix pinnings of  $\mathrm{GSpin}_{2n}$ ,  $\mathrm{GSO}_{2n}$ , and  $\mathrm{SO}_{2n}$  (which are  $\Gamma_F$ -equivariant if  $(V_{2n}, Q_{2n})$  is defined over  $F$ ) compatibly via  $\mathrm{pr}$  and  $\mathrm{pr}^\circ$ .

**Lemma 3.4.** *Via (3.5), the central embedding of scalar matrices  $\mathrm{cent}^\circ : \mathbb{G}_m \rightarrow \mathrm{GSO}_{2n}$  and  $\mathrm{sim} : \mathrm{GSO}_{2n} \rightarrow \mathbb{G}_m$  are dual to  $\mathcal{N} : \mathrm{GSpin}_{2n} \rightarrow \mathbb{G}_m$  and the central embedding  $\mathrm{cent} : \mathbb{G}_m \rightarrow \mathrm{GSpin}_{2n}$  of (3.2), respectively.*

*Remark 3.5.* The dual map of  $\mathrm{cent}^\circ$  was made explicit in (2.5). According to the present lemma, (2.5) gives an explicit formula for  $\mathcal{N}$  restricted to  $T_{\mathrm{GSpin}}$ .

*Proof.* Write  $Z^0$  for the identity component of the center of  $\mathrm{GSpin}_{2n}$ , consisting of  $(s_0, 1, \dots, 1)$  with  $s_0 \in \mathbb{G}_m$  in the notation of Lemma 2.5. The dual of  $\mathrm{sim} : \mathrm{GSO}_{2n} \rightarrow \mathbb{G}_m$  is calculated as the central cocharacter  $\mathbb{G}_m \rightarrow Z^0 \subset \mathrm{GSpin}_{2n}$ ,  $z \mapsto (z, 1, \dots, 1)$ . The inclusion  $\mathrm{cent} : \mathbb{G}_m \rightarrow \mathrm{GSpin}_{2n}$  identifies  $\mathbb{G}_m$  with  $Z^0$ . Thus  $\mathrm{cent}$  is dual to  $\mathrm{sim}$ .

Both  $\mathcal{N} \circ \mathrm{cent}$  and  $\mathrm{sim} \circ \mathrm{cent}^\circ$  are the squaring map on  $\mathbb{G}_m$ . Using the hat symbol to denote a dual morphism, we see that

$$\mathcal{N} \circ \mathrm{cent} = \widehat{\mathrm{cent}^\circ} \circ \widehat{\mathrm{sim}} = \widehat{\mathrm{cent}^\circ} \circ \mathrm{cent}$$

and that they are all equal to the squaring map. It follows that  $\mathcal{N}$  is dual to  $\mathrm{cent}^\circ$ .  $\square$

We have the morphism of quadratic spaces

$$\varphi : (\mathbb{C}^{2n-1}, Q_{2n-1}) \rightarrow (\mathbb{C}^{2n}, Q_{2n}), \quad y \mapsto (y_1, y_2, \dots, y_{n-1}, y_{2n-1}, y_n, y_{n+1}, \dots, y_{2n-1}).$$

Indeed,  $Q_{2n}\varphi = Q_{2n-1}$  as readily checked. We have the complementary embedding:

$$\varphi' : \mathbb{C} \rightarrow \mathbb{C}^{2n}, \quad u \mapsto x, \quad \text{where} \quad \begin{cases} x_k = 0 & k \neq n, 2n \\ x_k = (-1)^{k/n}u & \text{if } k = n \text{ or } k = 2n. \end{cases}$$

Write  $U := \varphi'(\mathbb{C}) = (e_n - e_{2n}) \cdot \mathbb{C}$  for the image. The induced quadratic form on  $U$  is then  $a \cdot (e_n - e_{2n}) \mapsto -a^2$ . This gives us an orthogonal decomposition of quadratic spaces  $\mathbb{C}^{2n} = \mathbb{C}^{2n-1} \hat{\oplus} U$ . Let  $\mathrm{PO}_m$  to denote the adjoint group of  $\mathrm{O}_m$ . The decomposition induces morphisms (cf. Lemmas 3.2, 3.3)

$$(3.6) \quad \begin{aligned} i_{\mathrm{std}} &:= C_{\varphi, \varphi'} : \mathrm{GSpin}_{2n-1} \times \mathrm{GSpin}_1 \rightarrow \mathrm{GSpin}_{2n}, \\ i_{\mathrm{std}}^\circ &:= i_{\mathbb{C}^{2n-1}, \mathbb{C}} : \mathrm{O}_{2n-1} \times \mathrm{O}_1 \rightarrow \mathrm{O}_{2n}, \text{ and} \\ \overline{i_{\mathrm{std}}} &:= \mathrm{PO}_{2n-1} \rightarrow \mathrm{PO}_{2n}, \end{aligned}$$

where  $\overline{i_{\mathrm{std}}}$  is induced from  $i_{\mathrm{std}} : \mathrm{GSpin}_{2n-1} \times \mathrm{GSpin}_1 \rightarrow \mathrm{GSpin}_{2n} \rightarrow \mathrm{PSO}_{2n} \subset \mathrm{PO}_{2n}$ . By Lemma 3.3, we have  $\mathrm{pr}^\circ \circ i_{\mathrm{std}} = i_{\mathrm{std}}^\circ \circ (\mathrm{pr}_{2n-1}^\circ \times \mathrm{pr}_U^\circ)$ .

Let  $1_{2n-1}, 1_U$  denote the identity map on  $\mathbb{C}^{2n-1}, U$ . Then (cf. (2.2))

$$i_{\mathrm{std}}^\circ(-1_{2n-1}, 1_U) = - \left( \begin{array}{c|c} 1_{n-1} & \\ \hline 0 & 1 \\ \hline 1 & 0 \\ \hline & 1_{n-1} \end{array} \right) = \vartheta^\circ \in \mathrm{O}_{2n}.$$

Fix  $\sqrt{-1} \in \mathbb{G}_m = Z(\mathrm{GPin}_{2n})$ . Define

$$(3.7) \quad \vartheta := \sqrt{-1} \cdot i_{\mathrm{std}}(1_{\mathbb{C}(\mathbb{C}^{2n-1})} \hat{\otimes} (e_n - e_{2n})) = \sqrt{-1}(e_n - e_{2n}) \in \mathrm{GPin}_{2n} \setminus \mathrm{GSpin}_{2n}.$$

**Lemma 3.6.** *We have*

- (i)  $\mathrm{pr}_{2n}^\circ(\vartheta) = \vartheta^\circ$  and  $\vartheta^2 = 1$ .
- (ii) *The conjugation action of  $\vartheta$  (resp.  $\vartheta^\circ$ ) fixes the subgroup  $i_{\mathrm{std}}(\mathrm{GSpin}_{2n-1} \times \mathrm{GSpin}_1) \subset \mathrm{GSpin}_{2n}$  via  $i_{\mathrm{std}}$  (resp.  $\mathrm{SO}_{2n-1} \times \mathrm{SO}_1 \subset \mathrm{SO}_{2n}$  via  $i_{\mathrm{std}}^\circ$ ) and induces the identity automorphism on that subgroup.*



(iii) The conjugation action of  $\vartheta$  (resp.  $\vartheta^\circ$ ) defines the outer automorphism  $\theta$  of  $\mathrm{GSpin}_{2n}$  (resp.  $\theta^\circ$  of  $\mathrm{GSO}_{2n}$ ) in Lemmas 2.3 and 2.4.

*Proof.* (i) Let  $w_1 \in \mathbb{C}^{2n-1}$  and  $w_2 := e_n - e_{2n} \in U$ . All of  $w_1, w_2, \vartheta$  have degree 1 in  $C(\mathbb{C}^{2n})$ . In either  $C(\mathbb{C}^{2n})$  or  $C(U)$ , we have  $w_2^2 = Q_{2n}(w_2) = -1$  and  $\vartheta^2 = -w_2^2 = 1$ . Thus  $\vartheta w_1 \vartheta^{-1} = -w_1 \vartheta \vartheta^{-1} = -w_1$  and  $\vartheta w_2 \vartheta^{-1} = w_2$ . Hence  $\mathrm{pr}_{2n}^\circ(\vartheta) = \vartheta^\circ$ .

(ii) This is obvious for  $\vartheta^\circ$ . The conjugation by  $\vartheta$  is the identity on  $C^+(\mathbb{C}^{2n-1})$  and  $C^+(U)$ , since  $\vartheta \perp \mathbb{C}^{2n-1}$  and  $C^+(U)$  is commutative, respectively. The assertion for  $\vartheta$  follows.

(iii) This is true by definition for  $\theta^\circ$ . Since  $\theta$  and the conjugation by  $\vartheta$  act trivially on the center of  $\mathrm{GSpin}_{2n}$ , it suffices to check that their actions are identical on the adjoint group. This reduces to the fact that  $\theta^\circ$  is given by the  $\vartheta^\circ$ -conjugation, as  $\theta$  and  $\theta^\circ$  (resp.  $\vartheta$  and  $\vartheta^\circ$ ) induce the same action on the adjoint group (thanks to part (i)).  $\square$

We have fixed pinnings of  $\mathrm{GSpin}_{2n}$ ,  $\mathrm{GSO}_{2n}$ , and  $\mathrm{SO}_{2n}$  compatibly via  $\mathrm{pr}$ . They are fixed by  $\theta \in \mathrm{Aut}(\mathrm{GSpin}_{2n})$  and  $\theta^\circ \in \mathrm{Aut}(\mathrm{GSO}_{2n})$ . It is easy to see that  $\theta$  and  $\theta^\circ$  induce automorphisms of based root data, which correspond to each other via duality of the two based root data. Thus letting  $E/F$  be a quadratic extension of fields of characteristic 0, and  $\mathrm{GSO}_{2n}^{E/F}$  an outer form of  $\mathrm{GSO}_{2n}$  over  $F$  with respect to the Galois action  $\Gamma_{E/F} = \{1, c\} \xrightarrow{\sim} \{1, \theta\}$ , we can identify

$$L(\mathrm{GSO}_{2n}^{E/F}) = \mathrm{GSpin}_{2n} \rtimes \{1, c\} = \mathrm{GPin}_{2n},$$

where the semi-direct product is given by  $cgc^{-1} = \theta(g)$ . (Of course  $c = c^{-1}$ .) The second identification above is via  $c \mapsto \vartheta$ . Similarly, for  $\mathrm{SO}_{2n}^{E/F}$  an outer form of  $\mathrm{SO}_{2n}$  with respect to  $\Gamma_{E/F} = \{1, c\} \xrightarrow{\sim} \{1, \theta^\circ\}$ , we have

$$L(\mathrm{SO}_{2n}^{E/F}) = \mathrm{SO}_{2n} \rtimes \{1, c\} = \mathrm{O}_{2n} \quad \text{via } c \mapsto \vartheta^\circ.$$

Let us describe the center  $Z(\mathrm{Spin}_{2n})$  of  $\mathrm{Spin}_{2n} = \mathrm{Spin}(V_{2n})$  explicitly as this is going to be useful for classifying inner twists of (quasi-split forms of)  $\mathrm{SO}_{2n}$  and  $\mathrm{GSO}_{2n}$  in §8. In what follows, we identify  $Z(\mathrm{GSpin}_{2n}) = \{(s_0, s_1) : s_0 \in \mathbb{G}_m, s_1 \in \{\pm 1\}\}$  as in Lemma 2.5 and write  $1, -1$  for  $(1, 1), (1, -1) \in Z(\mathrm{GSpin}_{2n})$ .

**Lemma 3.7.** *Let  $\zeta_4$  be a primitive fourth root of unity. We have  $Z(\mathrm{Spin}_{2n}) \subset Z(\mathrm{GSpin}_{2n})$  via  $T_{\mathrm{Spin}} \subset T_{\mathrm{GSpin}}$  explained above. Moreover, the following are true.*

- (i) *If  $n$  is even,  $Z(\mathrm{Spin}_{2n}) = \{1, -1, z_+, z_-\}$  and is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^2$ . If  $n$  is odd,  $Z(\mathrm{Spin}_{2n}) = \{1, -1, \zeta, -\zeta = \zeta^{-1}\}$  and is isomorphic to  $\mathbb{Z}/4\mathbb{Z}$ , where  $\zeta = (\zeta_4, -1)$ .*
- (ii) *The action of  $\theta$  is trivial on  $\{1, -1\}$  and permuting  $\{z_+, z_-\}$  (resp.  $\{\zeta, -\zeta\}$ ).*

*Proof.* We have  $Z(\mathrm{Spin}_{2n}) = Z(\mathrm{GSpin}_{2n}) \cap \mathrm{Spin}_{2n} = \{z \in Z(\mathrm{GSpin}_{2n}) : \mathcal{N}(z) = 1\}$ , where  $\mathcal{N}$  is described by (2.5) (Remark 3.5). It follows from Lemma 2.5 that

$$Z(\mathrm{Spin}_{2n}) = \{(s_0, s_1) : s_0^2 = s_1^n\},$$

which is alternatively described as in (i). Assertion (ii) is also clear from that lemma.  $\square$

#### 4. THE SPIN REPRESENTATIONS

We recall the definition of the spin representations defined using the Clifford algebra, and show that they coincide with the representations that we defined in Section 2 in terms of root data.

Consider the quadratic space  $V_{2n} := \mathbb{C}^{2n}$  from (3.4) with standard basis  $\{e_1, \dots, e_{2n}\}$  and quadratic form  $Q_{2n}$ . Define  $W_{2n} := \bigoplus_{i=1}^n \mathbb{C}e_i$  and  $W'_{2n} := \bigoplus_{i=n+1}^{2n} \mathbb{C}e_i$ . We often omit the subscript  $2n$  to lighten notation, when there is no danger of confusion. Since  $W$  is isotropic we obtain a morphism  $\bigwedge W \xrightarrow{\sim} C(W) \hookrightarrow C(V)$ . Through this injection we view  $\bigwedge W$  as a subspace of  $C(V)$ . The subspace  $\bigwedge W$  is a left  $C(V)$ -ideal, whose  $C(V)$ -module structure is uniquely characterized by the following:

- $w \in W \subset V$  acts through left multiplication,

• and  $w' \in W' \subset V$  acts as

$$(4.1) \quad w'(w_1 \wedge w_2 \wedge \cdots \wedge w_r) = \sum_{i=1}^r (-1)^{i+1} \langle w', w_i \rangle (w_1 \wedge w_2 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge w_r),$$

on  $w_1 \wedge \cdots \wedge w_r \in \bigwedge^r W \subset \bigwedge W$ .

The subspaces  $\bigwedge^+ W := \bigwedge_{i \in 2\mathbb{Z}_{\geq 0}} W$  and  $\bigwedge^- W := \bigwedge_{i \in 1+2\mathbb{Z}_{\geq 0}} W$  are stable under  $C^+(V)$ . By restriction we obtain the spin representations

$$(4.2) \quad \text{spin}: \text{GPin}_{2n} \rightarrow \text{GL}\left(\bigwedge W\right) \quad \text{and} \quad \text{spin}^\pm: \text{GSpin}_{2n} \rightarrow \text{GL}\left(\bigwedge^\pm W\right).$$

In (4.5) and (4.6) below, we will choose (ordered) bases for  $\bigwedge W$  and  $\bigwedge^\pm W$  coming from  $\{e_1, \dots, e_n\}$  to view  $\text{spin}$  and  $\text{spin}^\pm$  as  $\text{GL}_{2n}$  and  $\text{GL}_{2n-1}$ -valued representations, respectively. We had another definition of  $\text{spin}^\varepsilon$  as the representation with highest weight  $\mu_\varepsilon$  (Definition 2.6),  $\varepsilon \in \{+, -\}$ . Let us check that the two definitions coincide via (3.5).

**Lemma 4.1.** *The highest weight of the half spin representation  $\text{spin}^\varepsilon$  of  $\text{GSpin}_{2n}$  on  $\bigwedge^\varepsilon W$  is equal to  $\mu_\varepsilon$ .*

*Proof.* We may compare  $\mu_\varepsilon$  and the highest weight of  $\text{spin}^\varepsilon$  after pulling back along  $\text{Spin}_{2n} \times \mathbb{G}_m \rightarrow \text{GSpin}_{2n}$ . They coincide on  $\text{Spin}_{2n}$  by [FH91, Prop. 20.15] and evidently restrict to the weight 1 character on  $\mathbb{G}_m$ . The lemma follows.  $\square$

Let us introduce a bilinear pairing on  $\bigwedge W$  which is invariant under the spin representation up to scalars. Let  $\text{pr}_n: \bigwedge W \rightarrow \mathbb{C}$  denote the projection onto  $\bigwedge^n W$ , identified with  $\mathbb{C}$  via  $e_1 \wedge \cdots \wedge e_n \mapsto 1$ . Write  $\tau: \bigwedge W \xrightarrow{\sim} \bigwedge W$  for the  $\mathbb{C}$ -linear anti-automorphism  $w_1 \wedge \cdots \wedge w_r \mapsto w_r \wedge \cdots \wedge w_1$  for  $r \geq 1$  and  $w_1, \dots, w_r \in W$ . Define

$$((\dot{w}_1, \dot{w}_2)) := \text{pr}_n(\tau(\dot{w}_1) \wedge \dot{w}_2), \quad \dot{w}_1, \dot{w}_2 \in \bigwedge W.$$

**Lemma 4.2.** *The pairing  $((\ , \ ))$  is nondegenerate; it is alternating if  $n \equiv 2, 3 \pmod{4}$  and symmetric if  $n \equiv 0, 1 \pmod{4}$ . The restriction of  $((\ , \ ))$  to  $\bigwedge^+ W$  (resp.  $\bigwedge^- W$ ) is nondegenerate if  $n$  is even, and identically zero if  $n$  is odd. We have*

$$(4.3) \quad ((\text{spin}(g)\dot{w}_1, \text{spin}(g)\dot{w}_2)) = \mathcal{N}(g)((\dot{w}_1, \dot{w}_2)), \quad g \in \text{GPin}_{2n}(\mathbb{C}), \quad \dot{w}_1, \dot{w}_2 \in \bigwedge W.$$

In particular, we have  $\text{spin}^\varepsilon \simeq (\text{spin}^{(-1)^{n\varepsilon}})^\vee \otimes \mathcal{N}$ .

*Proof.* The first two assertions are elementary and left to the reader. The last assertion follows from the rest. For the equality (4.3), we claim that

$$(4.4) \quad ((c\dot{w}_1, \dot{w}_2)) = ((\dot{w}_1, \beta(c)\dot{w}_2)), \quad c \in C(V), \quad \dot{w}_1, \dot{w}_2 \in \bigwedge W.$$

Since  $\text{GPin}_{2n} \subset C(V)$ , this implies (4.3) as

$$((\text{spin}(g)\dot{w}_1, \text{spin}(g)\dot{w}_2)) = ((\dot{w}_1, \text{spin}(\beta(g)g)\dot{w}_2)) = \beta(g)g((\dot{w}_1, \dot{w}_2)).$$

It remains to prove the claim. The proof of (4.4) reduces to the case  $c \in V$ , then to the two cases  $c \in W$  and  $c \in W'$  by linearity. In both cases, (4.4) follows from the explicit description of the  $C(V)$ -action as in (4.1). Indeed, (4.4) is obvious if  $c \in W$ . When  $c \in W'$ , it is enough to show that for  $0 \leq r, s \leq n$ ,  $1 \leq i_1 < \cdots < i_r \leq n$ ,  $1 \leq j_1 < \cdots < j_s \leq n$ , and  $1 \leq k \leq n$ ,

$$\tau(e_{n+k}(e_{i_1} \wedge \cdots \wedge e_{i_r})) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_s}) = \tau(e_{i_1} \wedge \cdots \wedge e_{i_r}) \wedge (e_{n+k}(e_{j_1} \wedge \cdots \wedge e_{j_s})).$$

(This implies (4.4) by taking  $\text{pr}_n$ .) The equality is simply  $0 = 0$  unless  $k = r_0 = s_0$  for some  $1 \leq r_0 \leq r$  and  $1 \leq s_0 \leq s$ . In the latter case, the equality boils down to

$$(-1)^{r_0+1} e_{i_r} \wedge \cdots \wedge \widehat{e}_{i_{r_0}} \wedge \cdots \wedge e_{i_1} \wedge e_{j_1} \wedge \cdots \wedge e_{j_s} = (-1)^{s_0+1} e_{i_r} \wedge \cdots \wedge e_{i_1} \wedge e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_{s_0}} \wedge \cdots \wedge e_{j_s},$$

which is clear. The proof is complete.  $\square$

We also discuss the odd case. Equip  $V_{2n-1} = \mathbb{C}^{2n-1}$  with standard basis  $\{f_1, \dots, f_{2n-1}\}$  and quadratic form  $Q_{2n-1}$  of (3.4). As in [FH91, p. 306], we decompose

$$V_{2n-1} := \mathbb{C}^{2n-1} = W_{2n-1} \oplus W'_{2n-1} \oplus U_{2n-1},$$

where  $W_{2n-1} := \bigoplus_{i=1}^{n-1} \mathbb{C}f_i$ ,  $W'_{2n-1} := \bigoplus_{i=n}^{2n-2} \mathbb{C}f_i$ , and  $U_{2n-1} := \mathbb{C}f_{2n-1}$ . Again we omit the subscript  $2n-1$  when it is clear from the context. Then  $W$  and  $W'$  are  $(n-1)$ -dimensional isotropic subspaces, and  $U$  is a line perpendicular to them. As in the even case, each of  $\bigwedge W$  and  $\bigwedge^\pm W$  can be viewed as a subspace of  $C(V)$  and has a unique structure of left  $C(V)$ -module where:

- $w \in W \subset V$  acts on  $\bigwedge W$  through left multiplication,
- $w' \in W' \subset V$  acts as in (4.1) (cf. [FH91, 20.16]),
- $f_{2n-1}$  acts trivially on  $\bigwedge^+ W$  and as  $-1$  on  $\bigwedge^- W$ .

Consider the bijection

$$\psi: \bigwedge W_{2n-1} \xrightarrow{\sim} \bigwedge^+ W_{2n}, \quad w_1 \wedge \cdots \wedge w_r \mapsto \begin{cases} w_1 \wedge \cdots \wedge w_r \wedge e_n, & r \text{ odd} \\ w_1 \wedge \cdots \wedge w_r, & r \text{ even.} \end{cases}$$

**Lemma 4.3.** *For all  $g \in \mathrm{GSpin}_{2n-1}$  and all  $w \in \bigwedge W_{2n-1}$  we have  $i_{\mathrm{std}}(g)\psi(w) = \psi(gw)$ , where  $i_{\mathrm{std}}(g)$  and  $g$  act by  $\mathrm{spin}^+$  of  $\mathrm{GSpin}_{2n}$  and  $\mathrm{spin}$  of  $\mathrm{GSpin}_{2n-1}$ , respectively.*

*Proof.* We keep writing  $W = W_{2n-1}$ ,  $W' = W'_{2n-1}$ ,  $U = U_{2n-1}$ . We identify  $V_{2n} = (W \oplus U^1 \oplus W' \oplus U^2)$  via  $W_{2n} = W \oplus U^1$  and  $W'_{2n} = W' \oplus U^2$  with  $U^1 = \mathbb{C}e_n$  and  $U^2 = \mathbb{C}e_{2n}$ , mapping the basis of  $W$  (resp.  $W'$ ) onto the first  $n-1$  elements in the basis of  $W_{2n}$  (resp.  $W'_{2n}$ ). This also gives the embedding  $V_{2n-1} \subset V_{2n}$ , with  $U$  diagonally embedded in  $U^1 \oplus U^2$  (so  $f_{2n-1}$  maps to  $e_n + e_{2n}$ ), as in the formula below (3.4).

There is an obvious embedding  $\iota^+ : \bigwedge W \hookrightarrow \bigwedge(W \oplus U^1)$ . We also have  $\iota^- : \bigwedge W \hookrightarrow \bigwedge(W \oplus U^1)$  by  $(\cdot) \wedge e_n$ . Both  $\iota^+$  and  $\iota^-$  are  $C(W \oplus W')$ -equivariant, by using that left and right multiplications commute and that  $e_n$  is orthogonal to  $W \oplus W'$ . Furthermore,  $\iota^-$  intertwines the  $f_{2n-1}$ -action on  $\bigwedge^- W$ , which is by multiplication by  $-1$ , and the  $e_n + e_{2n}$ -action on  $\bigwedge^+(W \oplus U^1)$ , since  $w \wedge e_n = -e_n \wedge w$  if  $w \in \bigwedge^- W$  and since  $W \perp e_{2n}$  with respect to  $Q_{2n}$ .

Now we claim that  $\psi$  is  $C^+(W \oplus W' \oplus U)$ -equivariant, which implies the lemma by restricting from  $C^+(W \oplus W' \oplus U)$  to  $\mathrm{GSpin}_{2n-1}$ . It suffices to verify equivariance of  $\psi$  under  $C^+(W \oplus W')$  and  $C^-(W \oplus W') \otimes f_{2n-1}$ . But  $\psi$  is  $\iota^+$  on  $\bigwedge^+ W$  and  $\iota^-$  on  $\bigwedge^- W$ . Thus the claim is deduced by putting together the equivariance in the preceding paragraph.  $\square$

**Lemma 4.4.** *Let  $\vartheta \in \mathrm{GPin}_{2n}$  be the element from (3.7). We have  $\bigwedge^+ W_{2n} \xrightarrow{\sim} \bigwedge^- W_{2n}$ ,  $x \mapsto \vartheta x$ . We have  $\mathrm{spin}^+ \circ \theta = \mathrm{spin}^-$  via this isomorphism, i.e.,  $\vartheta(\mathrm{spin}^+(g)x) = \mathrm{spin}^-(\theta(g))\vartheta x$  for each  $g \in \mathrm{GSpin}_{2n}$ .*

*Proof.* Henceforth we omit the symbol  $\wedge$  for the wedge product in  $W_{2n}$ . Consider  $v = e_{k_1} \cdots e_{k_r} \in \bigwedge^+ W_{2n}$ , with  $k_1 < k_2 < \dots < k_r$  and  $r$  is even. Then

$$\vartheta v = \sqrt{-1}(e_n e_{k_1} \cdots e_{k_r} - e_{2n} \cdot e_{k_1} \cdots e_{k_r}) \in \bigwedge W_{2n},$$

where  $e_{2n}$  acts by (4.1). Thus the isomorphism follows from the following computations.

$$e_n e_{k_1} \cdots e_{k_r} = \begin{cases} 0, & k_r = n, \\ e_{k_1} \cdots e_{k_r} e_n, & k_r \neq n, \end{cases}$$

$$e_{2n} e_{k_1} \cdots e_{k_r} = \sum_{i=1}^r (-1)^{i+1} \langle e_{2n}, e_{k_i} \rangle e_{k_1} \cdots \widehat{e_{k_i}} \cdots e_{k_r} = \begin{cases} -e_{k_1} \cdots e_{k_{r-1}}, & k_r = n, \\ 0, & k_r \neq n. \end{cases}$$

The last assertion comes down to showing that  $\vartheta g x = \theta(g)\vartheta x$ , where  $\vartheta g, \theta(g)\vartheta \in C(V)$  act through the  $C(V)$ -module structure on  $x \in \bigwedge W_{2n}$ . But this is clear since  $\theta(g) = \vartheta g \vartheta^{-1}$ .  $\square$

Consider the basis  $\{b_U\}$  of  $\bigwedge W_{2n}$ , with

$$(4.5) \quad b_U = (-1)^{\#U} e_{k_1} \cdot e_{k_2} \cdots e_{k_r} \in \bigwedge W_{2n},$$

where  $U = \{k_1 < k_2 < \dots < k_r\}$  ranges over the subsets of  $\{1, 2, \dots, n\}$ . The  $U$  of even size form a basis for  $\bigwedge^+ W_{2n}$ ; and the  $U$  with odd size form a basis for  $\bigwedge^- W_{2n}$ . Order the  $b_U$  for  $U$  odd, and the  $b_U$  for  $U$  even in such a way that the ordering of  $\{b_U\}_{|U|:\text{even}}$  corresponds to that of  $\{b_U\}_{|U|:\text{odd}}$  via  $b_U \mapsto \vartheta b_U / \sqrt{-1}$ . Then these orderings of the  $b_U$  gives us two identifications

$$(4.6) \quad \mathrm{GL}\left(\bigwedge^+ W_{2n}\right) \xrightarrow{\sim} \mathrm{GL}_{2n-1} \quad \text{and} \quad \mathrm{GL}\left(\bigwedge^- W_{2n}\right) \xrightarrow{\sim} \mathrm{GL}_{2n-1},$$

such that the following proposition holds.

**Proposition 4.5.** *The following diagram commutes*

$$\begin{array}{ccccc} & & \mathrm{GSpin}_{2n} & & \\ & \nearrow^{i_{\mathrm{std}}} & \downarrow \theta & \searrow^{\mathrm{spin}^+} & \\ \mathrm{GSpin}_{2n-1} & \xrightarrow{\mathrm{spin}} & & \xrightarrow{\quad} & \mathrm{GL}_{2n-1} \\ & \searrow_{i_{\mathrm{std}}} & \downarrow & \nearrow_{\mathrm{spin}^-} & \\ & & \mathrm{GSpin}_{2n} & & \end{array}$$

*Proof.* This follows from Equation (4.5), Lemmas 4.3, and (proof of) Lemma 4.4.  $\square$

## 5. SOME SPECIAL SUBGROUPS OF $\mathrm{GSpin}_{2n}$

In this section, the base field of all algebraic groups is an algebraically closed field of characteristic 0 such as  $\mathbb{C}$  or  $\overline{\mathbb{Q}_\ell}$ . We begin with principal morphisms for  $\mathrm{GSpin}_{2n-1}$  and  $\mathrm{GSpin}_{2n}$ . (See [Pat16, Sect. 7] and [Gro97, Ser96] for general discussions.) The following notation will be convenient for us. Denote by

$$j_{\mathrm{reg}}: \mathbb{G}_m \times \mathrm{SL}_2 \rightarrow \mathrm{GSpin}_{2n-1}$$

the product of the central embedding  $\mathbb{G}_m \hookrightarrow \mathrm{GSpin}_{2n-1}$  and a fixed principal  $\mathrm{SL}_2$ -mapping. Note that  $j_{\mathrm{reg}}$  has the following kernel<sup>6</sup>

$$\begin{cases} \langle (-1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) \rangle, & \text{if } n(n-1)/2 \text{ is odd,} \\ \langle (1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) \rangle, & \text{if } n(n-1)/2 \text{ is even.} \end{cases}$$

We write  $G_{\mathrm{pri}} \subset \mathrm{GSpin}_{2n-1}$  for the image of  $j_{\mathrm{reg}}$ . The group  $G_{\mathrm{pri}}$  is isomorphic to  $\mathrm{GL}_2$  if  $n(n+1)/2$  is odd, and to  $\mathbb{G}_m \times \mathrm{PGL}_2$  otherwise. Using  $i_{\mathrm{std}}$  from (3.6), we define

$$i_{\mathrm{reg}} = i_{\mathrm{std}} \circ j_{\mathrm{reg}}: \mathbb{G}_m \times \mathrm{SL}_2 \rightarrow \mathrm{GSpin}_{2n}.$$

We also introduce the maps

$$\overline{j_{\mathrm{reg}}} = \mathrm{Ad} \circ j_{\mathrm{reg}}: \mathrm{PGL}_2 \rightarrow \mathrm{PSO}_{2n-1} \quad \text{and} \quad \overline{i_{\mathrm{reg}}} = \overline{i_{\mathrm{std}}} \circ \overline{j_{\mathrm{reg}}}: \mathrm{PGL}_2 \rightarrow \mathrm{PSO}_{2n},$$

where  $\mathrm{Ad}: \mathrm{SO}_{2n} \rightarrow \mathrm{PSO}_{2n}$  is the adjoint map. (Of course  $\mathrm{Ad}$  induces  $\mathrm{SO}_{2n-1} \xrightarrow{\sim} \mathrm{PSO}_{2n-1}$  when pulled back via  $i_{\mathrm{reg}}^\circ$ .)<sup>7</sup>

The  $\mathrm{spin}$  representation of  $\mathrm{Spin}_7$  is orthogonal ([KS16, Lem. 0.1]), yielding an embedding

$$\mathrm{spin}: \mathrm{Spin}_7 \hookrightarrow \mathrm{SO}_8.$$

Fixing a non-isotropic line in the underlying 8-dimensional space, the stabilizer of the line in  $\mathrm{Spin}_7$  is isomorphic to a group of type  $G_2$ , cf. [GS98, p.169, Prop. 2.2(4)]. Thereby we obtain an embedding  $j_{\mathrm{spin}}: G_2 \hookrightarrow \mathrm{Spin}_7$ . Alternatively, an embedding  $G_2 \hookrightarrow \mathrm{Spin}_7$  can be obtained via octonion algebras. The conjugacy class of  $j_{\mathrm{spin}}$  is canonical (i.e., independent of choices) by [Che19, Prop. 2.11]. Denote by

$$i_{\mathrm{spin}}: G_2 \hookrightarrow \mathrm{Spin}_8$$

<sup>6</sup>To see this, one can use Proposition 6.1 of [Gro00], where the  $\mathrm{SL}_2$ -representations appearing in the composition  $\mathrm{SL}_2 \xrightarrow{\mathrm{pri}} \mathrm{GSpin}_{2n-1} \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2n-1}$  are computed.

<sup>7</sup>When denoting the group standing alone, we prefer  $\mathrm{SO}_{2n-1}$  to  $\mathrm{PSO}_{2n-1}$ . When thinking of a projective representation or a subgroup of  $\mathrm{PSO}_{2n}$ , we usually write  $\mathrm{PSO}_{2n-1}$ .

the composite  $i_{\text{std}} \circ j_{\text{spin}}$ . The restriction of  $\text{spin}^\varepsilon : \text{Spin}_8 \rightarrow \text{GL}_8$  via  $i_{\text{spin}}$  is isomorphic to  $\mathbf{1} \oplus \text{std}$ , where  $\mathbf{1}$  and  $\text{std}$  are the trivial and the unique irreducible 7-dimensional representation, respectively. (This is easy to see by dimension counting, as the other irreducible representations have dimension  $\geq 14$ .)

**Lemma 5.1.** *The representation  $\text{spin} : \text{Spin}_7 \hookrightarrow \text{SO}_8$  is  $\text{O}_8$ -conjugate to  $\theta^\circ \text{spin}$  but not locally conjugate (thus not conjugate) as an  $\text{SO}_8$ -valued representation.*

*Proof.* Evidently  $\text{spin}$  and  $\theta^\circ \text{spin}$  are  $\text{O}_8$ -conjugate since  $\theta^\circ = \text{Int}(\vartheta^\circ)$  with  $\vartheta^\circ \in \text{O}_8$ . Let  $T_{\text{Spin}} \subset \text{Spin}_7$ ,  $T_{\text{SO}} \subset \text{SO}_8$ , and  $T_{\text{GL}} \subset \text{GL}_8$  be maximal tori such that  $\text{spin}(T_{\text{Spin}}) \subset T_{\text{SO}}$  and  $\text{std}(T_{\text{SO}}) \subset T_{\text{GL}}$ . (In this proof,  $T_{\text{SO}}$  need not coincide with that of §2.) Without loss of generality, we may assume that  $\theta^\circ(T_{\text{SO}}) = T_{\text{SO}}$ . Let  $\Omega_{\text{Spin}}, \Omega_{\text{SO}}, \Omega_{\text{GL}}$  denote the corresponding Weyl groups. Fix an isomorphism  $T_{\text{Spin}} \simeq \mathbb{G}_m^3$  and accordingly  $X_*(T_{\text{Spin}}) = \{(a_1, a_2, a_3) \in \mathbb{Z}^3\}$ . Then we have group morphisms

$$X_*(T_{\text{Spin}}) \xrightarrow{\text{spin}} X_*(T_{\text{SO}}) \xrightarrow{\text{std}} X_*(T_{\text{GL}}).$$

As we know the weights of the  $\text{spin}$  representation, we know that

$$\text{std}(\text{spin}(a_1, a_2, a_3)) \in \Omega_{\text{GL}}(a_1^{\varepsilon_1} + a_2^{\varepsilon_2} + a_3^{\varepsilon_3} : \varepsilon_i \in \{\pm 1\}).$$

(The  $\Omega_{\text{GL}}$ -orbit of tuples is simply an unordered tuple.) When  $a_1, a_2, a_3$  are distinct, the right hand side breaks up into exactly two  $\Omega_{\text{SO}}$ -orbits, which are permuted by  $\theta^\circ$ . It follows that  $\text{spin}(t)$  and  $\theta^\circ(\text{spin}(t))$  are not  $\text{SO}_8$ -conjugate for  $t = (t_1, t_2, t_3) \in T_{\text{Spin}}$  with distinct  $t_1, t_2, t_3$ .  $\square$

**Proposition 5.2.** *Let  $n \geq 3$ . Let  $\overline{H} \subset \text{PSO}_{2n}$  be a (possibly disconnected) reductive subgroup (over  $\mathbb{C}$  or  $\overline{\mathbb{Q}}_\ell$ ) containing a regular unipotent element. Up to a conjugation by an element of  $\text{PSO}_{2n}$ , the following holds (in particular  $\overline{H}$  is connected in all cases):*

- (1) if  $n > 4$ , then  $\overline{H} = \text{PSO}_{2n}$ ,  $\overline{H} = \overline{i_{\text{std}}(\text{PSO}_{2n-1})}$ , or  $\overline{H} = \overline{i_{\text{reg}}(\text{PGL}_2)}$ ;
- (2) if  $n = 4$ , then  $\overline{H}$  is either as in (1),  $\overline{H} = \overline{\text{spin}(\text{SO}_7)}$ ,  $\overline{H} = \overline{\theta^\circ \text{spin}(\text{SO}_7)}$ , or  $\overline{H} = \overline{i_{\text{spin}}(G_2)}$ .

If  $H \subset \text{SO}_{2n}$  is a (possibly disconnected) reductive subgroup containing a regular unipotent element, then  $H^0 \subset H \subset H^0 \cdot Z(\text{SO}_{2n})$  with  $H^0$  surjects onto  $\overline{H} \subset \text{PSO}_{2n}$  as in the list above.

*Proof.* We start by proving the assertion on  $H$ . We employ the classification of maximal reductive subgroups of  $\text{SO}_{2n}$  containing a regular unipotent element in [SS97, Thm. B], where only (i)(a) and (iv)(a)(e)(g) are relevant to us. Then one of the following holds up to conjugation:<sup>8</sup>

- $H = \text{SO}_{2n}$ .
- $H$  is a reducible subgroup of  $\text{GL}_{2n}$  via  $\text{std}$ :  $H^0$  is either  $i_{\text{std}}^\circ(\text{SO}_{2n-1})$ ,  $i_{\text{reg}}(\text{PGL}_2)$ , or  $n = 4$  and  $i_{\text{spin}}^\circ(G_2)$ .
- $H$  is an irreducible subgroup of  $\text{GL}_{2n}$  via  $\text{std}$ :  $n = 4$  and  $H^0 = \text{spin}(\text{Spin}_7)$  or  $H^0 = \theta^\circ \text{spin}(\text{Spin}_7)$ .

Here the cases  $i_{\text{spin}}^\circ(G_2)$  and  $i_{\text{reg}}(\text{PGL}_2)$  appear in (iv)(a) and (iv)(e) of *loc. cit.* as a maximal reductive subgroup of  $i_{\text{std}}^\circ(\text{SO}_{2n-1})$ . When  $n = 3$ ,  $\text{SO}_6$  is isogenous to  $\text{SL}_4$  and the above list can still be deduced from *loc. cit.*

In fact it is not immediately clear from [SS97, Thm. B] that  $H$  can be conjugated in  $\text{SO}_{2n}$  to one of the subgroups above, so let us explain this point. The case  $H = \text{SO}_{2n}$  is trivial. In the last case, *loc. cit.* tells us that  $H^0$  is conjugate to  $\text{std}(\text{spin}(\text{Spin}_7))$  in  $\text{GL}_8$ . Since  $\text{O}_8$  is acceptable, we see that  $H^0$  is  $\text{O}_8$ -conjugate to  $\text{spin}(\text{Spin}_7)$  so the result follows. In the second case, what *loc. cit.* gives us is that either  $H^0$  embeds in  $\text{SO}_{2n}$  via a principal  $\text{PGL}_2$ -morphism or in a way that  $\text{std}(H^0)$  decomposes the underlying  $2n$ -dimensional space into irreducible spaces of dimensions  $2n-1$  and  $1$ . In the former case, this is a special case of [GR10, Prop. 2.2] (when  $\rho$  and  $\varphi|_{\mathcal{W}}$  are trivial) since regular nilpotent elements are all conjugate in the Lie algebra (of  $H^0$ ). In the  $\text{SO}_{2n-1}$ -case,  $H^0$  is the stabilizer of a non-isotropic line in the underlying  $2n$ -dimensional

<sup>8</sup>The statement of [SS97, Thm. B] is not entirely clear on whether the list describes  $H^0$  or  $H$ . We interpret it as the former since that is what their proof shows. For instance, regarding (i)(a) of their theorem, a maximal reductive subgroup of type  $B_{n-1}$  in  $\text{SO}_{2n}$  is not  $i_{\text{std}}^\circ(\text{SO}_{2n-1})$  but  $Z(\text{SO}_{2n}) \times i_{\text{std}}^\circ(\text{SO}_{2n-1})$ , which is disconnected.

quadratic space. Since non-isotropic lines are in a single  $\mathrm{SO}_{2n}$ -orbit, we can conjugate  $H^0$  to  $i_{\mathrm{std}}^\circ(\mathrm{SO}_{2n-1})$  by making  $H^0$  stabilize a particular non-isotropic line. In the remaining  $G_2$ -case with  $n = 4$ , we may assume  $H^0 \subset i_{\mathrm{std}}^\circ(\mathrm{SO}_7)$  by the preceding argument. Since  $G_2$ -subgroups of  $\mathrm{SO}_7$  are conjugate by [Che19, Prop. 2.11], we are done.

Now that we have justified the above list, let us proceed to identify  $H$ . There is nothing to do when  $H = \mathrm{SO}_{2n}$ . In the second case,  $\mathrm{std}(H)$  is contained in a parabolic subgroup of  $\mathrm{GL}_{2n}$  with Levi component  $\mathrm{GL}_{2n-1} \times \mathrm{GL}_1$ . By reductivity  $\mathrm{std}(H)$  is contained in  $\mathrm{GL}_{2n-1} \times \mathrm{GL}_1$ , and it is an irreducible subgroup. We see that  $H \subset H^+ := i_{\mathrm{std}}^\circ(\mathrm{SO}_{2n-1}) \times Z(\mathrm{SO}_{2n})$ , and by Schur's lemma, the centralizer of  $H^0$  in  $H^+$  is  $Z(\mathrm{SO}_{2n})$ . Since  $H^0$  has no nontrivial outer automorphism, the conjugation by each  $h \in H$  on  $H^0$  are inner automorphisms. Thus there exists  $h' \in H^0$  such that  $h'h^{-1}$  centralizes  $H^0$ . It follows that  $H \subset H^0 \times Z(\mathrm{SO}_{2n})$ . In the last case, the centralizer of  $H^0$  in  $\mathrm{SO}_{2n}$  is  $Z(\mathrm{SO}_{2n})$  again by Schur's lemma, with no nontrivial outer automorphism for  $H^0$ . As in the second case, we deduce  $H^0 \subset H \subset H^0 \times Z(\mathrm{SO}_{2n})$ .

Finally the assertion on  $\overline{H}$  is implied by the description of its preimage in  $\mathrm{SO}_{2n}$ .  $\square$

**Lemma 5.3.** *Let  $r: \Gamma \rightarrow \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$  be a semisimple representation containing a regular unipotent element in its image. Let  $\chi: \Gamma \rightarrow \overline{\mathbb{Q}}_\ell^\times$  be a character and  $\epsilon \in \{+, -\}$ . If  $\chi \otimes \mathrm{spin}^\epsilon r \simeq \mathrm{spin}^\epsilon r$  then  $\chi = 1$ .*

*Proof.* Write  $\bar{r}: \Gamma \rightarrow \mathrm{PSO}_{2n}(\overline{\mathbb{Q}}_\ell)$  for the projectivization of  $r$ . By Proposition 5.2 we can distinguish between two cases for the Zariski closure of the image of  $\bar{r}$  in  $\mathrm{PSO}_{2n}(\overline{\mathbb{Q}}_\ell)$ . If the Zariski closure of  $\bar{r}$  is either  $\mathrm{PSO}_{2n}$ ,  $\overline{i_{\mathrm{std}}(\mathrm{PSO}_{2n-1})}$ , or (when  $n = 4$ )  $\overline{\mathrm{spin}(\mathrm{SO}_7)}$ , then  $r$  is strongly irreducible, and the statement follows from [KS16, Lem. 4.8(i)]. In the remaining cases, we may assume that the Zariski closure of  $\mathrm{Im}(\bar{r})$  is  $\overline{i_{\mathrm{reg}}(\mathrm{PGL}_2)}$  or  $\overline{i_{\mathrm{spin}}(G_2)}$ . Then  $\mathrm{Im}(\bar{r}) \subset \overline{i_{\mathrm{std}}(\mathrm{SO}_{2n-1}(\overline{\mathbb{Q}}_\ell))}$  so  $\mathrm{Im}(r)$  is contained in  $\overline{\mathbb{Q}}_\ell$ -points of  $\mathrm{GSpin}_{2n-1}Z(\mathrm{GSpin}_{2n})$  (which is the preimage of  $i_{\mathrm{std}}(\mathrm{SO}_{2n-1})$  in  $\mathrm{GSpin}_{2n}$ ). Then we show  $\chi = 1$  by the argument exactly as in Cases (i), (ii), (iv) in the proof of [KS16, Lem. 5.2], noting that  $\mathrm{spin}^\epsilon$  restricts to  $\mathrm{spin}$  on  $\mathrm{GSpin}_{2n-1}$ .  $\square$

Let  $H$  be a (possibly disconnected) reductive group over  $\overline{\mathbb{Q}}_\ell$ . Two Galois representations  $r_1, r_2: \Gamma \rightarrow H(\overline{\mathbb{Q}}_\ell)$  are *weakly conjugate* if their restrictions to an open subgroup of  $\Gamma$  are conjugate. We say that  $H$  is *weakly acceptable* if the following holds true: if two Galois representations  $r_1, r_2: \Gamma \rightarrow H(\overline{\mathbb{Q}}_\ell)$  are locally conjugate then they are weakly conjugate. Recall from [KS16, Lem. 4.4, Prop. B.1] that  $\mathrm{O}_{2n}$  is acceptable.

**Proposition 5.4.** *Let  $H$  be one of the following algebraic groups  $\mathrm{SO}_{2n}$ ,  $\mathrm{GSpin}_{2n}$ ,  $\mathrm{SO}_{2n} \rtimes \Gamma_{E/F}$ ,  $\mathrm{GSpin}_{2n} \rtimes \Gamma_{E/F}$ , where  $\Gamma_{E/F}$  acts through  $\theta^\circ$  or  $\theta$  in the semi direct products. We write  $H^0$  for the neutral component of  $H$ . Let*

$$r_1, r_2: \Gamma_F \rightarrow H(\overline{\mathbb{Q}}_\ell)$$

*be semisimple Galois representations such that*

- $r_1$  and  $r_2$  are locally conjugate and
- the Zariski closure of  $r_1(\Gamma)$  contains a regular unipotent element.

*Then  $r_1$  and  $r_2$  are  $H^0$ -conjugate.*

*Proof.* We first look at the case where  $H = \mathrm{SO}_{2n}$ . Write  $\bar{r}_1, \bar{r}_2: \Gamma \rightarrow \mathrm{PSO}_{2n}(\overline{\mathbb{Q}}_\ell)$  for the projectivizations of  $r_1, r_2$ . Since  $\mathrm{O}_{2n}$  is acceptable,  $r_1$  and  $r_2$  are conjugate by an element of  $\mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$ . In particular the Zariski closure of  $r_2(\Gamma)$  also contains a regular unipotent element. Thus the Zariski closures of  $\bar{r}_1(\Gamma)$  and  $\bar{r}_2(\Gamma)$  in  $\mathrm{PSO}_{2n}$  are conjugate to each other by Proposition 5.2. This is clear possibly except when  $\bar{r}_1(\Gamma)$  and  $\bar{r}_2(\Gamma)$  are conjugate to  $\overline{\mathrm{spin}(\mathrm{Spin}_7)}$  and  $\theta^\circ \overline{\mathrm{spin}(\mathrm{Spin}_7)}$  respectively, or vice versa. However Lemma 5.1 tells us that  $r_1$  and  $r_2$  cannot be locally conjugate in that case.

Conjugating  $r_2$  by an element of  $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ , we may assume that the Zariski closures are equal, to be denoted by  $I$ . Then  $I$  is one of the algebraic subgroups of  $\mathrm{PSO}_{2n}$  in Proposition 5.2. Fix an element  $w \in \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$  such that  $r_1 = \mathrm{Int}(w) \circ r_2$ , so that  $\mathrm{Int}(w)$  induces an automorphism

of  $I$ . There are now two cases by Proposition 5.2 (excluding  $\overline{\text{spin}}(\text{SO}_7)$  and its  $\theta^\circ$ -conjugate as explained above): either (A)  $I = \text{PSO}_{2n}$ , (B)  $I$  is  $\text{SO}_{2n-1}$ ,  $G_2$ , or  $\text{PGL}_2$ .

Case (A). If  $r_1$  has Zariski dense image in  $\text{SO}_{2n}$  then there exists  $\mathfrak{q}$  such that  $r_1(\text{Frob}_{\mathfrak{q}})$  and  $gr_1(\text{Frob}_{\mathfrak{q}})g^{-1}$  are not outer conjugate. Thus  $r_2(\text{Frob}_{\mathfrak{q}}) = gr_1(\text{Frob}_{\mathfrak{q}})g^{-1}$  leads us to a contradiction.

Case (B). We have  $\text{Out}(I) = \{1\}$ , so  $\text{Int}(w)$  is an inner automorphism of  $I$ . Multiplying  $w$  by an element of  $I \subset \text{PSO}_{2n}$ , we may assume that  $\text{Int}(w)$  is trivial on  $I$ . Thus we may assume  $\bar{r}_1 = \bar{r}_2$ . Then  $r_1 = \chi r_2$  for some  $\chi: \Gamma \rightarrow Z(\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)) = \{\pm 1\}$ . As  $r_1(\Gamma)$  has a regular unipotent element in its Zariski closure,  $\text{std} \circ r_1$  is irreducible or decomposes as a sum of a 1-dimensional representation with a  $2n - 1$ -dimensional representation. By [KS16, Prop 4.9],  $\chi = 1$ .

Now consider  $H = \text{SO}_{2n} \rtimes \Gamma_{E/F}$ . Let  $\bar{r}_1, \bar{r}_2$  for the composites of  $r_1, r_2$  with the surjection  $H \twoheadrightarrow \text{PSO}_{2n} \rtimes \Gamma_{E/F}$ . As in the  $\text{SO}_{2n}$ -case, we may assume that the Zariski closures of  $\bar{r}_1(\Gamma)$  and  $\bar{r}_2(\Gamma)$  in  $\text{PSO}_{2n} \rtimes \Gamma_{E/F}$  are equal. Let  $I$  denote this algebraic group. Then  $I \subset \text{PSO}_{2n} \rtimes \Gamma_{E/F}$  contains a regular unipotent and surjects onto  $\Gamma_{E/F}$ . Therefore either (A)'  $I = \text{PSO}_{2n} \rtimes \Gamma_{E/F}$ , or (B)'  $I$  is  $\text{PSO}_{2n}$ -conjugate to<sup>9</sup>

$$(5.1) \quad \text{PGL}_2 \times \Gamma_{E/F}, \quad G_2 \times \Gamma_{E/F}, \quad \text{or} \quad \text{SO}_{2n-1} \times \Gamma_{E/F}.$$

The last three groups have the property that  $\text{Out}(I) = 1$ . Arguing as in Cases (A) and (B) as above, we find that  $r_1$  and  $r_2$  are  $\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ -conjugate. (In case (B)', even though  $I$  is disconnected, the point is that an inner automorphism of  $I$  can be written as the conjugation by an element in the neutral component of  $I$ .)

We now treat the  $\text{GSpin}_{2n}$ -case. Write  $r_1^\circ, r_2^\circ$  for the composition of  $r_1, r_2$  with  $\text{pr}^\circ: \text{GSpin}_{2n} \rightarrow \text{SO}_{2n}$ . Then  $r_1^\circ$  and  $r_2^\circ$  are conjugate by the  $\text{SO}_{2n}$ -case treated above. Hence we may assume that  $r_2 = \chi r_1$  with a continuous character  $\chi: \Gamma \rightarrow \overline{\mathbb{Q}}_\ell^\times$ , where  $\overline{\mathbb{Q}}_\ell^\times = \ker(\text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rightarrow \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell))$  via Lemma 3.1 (ii). Since  $r_1$  and  $\chi \otimes r_1$  are locally conjugate by the initial assumption, we have

$$\text{spin}^\epsilon(r_1) \simeq \text{spin}^\epsilon(\chi \otimes r_1) \simeq \chi \otimes \text{spin}^\epsilon(r_1), \quad \epsilon \in \{\pm 1\}.$$

It follows from Lemma 5.3 that  $\chi = 1$ .

Finally, consider the group  $\text{GSpin}_{2n} \rtimes \Gamma_{E/F}$ . By the  $\text{GSpin}_{2n}$ -case above, we may assume that  $r_1|_{\Gamma_E} = r_2|_{\Gamma_E}$ . (Strictly speaking, we proved the  $\text{SO}_{2n}$ -case and  $\text{GSpin}_{2n}$ -case for  $\Gamma = \Gamma_F$ , but the proof goes through without change for  $\Gamma_E$ .) Writing  $r_i^\circ := \text{pr}^\circ \circ r_i$  for  $i = 1, 2$ , we have  $r_1^\circ|_{\Gamma_E} = r_2^\circ|_{\Gamma_E}$ . By the preceding argument, we deduce that  $r_1^\circ = r_2^\circ$ . On the other hand,  $r_1|_{\Gamma_E} = r_2|_{\Gamma_E}$  implies that  $r_1 \simeq r_2$  or  $r_1 \simeq r_2 \otimes \chi$  by Example A.6, with  $\chi$  as in that example. If  $r_1 \simeq r_2 \otimes \chi$  then we should have  $r_1^\circ \simeq r_2^\circ \otimes \chi_{E/F}$  for  $\chi_{E/F}: \Gamma_F \twoheadrightarrow \Gamma_{E/F} = \{\pm 1\}$ , but this is a contradiction as in the proof of case (B) above (or by Example A.5). Therefore  $r_1 \simeq r_2$ .  $\square$

## 6. ON $\text{SO}_{2n}$ -VALUED GALOIS REPRESENTATIONS

In this section we construct Galois representations associated with automorphic representations of even orthogonal groups over a totally real field  $F$ . More precisely, we will derive a weaker version of Conjecture 1 for such groups from the literature. Let either

- $E = F$ , or
- $E$  be a CM quadratic extension of  $F$ .

In the latter case write  $c$  for the nontrivial element of  $\Gamma_{E/F} := \text{Gal}(E/F)$ . Write  $\text{SO}_{2n}^{E/F}$  for the split group  $\text{SO}_{2n}$  if  $E = F$ , and the quasi-split outer form of  $\text{SO}_{2n}$  over  $F$  relative to  $E/F$  otherwise. To be precise, in the latter case,

$$(6.1) \quad \text{O}_{2n}^{E/F}(R) := \left\{ g \in \text{GL}_{2n}(E \otimes_F R) \mid c(g) = \vartheta^\circ g \vartheta^\circ, g^t \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \right\}$$

<sup>9</sup>Note that indeed  $\theta(\text{SO}_{2n-1}) = \text{SO}_{2n-1}$  and  $\theta$  is trivial on  $\text{SO}_{2n-1}$ , so these semi-direct products make sense (and are in fact direct products).

for  $F$ -algebras  $R$ , and  $\mathrm{SO}_{2n}^{E/F}$  is the connected component where  $\det(g) = 1$ . We can extend the standard embedding  $\mathrm{std} : \mathrm{SO}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$  to a map (still denoted  $\mathrm{std}$ )

$$(6.2) \quad \mathrm{std} : {}^L(\mathrm{SO}_{2n}^{E/F}) = \mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F} \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}),$$

whose image is  $\mathrm{SO}_{2n}(\mathbb{C})$  if  $E = F$  and  $\mathrm{O}_{2n}(\mathbb{C})$  if  $E \neq F$ . More precisely, when  $E \neq F$ , we fix the extended map  $\mathrm{std}$  by requiring  $c \mapsto \vartheta^\circ$ . (We defined  $\mathrm{O}_{2n}$  explicitly in the last section, and  $\vartheta^\circ$  was given in (2.2).)

Let  $\pi^b$  be a cuspidal automorphic representation of  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$ . The following will be key assumptions on  $\pi^b$ . (Recall from §1 that  $\mathrm{St}_{\mathrm{SO}, \mathfrak{q}_{\mathrm{St}}}$  denotes the Steinberg representation.)

**(coh $^\circ$ )**  $\pi_\infty^b$  is cohomological for an irreducible algebraic representation  $\xi^b = \otimes_{y \in \mathcal{V}_\infty} \xi_y^b$  of  $\mathrm{SO}_{2n, F \otimes \mathbb{C}}^{E/F}$ .

**(St $^\circ$ )** There exists a prime  $\mathfrak{q}_{\mathrm{St}}$  of  $F$  such that  $\pi_{\mathfrak{q}_{\mathrm{St}}}^b \simeq \mathrm{St}_{\mathrm{SO}, \mathfrak{q}_{\mathrm{St}}}$  up to a character twist.

Condition (coh $^\circ$ ) implies that  $\pi^b$  is  $\mathbb{C}$ -algebraic in the sense of Buzzard–Gee [BG14, Lem. 7.2.2], thus also  $L$ -algebraic as the half sum of positive (co)roots is integral for  $\mathrm{SO}_{2n}^{E/F}$ . In (St $^\circ$ ), characters of  $\mathrm{SO}_{2n}^{E/F}(F_{\mathfrak{q}_{\mathrm{St}}})$  are exactly the characters factoring through the cokernel of  $\mathrm{Spin}_{2n}^{E/F}(F_{\mathfrak{q}_{\mathrm{St}}}) \rightarrow \mathrm{SO}_{2n}^{E/F}(F_{\mathfrak{q}_{\mathrm{St}}})$ . Such characters are in a natural bijection with characters of  $F_{\mathfrak{q}_{\mathrm{St}}}^\times / (F_{\mathfrak{q}_{\mathrm{St}}}^\times)^2$ , since the group of such characters is classified by  $H^1(F_{\mathfrak{q}_{\mathrm{St}}}, \{\pm 1\})$ .

Write  $T_{\mathrm{SO}} := T_{\mathrm{GSO}} \cap \mathrm{SO}_{2n}$  over  $\mathbb{C}$  and choose the Borel subgroup containing  $T_{\mathrm{SO}}$  in  $\mathrm{SO}_{2n}^{E/F}$  as in the preceding section. For each  $y \in \mathcal{V}_\infty$ , the highest weight of  $\xi_y^b$  gives rise to a dominant cocharacter  $\lambda(\xi_y^b) \in X_*(T_{\mathrm{SO}})$ . Let  $\phi_{\pi_y^b} : W_{F_y} \rightarrow {}^L\mathrm{SO}_{2n}^{E/F}$  denote the  $L$ -parameter of  $\pi_y^b$  assigned by [Lan89]. Recall  $\mathrm{std} : \mathrm{SO}_{2n} \hookrightarrow \mathrm{GL}_{2n}$  denotes the standard embedding. We also consider the following conditions:

**(std-reg $^\circ$ )**  $\mathrm{std} \circ \phi_{\pi_y^b}|_{W_{\overline{F}_y}}$  is regular (i.e., the centralizer group in  $\mathrm{GL}_{2n}(\mathbb{C})$  is a torus) for every  $y \in \mathcal{V}_\infty$ .

**(disc- $\infty$ )** If  $n$  is odd then  $[E : F] = 2$ . If  $n$  is even then  $E = F$ .

When (coh $^\circ$ ) is satisfied, imposing (std-reg $^\circ$ ) amounts to requiring that  $\mathrm{std} \circ \lambda(\xi_y^b)$  is a regular cocharacter of  $\mathrm{GL}_{2n}$ . Since  $E$  is either  $F$  or a CM quadratic extension of  $F$ , condition (disc- $\infty$ ) is equivalent to requiring  $\mathrm{SO}_{2n}^{E/F}(F_y)$  to admit discrete series at all infinite places  $y$  of  $F$  (or equivalently, to admit compact maximal tori).

When  $\mathfrak{q}$  is a prime of  $F$ , write  $\phi_{\pi_{\mathfrak{q}}^b} : W_{F_{\mathfrak{q}}} \rightarrow {}^L\mathrm{SO}_{2n}^{E/F}$  for the  $L$ -parameter of  $\pi_{\mathfrak{q}}^b$  as given by [Art13, Thm 1.5.1]. (By the Langlands quotient theorem,  $\pi_{\mathfrak{q}}^b$  is the unique quotient of an induced representation from a character twist of a tempered representation on a Levi subgroup. Apply Arthur's theorem to this tempered representation.) Note that  $\phi_{\pi_{\mathfrak{q}}^b}$  is well-defined up to the outer action automorphism action when  $\mathfrak{q}$  does not split in  $E/F$ .

Let  $\mathrm{Unr}(\pi^b)$  denote the set of finite primes  $\mathfrak{q}$  of  $F$  such that  $\mathfrak{q}$  is unramified in  $E$  and  $\pi_{\mathfrak{q}}^b$  is unramified. In this case, the unramified  $L$ -parameter  $\phi_{\pi_{\mathfrak{q}}^b}$  is determined (up to  $\mathrm{SO}_{2n}(\mathbb{C})$ -conjugacy, not just up to outer automorphism) by the Satake isomorphism.

Thanks to Arthur, we can lift  $\pi^b$  to an automorphic representation of  $\mathrm{GL}_{2n}$ .

**Proposition 6.1** (Arthur). *Assume that  $\pi^b$  satisfies (St $^\circ$ ). Then there exists a self-dual automorphic representation  $\pi^\sharp$  of  $\mathrm{GL}_{2n}(\mathbb{A}_F)$ , which is either cuspidal or the isobaric sum of two cuspidal self-dual representations of  $\mathrm{GL}_{2n-1}(\mathbb{A}_F)$  and  $\mathrm{GL}_1(\mathbb{A}_F)$ , such that*

(Ar1)  $\pi_{\mathfrak{q}}^\sharp$  is unramified at every  $\mathfrak{q} \in \mathrm{Unr}(\pi^b)$ , and  $\phi_{\pi_{\mathfrak{q}}^\sharp} \simeq \mathrm{std} \circ \phi_{\pi_{\mathfrak{q}}^b}$ .

(Ar2)  $\pi_{\mathfrak{q}_{\mathrm{St}}}^\sharp \simeq \mathrm{St}_{2n-1} \boxplus \mathbf{1}$  up to a quadratic character of  $\mathrm{GL}_{2n}(\mathbb{A}_F)$ .

(Ar3)  $\phi_{\pi_v^\sharp} \simeq \mathrm{std} \circ \phi_{\pi_v^b}$  at every  $F$ -place  $v$ .

If  $\pi^b$  satisfies both (St $^\circ$ ) and (coh $^\circ$ ) then we furthermore have

(Ar4)  $\pi_y^\sharp$  and  $\pi_y^b$  are tempered for all infinite  $F$ -places  $y$ .

If  $\pi^b$  has properties (coh $^\circ$ ), (St $^\circ$ ), and (std-reg $^\circ$ ), then the following strengthening holds:



(Ar4)+  $\pi_v^\#$  and  $\pi_v^b$  are tempered for all  $F$ -places  $v$ .

*Proof.* Consider  $\pi^b$  satisfying (St $^\circ$ ). For notational convenience, we assume  $\pi_{\mathfrak{qSt}}^b \simeq \text{St}_{\text{SO}, \mathfrak{qSt}}$  (not just up to a quadratic character twist) as the general case works in the same way. By [Art13, Thm. 1.5.2] (using the notation there),<sup>10</sup> we have a formal global parameter  $\psi$  (as in [Art13, 1.4]) such that  $\pi^b$  appears as a direct summand of a member of  $\tilde{\Pi}(\psi)$ . (It is a direct summand since the discrete  $L^2$ -spectrum is semisimple.) In particular  $\pi_{\mathfrak{qSt}}^b$  is a direct summand of a member of  $\tilde{\Pi}(\psi_{\mathfrak{qSt}})$ . Proposition B.1 implies that  $\psi_{\mathfrak{qSt}} \simeq \psi_{\text{St}, \mathfrak{qSt}}$ , where  $\psi_{\text{St}}$  is defined above the proposition. Thus

$$(6.3) \quad \psi_{\text{St}, \mathfrak{qSt}} \simeq \psi_{\text{St}_{2n-1}, \mathfrak{qSt}} \oplus \psi_{\mathbf{1}, \mathfrak{qSt}},$$

where  $\psi_{\text{St}_{2n-1}, \mathfrak{qSt}}$  (resp.  $\psi_{\mathbf{1}, \mathfrak{qSt}}$ ) denotes the  $A$ -parameter for the Steinberg (resp. trivial) representation  $\text{St}_{2n-1}$  of  $\text{GL}_{2n-1}(F_{\mathfrak{qSt}})$  (resp.  $\text{GL}_1(F_{\mathfrak{qSt}})$ ). It follows that either  $\psi = \pi^\#$  or  $\psi = \pi_1^\# \boxplus \pi_2^\#$ , where  $\pi^\#$ ,  $\pi_1^\#$ , and  $\pi_2^\#$  are cuspidal self-dual automorphic representations of  $\text{GL}_{2n}(\mathbb{A}_F)$ ,  $\text{GL}_{2n-1}(\mathbb{A}_F)$ , and  $\text{GL}_1(\mathbb{A}_F)$ , respectively. (In particular only the trivial  $\text{SU}(2)$ -representation occurs in the global parameter  $\psi$ .) In the second case, we take  $\pi^\#$  to be the isobaric sum of  $\pi_1^\#$  and  $\pi_2^\#$ . Now (Ar2) follows from (6.3). We define  $\phi_{\mathfrak{q}} \in \tilde{\Phi}(\text{SO}_{2n, F_{\mathfrak{q}}}^{E/F})$  as the restriction of  $\psi_v \in \tilde{\Psi}(\text{SO}_{2n, F_{\mathfrak{q}}}^{E/F})$  from  $\mathcal{L}_{F_{\mathfrak{q}}} \times \text{SU}(2)$  to  $\mathcal{L}_{F_{\mathfrak{q}}}$ . Then Properties (Ar1) and (Ar3) with  $\phi_{\mathfrak{q}}$  in place of  $\phi_{\pi_{\mathfrak{q}}^b}$  are part of Arthur's result already cited.

To complete the proof of (Ar1) and (Ar3), it suffices to verify that  $\phi_v = \phi_{\pi_v^b}$  in  $\tilde{\Phi}(G_{F_v})$ . In the notation of [Art13] (between Theorems 1.5.1 and 1.5.2),  $\phi_v$  gives rise to

- a  $F_v$ -rational parabolic subgroup  $P_v \subset G_{F_v}$  with a Levi factor  $M_v$ ,
- a bounded parameter  $\phi_{M_v} \in \tilde{\Phi}(M_v)$ ,
- a point  $\lambda$  in the open chamber for  $P_v$  in  $X_*(M_v)_{F_v} \otimes_{\mathbb{Z}} \mathbb{R}$ ,

such that  $\phi_v$  comes from the  $\lambda$ -twist  $\phi_{M_v, \lambda}$  of  $\phi_{M_v}$ . (This is the counterpart of the Langlands quotient construction for  $L$ -parameters.) The statement of [Art13, Thm. 1.5.2] tells us that  $\pi_v^b$  is a subrepresentation of the normalized induction  $\text{Ind}_{P_v(F_v)}^{G(F_v)}(\sigma_{v, \lambda})$  for some  $\sigma_v \in \tilde{\Pi}(M_v)$ , where  $\sigma_{v, \lambda}$  denotes the  $\lambda$ -twist of  $\sigma_v$ , since  $\pi_v^b$  appears in the packet of  $\psi_v$  in *loc. cit.* According to the same theorem,  $\text{Ind}_{P_v(F_v)}^{G(F_v)}(\sigma_{v, \lambda})$  must be completely reducible since it appears in the  $L^2$ -discrete spectrum. This means that  $\pi_v^b$  is irreducible and the Langlands quotient of  $\text{Ind}_{P_v(F_v)}^{G(F_v)}(\sigma_{v, \lambda})$  (thus  $\pi_v^b$  is isomorphic to the latter). Since the formation of Langlands parametrization is compatible with the Langlands quotient, it follows that  $\phi_v$  is the  $L$ -parameter of  $\pi_v^b$ , namely that  $\phi_v = \phi_{\pi_v^b}$ .

It remains to check (Ar4) and (Ar4)+. Assume (coh $^\circ$ ) in addition to (St $^\circ$ ). Thanks to (Ar3),  $\pi^\#$  is  $L$ -algebraic since  $L$ -algebraicity is preserved by *std.* Applying [Clo90, Lem. 4.9] to  $\pi^\# \otimes |\det|^{1/2}$  if  $\pi^\#$  is cuspidal, and  $\pi_1^\#$  and  $\pi_2^\#$  otherwise, to deduce that  $\pi_v^\#$  is essentially tempered at all  $y|_\infty$ . Since  $\pi^\#$  is self-dual,  $\pi_y^\#$  are a fortiori tempered. Now suppose furthermore that  $\pi_y^b$  has property (std-reg $^\circ$ ). Then  $\pi^\#$  is regular  $L$ -algebraic. Arguing as above but applying [Car12, Thm. 1.2] to  $\pi^\#$  at finite places, in place of [Clo90, Lem. 4.9] at infinite places, we deduce (Ar4)+. Finally, whenever  $\pi_v^\#$  is tempered (for finite or infinite  $v$ ), this implies that  $\psi_v$  is bounded, hence that  $\pi_v^b$  is tempered by [Art13, Thm. 1.5.1].  $\square$

**Corollary 6.2.** *Assume that  $n$  is even (resp. odd) if  $E = F$  (resp. if  $E \neq F$ ). If  $\pi^b$  satisfies (St $^\circ$ ) and (coh $^\circ$ ) then  $\pi_y^b$  is a discrete series representation for every infinite place  $y$ .*

*Proof.* Note that the parity condition exactly guarantees that  $\text{SO}_{2n}^{E/F}(F_y)$  contains an elliptic maximal torus at infinite places  $y$ , so that it admits discrete series. In this case, a tempered

<sup>10</sup>E.g.  $\tilde{\Phi}(\text{SO}_{2n, F_{\mathfrak{q}}}^{E/F})$  means the set of isomorphism classes of  $L$ -parameters for  $\text{SO}_{2n, F_{\mathfrak{q}}}^{E/F}$  modulo the action of the outer automorphism group  $\tilde{\text{Out}}_{2n}(G)$  as defined in [Art13, 1.2]. Similarly  $\tilde{\Pi}(\cdot)$  denotes a packet consisting of finitely many isomorphism classes of representations up to the same outer automorphisms. By abuse of terminology, a representation will often mean the outer automorphism orbit of representations in this proof.

$\xi$ -cohomological representation is a discrete series representation by [BW00, Thm. III.5.1]. Thus the corollary follows from (Ar4) of the preceding proposition.  $\square$

Continue to assume  $(\text{St}^\circ)$  and  $(\text{coh}^\circ)$  for  $\pi^b$ . For each infinite place  $y$  of  $F$ , write  $\phi_{\pi_y^b}: W_{F_y} \rightarrow {}^L\text{SO}_{2n}$  for the  $L$ -parameter of  $\pi_y^b$  as given by [Art13, Thm 1.5.1]. Let us describe  $\phi_{\pi_y^b}|_{W_{\overline{F}_y}}$  explicitly. The half sum of positive coroots  $\rho_{\text{SO}} \in X_*(\text{T}_{\text{SO}})$  is equal to  $(n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1}$ . Fix an  $\mathbb{R}$ -isomorphism  $\overline{F}_y \simeq \mathbb{C}$  once and for all, so that we can identify  $W_{\overline{F}_y} = \mathbb{C}^\times$ . Possibly after  $\text{SO}_{2n}^{E/F}(\mathbb{C})$ -conjugation, we have (as following from the construction of discrete series  $L$ -packets in [Lan89, p.134])

$$(6.4) \quad \phi_{\pi_y^b}(z) = (z/\overline{z})^{\rho_{\text{SO}} + \lambda(\xi_y^b)}, \quad z \in W_{\overline{F}_y}.$$

We noted that  $\pi^b$  is both  $L$ -algebraic thanks to  $(\text{coh}^\circ)$ . Then Conjecture 1 predicts the existence of an  ${}^L\text{SO}_{2n}^{E/F}$ -valued Galois representation attached to  $\pi^b$ . When  $(\text{std-reg}^\circ)$  is also assumed (in addition to  $(\text{St}^\circ)$  and  $(\text{std-reg}^\circ)$ ), Theorem 6.3 below proves the conjecture modulo outer automorphisms in that (SO-i) is weaker than what is predicted. (This is to be upgraded by (SO-i+) in §13; also see Remark 13.2.) The proof is carried out by reducing to the known results for  $\pi^\#$  on  $\text{GL}_{2n}$ .

**Theorem 6.3.** *Let  $\pi^b$  be a cuspidal automorphic representation of  $\text{SO}_{2n}^{E/F}(\mathbb{A}_F)$  satisfying  $(\text{coh}^\circ)$ ,  $(\text{St}^\circ)$ , and  $(\text{std-reg}^\circ)$ . Then there exists a semisimple Galois representation (depending on  $\iota$ )*

$$\rho_{\pi^b} = \rho_{\pi^b, \iota}: \Gamma_F \rightarrow \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F},$$

whose restriction to  $\Gamma_{F_{\mathfrak{q}}}$  at every  $F$ -place  $\mathfrak{q}|\ell$  is potentially semistable, such that the following hold.

(SO-i) For every finite  $F$ -place  $\mathfrak{q}$  (including  $\mathfrak{q}|\ell$ ),

$$\iota\phi_{\pi_{\mathfrak{q}}^b} \overset{\text{out}}{\sim} \text{WD}(\rho_{\pi^b}|_{\Gamma_{F_{\mathfrak{q}}}})^{F\text{-ss}}.$$

(SO-ii) Let  $\mathfrak{q} \in \text{Unr}(\pi^b)$ . If  $\mathfrak{q} \nmid \ell$  then  $\rho_{\pi^b, \mathfrak{q}}$  is unramified at  $\mathfrak{q}$ , and for all eigenvalues  $\alpha$  of  $\text{std}(\rho_{\pi^b}(\text{Frob}_{\mathfrak{q}}))_{\text{ss}}$  and all embeddings  $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$  we have  $|\alpha| = 1$ .

(SO-iii) For each  $\mathfrak{q}|\ell$ , and for each  $y: F \hookrightarrow \mathbb{C}$  such that  $\iota y$  induces  $\mathfrak{q}$ , we have  $\mu_{\text{HT}}(\rho_{\pi^b, \mathfrak{q}}, \iota y) \overset{\text{out}}{\sim} \mu_{\text{Hodge}}(\xi_y^b, y)$ .

(SO-iv) If  $\pi_{\mathfrak{q}}$  is unramified at  $\mathfrak{q}|\ell$ , then  $\rho_{\pi^b, \mathfrak{q}}$  is crystalline. If  $\pi_{\mathfrak{q}}$  has a non-zero Iwahori fixed vector at  $\mathfrak{q}|\ell$ , then  $\rho_{\pi^b, \mathfrak{q}}$  is semistable.

(SO-v) Assume  $(\text{disc-}\infty)$ . Then  $\rho_{\pi^b}$  is totally odd. More explicitly, for each real place  $y$  of  $F$  and the corresponding complex conjugation  $c_y \in \Gamma_F$  (well-defined up to conjugacy),

$$\rho_{\pi^b}(c_y) \sim \begin{cases} \text{diag}(\underbrace{1, \dots, 1}_{n/2}, \underbrace{-1, \dots, -1}_{n/2}, \underbrace{1, \dots, 1}_{n/2}, \underbrace{-1, \dots, -1}_{n/2}), & n : \text{even}, \\ \text{diag}(\underbrace{1, \dots, 1}_{(n-1)/2}, \underbrace{-1, \dots, -1}_{(n-1)/2}, \underbrace{1, 1, \dots, 1}_{(n-1)/2}, \underbrace{-1, \dots, -1}_{(n-1)/2}, 1) \rtimes c, & n : \text{odd}. \end{cases}$$

Condition (SO-i) characterizes  $\rho_{\pi^b}$  uniquely up to  $\text{O}_{2n}(\overline{\mathbb{Q}}_\ell)$ -conjugation.

*Remark 6.4.* Since  $\pi_\infty^b$  is a discrete series representation, the conjugation by  $\phi_{\pi_y^b}(j)$  on  $\text{T}_{\text{SO}}$  is the inverse map, where  $j$  denotes the usual element of the real Weil group. Thus (SO-v) and (6.4) imply Buzzard–Gee’s prediction on the image of complex conjugation in [BG14, Conj. 3.2.1, 3.2.2]. When  $n$  is odd, we also observe that (SO-v) is equivalent to

$$\rho_{\pi^b}(c_y) \sim \text{diag}(\underbrace{1, \dots, 1}_{(n-1)/2}, \underbrace{-1, \dots, -1}_{(n-1)/2}, a, \underbrace{1, \dots, 1}_{(n-1)/2}, \underbrace{-1, \dots, -1}_{(n-1)/2}, a^{-1}) \rtimes c, \quad \forall a \in \overline{\mathbb{Q}}_\ell^\times.$$

*Remark 6.5.* Without  $(\text{St}^\circ)$ , an analogous theorem can be proved only under  $(\text{coh}^\circ)$ , with or without assuming  $(\text{std-reg})$ , but in a weaker and less precise form. The strategy is similar:

transfer  $\pi^b$  to an automorphic representation of  $\mathrm{GL}_{2n}(\mathbb{A}_F)$ , which is an isobaric sum of cuspidal self-dual automorphic representations, and apply the known results on associating Galois representations.

*Remark 6.6.* If we assume  $(\mathrm{coh}^\circ)$  and  $(\mathrm{St}^\circ)$  but not  $(\mathrm{std}\text{-}\mathrm{reg}^\circ)$ , then a congruence argument (e.g., using [GK19]) would allow one to prove only a weaker version of the theorem. Namely a standard method would show (SO-i) only up to semisimplification, but that would not be enough to translate  $(\mathrm{St}^\circ)$  into the property of  $\rho_{\pi^b}$  having a regular unipotent element in the image. (Compare with Corollary 6.7 below.) As a result, our main argument would not go through.

*Proof of Theorem 6.3.* Let  $\pi^\#$  be as in Proposition 6.1 so that

Case 1:  $\pi^\#$  is cuspidal, or

Case 2:  $\pi^\# = \pi_1^\# \boxplus \pi_2^\#$ , with  $\pi_1^\#$  (resp.  $\pi_2^\#$ ) a cuspidal automorphic representation of  $\mathrm{GL}_{2n-1}(\mathbb{A}_F)$  (resp.  $\mathrm{GL}_1(\mathbb{A}_F)$ ).

As in the proof there, we know that  $\pi^\#$  is  $L$ -algebraic.

In Case 1, consider the  $C$ -algebraic twist  $\Pi := \pi^\# \otimes |\det|^{(1-2n)/2}$  is  $C$ -algebraic, which is regular by  $(\mathrm{std}\text{-}\mathrm{reg})$ , and essentially self-dual (“essentially” means up to a character twist). Applying the well-known construction of Galois representations (see [BLGGT14, Thm. 2.1.1] for a summary and further references) to  $\Pi$ , we obtain a semisimple Galois representation (recall  $\Gamma = \Gamma_F$  by convention)

$$\rho_\Pi : \Gamma \rightarrow \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell),$$

satisfying the obvious analogues of properties (SO-i) through (SO-v) for  $\mathrm{GL}_{2n}$ , with  $\rho_\Pi$  and  $\mathrm{GL}_{2n}$  in place of  $\rho_{\pi^b}$  and  $\mathrm{O}_{2n}$ ; call these analogues (GL-i), ..., (GL-vi). By ‘obvious’, we mean for instance that (GL-ii) is about the eigenvalues of  $\rho_\Pi(\mathrm{Frob}_q)$  having absolute value 1. We also spell out (GL-i), which states that

$$(6.5) \quad \iota\phi_{\Pi \otimes |\det|^{(1-2n)/2}} \sim \mathrm{WD}(\rho_\Pi|_{\Gamma_{F_q}})^{\mathrm{F}\text{-ss}}, \quad \mathfrak{q} \nmid \ell.$$

In particular, for all finite places  $\mathfrak{q} \nmid \ell$  where  $\Pi$  is unramified,

$$(6.6) \quad \rho_\Pi(\mathrm{Frob}_q)_{\mathrm{ss}} \sim \iota\phi_{\Pi \otimes |\det|^{(1-2n)/2}}(\mathrm{Frob}_q) \sim \iota\phi_{\pi^\#}(\mathrm{Frob}_q) \sim \mathrm{std}(\phi_{\pi^\#}(\mathrm{Frob}_q)).$$

Since each  $\pi_q^\#$  is self-dual, we see that  $\rho_\Pi$  is self-dual. By (Ar2) and (6.5) at  $\mathfrak{q} = \mathfrak{q}_{\mathrm{St}}$  as well as semisimplicity of  $\rho_\Pi$ , we see that either

- $\rho_\Pi$  is strongly irreducible<sup>11</sup>, or
- $\rho_\Pi = \rho_1 \oplus \rho_2$  for self-dual strongly irreducible subrepresentations  $\rho_1$  and  $\rho_2$  with  $\dim \rho_1 = n - 1$  and  $\dim \rho_2 = 1$ .

(To see the strong irreducibility, as opposed to irreducibility, notice that the restriction of  $\rho_\Pi$  to an open finite-index subgroup of  $\Gamma$  still contains a regular unipotent of  $\mathrm{GL}_{2n-1} \times \mathrm{GL}_1$  in the image.) Either way, it follows from [BC11, Cor. 1.3] that every irreducible constituent of  $\rho_\Pi$  is orthogonal in the sense of *loc. cit.* (As we are in Case 1, apply their corollary with  $\eta = |\cdot|^{2n-1}$ , in which case  $\eta_\lambda(c) = -1$  in their notation.)

Now we turn to Case 2. Take  $\Pi_1 := \pi_1^\# |\det|^{1-n}$  and  $\Pi_2 := \pi_2^\#$ . Each of  $\Pi_1$  and  $\Pi_2$  is cuspidal, regular  $C$ -algebraic, and essentially self-dual, so the same construction yields  $\rho_{\Pi_1}$  and  $\rho_{\Pi_2}$ , which are  $2n - 1$  and 1-dimensional, respectively. Then put  $\rho_\Pi := \rho_{\Pi_1} \oplus \rho_{\Pi_2}$ . As before, (GL-i), ..., (GL-v) hold true for  $\rho_\Pi$ . Moreover an argument as in Case 1 shows that  $\rho_{\Pi_1}$  and  $\rho_{\Pi_2}$  are self-dual and orthogonal. It follows from (Ar2) and (6.5) at  $v = \mathfrak{q}_{\mathrm{St}}$  that  $\rho_{\Pi_1}$  and  $\rho_{\Pi_2}$  are strongly irreducible.

From here on, we treat the two cases together. Since  $\rho_\Pi$  is self-dual and orthogonal, after conjugating  $\rho_\Pi$  by an element of  $\mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$ , we can ensure that  $\rho_\Pi(\Gamma) \subset \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$ . Write

$$\rho_{\pi^b} : \Gamma \rightarrow \mathrm{O}_{2n}(\overline{\mathbb{Q}}_\ell)$$

<sup>11</sup>We call a Galois representation *strongly irreducible* if its restriction to any open subgroup is irreducible. See [KS16, Def. 3.2] for details.

for the  $O_{2n}(\overline{\mathbb{Q}}_\ell)$ -valued representation that  $\rho_\Pi$  factors through. (In case  $\rho_\Pi$  is reducible, we even have  $\rho_\Pi(\Gamma) \subset (O_{2n-1} \times O_1)(\overline{\mathbb{Q}}_\ell)$ .) Let us check that this is the desired Galois representation and deduce properties (SO-i) through (SO-v) from (GL-i) through (GL-v).

We start with the case  $E = F$ . Then  $\phi_{\pi^\flat}(\text{Frob}_\mathfrak{q}) \in \text{SO}_{2n}(\mathbb{C})$  in (6.6), so we deduce via the Chebotarev density theorem that  $\rho_{\pi^\flat}$  has image in  $\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ . Note that (GL-ii) is the same statement as (SO-ii). The Hodge-theoretic properties at  $\ell$  in (SO-iii) and (SO-iv) may be checked after composing with a faithful representation, so these properties hold. One sees from [KS16, Appendix B] (for  $O_{2n}$ ) that (GL-i) implies (SO-i). (Alternatively, one can appeal to [GGP12, Thm. 8.1].) The assertion on the cocharacters in (SO-iii) also follows (GL-iii) that the two cocharacters become conjugate in  $\text{GL}_{2n}$ . Finally (GL-v), namely the (total) oddness of  $\text{std}(\rho_{\pi^\flat})$ , tells us that  $\text{std}(\rho_{\pi^\flat}(c_y)) \in \text{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$  has eigenvalues 1 and  $-1$  with multiplicity  $n$  each, for every  $y \in \mathcal{V}_\infty$ . As  $\rho_{\pi^\flat}(c_y) \in \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$  has order 2, we have

$$\rho_{\pi^\flat}(c_y) \sim \text{diag}(\underbrace{1, \dots, 1}_{a_y}, \underbrace{-1, \dots, -1}_{b_y}, \underbrace{1, \dots, 1}_{a_y}, \underbrace{-1, \dots, -1}_{b_y}), \quad a_y + b_y = n, \quad a_y, b_y \in \mathbb{Z}_{\geq 0}.$$

So (GL-v) implies that  $a_y = b_y$ . (This is possible as  $n$  is even.) From this, one computes the adjoint action of  $\rho_{\pi^\flat}(c_y)$  on  $\text{Lie SO}_{2n}(\overline{\mathbb{Q}}_\ell)$  to be  $-n$ . (A similar computation is done in the proof of [KS16, Lem. 1.9] for  $\text{GSp}_{2n}$ .) Thus  $\rho_{\pi^\flat}$  is totally odd.

It remains to treat the case  $E \neq F$ . In this case, the standard embedding  $\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F} \hookrightarrow \text{GL}_{2n} \rtimes \Gamma_{E/F}$  identifies  $\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F} \xrightarrow{\sim} O_{2n}(\overline{\mathbb{Q}}_\ell)$ . The composition of  $\rho_{\pi^\flat}$  with this isomorphism is still to be denoted by  $\rho_{\pi^\flat}$ . Since  $\phi_{\pi^\flat}(\text{Frob}_\mathfrak{q}) \in O_{2n}(\mathbb{C}) \setminus \text{SO}_{2n}(\mathbb{C})$  (resp.  $\phi_{\pi^\flat}(\text{Frob}_\mathfrak{q}) \in \text{SO}_{2n}(\mathbb{C})$ ) in (6.6) when  $\mathfrak{q}$  is inert (resp. split) in  $E$  by the unramified Langlands correspondence, we see that

$$\rho_{\pi^\flat} : \Gamma \rightarrow \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$$

commutes with the natural projections onto  $\Gamma_{E/F}$ . (By continuity it suffices to check the commutativity on Frobenius conjugacy classes.) Thus  $\rho_{\pi^\flat}$  is a Galois representation valued in  $L(\text{SO}_{2n}^{E/F})$ . Properties (SO-i) through (SO-iv) follow from (GL-i) through (GL-iv) in the same way as for the  $E = F$  case. Here is a proof of (SO-v). When  $n$  is odd, we have

$$(6.7) \quad \text{std} \rho_{\pi^\flat}(c_y) \sim \text{diag}(\underbrace{1, \dots, 1}_n, \underbrace{-1, \dots, -1}_n)$$

$$(6.8) \quad \sim \text{diag}(\underbrace{1, \dots, 1}_{\frac{n-1}{2}}, \underbrace{-1, \dots, -1}_{\frac{n+1}{2}}, \underbrace{1, \dots, 1}_{\frac{n-1}{2}}, \underbrace{-1, \dots, -1}_{\frac{n+1}{2}}) \cdot \text{std}(c) \quad \text{in } \text{GL}_{2n}(\overline{\mathbb{Q}}_\ell).$$

(Recall that  $\text{std}(c) = \vartheta^\circ$  is the  $2n \times 2n$  permutation matrix switching  $n$  and  $2n$ .) Therefore

$$\rho_{\pi^\flat}(c_y) \sim \text{diag}(\underbrace{1, \dots, 1}_{\frac{n-1}{2}}, \underbrace{-1, \dots, -1}_{\frac{n+1}{2}}, \underbrace{1, \dots, 1}_{\frac{n-1}{2}}, \underbrace{-1, \dots, -1}_{\frac{n+1}{2}}) \rtimes c \quad \text{in } L\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell).$$

From this, it follows that the adjoint action of  $\rho_{\pi^\flat}(c_y)$  on  $\text{Lie SO}_{2n}(\overline{\mathbb{Q}}_\ell)$  has trace equal to  $-n$ . Hence  $\rho_{\pi^\flat}$  is totally odd.  $\square$

The following corollary allows us to apply Proposition 5.2 to identify the Zariski closure of the image of  $\rho_{\pi^\flat}$ .

**Corollary 6.7.** *In the setup of Theorem 6.3, the image of  $\rho_{\pi^\flat}$  (thus also  $\rho_{\pi^\flat}(\Gamma_E)$ ) contains a regular unipotent element of  $\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ .*

*Proof.* Suppose that  $\mathfrak{q}_{\text{St}} \nmid \ell$ . Then  $\iota \phi_{\pi^\flat} |_{W_{F_\mathfrak{q}}} \overset{\text{out}}{\sim} \text{WD}(\rho_{\pi^\flat} |_{\Gamma_{F_\mathfrak{q}}})^{\text{F-ss}}$  by (SO-i). Since  $\phi_{\pi^\flat}$  contains a regular unipotent element in the image, so does  $\text{WD}(\rho_{\pi^\flat} |_{\Gamma_{F_\mathfrak{q}}})$ . Therefore  $\rho_{\pi^\flat} |_{\Gamma_{F_\mathfrak{q}}}$  has a regular unipotent in the image. If  $\mathfrak{q}_{\text{St}} | \ell$  then the same is shown following the argument of [KS16, Lem. 3.2].  $\square$

The next corollary is solely about automorphic representations, but proved by means of Galois representations. Interestingly we do not know how to derive it within the theory of automorphic forms. The corollary is not needed in this paper (except in Remark 14.2) as (disc- $\infty$ ) will be imposed in the main case of interest.

**Corollary 6.8.** *Let  $\pi^b$  be a cuspidal automorphic representation of  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$  satisfying (coh $^\circ$ ), (St $^\circ$ ), and (std-reg $^\circ$ ). If (disc- $\infty$ ) is false (i.e.,  $n$  is odd and  $E = F$ , or  $n$  is even and  $[E : F] = 2$ ), then  $\pi^\#$  in Proposition 6.1 (the functorial lift of  $\pi^b$  to  $\mathrm{GL}_{2n}$ ) is the isobaric sum of cuspidal self-dual automorphic representations of  $\mathrm{GL}_{2n-1}(\mathbb{A}_F)$  and  $\mathrm{GL}_1(\mathbb{A}_F)$ .*

*Proof.* Fix a real place  $y$  of  $F$ . Up to conjugation, we may assume that

$$\rho_{\pi^b}(c_y) = \mathrm{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}) \rtimes c_y,$$

where the latter  $c_y$  means its image in  $\Gamma_{E/F}$ ; so  $\mathrm{std}(c_y) = 1$  if  $E = F$  and  $\mathrm{std}(c_y) = \vartheta^\circ$  if  $[E : F] = 2$ . The proof of Theorem 6.3 shows that  $\mathrm{std}(\rho_{\pi^b}(c_y)) \in \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$  is odd for every real place  $y$ . That is,  $\mathrm{std}(\rho_{\pi^b}(c_y))$  has each of the eigenvalues 1 and  $-1$  with multiplicity  $n$ . It is elementary to see that this is impossible when (disc- $\infty$ ) is false. Indeed, if  $n$  is odd and  $E = F$ , then the number of 1's on the diagonal of  $\rho_{\pi^b}(c_y)$  is obviously even (so cannot equal  $n$ ). If  $n$  is even and  $[E : F] = 2$ , this is elementary linear algebra.  $\square$

*Remark 6.9.* The corollary suggests that in that setup,  $\pi^b$  should come from an automorphic representation on  $\mathrm{Sp}_{2n-2}(\mathbb{A}_F)$ , where  $\mathrm{Sp}_{2n-2}$  is viewed as a twisted endoscopic group for  $\mathrm{SO}_{2n}^{E/F}$  (see the paragraph containing (1.2.5) in [Art13]).

If we assume (coh $^\circ$ ) and (St $^\circ$ ) but not (std-reg $^\circ$ ), then some expected properties to be needed in our arguments are not known. We formulate them as a hypothesis so that our results become unconditional once the hypothesis is verified. (In the preceding arguments in this section, (std-reg $^\circ$ ) allowed us to apply the results on the Ramanujan conjecture and construction of automorphic Galois representations for *regular* algebraic cuspidal automorphic representations of  $\mathrm{GL}_n$  which are self-dual.)

**Hypothesis 6.10.** *Assume (disc- $\infty$ ). When  $\pi^b$  satisfies (coh $^\circ$ ) and (St $^\circ$ ) but not (std-reg $^\circ$ ), the following hold true.*

- (1)  $\pi_q^b$  is tempered at every finite prime  $q$  where  $\pi_q^b$  is unramified.
- (2) There exists a semisimple Galois representation  $\rho_{\pi^b} : \Gamma_F \rightarrow \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$  satisfying (SO-i) at every  $q$  where  $\pi_q^b$  is unramified as well as (SO-iii), (SO-iv), and (SO-v). Moreover  $\rho_{\pi^b}(\Gamma_F)$  contains a regular unipotent element.

The hypothesis readily implies (SO-ii) for  $\rho_{\pi^b}$ . We expect that this hypothesis is accessible via suitable orthogonal Shimura varieties. If one is only interested in constructing the  $\mathrm{GSpin}_{2n}$ -valued representation  $\rho_\pi$  without proving its  $\ell$ -adic Hodge-theoretic properties, then (SO-iii) and (SO-iv) may be dropped from the hypothesis.

*Remark 6.11.* Corollary 6.7 (or the above hypothesis, if (std-reg $^\circ$ ) fails) tells us that the Zariski closure of  $\rho_{\pi^b}(\Gamma_F)$  belongs to the list of subgroups of  $\mathrm{SO}_{2n}$  in Proposition 5.2. In the list, the  $\mathrm{PGL}_2$ ,  $G_2$ , and  $\mathrm{PSO}_{2n-1}$  cases can only occur when (std-reg $^\circ$ ) is not satisfied. Since  $\mathrm{PGL}_2$  and  $G_2$  are contained in  $\mathrm{PSO}_{2n-1}$  (up to conjugation), we only need to observe this for  $\mathrm{PSO}_{2n-1}$ . In this case,  $\mu_{\mathrm{HT}}(\rho_{\pi^b, q}, \iota y)$  of Theorem 6.3 must factor through  $i_{\mathrm{std}}^\circ : \mathrm{SO}_{2n-1} \hookrightarrow \mathrm{SO}_{2n}$ , thus cannot be regular as a cocharacter of  $\mathrm{GL}_{2n}$ . By (SO-iii) of the theorem,  $\mathrm{std}(\mu_{\mathrm{Hodge}}(\xi^b, y))$  is not regular either, contradicting (std-reg $^\circ$ ). This observation can be used to skip Cases 2 and 3 in the proof of Proposition 10.3 when (std-reg $^\circ$ ) is assumed.

## 7. EXTENSION AND RESTRICTION

Let  $\pi$  be a cuspidal automorphic representation of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ . In this section we study irreducible subrepresentations  $\pi^b \subset \pi$  and compare the conditions (St), (coh) on  $\pi$  with (St $^\circ$ ) and (Coh $^\circ$ ) on  $\pi^b$ .

**Lemma 7.1.** *Let  $\mathfrak{q}$  be a finite place of  $F$  where  $E/F$  is unramified. Let  $\pi$  be an irreducible admissible representation of  $\mathrm{GSO}_{2n}^{E/F}(F_{\mathfrak{q}})$ , and let  $\pi^{\flat} \subset \pi$  be an irreducible subrepresentation. Then (St) holds for  $\pi$  if and only if  $(\mathrm{St}^{\circ})$  holds for  $\pi^{\flat}$ .*

*Proof.* Write  $G = \mathrm{GSO}_{2n}^{E/F}(F_{\mathfrak{q}})$  and  $G_0 = \mathrm{SO}_{2n}^{E/F}(F_{\mathfrak{q}})$ .

( $\Leftarrow$ ) Write  $G' = \mathrm{GSpin}_{2n}^{E/F}(F_{\mathfrak{q}})$  and  $G'_0 = \mathrm{Spin}_{2n}^{E/F}(F_{\mathfrak{q}})$ . By abuse of notation, write  $G_0/G'_0 := \mathrm{coker}(\mathrm{pr} : G'_0 \rightarrow G_0)$  and likewise for  $G/G'$ . These are finite abelian groups. We claim that any smooth character  $G_0 \rightarrow \mathbb{C}^{\times}$  can be extended to a smooth character  $G \rightarrow \mathbb{C}^{\times}$ . Since such characters factor through  $G_0/G'_0$  and  $G/G'$ , respectively (see e.g., [KS, Cor. 2.6]) the claim would follow once we verify that  $G_0/G'_0 \rightarrow G/G'$  is injective. So let  $g_0 \in G_0$  and suppose that  $g_0 = \mathrm{pr}(g)$  for  $g \in G$ . Then  $1 = \mathrm{sim}(g_0) = \mathrm{sim}(\mathrm{pr}(g)) = \mathcal{N}(g)^2$  by Lemma 3.1 (iii). If  $\mathcal{N}(g) = -1$  then we replace  $g$  with  $zg$  using  $z \in Z_{\mathrm{GSpin}}$  such that  $\mathcal{N}(z) = -1$  (in the coordinates of Lemma 2.5, choose  $z = (1, -1)$  if  $n$  is odd, and  $z = (\zeta_4, -1)$  if  $n$  is even); so we may assume that  $\mathcal{N}(g) = 1$ . But this means that  $g_0$  is trivial in  $G_0/G'_0$ . The claim has been proved.

Thanks to the claim, we may assume  $\pi^{\flat} = \mathrm{St}_{G_0}$ . Write  $B = TN \subset \mathrm{GSO}_{2n}^{E/F}$  for a Borel subgroup (see below (8.4)). We write  $B_0, T_0, N_0$  for the intersections of  $B, T, N$  with  $G_0$ . Then  $N_0 = N$ , and the space of  $N_0$ -coinvariants in  $\pi|_{G_0}$  is the same as the  $N$ -coinvariants. In particular we have the composition

$$(\mathrm{St}_{G_0})_{N_0} = \mathbf{1}(\delta_{B_0}) \subset \pi_{N_0} \xrightarrow{\sim} \pi_N|_{T_0}.$$

The image of  $\mathbf{1}(\delta_{B_0})$  in  $\pi_N$  is a one-dimensional vector space  $V$  that is  $T$  stable, on which  $T$  acts by some character  $\chi$  with  $\chi|_{T_0} = \delta_{B_0}$ . As  $\delta_B$  extends  $\delta_{B_0}$ , we may write  $\chi = \delta_B \cdot \xi$ , where  $\xi$  is a character of  $T/T_0$ . The projection map  $\pi_N \rightarrow \mathbf{1}(\delta_B \xi)$  now yields

$$(7.1) \quad 0 \neq \mathrm{Hom}_T(\pi_N, \mathbf{1}(\delta_B \xi)) = \mathrm{Hom}_G(\pi, \mathrm{Ind}_T^G(\delta_B \xi)).$$

Let  $\eta : G/G_{\mathrm{der}} \rightarrow \mathbb{C}^{\times}$  be a character of the cocenter so that  $T/T_0 \rightarrow G/G_{\mathrm{der}} \rightarrow \mathbb{C}^{\times}$  coincides with  $\xi$ . Then  $\mathrm{Ind}_T^G(\delta_B \xi) = \mathrm{Ind}_T^G(\delta_B)\eta$ , and we find from (7.1) an equivariant mapping  $\pi\eta^{-1} \rightarrow \mathrm{Ind}_T^G(\delta_B)$ . As the representation  $\mathrm{Ind}_T^G(\delta_B)$  has the Steinberg as a unique irreducible subrepresentation,  $\pi\eta^{-1} \simeq \mathrm{St}_G$ .

( $\Rightarrow$ ) Assume  $\pi = \mathrm{St}_G$ . As before we have  $\pi_{N_0} \xrightarrow{\sim} \pi_N|_{T_0} = \mathbf{1}(\delta_B^{-1}|_{T_0})$ . As  $\delta_B^{-1}|_{T_0} = \delta_{B_0}^{-1}$ ,  $\pi$  contains the Steinberg representation of  $G_0$ . On the other hand,  $\pi$  has a nonzero fixed vector for an Iwahori subgroup  $I$  of  $G$ , and so every irreducible  $G_0$  submodule has a nonzero  $G_0$ -Iwahori fixed vector as well. By Clifford theory,  $\pi|_{G_0}$  is semi-simple. Each semi-stable representation  $\tau$  of  $\pi|_{G_0}$  has  $\tau^{I_0} \simeq (\tau_{N_0})^{T_0^{\circ}}$  where  $T_0^{\circ} \subset T_0$  is the maximal compact subgroup (Casselman [Cas80, Prop. 2.3]), and  $I_0 := G_0 \cap I$ . Thus  $\mathrm{St}_G$  is irreducible as  $G_0$ -representation. As  $\pi|_{G_0}$  contains the Steinberg representation of  $G_0$ , it is thus isomorphic to the Steinberg representation.  $\square$

**Lemma 7.2.** *Let  $\pi$  be an irreducible admissible representation of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{R})$  with central character  $\omega_{\pi}$ . Let  $\pi^{\flat}$  be an irreducible  $\mathrm{SO}_{2n}^{E/F}(\mathbb{R})$  subrepresentation. Let  $\xi$  be an irreducible algebraic representation of  $\mathrm{GSO}_{2n}^{E/F}$ , and  $\xi^{\flat}$  its pullback to  $\mathrm{SO}_{2n}^{E/F}$ . Then:*

- (1) *The representation  $\pi$  is essentially unitary if and only if  $\pi^{\flat}$  is unitary.*
- (2) *The representation  $\pi$  is a discrete series representation if and only if  $\pi^{\flat}$  is a discrete series representation.*
- (3) *Assume  $\pi$  is essentially unitary. Then  $\pi$  is  $\xi$ -cohomological if and only if  $\pi^{\flat}$  is  $\xi^{\flat}$ -cohomological and  $\omega_{\pi} = \omega_{\xi}^{-1}$ , where  $\omega_{\xi}$  is the central character of  $\xi$  on  $Z(\mathrm{GSO}_{2n}^{E/F})(\mathbb{R})$ .*

*Proof.* Write  $G = \mathrm{GSO}_{2n}^{E/F}(\mathbb{R})$  and  $G_0 = \mathrm{SO}_{2n}^{E/F}(\mathbb{R})$ .

(1).  $\Rightarrow$  is obvious.  $\Leftarrow$ : Assume  $\pi^{\flat}$  is unitary. We may assume  $\omega_{\pi} = 1$ . Choose a Hermitian form  $h(\cdot, \cdot)$  on  $\pi$ , extending the  $G_0$ -equivariant one on  $\pi^{\flat}$ . Choose representatives  $\{g_1, \dots, g_r\}$  for the quotient  $G/F_{\mathbb{R}}^{\times}G_0$  and define  $h'(\cdot, \cdot) = \sum_{i=1}^r h(g_i \cdot, g_i \cdot)$ . Then  $h'(\cdot, \cdot)$  is a  $G$ -equivariant Hermitian form on  $\pi$ .

(2). This follows directly from the characterization of discrete series representations through the  $L^2$ -property (modulo center) of their matrix coefficients.

(3). This is implied by Salamanca-Riba [SR99, Thm. 1.8] that a unitary representation is cohomological if its central character and infinitesimal character coincides with that of an algebraic representation.  $\square$

## 8. CERTAIN FORMS OF $\mathrm{GSO}_{2n}$ AND OUTER AUTOMORPHISMS

In this section we introduce a certain form of the split group  $\mathrm{GSO}_{2n}$  over a totally real field  $F$ , to be used to construct Shimura varieties. We start by considering real groups. Let  $\mathrm{GO}_{2n}^{\mathrm{cpt}}$ ,  $\mathrm{O}_{2n}^{\mathrm{cpt}}$ ,  $\mathrm{SO}_{2n}^{\mathrm{cpt}}$ ,  $\mathrm{PSO}_{2n}^{\mathrm{cpt}}$  and  $\mathrm{GSO}_{2n}^{\mathrm{cpt}}$  be the various versions of the orthogonal group defined by the quadratic form  $x_1^2 + x_2^2 + \cdots + x_{2n}^2$  on  $\mathbb{Q}^{2n}$ . Consider the matrix  $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$ . We define the group  $\mathrm{GSO}_{2n}^J$  over  $\mathbb{R}$  to be the inner form of  $\mathrm{GSO}_{2n, \mathbb{R}}^{\mathrm{cpt}}$  defined by  $J$ . Thus, for all  $\mathbb{R}$ -algebras  $R$  we have

$$(8.1) \quad \mathrm{GSO}_{2n}^J(R) = \{g \in \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{C} \otimes_{\mathbb{R}} R) \mid J\bar{g}J^{-1} = g\}.$$

For  $g \in \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{C} \otimes_{\mathbb{R}} R)$  we have  $g^t J\bar{g} = \mathrm{sim}(g)J$  if and only if  $J\bar{g}J^{-1} = g$ , and thus  $\mathrm{GSO}_{2n}^J(\mathbb{R})$  is the group of matrices  $g \in \mathrm{GL}_{2n}(\mathbb{C})$  preserving the forms

$$(8.2) \quad \begin{cases} x_1^2 + x_2^2 + \cdots + x_{2n}^2 \\ -x_1\bar{x}_{n+1} + x_{n+1}\bar{x}_1 - x_2\bar{x}_{n+2} + x_{n+2}\bar{x}_2 - \cdots - x_n\bar{x}_{2n} + x_{2n}\bar{x}_n \end{cases}$$

up to the scalar  $\mathrm{sim}(g) \in \mathbb{R}^\times$  (the scalar is required to be the same for both forms), and such that  $g$  satisfies the condition  $\det(g) = \mathrm{sim}(g)^n$ .

In a similar way we define the inner forms  $\mathrm{GO}_{2n}^J, \mathrm{SO}_{2n}^J, \mathrm{O}_{2n}^J, \mathrm{PSO}_{2n}^J$  of  $\mathrm{SO}_{2n}^{\mathrm{cpt}}, \mathrm{O}_{2n}^{\mathrm{cpt}}, \mathrm{PSO}_{2n}^{\mathrm{cpt}}$ . Then  $\mathrm{SO}_{2n}^J(\mathbb{R})$  is the real Lie group which is often denoted  $\mathrm{SO}^*(2n)$  in the literature (e.g., [Hel01, Sect. X.2, p. 445]). Note that  $\mathrm{SO}_{2n}^J(\mathbb{R})$  is not isomorphic to any of the classical groups  $\mathrm{SO}(p, q)$ , where  $2n = p + q$  (see [Kna02, thm 6.105(c)]). The group  $\mathrm{SO}(p, q)$  with  $2n = p + q$  is quasi-split if and only if  $|n - p| \leq 1$ , giving rise to two classes of inner twists (recall that  $\mathrm{SO}(p, q)$  and  $\mathrm{SO}(p', q')$  lie in the same inner class if and only if  $p \equiv p' \pmod{2}$ ). The group  $\mathrm{SO}_{2n}^J$ , and hence the group  $\mathrm{GSO}_{2n}^J$ , is not quasi-split since  $\mathrm{SO}_{2n}^J$  is not isomorphic to any group of the form  $\mathrm{SO}(p, q)$ .

We pin down the isomorphisms

$$(8.3) \quad \begin{aligned} C_X: \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{C}) &\xrightarrow{\sim} \mathrm{GSO}_{2n}(\mathbb{C}), & g &\mapsto X^{-1}gX, & X &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \\ \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{C}) &\xrightarrow{\sim} \mathrm{GSO}_{2n}^J(\mathbb{C}), & g &\mapsto (g, J^{-1}gJ) \in \mathrm{GSO}_{2n}(\mathbb{C})^2 = \mathrm{GSO}_{2n}(\mathbb{C} \otimes \mathbb{C}). \end{aligned}$$

**Lemma 8.1.** (i) *The group  $\mathrm{GSO}_{2n}^J$  is an inner form of  $\mathrm{GSO}(n, n)$  if  $n$  is even, and an outer form otherwise.*

(ii) *Explicitly,*

$$\mathrm{GO}_{2n}^J(\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{C}) \left| \begin{array}{l} A, B \in M_n(\mathbb{C}) \text{ such that} \\ A^t A + \bar{B}^t \bar{B} = \lambda \cdot 1_n \text{ (where } \lambda = \mathrm{sim}(g) \in \mathbb{C}^\times) \\ A^t B = \bar{B}^t \bar{A} \end{array} \right. \right\}.$$

(iii) *The groups  $\mathrm{SO}_{2n}^J(\mathbb{R}), \mathrm{O}_{2n}^J(\mathbb{R})$  are connected and  $|\pi_0(\mathrm{GSO}_{2n}^J(\mathbb{R}))| = 2$ .*

(iv) *The mapping*

$$\theta^J: \mathrm{GSO}_{2n}^J(\mathbb{R}) \rightarrow \mathrm{GSO}_{2n}^J(\mathbb{R}), \quad g = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \mapsto TgT^{-1} = \begin{pmatrix} A & -B \\ \bar{B} & \bar{A} \end{pmatrix}$$

*for  $T = i \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GO}_{2n}^J(\mathbb{R})$  is an automorphism of  $\mathrm{GSO}_{2n}^J$  over  $\mathbb{R}$ . It is outer if and only if  $n$  is odd.*

(v) *The groups  $\mathrm{SO}_{2n}^J$  and  $\mathrm{GSO}_{2n}^J$  have an outer automorphism defined over  $\mathbb{R}$  if and only if  $n$  is odd.*

(vi) *The groups  $\mathrm{SO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$  and  $\mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$  are connected.*

*Proof.* (i). The group  $\mathrm{SO}_{2n}^J$  is an inner form of  $\mathrm{SO}_{2n,\mathbb{R}}^{\mathrm{cpt}}$  and the compact form lies in the split inner class if and only if  $n$  is even.

(ii). Let  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{C})$ . Write  $\lambda = \mathrm{sim}(g)$ . We compute

$$\begin{aligned} J\bar{g} = gJ &\Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -\bar{C} & -\bar{D} \\ \bar{A} & \bar{B} \end{pmatrix} = \begin{pmatrix} B & -A \\ D & -C \end{pmatrix} \Leftrightarrow g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \\ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^t \begin{pmatrix} A & B \\ -B & A \end{pmatrix} &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A^t & -\bar{B}^t \\ B^t & \bar{A}^t \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} A^t A + \bar{B}^t \bar{B} & A^t B - \bar{B}^t \bar{A} \\ B^t A - \bar{A}^t \bar{B} & B^t B + \bar{A}^t \bar{A} \end{pmatrix}. \end{aligned}$$

These identities are equivalent to the stated conditions on  $g$ .

(iii) By [Zha97, Cor. 6.3],  $\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \geq 0$  for all  $A, B \in \mathrm{M}_n(\mathbb{C})$ . By Lemma 8.1(i) any  $g \in \mathrm{O}_{2n}^J(\mathbb{R})$  has  $\det(g) \geq 0$  and thus  $\det(g) = 1$ . Thus  $\mathrm{O}_{2n}^J(\mathbb{R}) = \mathrm{SO}_{2n}^J(\mathbb{R})$ . By [Kna02, prop I.1.145] the group  $\mathrm{SO}_{2n}^J(\mathbb{R})$  (and hence  $\mathrm{O}_{2n}^J(\mathbb{R})$ ) is connected. The similitudes factor  $\mathrm{sim}$  equals  $x \mapsto x^2$  on  $\mathbb{R}^\times \subset \mathrm{GSO}_{2n}^J(\mathbb{R})$ . In particular  $\mathrm{GSO}_{2n}^J(\mathbb{R}) \rightarrow \mathbb{R}^\times / \mathbb{R}_{>0}^\times$  has connected kernel  $\mathbb{R}_{>0}^\times \times \mathrm{SO}_{2n}^J(\mathbb{R})$ . Hence  $\pi_0(\mathrm{GSO}_{2n}^J(\mathbb{R})) \simeq \mathbb{R}^\times / \mathbb{R}_{>0}^\times$ .

(iv) We have  $T^t T = -1$  and  $J\bar{T}J^{-1} = J$ , so indeed  $T \in \mathrm{GO}_{2n}^J(\mathbb{R})$ . As  $\mathrm{sim}(T) = -1$  and  $\det(T) = i^{2n}(-1)^n = 1$ , we have  $\mathrm{sim}(T)^n \neq \det(T)$  if and only if  $n$  is odd.

(v) By the example in (iv) we may assume  $n$  even. Any  $\mathbb{R}$ -automorphism  $\theta \in \mathrm{Aut}(\mathrm{GSO}_{2n}^J)$  is given by  $\theta: g \mapsto YgY^{-1}$  for some  $Y \in \mathrm{GO}_{2n}^J(\mathbb{C})$ . Replacing  $Y$  with  $tY$  for some  $t \in \mathbb{C}^\times$  we may assume that  $\mathrm{sim}(Y) = 1$  (as  $\theta$  does not change, it is still defined over  $\mathbb{R}$ ). Write  $\sigma: \mathrm{GO}_{2n}^{\mathrm{cpt}}(\mathbb{C}) \rightarrow \mathrm{GO}_{2n}^{\mathrm{cpt}}(\mathbb{C})$  for the automorphism  $g \mapsto J\bar{g}J^{-1}$ , so that  $\mathrm{GO}_{2n}^J(\mathbb{R}) = \mathrm{GO}_{2n}^{\mathrm{cpt}}(\mathbb{C})^{\sigma=\mathrm{id}}$ . As  $\theta$  is defined over  $\mathbb{R}$ ,

$$\theta(\sigma g) = \sigma\theta(g) \quad \forall g \in \mathrm{GO}_{2n}^J(\mathbb{C}),$$

and therefore  $YJ \cdot \bar{g} \cdot J^{-1}Y^{-1} = J\bar{Y} \cdot \bar{g} \cdot \bar{Y}^{-1}J^{-1}$ , so  $\bar{Y}^{-1}J^{-1}YJ \cdot \bar{g} = \bar{g} \cdot \bar{Y}^{-1}J^{-1}YJ$ . Thus

$$\lambda \cdot YJ = J\bar{Y} \quad \text{for some } \lambda \in Z(\mathrm{GSO}_{2n}^J(\mathbb{C})) = \mathbb{C}^\times.$$

We have  $Y^t Y = 1$ , so we compute as follows using  $J^t J = 1$ :

$$1 = (\bar{Y})^t \bar{Y} = (\lambda J^{-1}YJ)^t (\lambda J^{-1}YJ) = \lambda^2 (J^t Y^t Y J) = \lambda^2.$$

Therefore  $\lambda \in \{\pm 1\}$ . If  $\lambda = 1$  then  $Y \in \mathrm{O}_{2n}^J(\mathbb{R}) = \mathrm{SO}_{2n}^J(\mathbb{R})$ , and  $\theta$  is inner. If  $\lambda = -1$  then  $\sigma(Y) = -Y$ . Thus  $\sigma(iY) = iY$  and  $Y' = iY \in \mathrm{O}_{2n}^J(\mathbb{R}) = \mathrm{SO}_{2n}^J(\mathbb{R})$ . Thus  $\theta = (g \mapsto Y'gY'^{-1})$  is inner.

(vi) It is standard that  $\mathrm{SO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$  is connected. Let us show that  $\mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$  is connected from this. The multiplication map  $\mathrm{SO}_{2n}^{\mathrm{cpt}}(\mathbb{R}) \times \mathbb{R}^\times \rightarrow \mathrm{GSO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$  has connected image since  $\mathrm{SO}_{2n}^{\mathrm{cpt}}(\mathbb{R})$  meets both connected components of  $\mathbb{R}^\times$ . So we will be done if we check the surjectivity. This is equivalent to the injectivity of  $H^1(\mathbb{R}, \{\pm 1\}) \rightarrow H^1(\mathbb{R}, \mathrm{SO}_{2n}^{\mathrm{cpt}} \times \mathrm{GL}_1)$ , which follows from the fact that there is no  $g \in \mathrm{SO}_{2n}^{\mathrm{cpt}}(\mathbb{C})$  with  $g^{-1}\bar{g} = -1$ . (Via  $h = \sqrt{-1}g$ , the latter is equivalent to non-existence of  $h \in \mathrm{GL}_{2n}(\mathbb{R})$  with  $h^t h = -1$ , which is clear.)  $\square$

Now we turn to the global setup. Let  $n$  and  $E/F$  be as in §6 and impose condition (disc- $\infty$ ) from now on. In analogy with the  $\mathrm{SO}_{2n}$ -case, we introduce a quasi-split form  $G^*$  of  $\mathrm{GSO}_{2n}$  over  $F$ . Concretely, if  $n$  is even, we take the split form  $G^* := \mathrm{GSO}_{2n,F}$  (or simply  $\mathrm{GSO}_{2n}$ ). If  $n$  is odd, we fix an imaginary quadratic extension  $E/F$  and let  $G^*$  be the quasi-split form  $\mathrm{GSO}_{2n,F}^{E/F}$  of  $\mathrm{GSO}_{2n,F}$  (up to  $F$ -automorphism) given by the 1-cocycle  $\mathrm{Gal}(E/F) \rightarrow \mathrm{Aut}(\mathrm{GSO}_{2n,E})$  sending the nontrivial element to  $\theta^\circ = \mathrm{Int}(\vartheta^\circ)$ . Since  $\vartheta^\circ \in \mathrm{O}_{2n}(E)$ , this cocycle comes from the  $\mathrm{Aut}(\mathrm{SO}_{2n,E})$ -valued cocycle determining  $\mathrm{SO}_{2n}^{E/F}$  as an outer form of  $\mathrm{SO}_{2n}$ , thus we have  $\mathrm{SO}_{2n}^{E/F} \hookrightarrow \mathrm{GSO}_{2n}^{E/F}$ . Concretely, in analogy with (6.1),

$$(8.4) \quad \mathrm{GSO}_{2n}^{E/F}(R) = \{g \in \mathrm{GL}_{2n}(E \otimes_F R) \mid c(g) = \vartheta^\circ g \vartheta^\circ, g^t \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} g = \lambda \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \det(g) = \lambda^n\},$$

and  $\mathrm{GO}_{2n}^{E/F}(R)$  is defined by removing the condition  $\det(g) = \lambda^n$ . We write  $G^* = \mathrm{GSO}_{2n}^{E/F}$  in both cases, understanding that  $E = F$  if  $n$  is even, for a streamlined exposition. In both cases, we have an exact sequence

$$(8.5) \quad 1 \rightarrow \mathrm{SO}_{2n}^{E/F} \rightarrow \mathrm{GSO}_{2n}^{E/F} \rightarrow \mathbb{G}_m \rightarrow 1,$$



where the similitude map  $\mathrm{GSO}_{2n}^{E/F} \rightarrow \mathbb{G}_m$  is the usual one if  $E = F$ , and  $g \mapsto \lambda$  in (8.4) if  $E \neq F$ .

Write  $(\cdot)^D$  for the Pontryagin dual of an abelian group. By [Kot86, Thm 1.2] we have for each  $F$ -place  $v$  a map<sup>12</sup>

$$\alpha_v : H^1(F_v, G_{\mathrm{ad}}^*) \rightarrow \pi_0(Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma_v})^D,$$

which is an isomorphism if the place  $v$  is finite (but not if  $v$  is infinite). Note that  $\widehat{G_{\mathrm{ad}}^*}$  is isomorphic to  $\mathrm{Spin}_{2n}(\mathbb{C})$ , on which  $\Gamma$  acts non-trivially via  $\mathrm{Gal}(\overline{F}/F)$  and  $\theta$  (resp. trivially) if  $n$  is odd (resp. even).

**Lemma 8.2.** *We have*

$$Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma_v} \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & n \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} & n \text{ is odd, } v \text{ is non-split in } E/F \\ \mathbb{Z}/4\mathbb{Z} & n \text{ is odd, } v \text{ is split in } E/F \end{cases}$$

*Proof.* This follows from Lemma 3.7.  $\square$

By [Kot86, Prop. 2.6] and the Hasse principle from [PR94, Thm. 6.22] we have an exact sequence of pointed sets

$$(8.6) \quad 1 \rightarrow H^1(F, G_{\mathrm{ad}}^*) \rightarrow \bigoplus_v H^1(F_v, G_{\mathrm{ad}}^*) \xrightarrow{\sum_v \alpha_v} \pi_0(Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma})^D \rightarrow 1.$$

Since  $Z(\widehat{G_{\mathrm{ad}}^*})$  is finite, we may forget  $\pi_0(\cdot)$  in (8.6) and the proof of the lemma below.

**Lemma 8.3.** *There exists an inner twist  $G$  of  $G^*$  such that for all  $F$ -places  $v \neq \mathfrak{q}_{\mathrm{St}}$ , we have*

$$(8.7) \quad G_v \simeq \begin{cases} \mathrm{GSO}_{2n, F_v}^J & v = y_{\infty} \\ \mathrm{GSO}_{2n, F_v}^{\mathrm{cpt}} & v \in \mathcal{V}_{\infty} \setminus \{y_{\infty}\} \\ G_{F_v}^* & v \nmid \infty, v \neq \mathfrak{q}_{\mathrm{St}}. \end{cases}$$

*This inner twist  $G$  is unique up to isomorphism if  $n$  is even or  $v$  is split; otherwise there are two choices for  $G$ .*

*Proof.* Put

$$(8.8) \quad a_{\mathfrak{q}_{\mathrm{St}}} := -\alpha_{y_{\infty}}(\mathrm{GSO}_{2n, F_{y_{\infty}}}^J) - \sum_{v \neq y_{\infty}} \alpha_v(\mathrm{GSO}_{2n, F_{y_{\infty}}}^{\mathrm{cpt}}) \in (Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma})^D.$$

By duality, the inclusion  $Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma} \subset Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma_v}$  induces a surjection  $(Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma_v})^D \rightarrow (Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma})^D$ . Hence we can choose some invariant  $\widetilde{a_{\mathfrak{q}_{\mathrm{St}}}} \in (Z(\widehat{G_{\mathrm{ad}}^*})^{\Gamma_v})^D$  mapping to the expression on the right hand side of (8.8). Let  $G_{\mathfrak{q}_{\mathrm{St}}}$  be the inner twist of  $G^*$  over  $F_{\mathfrak{q}_{\mathrm{St}}}$  corresponding to  $\widetilde{a_{\mathfrak{q}_{\mathrm{St}}}}$ . Then, by (8.6) the collection of local inner twists  $\{G_v\}_{\text{places } v}$  comes from a global inner twist  $G/F$ , unique up to isomorphism. Conversely any  $G$  as in the lemma satisfies  $\alpha_{\mathfrak{q}_{\mathrm{St}}}(G) = a_{\mathfrak{q}_{\mathrm{St}}}$  by (8.6). Therefore the number of choices for  $G$  equals the number of choices for  $\widetilde{a_{\mathfrak{q}_{\mathrm{St}}}}$ , which can be computed using Lemma 8.2.  $\square$

*Remark 8.4.* The group  $G_{\mathfrak{q}_{\mathrm{St}}}$  in the lemma is never quasi-split, regardless of the parity of  $[F : \mathbb{Q}]$ . It is always a unitary group for a Hermitian form over a quaternion algebra. This corresponds to the “ $d = 2$  case” in [Art13, §9.1]. In this case the rank of  $G_{\mathfrak{q}_{\mathrm{St}}}$  is roughly  $n/2$  (see [Art13] for precise information).

**Lemma 8.5.** *There exists an outer automorphism  $\theta^*$  of  $G^*$  defined over  $F$  coming from the conjugation by an element of  $\mathrm{O}_{2n}^{E/F}(F) - \mathrm{SO}_{2n}^{E/F}(F)$ . (In particular, the latter is nonempty.) The integer  $n$  is odd if and only if there exists an outer automorphism of  $G$  over  $F$  that induces  $\theta^*$  (up to inner automorphism) via inner twisting  $G_{\overline{F}} \simeq G_{\overline{F}}^*$ .*

<sup>12</sup>This map has been computed explicitly by Arthur [Art13, Section 9.1] for all possible inner forms of classical groups.

Obviously  $\theta^*$  is well-defined up to inner automorphism. Henceforth we will fix an  $F$ -automorphism  $\theta^*$  of  $G$  (resp.  $\theta$  of  $G$  if  $n$  is odd) as above.<sup>13</sup> Evidently  $\theta^*$  (resp.  $\theta$  if  $n$  is odd) induces  $\theta^\circ$  on  $\text{GSO}_{2n, \mathbb{C}}$  (defined below (2.2)) up to inner automorphism under every embedding  $F \hookrightarrow \mathbb{C}$ .

*Proof.* The proof is straightforward for  $G^*$ . Indeed, the conjugation by  $\vartheta^\circ \in \text{O}_{2n}^{E/F}(F) - \text{SO}_{2n}^{E/F}(F)$  is such an automorphism, where  $\vartheta^\circ$  is the matrix as in (2.2).

We deal with the case of  $G$ . If such an automorphism exists then clearly  $n$  is odd by Lemma 8.1. From now we assume  $n$  is odd. Write  $G^b$  for the kernel of  $\text{sim}: G \rightarrow \mathbb{G}_m$ . If  $\theta$  is an outer automorphism of  $G^b$  defined over  $F$ , then  $\theta(zx) := z\theta(x)$  ( $z \in \mathbb{G}_m, x \in G^b$ ) provides an extension of  $\theta$  to  $G$ . In particular it is enough to prove the lemma for  $G^b$ . We have a short exact sequence of group schemes over  $F$

$$1 \rightarrow \text{Inn}(G^b) = G_{\text{ad}} \rightarrow \text{Aut}(G^b) \rightarrow \text{Out}(G^b) = \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

(For the latter equality, notice that the  $\Gamma$ -action is necessarily trivial on the order 2 group  $\text{Out}(G^b)(\overline{F})$ .) The associated long exact sequence over  $F$  and its localization over  $F_v$  yield a commutative diagram:

$$\begin{array}{ccccc} \text{Aut}(G^b)(F) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\delta} & H^1(F, G_{\text{ad}}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Aut}(G^b)(F_v) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\delta_v} & H^1(F_v, G_{\text{ad}}), \end{array}$$

where  $\delta$  and  $\delta_v$  denote the connecting morphisms. We need to prove that  $\delta(-1)$  is trivial, or equivalently that  $\delta_v(-1)$  is trivial for all  $v$ , by the Hasse principle for adjoint groups.

We know that  $\delta_v(-1)$  is trivial if  $G_v^b$  is quasi-split (since  $\text{O}_{2n}(F_v) - \text{SO}_{2n}(F_v)$  and  $\text{O}_{2n}^{E_v/F_v}(F_v) - \text{SO}_{2n}^{E_v/F_v}(F_v)$  are nonempty by the earlier part of this proof) or if  $v$  is a real place (by Lemma 8.1 at  $v = y_\infty$  and the fact that  $\text{O}(2n, \mathbb{R}) - \text{SO}(2n, \mathbb{R})$  is visibly nonempty at  $v \neq y_\infty$ ). Therefore  $\alpha_v(\delta_v(-1)) = 0$  for  $v \neq \mathfrak{q}_{\text{St}}$ . So it remains to see  $\alpha_{\mathfrak{q}_{\text{St}}}(\delta_{\mathfrak{q}_{\text{St}}}(-1)) = 0$ .

On the other hand, the exact sequence (8.6) holds for  $G^b$  in place of  $G^{*\flat}$ , thus  $\sum_v \alpha_v(\delta_v(-1)) = 0$  in  $\pi_0(Z(\widehat{G_{\text{ad}}})^\Gamma)^D$ . It follows that  $\alpha_{\mathfrak{q}_{\text{St}}}(\delta_{\mathfrak{q}_{\text{St}}}(-1))$  maps trivially under the restriction map

$$(8.9) \quad (Z(\widehat{G_{\text{ad}}})^{\Gamma_{\mathfrak{q}_{\text{St}}}})^D \rightarrow (Z(\widehat{G_{\text{ad}}})^\Gamma)^D.$$

This implies  $\alpha_{\mathfrak{q}_{\text{St}}}(\delta_{\mathfrak{q}_{\text{St}}}(-1)) = 0$  if the above map is an isomorphism. This is the case unless  $G^{*\flat}$  is non-split but  $G_{\mathfrak{q}_{\text{St}}}^{*\flat}$  is split (see Lemma 8.2). In the latter case, we can globalize  $F_{\mathfrak{q}_{\text{St}}}$  to a totally real field  $F'$  with a finite place  $\mathfrak{q}'$  such that  $F'_{\mathfrak{q}'} \simeq F_{\mathfrak{q}_{\text{St}}}$ , and globalize  $G_{\mathfrak{q}_{\text{St}}}^b$  to an inner form  $G^b$  of the split group  $\text{SO}_{2n}$  over  $F'$  such that  $G_{\mathfrak{q}'}^b \simeq G_{\mathfrak{q}_{\text{St}}}^b$  and such that  $G^b$  is quasi-split at all finite places away from  $\mathfrak{q}'$ , and either compact or  $\text{SO}_{2n}^J$  at each real place. Such a  $G^b$  exists by adapting the proof of Lemma 8.3. (We have switched the base field from  $F$  to  $F'$  to possibly increase the number of real places to kill the obstruction to the existence of  $G^b$ .) Applying the preceding argument to  $G^b$ , we have  $\alpha_{\mathfrak{q}_{\text{St}}}(\delta_{\mathfrak{q}_{\text{St}}}(-1))$  mapped trivially under the analogue of (8.9) for  $G^b$  at  $\mathfrak{q}'$ , which is now an isomorphism. We conclude that  $\alpha_{\mathfrak{q}_{\text{St}}}(\delta_{\mathfrak{q}_{\text{St}}}(-1)) = 0$ .  $\square$

*Remark 8.6.* It may be possible to directly show that  $\delta_{\mathfrak{q}_{\text{St}}}(-1)$  vanishes by choosing an explicit model for  $G_{\mathfrak{q}_{\text{St}}}$  and proving the analogue of Lemma 8.1 at  $\mathfrak{q}_{\text{St}}$ . We have circumvented this via the Hasse principle.

## 9. SHIMURA VARIETIES OF TYPE $D$ CORRESPONDING TO $\text{spin}^\pm$

We introduced an inner form  $G$  of a quasi-split form  $G^*$  of  $\text{GSO}_{2n}$  over a totally real field  $F$ . The groups  $G$  and  $G^*$  are equipped with outer automorphisms  $\theta$  and  $\theta^*$  over  $F$ . We are going

<sup>13</sup>As introduced in §2,  $\theta$  also denotes an automorphism of  $\text{GSpin}_{2n}$ . It will be clear from the context which group  $\theta$  is acting on.

to construct Shimura data associated with  $\text{Res}_{F/\mathbb{Q}}G$  by giving an  $\mathbb{R}$ -morphism  $\text{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \rightarrow (\text{Res}_{F/\mathbb{Q}}G) \otimes_{\mathbb{Q}} \mathbb{R}$ . We define

$$\begin{aligned} h_{(-1)^n} : \mathbb{S} &\rightarrow \text{GSO}_{2n}^J, & x + yi &\mapsto \begin{pmatrix} x1_n & y1_n \\ -y1_n & x1_n \end{pmatrix} \\ h_{(-1)^{n+1}} : \mathbb{S} &\rightarrow \text{GSO}_{2n}^J, & x + yi &\mapsto \begin{pmatrix} & x1_n & & \text{diag}(y1_{n-1}, -y) \\ \text{diag}(-y1_{n-1}, y) & & & x1_n \end{pmatrix}. \end{aligned}$$

We will often omit  $1_n$  in the cases similar to the above if a matrix is clearly  $2n \times 2n$  in the context. Recall the cocharacters  $\mu_+, \mu_-$  from (2.7). They are outer conjugate as  $\mu_+ = \vartheta^\circ \mu_- (\vartheta^\circ)^{-1}$  (but not inner, cf. (2.6)).

**Lemma 9.1.** *Let  $\varepsilon \in \{+, -\}$ .*

- (i) *Consider the inclusion of  $\mathbb{C}^\times$  in  $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{C}^{\times, \text{Gal}(\mathbb{C}/\mathbb{R})}$  indexed by the identity morphism  $\text{id}_{\mathbb{C}/\mathbb{R}} \in \text{Gal}(\mathbb{C}/\mathbb{R})$ . Then  $C_X h_{\varepsilon, \mathbb{C}}|_{\mathbb{C}^\times} = \mu_\varepsilon$ .*
- (ii) *The complex conjugate morphism  $\bar{h}_\varepsilon : z \mapsto h_\varepsilon(\bar{z})$  is  $\text{GSO}_{2n}^J(\mathbb{R})$ -conjugate to  $h_{\pm(-1)^n \varepsilon}$ .*

*Proof.* In the proof, put  $\varepsilon = (-1)^n$ . (i). Recall  $C_X$  from (8.3), which induces  $\text{GSO}_{2n}^J(\mathbb{R}) \hookrightarrow \text{GSO}_{2n}^{\text{cpt}}(\mathbb{C}) \xrightarrow{C_X} \text{GSO}_{2n}(\mathbb{C})$ . The morphism  $C_X h_\varepsilon$  equals  $x + yi \mapsto \begin{pmatrix} x+yi & 0 \\ 0 & x-yi \end{pmatrix}$ . The holomorphic part of this morphism is  $z \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ , which is  $\mu_\varepsilon$ . Then  $h_\varepsilon = \vartheta^\circ h_{-\varepsilon} \vartheta^\circ$ , where  $\vartheta^\circ$  is as in (2.2). Write  $\vartheta^c = \begin{pmatrix} -1_{2n-1} & 0 \\ 0 & 1 \end{pmatrix}$ . Note that  $C_X(w^c) = \vartheta^\circ$ , so  $\vartheta^c$ -conjugation becomes  $\vartheta^\circ$ -conjugation under  $C_X$ . As  $\vartheta^\circ$  swaps  $\mu_\varepsilon$  and  $\mu_{-\varepsilon}$ , we obtain  $C_X h_{-\varepsilon}|_{\mathbb{C}^\times} = \mu_{-\varepsilon}$ .

(ii). Write  $z = x + yi \in \mathbb{C}$ . Using Lemma 8.1 we compute

$$h_\varepsilon(\bar{z}) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = T h_+(z) T^{-1} = \theta^J h_\varepsilon(z) \sim h_+(z) \in \text{GSO}_{2n}^J(\mathbb{R}).$$

The proof for  $h_{-\varepsilon}$  is the same.  $\square$

**Lemma 9.2.** *Let  $\varepsilon \in \{+, -\}$  and put  $K_\varepsilon := \text{Cent}_{\text{GSO}_{2n}^J(\mathbb{R})}(h_\varepsilon)$ . The following hold.*

- (i) *Let  $\mathbb{C}^\times$  act on  $\text{Lie GSO}_{2n}^J(\mathbb{R})$  via  $\text{Ad} \circ h_\varepsilon$ . Then only the characters  $z \mapsto z^{-1}\bar{z}$ ,  $z \mapsto 1$ , and  $z \mapsto z\bar{z}^{-1}$  appear in the representation  $\text{Lie GSO}_{2n}^J(\mathbb{R})$  of  $\mathbb{C}^\times$ .*
- (ii) *The involution  $\text{Ad } h_\varepsilon(i)$  is a Cartan involution of  $\text{PSO}_{2n}^J$ .*
- (iii)  *$K_+$  and  $K_-$  are  $\text{GSO}_{2n}^J(\mathbb{R})$ -conjugate.*

*Proof.* For (i) and (ii), we only treat the case of  $\varepsilon = (-1)^n$  as the argument for  $-\varepsilon$  is the same. Let  $z = x + yi \in \mathbb{C}$  and consider the left-multiplication action of the matrix  $h_\varepsilon(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$  on  $M_{2n}(\mathbb{C})$ . The matrix  $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$  is conjugate to  $\begin{pmatrix} x+yi & \\ & x-yi \end{pmatrix}$  via  $\begin{pmatrix} 1 & \\ & -i \end{pmatrix}$ . Thus, for this left multiplication only the characters  $z$  and  $\bar{z}$  appear in  $M_{2n}(\mathbb{C})$ . Similarly, for the right multiplication, only the characters  $z$  and  $\bar{z}$  appear. Hence for the conjugation action only the characters  $z\bar{z}^{-1}$ ,  $\bar{z}z^{-1}$  and  $1$  appear in  $M_{2n}(\mathbb{C})$ . Since  $\text{Lie GSO}_{2n}^J(\mathbb{R})$  is contained in  $M_{2n}(\mathbb{C})$  via the standard representation, (i) is true for  $h_\varepsilon(z)$ . Since  $J^{-1} = h_\varepsilon(i)$  the inner form of  $\text{GSO}_{2n}^J$  defined by  $h_\varepsilon(i)$  is the compact modulo center form  $\text{GSO}_{2n, \mathbb{R}}^{\text{cpt}}$ , so part (ii) follows.

Let us prove (iii). The adjoint mapping  $\text{Ad} : \text{GSO}_{2n}^J \rightarrow \text{GSO}_{2n, \text{ad}}^J$  is surjective on  $\mathbb{R}$ -points by Hilbert 90. Clearly  $\text{Ad}(K_\varepsilon) \subset \text{Cent}_{\text{GSO}_{2n, \text{ad}}^J(\mathbb{R})}(h_\varepsilon)$ . Write  $\bar{h}_\varepsilon := \text{Ad} \circ h_\varepsilon$ . The Lie algebra  $\text{Lie}(K_\varepsilon)$  (resp. the Lie algebra of  $\text{Cent}_{\text{GSO}_{2n, \text{ad}}^J(\mathbb{R})}(\bar{h}_\varepsilon)$ ) is the  $(0, 0)$  part of  $\text{Lie}(\text{GSO}_{2n}^J)$  (resp.  $\text{Lie}(\text{GSO}_{2n, \text{ad}}^J)$ ) via  $h_\varepsilon$ , in the sense of [Del79]. In particular

$$\text{ad} : \text{Lie}(K_\varepsilon) \rightarrow \text{Lie}(\text{Cent}_{\text{GSO}_{2n, \text{ad}}^J(\mathbb{R})}(\bar{h}_\varepsilon))$$

is surjective. Since  $\text{Cent}_{\text{GSO}_{2n, \text{ad}}^J(\mathbb{R})}(\bar{h}_\varepsilon)$  is connected by [Del79, proof of Prop. 1.2.7], we have  $\text{Ad}(K_\varepsilon) = \text{Cent}_{\text{GSO}_{2n, \text{ad}}^J(\mathbb{R})}(\bar{h}_\varepsilon)$ . The latter is the identity component of a maximal compact subgroup by *loc. cit.* so  $\text{Ad}(K_-)$  and  $\text{Ad}(K_+)$  are conjugate in  $\text{GSO}_{2n, \text{ad}}^J(\mathbb{R})$ . Since  $K_\varepsilon = \text{Ad}^{-1}(\text{Ad}(K_\varepsilon))$  and since  $\text{Ad}$  is surjective on real points, we lift a conjugating element to see that  $K_+$  and  $K_-$  are conjugate in  $\text{GSO}_{2n}^J(\mathbb{R})$ .  $\square$

Let  $X^\varepsilon$  be the  $G(\mathbb{R})$ -conjugacy class of the morphism

$$(9.1) \quad h^\varepsilon : \mathbb{S} \rightarrow (\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathbb{R}}, \quad z \mapsto (h_\varepsilon(z), 1, \dots, 1) \in \prod_{v \in \mathcal{V}_\infty} G_{F_v},$$

where the non-trivial component corresponds to the place  $v_\infty$ . Then  $\mu = (\mu_\varepsilon, 1, \dots, 1) \in X_*((\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathbb{C}}) = X_*(\mathrm{GSO}_{2n, \mathbb{C}})^{\mathcal{V}_\infty}$  is the cocharacter attached to  $h_\varepsilon$ . The reflex field of  $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$  means the field of definition for the conjugacy class of  $\mu$ , as a subfield of  $\mathbb{C}$ .

**Lemma 9.3.** *Let  $\varepsilon \in \{\pm 1\}$ . Then*

- (i) *The pair  $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$  is a Shimura datum of abelian type.*
- (ii) *The Shimura data  $(\mathrm{Res}_{F/\mathbb{Q}}G, X^+)$  and  $(\mathrm{Res}_{F/\mathbb{Q}}G, X^-)$  are isomorphic if and only if  $n$  is odd.*
- (iii) *If  $F \neq \mathbb{Q}$ , the Shimura varieties attached to  $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$  are projective.*
- (iv) *The reflex field of the datum  $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$  is equal to  $E$ , equipped with an embedding  $x_\infty : E \hookrightarrow \mathbb{C}$  extending  $y_\infty : F \hookrightarrow \mathbb{C}$ .*

*Remark 9.4.* No Shimura data arise from the ‘‘opposite’’ case, namely when  $n$  is even and  $G^*$  is a non-split outer form  $\mathrm{GSO}_{2n, F}^{E/F}$ , or when  $n$  is odd and  $G^*$  is the split  $\mathrm{GSO}_{2n, F}$ , as the group  $G^*(F_\infty)$  and its inner forms do not admit discrete series in that case.

*Remark 9.5.* When  $F = \mathbb{Q}$ , the Shimura datum  $(G, X^\varepsilon)$  can be shown to be of Hodge type but we do not need this fact.

*Proof.* (i) Clearly,  $(\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathrm{ad}}$  has no compact factor defined over  $\mathbb{Q}$ , which is one of Deligne’s axioms of Shimura datum. The remaining two axioms follow from Lemma 9.2, and hence  $(\mathrm{Res}_{F/\mathbb{Q}}G, X^\varepsilon)$  is a Shimura datum. In the terminology of Deligne [Del79],  $(G, X^\varepsilon)$  is of type  $D_l^{\mathbb{H}}$ . By [Del79, Prop. 2.3.10], a datum  $(G, X)$  of type  $D_l^{\mathbb{H}}$  is of abelian type if the derived group of  $G_{\mathbb{C}}$  is  $\mathrm{SO}_{2n, \mathbb{C}}$ . (Not all Shimura data of type  $D_l^{\mathbb{H}}$  are of abelian type.)

(ii) If  $n$  is even then every automorphism of  $G \times_F F_v$  is inner at each real place  $v$  of  $F$ . This together with Lemma 8.1 implies that no automorphism of  $\mathrm{Res}_{F/\mathbb{Q}}G$  carries  $X^+$  onto  $X^-$ . Now suppose that  $n$  is odd. By Lemma 8.5 there exists an outer automorphism  $\theta \in \mathrm{Aut}(G)$  defined over  $F$ . We have  $\theta_{y_\infty} = g\theta^J g^{-1}$ , for some  $g \in \mathrm{GSO}_{2n}^J(\mathbb{R})$ . In particular  $\mathrm{Res}_{F/\mathbb{Q}}\theta$  defines an isomorphism of Shimura data  $(\mathrm{Res}_{F/\mathbb{Q}}G, X^+) \xrightarrow{\sim} (\mathrm{Res}_{F/\mathbb{Q}}G, X^-)$ .

(iii) If  $F \neq \mathbb{Q}$  there exists some real place  $y'_\infty \in \mathcal{V}_\infty$  of  $F$  different from  $y_\infty$ . Since  $G_{y'_\infty}$  is compact modulo center,  $G$  is compact modulo center as well, and hence the associated Shimura varieties are projective by Bailey-Borel [BB64, Thm. 1].

(iv) Assume that  $n$  is odd (thus  $[E : F] = 2$ ). Suppose that  $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$  stabilizes the conjugacy class of  $\mu$ . Since  $\sigma(\mu) \sim \mu$  we have  $\sigma(y_\infty) = y_\infty$ , so  $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$  with respect to  $y_\infty : F \hookrightarrow \mathbb{C}$ . If  $\sigma$  is non-trivial in  $\mathrm{Gal}(E/F)$ , then Lemma 9.1 (iii) tells us that  $\sigma(\mu) \sim (\mu_{-\varepsilon}, 1, \dots, 1)$ , which is not  $\mathrm{GSO}_{2n}(\mathbb{C})$ -conjugate to  $\mu$ . Thus  $\sigma$  is trivial on  $E$  (embedded in  $\mathbb{C}$  extending  $y_\infty$ ). Conversely, if  $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ , then  $\sigma(\mu) = \mu$ . Hence the reflex field is  $E$ . When  $n$  is even (thus  $E = F$ ), the preceding argument shows that the reflex field is  $F$ .  $\square$

We introduce the following notation. Let  $\varepsilon \in \{+, -\}$ .

- Taking an algebraic closure of  $E$  in  $\mathbb{C}$  via  $x_\infty : E \hookrightarrow \mathbb{C}$ , we fix  $\overline{F} = \overline{E} \hookrightarrow \mathbb{C}$ .
- We fix an isomorphism  $G \otimes_F \mathbb{A}_F^{\infty, \mathrm{qst}} \simeq G^* \otimes_F \mathbb{A}_F^{\infty, \mathrm{qst}}$ .
- $Z$  is the center of  $G$ .
- $\xi$  is an irreducible algebraic representation of  $(\mathrm{Res}_{F/\mathbb{Q}}G) \times_{\mathbb{Q}} \mathbb{C}$ .
- $\Pi_\xi^{G(F_\infty)}$  is the set of isomorphism classes of (irreducible) discrete series representations of  $G(F_\infty)$  which have the same infinitesimal and central characters as  $\xi^\vee$ .
- $K_\infty^\varepsilon$  is the centralizer of  $h^\varepsilon$  in  $G(F_\infty)$ .
- For irreducible admissible representations  $\tau_\infty$  of  $G(F_\infty)$ , put

$$(9.2) \quad \mathrm{ep}^\varepsilon(\tau_\infty \otimes \xi) := \sum_{i=1}^{n(n-1)} (-1)^i \dim \mathrm{H}^i(\mathrm{Lie} G(F_\infty), K_\infty^\varepsilon; \tau_\infty \otimes \xi)$$

Let  $\pi^{\natural}$  be a cuspidal automorphic representation of  $G(\mathbb{A}_F)$  such that

- (St) at a finite place  $\mathfrak{q}_{\text{St}}$  and (std-reg) are satisfied, and
- $\pi^{\natural}$  is  $\xi$ -cohomological.

The latter requirement implies via (the proof of) [KS16, Lem. 7.1] the following condition:

**(cent)** there exists an integer  $w \in \mathbb{Z}$ , called the *weight*, such that for every infinite  $F$ -place  $y|\infty$  the central character of  $\xi \otimes_F F_y$  is of the form  $\mathbb{C}^\times \ni x \mapsto x^w$ .

Let  $A(\pi^{\natural})$  be the set of (isomorphism classes of) cuspidal automorphic representations  $\tau$  of  $G(\mathbb{A}_F)$  such that

- (i)  $\tau_{\mathfrak{q}_{\text{St}}} \simeq \pi_{\mathfrak{q}_{\text{St}}}^{\natural} \otimes \delta$  for an unramified character  $\delta$  of the group  $G(F_{\mathfrak{q}_{\text{St}}})$ ,
- (ii)  $\tau^{\infty, \mathfrak{q}_{\text{St}}} \simeq \pi^{\natural, \infty, \mathfrak{q}_{\text{St}}}$ , and
- (iii)  $\tau_{\infty}$  is  $\xi$ -cohomological.

Define a rational number  $a^{\varepsilon}(\pi^{\natural})$  by

$$(9.3) \quad a^{\varepsilon}(\pi^{\natural}) := (-1)^{n(n-1)/2} N_{\infty}^{-1} \sum_{\tau \in A(\pi^{\natural})} m(\tau) \cdot \text{ep}^{\varepsilon}(\tau_{\infty} \otimes \xi),$$

where  $m(\tau)$  is the multiplicity of  $\tau$  in the discrete automorphic spectrum of  $G$ , and

$$(9.4) \quad N_{\infty} := |\Pi_{\xi}^{G(F_{\infty})}| \cdot |\pi_0(G(F_{\infty})/Z(F_{\infty}))| = 2^{n-1} \cdot 2,$$

where we know  $|\pi_0(G(F_{\infty})/Z(F_{\infty}))| = 2$  from Lemma 8.1 (iii) and (vi).

**Lemma 9.6.** *The groups  $K_{\infty}^+$  and  $K_{\infty}^-$  are  $G(F_{\infty})$ -conjugate. In particular  $a^-(\pi^{\natural}) = a^+(\pi^{\natural})$ .*

Henceforth we will write  $a(\pi^{\natural}) \in \mathbb{Q}$  for the common value of  $a^{\pm}(\pi^{\natural})$ .

*Proof.* The  $y_{\infty}$ -components of  $K_{\infty}^{\pm}$  is  $K_{\pm}$ , which are conjugate to each other by Lemma 9.2. The components of  $K_{\infty}^{\pm}$  at the other real places  $y$  equal  $G(F_y) \simeq \text{GSO}_{2n}^{\text{cpt}}(\mathbb{R})$ , which is connected. Therefore  $K_{\infty}^+$  and  $K_{\infty}^-$  are connected and  $G(F_{\infty})$ -conjugate. It then follows that  $\text{ep}^+(\tau_{\infty} \otimes \xi) = \text{ep}^-(\tau_{\infty} \otimes \xi)$  for all  $\tau_{\infty}$ . Thus  $a^+(\pi^{\natural}) = a^-(\pi^{\natural})$ .  $\square$

Since condition (cent) holds, we can attach to  $\xi$  an  $\ell$ -adic sheaf  $\mathcal{L}_{\iota\xi}$  on  $\text{Sh}_K^{\varepsilon}$  as in [KS16, below Lem. 7.1] and [Car86, Sect. 2.1, 2.1.4]. We have a canonical model  $\text{Sh}_K^{\varepsilon}$  over  $E$  for each sufficiently open compact subgroup  $K \subset G(\mathbb{A}_F^{\infty})$  and a distinguished embedding  $E \subset \overline{F}$ . We take the limit over  $K$  of the étale cohomology of with compact support

$$H_c^i(\text{Sh}^{\varepsilon}, \mathcal{L}_{\iota\xi}) := \varinjlim_K H_c^i(\text{Sh}_K^{\varepsilon} \times_E \overline{F}, \mathcal{L}_{\iota\xi}),$$

equipped with commuting actions of  $\Gamma_E = \text{Gal}(\overline{F}/E)$  and  $G(\mathbb{A}_F^{\infty})$ . The two groups act continuously and admissibly, respectively. Write  $H_c^i(\text{Sh}_K^{\varepsilon}, \mathcal{L}_{\iota\xi})^{\text{ss}}$  for the semisimplification as a  $\Gamma_E \times G(\mathbb{A}_F^{\infty})$ -module. (No semisimplification is necessary for the  $G(\mathbb{A}_F^{\infty})$ -action if  $F \neq \mathbb{Q}$ , in which case  $\text{Sh}_K^{\varepsilon}$  is projective. This can be seen from the semisimplicity of the discrete  $L^2$ -automorphic spectrum via Matsushima's formula.)

We construct Galois representations of  $\Gamma_E$  by taking the  $\iota\tau^{\infty}$ -isotypic part in the cohomology as follows. We consider  $\tau_1, \tau_2 \in A(\pi^{\natural})$  are equivalent and write  $\tau_1 \sim \tau_2$  if  $\tau_1^{\infty} \simeq \tau_2^{\infty}$ . Let  $A(\pi^{\natural})/\sim$  denote the set of (representatives for) equivalence classes. Let  $\tau \in A(\pi^{\natural})$ . Define

$$H_c^i(\text{Sh}^{\varepsilon}, \mathcal{L}_{\iota\xi})[\iota\tau^{\infty}] := \text{Hom}_{G(\mathbb{A}_F^{\infty})}(\iota\tau^{\infty}, H_c^i(\text{Sh}^{\varepsilon}, \mathcal{L}_{\iota\xi})^{\text{ss}}),$$

$$(9.5) \quad \rho_{\pi^{\natural}}^{\text{Sh}, \varepsilon} := (-1)^{n(n-1)/2} \sum_{\tau \in A(\pi^{\natural})/\sim} \sum_{i=0}^{n(n-1)} (-1)^i H_c^i(\text{Sh}^{\varepsilon}, \mathcal{L}_{\iota\xi})[\iota\tau^{\infty}].$$

A priori  $\rho_{\pi^{\natural}}^{\text{Sh}, \varepsilon}$  is an alternating sum of semisimple representations of  $\Gamma_E$ , thus a virtual representation (but see Theorem 9.7 below). Fix a sufficiently small compact open subgroup

$$K = \prod_{\mathfrak{q}|\infty} K_{\mathfrak{q}} \subset G(\mathbb{A}_F^{\infty}) \quad \text{such that} \quad (\pi^{\natural, \infty})^K \neq 0,$$

and also such that  $K_{\mathfrak{q}}$  is hyperspecial whenever  $\pi_{\mathfrak{q}}^{\natural}$  (or equivalently  $\pi_{\mathfrak{q}}$ ) is unramified. Let  $S_{\text{bad}}$  be the set of rational primes  $p$  for which either

- $p = 2$ ,
- $\text{Res}_{F/\mathbb{Q}}G$  is ramified, or
- $K_p = \prod_{\mathfrak{q}|p} K_{\mathfrak{q}}$  is not hyperspecial.

We write  $S_{\text{bad}}^F$  (resp.  $S_{\text{bad}}^E$ ) for the  $F$ -places (resp.  $E$ -places) above  $S_{\text{bad}}$ . We apply the Langlands–Kottwitz method at level  $K$  to compute the image of Frobenius elements under  $\rho_{\pi_{\mathfrak{q}}^{\natural}}^{\text{Sh},\varepsilon}$  at almost all primes.

**Theorem 9.7.** *There exists a finite set of rational primes  $\Sigma$  containing  $S_{\text{bad}}$ , such that for all  $\mathfrak{p}$  not above  $\Sigma$  and all sufficiently large integers  $j$ , we have*

$$(9.6) \quad \text{Tr} \rho_{\pi_{\mathfrak{q}}^{\natural}}^{\text{Sh},\varepsilon}(\text{Frob}_{\mathfrak{p}}^j) = \iota a(\pi_{\mathfrak{q}}^{\natural}) q_{\mathfrak{p}}^{jn(n-1)/4} \cdot \text{Tr}(\text{spin}^{\varepsilon}(\phi_{\pi_{\mathfrak{q}}^{\natural}}))(\text{Frob}_{\mathfrak{p}}^j).$$

Moreover the summand of (9.5) is nonzero only if  $i = n(n-1)/2$ . In particular the virtual representation  $\rho_{\pi_{\mathfrak{q}}^{\natural}}^{\text{Sh},\varepsilon}$  is a true representation,

*Proof.* We mimic the proof of [KS16, Prop. 8.2] closely. Note that our  $\rho_{\pi_{\mathfrak{q}}^{\natural}}^{\text{Sh},\varepsilon}$  corresponds to  $\rho_2^{\text{shim}}$  there. Another difference is that we use  $\Sigma$  to denote a set of primes of  $\mathbb{Q}$  (not  $F$  or  $E$ ). We suppose that  $F \neq \mathbb{Q}$  so that our Shimura varieties are proper. The case  $F = \mathbb{Q}$  will be addressed at the end of proof.

Let  $f_{\infty} = N_{\infty}^{-1} f_{\xi}$ , where  $f_{\xi}$  is the Euler–Poincaré (a.k.a. Lefschetz) function for  $\xi$  on  $G(F_{\infty})$  as recalled in [KS16, Appendix A], so that

$$\text{Tr} \tau_{\infty}(f_{\infty}) = N_{\infty}^{-1} \text{ep}(\tau_{\infty} \otimes \xi) = N_{\infty}^{-1} \sum_{i=0}^{\infty} (-1)^i \dim H^i(\mathfrak{g}, K_{\infty}; \tau_{\infty} \otimes \xi).$$

Choose a decomposable Hecke operator  $f^{\infty, \mathfrak{qSt}} = \prod_{\mathfrak{q} \neq \mathfrak{qSt}} f_{\mathfrak{q}} \in \mathcal{H}(G(\mathbb{A}_F^{\infty, \mathfrak{qSt}}) // K^{\mathfrak{qSt}})$  such that for all automorphic representations  $\tau$  of  $G(\mathbb{A}_F)$  with  $\tau_{\infty, K} \neq 0$  and  $\text{Tr} \tau_{\infty}(f_{\infty}) \neq 0$  we have

$$\text{Tr} \tau^{\infty, \mathfrak{qSt}}(f^{\infty, \mathfrak{qSt}}) = \begin{cases} 1 & \text{if } \tau^{\infty, \mathfrak{qSt}} \simeq \pi_{\mathfrak{q}}^{\natural, \infty, \mathfrak{qSt}} \\ 0 & \text{otherwise.} \end{cases}$$

This is possible since there are only finitely many such  $\tau$  (one of which is  $\pi_{\mathfrak{q}}^{\natural}$ ). Let  $f_{\mathfrak{qSt}}$  be a Lefschetz function from [KS16, Eq. (A.4)]. There exists a finite set of primes  $\Sigma \supset S_{\text{bad}} \cup \{\ell\}$  such that  $f_{\mathfrak{p}'}$  is the characteristic function of  $K_{\mathfrak{p}'}$  (which is hyperspecial) for every  $\mathfrak{p}'$  not above  $\Sigma$ . We fix  $\Sigma$  and  $f^{\infty, \mathfrak{qSt}}$  as above.

In the rest of the proof we fix a prime  $\mathfrak{p}$  not above  $\Sigma \cup \{\ell\}$ . Write  $\mathfrak{q} := \mathfrak{p} \cap F$ , and  $p$  for the rational prime below  $\mathfrak{p}$ . To apply the Langlands–Kottwitz method, we need an integral model for  $\text{Sh}^{\varepsilon}$  over  $\mathcal{O}_{E_{\mathfrak{p}}}$ . Thus we choose an isomorphism  $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$  such that the valuation on  $\overline{\mathbb{Q}}_p$  restricts to the  $\mathfrak{p}$ -adic valuation via  $\iota_p x_{\infty} : E \hookrightarrow \overline{\mathbb{Q}}_p$ . (Recall  $x_{\infty}$  from Lemma 9.3 (iv).) The  $(\text{Res}_{F/\mathbb{Q}}G)(\overline{\mathbb{Q}}_p)$ -conjugacy class of  $\iota_p \mu : \mathbb{G}_m \rightarrow (\text{Res}_{F/\mathbb{Q}}G)_{\overline{\mathbb{Q}}_p}$  is defined over  $E_{\mathfrak{p}}$ .

For  $j \in \mathbb{Z}_{\geq 1}$ , let  $f_p^{(j)}$  denote the function in the unramified Hecke algebra of  $G(F_p)$  constructed in [Kot90, §7] for the endoscopic group  $H = G^*$ , which is isomorphic to  $G$  over  $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ . (This is the function  $h_p$  in *loc. cit.* We take  $s$  and  $t_i$ 's on p.179 there to be trivial, so that  $h_p$  is the image of  $\phi_j$  under the standard base change map on p.180.) The  $L$ -group for  $(\text{Res}_{F/\mathbb{Q}}G)_{E_{\mathfrak{p}}}$  (with coefficients in  $\mathbb{C}$ ) can be identified as

$${}^L(\text{Res}_{F/\mathbb{Q}}G)_{E_{\mathfrak{p}}} = \left( \prod_{\sigma \in \text{Hom}(F, \overline{\mathbb{Q}}_p)} \widehat{G} \right) \rtimes \Gamma_{E_{\mathfrak{p}}},$$

where  $\Gamma_{E_{\mathfrak{p}}}$  acts trivially on the factor for  $\sigma = \iota_p y_{\infty}$ . (The Galois action may permute the other factors via its natural action on  $\text{Hom}(F, \overline{\mathbb{Q}}_p)$  but this does not matter to us.) The representation

of  ${}^L(\mathrm{Res}_{F/\mathbb{Q}}G)_{E_p}$  of highest weight  $\iota_p\mu$  is the representation  $(\mathrm{spin}^\varepsilon, 1, \dots, 1)$ . Here  $\mathrm{spin}^\varepsilon$  is on the factor for  $\sigma = \iota_p y_\infty$ , where we identify

$$G \times_{F,\sigma} \overline{\mathbb{Q}}_p = \mathrm{GSO}_{2n}^{E/F} \times_{F,\sigma} \overline{\mathbb{Q}}_p \xrightarrow{\text{via } \iota_p x_\infty} \mathrm{GSO}_{2n, \overline{\mathbb{Q}}_p},$$

(in the ambient group  $\mathrm{GL}_{2n}(E \otimes_F \overline{\mathbb{Q}}_p) \simeq \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p) \times \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$  of the left hand side, we project onto the  $\iota_p x_\infty$ -component) thus identify  $\widehat{G} = \mathrm{GSpin}_{2n}$  on the  $\iota_p y_\infty$ -component. Now let  $\tau_p = \prod_{q'|p} \tau_{q'}$  be an unramified representation of  $G(F_p) = (\mathrm{Res}_{F/\mathbb{Q}}G)(\mathbb{Q}_p) = \prod_{q'|p} G(F_{q'})$ , and denote by  $\phi_{\tau_p} : W_{\mathbb{Q}_p} \rightarrow {}^L(\mathrm{Res}_{F/\mathbb{Q}}G)_{\mathbb{Q}_p}$  its  $L$ -parameter. Then the  $\iota_p y_\infty$ -component of  $\phi_{\tau_p}|_{W_{E_p}}$  is given by  $\phi_{\tau_q}|_{W_{E_p}}$ . All in all, we can explicate [Kot84, (2.2.1)] in our setup as<sup>14</sup>

$$(9.7) \quad \mathrm{Tr} \tau_p(f_p^{(j)}) = q_p^{jn(n-1)/4} \mathrm{Tr}(\mathrm{spin}^\varepsilon(\phi_{\tau_q})(\mathrm{Frob}_p^j)).$$

As in the proof of [KS16, Prop. 8.2] (where our  $f_p^{(j)}$  is denoted by  $h_p^{G^*}$ ), the Lefschetz functions  $f_\infty$  and  $f_{\mathrm{qst}}$  allow us to simplify the stabilized Langlands-Kottwitz formula [KSZ, Thm. 11.3.9] (recalled in [KS16, Thm. 7.3]) and obtain a simple stabilization of the trace formula for  $G$ ; the outcomes are formulas (8.5) and (8.6) of [KS16]. Combining them, we obtain

$$(9.8) \quad \iota^{-1} \mathrm{Tr}(\iota f^{\infty,p} f_p \times \mathrm{Frob}_p^j, \mathrm{H}_c(\mathrm{Sh}^\varepsilon, \mathcal{L}_\xi)) = T_{\mathrm{cusps}, \chi}^G(f^{\infty,p} f_p^{(j)} f_\infty), \quad j \gg 1.$$

Note that  $f_p$  is the characteristic function of the hyperspecial subgroup  $K_p = \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}$ . Following the argument from Equation (8.7) to (8.10) in [KS16], we compute

$$(9.9) \quad \iota^{-1} \mathrm{Tr}(\mathrm{Frob}_p^j, \rho_{\pi_{\mathfrak{h}}}^{\mathrm{Sh}, \varepsilon}) = a(\pi_{\mathfrak{h}}) \mathrm{Tr} \pi_{\mathfrak{h}}^{\mathfrak{h}}(f_p^{(j)}) \stackrel{(9.7)}{=} a(\pi_{\mathfrak{h}}) q_p^{jn(n-1)/4} \cdot \mathrm{Tr}(\mathrm{spin}^\varepsilon(\phi_{\pi_{\mathfrak{h}}}))(\mathrm{Frob}_p^j).$$

Let us show that  $\rho_{\pi_{\mathfrak{h}}}^{\mathrm{Sh}, \varepsilon}$  is a true representation by showing we have only contribution to  $\rho_{\pi_{\mathfrak{h}}}^{\mathrm{Sh}, \varepsilon}$  from the middle cohomology. Since  $\mathrm{Sh}_K^\varepsilon$  is projective for each  $K$ , the action of  $\mathrm{Frob}_p$  on  $\mathrm{H}_c^i(\mathrm{Sh}_K, \mathcal{L}_\xi)$  is pure of weight  $-w + i$ , see Deligne [Del80, Cor. 3.3.6]. The argument of Part (2) of [KS16, Lem. 8.1] (replacing Lemma 2.7 in the proof there with Proposition 6.1 for  $\mathrm{SO}_{2n}$ ) implies that  $\tau_q|\mathrm{sim}|^{w/2} = \pi_{\mathfrak{h}}^{\mathfrak{h}}|\mathrm{sim}|^{w/2}$  is tempered and unitary. Combining with (9.9) we conclude that  $\mathrm{H}_c^i(\mathrm{Sh}^\varepsilon, \mathcal{L}_\xi)[\iota\tau^\infty] = 0$  unless  $i = n(n-1)/2$ .

Finally, the case  $F = \mathbb{Q}$  is handled via intersection cohomology as in the proof of [KS16, Prop. 8.2]. Thus we content ourselves with giving a sketch. For each  $\tau \in A(\pi_{\mathfrak{h}})$ , one observes as in [KS16, Lem. 8.1] that  $\mathrm{H}_c^i(\mathrm{Sh}^\varepsilon, \mathcal{L}_\xi)[\iota\tau^\infty]$  is isomorphic to the  $\iota\tau^\infty$ -isotypic part of the intersection cohomology as  $\Gamma_E$ -representations. The point is that  $\tau^\infty$  does not appear in any parabolic induction of an automorphic representation on a proper Levi subgroup of  $G(\mathbb{A}_F)$ . (If it does appear, then restricting  $\tau$  from  $G$  to isometry group  $G^b(\mathbb{A})$  and transferring to a quasi-split inner form  $G^{*b} = \mathrm{SO}_{2n}^{E/F}$  via [KS16, Prop. 6.3], we would have a cohomological automorphic representation  $\tau^{*b}$  of  $G^{*b}(\mathbb{A})$  with a Steinberg component that appears as a constituent in a parabolically induced representation. Then the Arthur parameter for  $\tau^{*b}$  cannot have the shape described in Proposition 6.1, leading to a contradiction.) The rest of the proof of [KS16, Prop. 8.2] carries over, via the analogue of part 2 of [KS16, Lem. 8.1] (the latter is proved using temperedness (Ar4)+ of Proposition B.1 in place of [KS16, Lem. 2.7] if (std-reg) is assumed; otherwise the temperedness is built into Hypothesis 6.10), bearing in mind that the middle degree is  $n(n-1)/2$  for us (which was  $n(n+1)/2$  for the group  $\mathrm{GSp}_{2n}$ ).  $\square$

**Corollary 9.8.** *Let  $\pi_{\mathfrak{h}}$  be as above. If  $\tau \in A(\pi_{\mathfrak{h}})$  then*

- (1)  $\tau_\infty$  belongs to the discrete series  $L$ -packet  $\Pi_\xi^{G(F_\infty)}$ ,
- (2)  $\tau^\infty \tau'_\infty \in A(\pi_{\mathfrak{h}})$  and  $m(\tau) = m(\tau^\infty \tau'_\infty)$  for all  $\tau'_\infty \in \Pi_\xi^{G(F_\infty)}$ .

Moreover  $a(\pi_{\mathfrak{h}}) = \sum_{\tau \in A(\pi_{\mathfrak{h}})/\sim} m(\tau) \in \mathbb{Z}_{>0}$ .

<sup>14</sup>A word on the sign convention is appropriate here. The sign of [Kot84, (2.2.1)] was flipped on [Kot90, p.193], meaning that the highest weight  $-\iota_p\mu$  should be used in (9.7). This was caused by the arithmetic vs geometric convention for Frobenius. However the sign of  $\mu$  was changed when going from [Kot90] to [Kot92], to be consistent with [Del79]. Thus it is correct to use a representation of highest weight  $\iota_p\mu$  after all.

*Proof.* This is the exact analogue of [KS16, Cor. 8.4, Cor. 8.5] and the same proof applies. The last equality is not explicitly stated there, but follows easily from the other parts of the corollary and Theorem 9.7. Indeed, we rewrite (9.3) as

$$a^\pm(\pi^\natural) = (-1)^{n(n-1)/2} N_\infty^{-1} \sum_{\tau \in A(\pi^\natural)/\sim} \sum_{\tau'_\infty \in \Pi_\xi^G} m(\tau^\infty \otimes \tau'_\infty) \text{ep}^\pm(\tau'_\infty \otimes \xi)$$

and observe that  $m(\tau^\infty \otimes \tau'_\infty) = m(\tau)$  and  $\text{ep}^\pm(\tau'_\infty \otimes \xi) = (-1)^{n(n-1)/2} |\pi_0(G(F_\infty)/Z(F_\infty))|$  for every  $\tau'_\infty$ .  $\square$

**Proposition 9.9.** *Assume that  $F \neq \mathbb{Q}$ . For each embedding  $x : E \hookrightarrow \mathbb{C}$  and  $\epsilon \in \{\pm 1\}$ , writing  $y_0$  for the  $F$ -place below  $x$ , we have*

$$\mu_{\text{HT}}(\rho_{\pi^\natural}^{\text{Sh}, \epsilon}, \iota x) \sim i_{a(\pi^\natural)} \circ \text{spin}^\epsilon \circ \left( \mu_{\text{Hodge}}(\xi_{y_0}) + \frac{n(n-1)}{4} \text{sim} \right).$$

*Proof.* We have a fixed embedding  $\bar{F} \hookrightarrow \mathbb{C}$  restricting to  $x_\infty : E \subset \mathbb{C}$ . We can reduce to the case  $x = x_\infty$  by Milne–Shih’s proof of Langlands’s conjecture on conjugation of Shimura varieties [MS82, Thm. 0.9]. To see this, choose an automorphism  $\tau \in \text{Aut}(\mathbb{C})$  such that  $x = \tau x_\infty$  and  $\tau$  fixes a special  $h$  and  $\mu = \mu_h$  as in *loc. cit.* Consider the conjugate Shimura datum  $\mathcal{D} = (\tau, \mu(\text{Res}_{F/\mathbb{Q}} G), \tau, \mu(X^\epsilon))$ . The point is that  $\mathcal{D}$  is isomorphic to  $(\text{Res}_{F/\mathbb{Q}} G', X'^{\epsilon})$ , where  $G'$  is given as in Lemma 8.3 except that  $y_\infty$  is replaced with the real place of  $F$  induced by  $x$ , and  $X'^{\epsilon}$  is constructed from  $G'$  by the recipe earlier in this section. The reflex field of  $\mathcal{D}$  is  $\tau(x_\infty(E)) = x(E) \subset \mathbb{C}$ , that is,  $x$  plays the role of  $x_\infty$  when working with  $\mathcal{D}$ . Thus we are indeed reduced to the setup of  $x = x_\infty$  by replacing  $(\text{Res}_{F/\mathbb{Q}} G, X^\epsilon)$  with  $\mathcal{D}$ .

Henceforth we suppose  $x = x_\infty$ . We introduce some notation. Let  $\mathfrak{p}$  be a prime of  $E$  above  $\ell$ , and  $\sigma : E \hookrightarrow \overline{\mathbb{Q}}_\ell$  an embedding inducing the  $\mathfrak{p}$ -adic valuation. Let  $r$  be a Galois representation of  $\Gamma_E$  on a  $\overline{\mathbb{Q}}_\ell$ -vector space. Write  $D_{\text{dR}, \sigma}(r)$  for the filtered  $\overline{\mathbb{Q}}_\ell$ -vector space associated with  $r|_{\Gamma_{E_{\mathfrak{p}}}}$  with respect to  $\sigma$  (as on [HT01, p.99]). Define  $\text{HT}_\sigma(r)$  to be the multi-set containing each  $j \in \mathbb{Z}$  with multiplicity  $\dim \text{gr}^j(D_{\text{dR}, \sigma}(r))$ . (So the cardinality of  $\text{HT}_\sigma(r)$  equals  $\dim \rho$ .) When  $a \in \mathbb{Z}_{>0}$  and  $A$  is a multi-set, we write  $A^{\oplus a}$  to denote the multi-set such that the multiplicity of each element in  $A^{\oplus a}$  is  $a$  times that in  $A$ .

Write  $\lambda(\xi) = \{\lambda(\xi_y)\}_{y|\infty}$  for the highest weight of  $\xi = \otimes_{y|\infty} \xi_y$ . In the basis of §2 for  $X^*(T_{\text{GSO}}) = X_*(T_{\text{GSpin}}) = \mathbb{Z}^{n+1}$ , we write  $\xi_x$  and the half sum of positive roots  $\rho$  for  $\text{GSO}_{2n}$  as

$$\begin{aligned} \lambda(\xi_{y_0}) &= (a_0, a_1, \dots, a_n), & a_1 \geq a_2 \geq \dots \geq |a_n| > 0, \\ \rho &= (-n(n-1)/4, n-1, n-2, \dots, 1, 0). \end{aligned}$$

Here  $|a_n| > 0$  comes from (std-reg). Let  $\mathcal{P}^\epsilon(n)$  denote the collection of subsets of  $\{1, 2, \dots, n\}$  whose cardinality is even if  $\epsilon = (-1)^\epsilon$  and odd if  $\epsilon = (-1)^{\epsilon+1}$ . Put

$$\begin{aligned} (b_0, b_1, \dots, b_n) &:= (a_0, a_1 + n - 1, a_2 + n - 2, \dots, a_{n-1} + 1, a_n) \\ &= \lambda(\xi_{y_0}) + \rho + (n(n-1)/4, 0, 0, \dots, 0), \end{aligned}$$

which equals  $\mu_{\text{Hodge}}(\xi_x) + \frac{n(n-1)}{4} \text{sim}$ . Via the description of weights in the representation  $\text{spin}^\epsilon$  in (2.8), the lemma amounts to the assertion that

$$(9.10) \quad \text{HT}_{\iota x}(\rho_{\pi^\natural}^{\text{Sh}, \epsilon}) = \left\{ b_0 + \sum_{i \in I} b_i \mid I \in \mathcal{P}^\epsilon(n) \right\}^{\oplus a(\pi^\natural)}.$$

We prove this assertion following the argument in [HT01, pp.99–104] partly based on [Fal83]. Let us set up some more notation. Write  $\text{Sh}_K^\epsilon(\mathbb{C})$  for the complex manifold obtained from  $\text{Sh}_K^\epsilon$  by base change via  $x_0 : E \hookrightarrow \mathbb{C}$ , and  $\mathcal{L}_\xi^{\text{top}}$  for the topological local system on  $\text{Sh}_K^\epsilon(\mathbb{C})$  coming from  $\xi$ . Writing  $K^\epsilon$  (Lemma 9.2) as  $K^\epsilon = \prod_y K_y^\epsilon$ , we have  $K_{y_0}^\epsilon = K_\epsilon$  and  $K_y^\epsilon = G_{F_y} \simeq \text{GSO}_{2n}^{\text{cpt}}$  for  $y \neq y_0$ . Restricting  $h_\mathbb{C}^\epsilon$  to  $\mathbb{S}_\mathbb{C} = \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$ , we obtain a cocharacter  $\mu^\epsilon : \mathbb{G}_{m, \mathbb{C}} \rightarrow K_\mathbb{C}^\epsilon$ . We also have a parabolic subgroup  $Q \subset G_\mathbb{C}$  with Levi component  $K_\mathbb{C}^\epsilon$  as [Fal83, p.57] (such that the Borel embedding goes into  $G(\mathbb{C})/Q$ ). Fix an elliptic maximal torus  $T_\infty \subset K^\epsilon$  and a Borel subgroup  $B \subset G_\mathbb{C}$  contained in  $Q$ . Let  $R^+$  (resp.  $R_{\text{nc}}^+$ ) denote the set of positive roots of  $T_\infty$



in  $B$  (resp.  $Q$ ). By  $R^-$  we denote the set of roots of  $T_\infty$  in the opposite Borel subgroup. Note that  $|R_{\text{nc}}^+| = \dim \text{Sh}_K^\varepsilon = n(n-1)/2$ . Write  $\Omega$  for the Weyl group of  $T_{\infty, \mathbb{C}}$  in  $G_{\mathbb{C}}$ , and  $\Omega_{\text{nc}}$  for the subset of  $\omega \in \Omega$  such that  $\omega\lambda$  is  $B \cap K_{\mathbb{C}}^\varepsilon$ -dominant whenever  $\lambda \in X^*(T_\infty)$  is  $B$ -dominant. Write  $\rho_G \in X^*(T)$  for the half sum of all  $B$ -positive roots, and define  $\omega \star \lambda := \omega(\lambda + \rho_G) - \rho_G$  for  $\lambda \in X^*(T_\infty)$ . We parametrize members of the discrete series  $L$ -packet  $\Pi_\xi^{G(F_\infty)}$  as  $\{\pi(\omega) | \omega \in \Omega_{\text{nc}}\}$  following [Har90, 3.3]. (Our  $\pi(\omega)$  is  $\pi(\omega\lambda, \omega R^+)$  in their notation.)

An irreducible representation  $V_\lambda$  of  $K_{\mathbb{C}}^\varepsilon$  of highest weight  $\lambda \in X^*(T_\infty)$  gives rise to an automorphic vector bundle, to be denoted  $\mathcal{E}_\lambda$ . Define a complex

$$\mathcal{K}_\lambda^\bullet = \bigoplus_{\omega \in \Omega_{\text{nc}}} \mathcal{E}_{\omega \star \lambda}[-l(\omega)],$$

where the summand means that  $\mathcal{E}_{\omega \star \lambda}$  is placed in degree  $l(\omega)$ . For  $j \in \mathbb{Z}$ , let  $\Omega_{\text{nc}}(j)$  denote the subset of  $\omega \in \Omega_{\text{nc}}$  such that the composition  $\mathbb{G}_m \xrightarrow{\mu^\varepsilon} T_\infty \xrightarrow{\omega \star \lambda} \mathbb{G}_m$  equals  $z \mapsto z^j$ . This defines a grading on  $\mathcal{K}_\lambda^\bullet$  with

$$\text{gr}_F^j(\mathcal{K}_\lambda^\bullet) = \bigoplus_{\omega \in \Omega_{\text{nc}}(j)} \mathcal{E}_{\omega \star \lambda}[-l(\omega)].$$

We apply the comparison theorem of [DLLZ, Thm. 1.1, Thm. 5.3.1] to our compact Shimura varieties,

$$\text{gr}^j D_{dR, \iota x_0}(H^i(\text{Sh}_{K, \overline{F}}, \mathcal{L}_{\iota\xi})) \simeq \text{gr}_F^j H^i(\text{Sh}_K(\mathbb{C}), \mathcal{L}_\xi^{\text{top}}).$$

(The two theorems here tell us that  $D_{dR, \iota x_0}(H^i(\text{Sh}_{K, \overline{F}}, \text{ét} \underline{V}_{\overline{\mathbb{Q}}_p})) \simeq H^i(\text{Sh}_{K, \mathbb{C}}, \text{dR} \underline{V}_{\mathbb{C}})$  compatible with filtrations, in their notation for coefficient sheaves. The latter space is isomorphic to  $H^i(\text{Sh}_K(\mathbb{C}), \text{B} \underline{V}_{\mathbb{C}})$  by the classical Riemann–Hilbert. Their  $\text{ét} \underline{V}_{\overline{\mathbb{Q}}_p}$  and  $\text{B} \underline{V}_{\mathbb{C}}$  are our  $\mathcal{L}_{\iota\xi}$  and  $\mathcal{L}_\xi^{\text{top}}$ ; their  $p$  is our  $\ell$ .) Using this isomorphism in place of the isomorphisms in the first display of [HT01, p.102], we obtain the following isomorphisms via Faltings’s dual BGG construction (as in the third display from the bottom on p.102 and the third and fourth displays from the bottom on p.103 of *loc. cit.*):

$$(9.11) \quad \text{gr}^j D_{dR, \iota x_0}(H^i(\text{Sh}_{K, \overline{F}}, \mathcal{L}_{\iota\xi})) \simeq H^i(\text{Sh}_K(\mathbb{C}), \text{gr}_F^j \mathcal{K}_\lambda^\bullet) \simeq \bigoplus_{\omega \in \Omega_{\text{nc}}(j)} H^{i-l(\omega)}(\text{Sh}_K(\mathbb{C}), \mathcal{E}_{\omega \star \lambda}),$$

equivariant with respect to the Hecke correspondences. For  $k \in \mathbb{Z}_{\geq 0}$ , write  $H^k(\text{Sh}(\mathbb{C}), \mathcal{E}_{\omega \star \lambda})$  for the direct limit of  $H^k(\text{Sh}_K(\mathbb{C}), \mathcal{E}_{\omega \star \lambda})$  over all sufficiently small open compact subgroups  $K$ . This is an admissible  $G(\mathbb{A}_F^\infty)$ -module. Since  $F \neq \mathbb{Q}$ , our Shimura varieties are compact. From [Har90, §3] we have,

$$H^k(\text{Sh}(\mathbb{C}), \mathcal{E}_{\omega \star \lambda}) \simeq \bigoplus_{\tau} m(\tau) \tau^\infty \otimes H^k(\text{Lie } Q, K^\varepsilon, \tau_\infty \otimes V_{\omega \star \lambda}).$$

We pass to the  $\tau^\infty$ -isotypic parts to obtain

$$H^k(\text{Sh}(\mathbb{C}), \mathcal{E}_{\omega \star \lambda})[\tau^\infty] \simeq \bigoplus_{\tau'_\infty} m(\tau^\infty \otimes \tau'_\infty) H^k(\text{Lie } Q, K^\varepsilon, \tau'_\infty \otimes V_{\omega \star \lambda}),$$

where the sum runs over irreducible unitary representations of  $G(F_\infty)$ . By [Har90, Thm. 3.4], the cohomology on the right hand side is nonvanishing exactly when  $\tau'_\infty = \pi(\omega)$  and  $k = |\omega R^+ \cap R_{\text{nc}}^+|$ , in which case it is one-dimensional. In that case  $\tau'_\infty \in \Pi_\xi^{G(F_\infty)}$  in particular, so  $m(\tau^\infty \otimes \tau'_\infty) = m(\tau)$  by Corollary 9.8.

Our case of interest is when  $k = i - l(\omega)$ . Since  $l(\omega) = |R^+ \cap \omega R^-|$ , we see that  $i - l(\omega) = |\omega R^+ \cap R_{\text{nc}}^+|$  if and only if  $i = |R_{\text{nc}}^+|$ . Therefore taking the direct limit of (9.11) over  $K$  and restricting to  $i = |R_{\text{nc}}^+| = n(n-1)/2$ , we obtain

$$\dim \text{gr}^j D_{dR, \iota x_0}(H^{\frac{n(n-1)}{2}}(\text{Sh}_{\overline{F}}, \mathcal{L}_\xi))[\tau^\infty] = \sum_{\omega \in \Omega_{\text{nc}}(j)} m(\tau).$$

Summing over  $\tau \in A(\pi^{\natural})/\sim$ , we see that the multiplicity of  $j$  in  $\mathrm{HT}_{\iota x_0}(\rho_{\pi^{\natural}}^{\mathrm{Sh}, \epsilon})$  equals

$$a(\pi^{\natural}) \cdot |\Omega_{\mathrm{nc}}(j)|.$$

To conclude (9.10), it remains to prove the following claim: that  $|\Omega_{\mathrm{nc}}(j)|$  is precisely the number of ways  $j$  can be written as  $b_0 + \sum_{i \in I} b_i$  with  $I \in \mathcal{P}^{\epsilon}(n)$ .

Let us prove the claim. For  $I \in \mathcal{P}^{\epsilon}(n)$ , let  $\omega_I$  denote the action on  $(t_0, t_1, \dots, t_n) \in X^*(\mathrm{T}_{\mathrm{GSO}})$  by  $t_i \mapsto -t_i$  for every  $i \in I$  and  $t_0 \mapsto t_0 + \sum_{i \in I} t_i$ . The association  $I \mapsto \omega_I$  allows us to identify  $\mathcal{P}^{\epsilon}(n) = \Omega_{\mathrm{nc}}$ . The cocharacter  $\mu^{\epsilon}$  is described (up to the action of the Weyl group of  $K_{\mathbb{C}}$ ) as  $z \mapsto (z, z, \dots, z, z) \in X^*(\mathrm{T}_{\mathrm{GSO}})$  if  $\epsilon = (-1)^n$  and  $z \mapsto (z, z, \dots, z, 1)$  if  $\epsilon = (-1)^{n+1}$ . From this, we can compute that if  $\lambda = (a_0, a_1, \dots, a_n)$  then  $(\omega_I \star \lambda) \circ \mu^{\epsilon}(z) = z^{a_0 + \sum_{i \in I} (a_i + (n-i))}$ . The claim follows.  $\square$

## 10. CONSTRUCTION OF THE $\mathrm{GSpin}_{2n}$ -VALUED GALOIS REPRESENTATIONS

We continue in the setting of §8 and §9. The goal of this section is to attach  $\mathrm{GSpin}_{2n}$ -valued Galois representations of  $\Gamma_E$  to automorphic representations of  $G^* = \mathrm{GSO}_{2n}^{E/F}$ . The main input is the cohomology of Shimura varieties studied in the last section. Throughout, we assume  $F \neq \mathbb{Q}$  as we rely on Theorem 9.7. Write  $\mathrm{std}: \mathrm{GSpin}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$  for the composite of  $\mathrm{pr}: \mathrm{GSpin}_{2n} \rightarrow \mathrm{GSO}_{2n}$  and the inclusion  $\mathrm{GSO}_{2n} \subset \mathrm{GL}_{2n}$ .

Let  $\pi$  be a cuspidal automorphic representation of  $G^*(\mathbb{A}_F)$ . Let  $\phi_{\pi_y}$  denote the  $L$ -parameter of  $\pi_y$  for  $y \in \mathcal{V}_{\infty}$ . Throughout this section, we assume that

- (St) for some finite  $F$ -place  $\mathfrak{q}_{\mathrm{St}}$  the local representation  $\pi_{\mathfrak{q}_{\mathrm{St}}}$  is isomorphic to the Steinberg representation up to character twist,
- (coh) the representation  $\pi_{\infty}$  is cohomological for some representation  $\xi$  of  $(\mathrm{Res}_{F/\mathbb{Q}} G^*) \otimes_{\mathbb{Q}} \mathbb{C}$  (then  $\xi$  satisfies condition (cent) by [KS16, Lem 7.1] as before).

Write  $G^{*\flat} := \mathrm{SO}_{2n}^{E/F} \subset G^*$  (see (8.5)) and choose  $\pi^{\flat}$  a cuspidal automorphic representation of  $G^{*\flat}(\mathbb{A}_F)$  in  $\pi|_{G^{*\flat}(\mathbb{A}_F)}$  (see [LS19]). We observe that  $\pi^{\flat}$  satisfies conditions (St $^{\circ}$ ) and (coh $^{\circ}$ ) of §6 thanks to Lemma 7.1 and 7.2. We assume either Hypothesis 6.10 for  $\pi^{\flat}$ , or the following analogue of (std-reg $^{\circ}$ ) for  $\pi$ :

**(std-reg)**  $\mathrm{std} \circ \phi_{\pi_y}|_{W_{\overline{F}_y}}$  is regular at every  $y \in \mathcal{V}_{\infty}$ .

If (std-reg) is imposed on  $\pi$ , then (std-reg $^{\circ}$ ) follows from (std-reg) and (coh $^{\circ}$ ). By [Lan89, §3, (iv)], we have that  $\phi_{\pi^{\flat}, y} = \mathrm{pr}^{\circ} \circ \phi_{\pi, y}$  at each  $y \in \mathcal{V}_{\infty}$ . We can also see (std-reg $^{\circ}$ ) from this and (std-reg).

Thanks to Theorem 6.3 if (std-reg) is assumed, or instead by Hypothesis 6.10, we have a Galois representation

$$\rho_{\pi^{\flat}}: \Gamma_{F, S_{\mathrm{bad}}} \rightarrow \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_{\ell}) \rtimes \mathrm{Gal}(E/F),$$

whose restriction to  $\Gamma_{E, S_{\mathrm{bad}}}$  satisfies

$$(10.1) \quad \rho_{\pi^{\flat}}(\mathrm{Frob}_{\mathfrak{p}})_{\mathrm{ss}} \overset{\mathrm{out}}{\sim} \iota \phi_{\pi^{\flat}}(\mathrm{Frob}_{\mathfrak{p}}) \in \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_{\ell}),$$

for all  $E$ -places  $\mathfrak{p} \notin S_{\mathrm{bad}}^E$ . Here  $\overset{\mathrm{out}}{\sim}$  indicates  $\mathrm{O}_{2n}$ -conjugacy (instead of  $\mathrm{SO}_{2n}$ -conjugacy).

Let  $H \subset \mathrm{SO}_{2n}$  denote the Zariski closure of the image of  $\rho_{\pi^{\flat}}: \Gamma_{E, S} \rightarrow \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_{\ell})$ . By Proposition 5.2, either  $H$  is connected or  $H = H^0 \times Z(\mathrm{SO}_{2n})$ . Therefore we can find a Galois character

$$\eta: \Gamma_{E, S} \rightarrow \{\pm 1\}$$

such that  $\eta \otimes \rho_{\pi^{\flat}}$  has Zariski dense image in  $H^0$ , where  $\otimes$  is taken via  $\{\pm 1\} = Z(\mathrm{SO}_{2n})$ . (Choose  $\eta = 1$  if  $H = H^0$ .) The element  $z_+ \in Z(\mathrm{GSpin}_{2n})$  of Lemma 2.7 is a lift of  $-1 \in Z(\mathrm{SO}_{2n})$ , satisfies  $z_+^2 = 1$ , and acts by the scalar  $\epsilon$  under  $\mathrm{spin}^{\epsilon}$  for both  $\epsilon \in \{\pm\}$ . Let

$$\tilde{\eta}: \Gamma_{E, S} \rightarrow \{1, z_+\}$$

denote the unique lift of  $\eta$ .

Recall that  $G$  is an inner form of  $G^* = \text{GSO}_{2n}^{E/F}$  giving rise to the Shimura data  $(\text{Res}_{F/\mathbb{Q}}, X^\pm)$  studied earlier. By [KS16, Prop. 6.3], there exists a cuspidal automorphic representation  $\pi^\natural$  of  $G(\mathbb{A}_F)$  such that

- $\pi^\natural$  is unramified at every place outside of  $S_{\text{bad}}$ ,
- $\pi_{\text{qst}}^\natural$  is an unramified character twist of the Steinberg representation,
- $\pi_\infty^\natural$  is  $\xi$ -cohomological.

Theorem 9.7 yields representations  $\rho_{\pi^\natural}^{\text{Sh}, \varepsilon}$  for  $\varepsilon \in \{\pm 1\}$  such that

$$(10.2) \quad \rho_{\pi^\natural}^{\text{Sh}, \varepsilon}(\text{Frob}_{\mathfrak{p}})_{\text{ss}} \sim \iota_{\mathfrak{p}}^{n(n-1)/4} (i_{a_\pi} \circ \text{spin}^\varepsilon(\phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}}))) \in \text{GL}_{a_\pi 2^{n-1}}(\overline{\mathbb{Q}}_\ell).$$

We define  $\rho_\pi^{\text{Sh}, \varepsilon} := \rho_{\pi^\natural}^{\text{Sh}, \varepsilon}$  for  $\varepsilon \in \{\pm\}$  and

$$\tilde{\rho}_\pi^{\text{Sh}} := \rho_\pi^{\text{Sh}, +} \oplus (\eta \otimes \rho_\pi^{\text{Sh}, -}).$$

Then  $\tilde{\rho}_\pi^{\text{Sh}}$  is a  $\Gamma_{E,S}$ -representation of dimension  $a_\pi 2^n$ , where  $a_\pi := a(\pi^\natural)$ . We set

$$\widetilde{\text{spin}}(\cdot) := \text{spin}^+(\cdot) \oplus (\eta \otimes \text{spin}^-(\cdot))$$

when the input is a  $\text{GSpin}_{2n}$ -valued Galois representation or a local  $L$ -parameter, and write  $\widetilde{\text{spin}}^a(\cdot)$  for the  $a$ -fold self-direct sum of  $\widetilde{\text{spin}}(\cdot)$ . (So  $\widetilde{\text{spin}} = \text{spin}$  if  $\eta = 1$ .) We have

$$(10.3) \quad \tilde{\rho}_\pi^{\text{Sh}}(\text{Frob}_{\mathfrak{p}})_{\text{ss}} \sim \iota_{\mathfrak{p}}^{n(n-1)/4} \widetilde{\text{spin}}^{a_\pi}(\phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}})) \in \text{GL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell)$$

for all finite  $E$ -places  $\mathfrak{p}$  not above some fixed finite set of places  $S \supset S_{\text{bad}}$ .

**Proposition 10.1.** *There exists a continuous semisimple representation*

$$\rho_\pi^C : \Gamma_{E,S} \rightarrow \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$$

such that we have

$$(10.4) \quad \forall \mathfrak{p} \notin S : \quad \widetilde{\text{spin}}(\rho_\pi^C(\text{Frob}_{\mathfrak{p}})_{\text{ss}}) \sim \iota_{\mathfrak{p}}^{n(n-1)/4} \widetilde{\text{spin}}(\phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}})) \in \text{GL}_{2^n}(\overline{\mathbb{Q}}_\ell),$$

$$(10.5) \quad \forall \mathfrak{p} \notin S_{\text{bad}} : \quad \text{pr}^\circ \rho_\pi^C(\text{Frob}_{\mathfrak{p}})_{\text{ss}} \overset{\text{out}}{\sim} \iota_{\mathfrak{p}} \text{pr}^\circ \phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}}) \in \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell).$$

*Proof.* When  $*$  is a map (resp. an element), we use  $\bar{*}$  to denote the composition with the adjoint map (resp. the image under the adjoint map) that is clear from the context.

The representation  $\text{spin}^{a_\pi} : \text{GSpin}_{2n} \rightarrow \text{GL}_{a_\pi 2^n}$  (resp.  $\overline{\text{spin}}^{a_\pi} : \text{SO}_{2n} \rightarrow \text{PGL}_{a_\pi 2^n}$ ) extends to the group  $\text{GPin}_{2n}$  (resp.  $\text{O}_{2n}$ ). Consider the diagram

$$(10.6) \quad \begin{array}{ccccc} & & \rho_\pi^{\text{Sh}} & & \\ & & \curvearrowright & & \\ & \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & & \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) & \xrightarrow{\text{spin}^{a_\pi}} & \text{GL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell) \\ & & & \downarrow \text{pr}^\circ & & \downarrow \\ & & & \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell) & \xrightarrow{\overline{\text{spin}}^{a_\pi}} & \text{PGL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell). \\ & & \rho_{\pi^b} & & & \end{array}$$

At each prime  $\mathfrak{p}$  of  $E$  not above  $S$ , we obtain from (10.1) that

$$(10.7) \quad \begin{aligned} & \overline{\text{spin}}^{a_\pi}((\eta \otimes \rho_{\pi^b})(\text{Frob}_{\mathfrak{p}})_{\text{ss}}) \sim \overline{\text{spin}}^{a_\pi}((\eta \otimes \phi_{\pi_{\mathfrak{p}}})(\text{Frob}_{\mathfrak{p}})) \\ & = \overline{\text{spin}}^{a_\pi}((\tilde{\eta} \otimes \phi_{\pi_{\pi_{\mathfrak{p}}}})(\text{Frob}_{\mathfrak{p}})) \sim \overline{\tilde{\rho}_\pi^{\text{Sh}}}(\text{Frob}_{\mathfrak{p}})_{\text{ss}} \in \text{PGL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell). \end{aligned}$$

Recall that  $\eta \otimes \rho_{\pi^b}$  has connected image. So (10.7) implies, via [KS16, Prop. 4.6, Ex. 4.7], the existence of  $g \in \text{GL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell)$  such that

$$\overline{\tilde{\rho}_\pi^{\text{Sh}}} = g(\overline{\text{spin}}^{a_\pi}(\eta \otimes \rho_{\pi^b}))g^{-1} : \Gamma_{E,S} \rightarrow \text{PGL}_{a_\pi 2^n}(\overline{\mathbb{Q}}_\ell).$$

Replace  $\tilde{\rho}_\pi^{\text{Sh}}$  by  $g^{-1}\tilde{\rho}_\pi^{\text{Sh}}g$  so that  $\overline{\tilde{\rho}_\pi^{\text{Sh}}} = \overline{\text{spin}}^{a_\pi}(\eta \otimes \rho_{\pi^b})$ . From Diagram (10.6) we deduce that

$$\tilde{\rho}_\pi^{\text{Sh}}(\Gamma_{E,S}) \subset \text{pr}^{\circ, -1}((\eta \otimes \rho_{\pi^b})(\Gamma_{E,S})) \subset \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell),$$

where  $\mathrm{GSpin}_{2n}$  is viewed as a subgroup of  $\mathrm{GL}_{a_\pi 2^n}$  via  $\mathrm{spin}^{a_\pi}$ . That is, there exists a representation  $\tilde{\rho}_\pi^C : \Gamma_{E,S} \rightarrow \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$  such that

$$\mathrm{spin}^{a_\pi}(\tilde{\rho}_\pi^C) = \tilde{\rho}_\pi^{\mathrm{Sh}} \quad \text{and} \quad \mathrm{pr}^\circ \tilde{\rho}_\pi^C = \eta \otimes \rho_{\pi^{\flat}}.$$

Define  $\rho_\pi^C := \tilde{\eta} \otimes \tilde{\rho}_\pi^C$ . Then it follows that

$$\widetilde{\mathrm{spin}}^{a_\pi}(\rho_\pi^C) = \tilde{\rho}_\pi^{\mathrm{Sh}} \quad \text{and} \quad \mathrm{pr}^\circ \rho_\pi^C = \rho_{\pi^{\flat}}.$$

Thanks to (10.1) and (10.3),  $\rho_\pi^C$  satisfies (10.5) and (10.4). The proof is complete.  $\square$

*Remark 10.2.* The bottom row in (10.6) cannot be replaced with  $\mathrm{PSO}_{2n}$ . (If it did, since  $\bar{\rho}_{\pi^{\flat}}$  has connected image in  $\mathrm{PSO}_{2n}$  by Proposition 5.2, the argument above would work without introducing the  $\eta$ -twist.) For instance, observe that  $\mathrm{GSpin}_{2n} \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^n} \rightarrow \mathrm{PGL}_{2^n}$  does not factor through  $\mathrm{PSO}_{2n}$  since  $\mathrm{spin}^+$  and  $\mathrm{spin}^-$  have different central characters.

We can refine (10.4) by separating  $\mathrm{spin}^+$  and  $\mathrm{spin}^-$ , which is a key intermediate step towards the main theorem. Our argument is quite delicate and sensitive to the underlying group-theoretic structures.

**Proposition 10.3.** *For all  $\mathfrak{p} \notin S$  and  $\epsilon \in \{+, -\}$ , conjugating  $\rho_\pi^C$  by an element of  $\mathrm{GPin}_{2n}$  if necessary, we have the following.*

$$(10.8) \quad \forall \mathfrak{p} \notin S : \quad \mathrm{spin}^\epsilon \rho_\pi^C(\mathrm{Frob}_{\mathfrak{p}})_{\mathrm{ss}} \sim \iota q_{\mathfrak{p}}^{n(n-1)/4} \mathrm{spin}^\epsilon \phi_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{p}}) \in \mathrm{GL}_{2^{n-1}}(\overline{\mathbb{Q}}_\ell),$$

$$(10.9) \quad \forall \mathfrak{p} \notin S_{\mathrm{bad}} : \quad \mathrm{pr}^\circ \rho_\pi^C(\mathrm{Frob}_{\mathfrak{p}})_{\mathrm{ss}} \overset{\mathrm{out}}{\sim} \iota \mathrm{pr}^\circ \phi_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{p}}) \in \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell).$$

*Proof.* The assertion (10.9) follows from (10.5) (and it is invariant under conjugation by an element of  $\mathrm{GPin}_{2n}$ ). The main thing to prove is (10.8). For simplicity, write  $\rho := \rho_\pi^C$ ,  $\rho^\circ := \mathrm{pr}^\circ \rho_\pi^C$ , and  $a := a_\pi$ . Recall from §1 that we often write  $G_0$  to mean  $G_0(\overline{\mathbb{Q}}_\ell)$  when  $G_0$  is a reductive group over  $\overline{\mathbb{Q}}_\ell$ . Moreover we assume  $\mathfrak{p} \notin S^E$  throughout, without repeating this condition. (When applying the Chebotarev density theorem, we can always ignore finitely many places.) From (10.4) we have

$$(10.10) \quad \rho^{\mathrm{Sh},+} \oplus (\eta \otimes \rho^{\mathrm{Sh},-}) \simeq (\mathrm{spin}^+ \rho \oplus (\eta \otimes \mathrm{spin}^- \rho))^{\oplus a}.$$

Write  $Z := Z(\mathrm{GSpin}_{2n})$  and  $H$  for the Zariski closure of  $\mathrm{im}(\rho^\circ)$  in  $\mathrm{SO}_{2n}$ . Then  $H$  contains a regular unipotent element by Corollary 6.7. We will divide into three cases based on Proposition 5.2.

**Case 1.**  $\mathrm{spin}^\epsilon \rho_\pi^C$  is irreducible for both  $\epsilon \in \{+, -\}$ . This happens when  $H^0$  is  $\mathrm{SO}_{2n}$ ,  $i_{\mathrm{std}}^\circ(\mathrm{SO}_{2n-1})$ , or  $\mathrm{spin}(\mathrm{Spin}_7)$  (possibly after conjugation in  $\mathrm{GSpin}_{2n}$ ). If  $\mathrm{spin}^+ \rho \simeq \eta \otimes \mathrm{spin}^- \rho$  then it is clear from (10.10) that  $\rho^{\mathrm{Sh},+} \simeq \eta \otimes \rho^{\mathrm{Sh},-} \simeq (\mathrm{spin}^+ \rho)^{\oplus a} \simeq (\eta \otimes \mathrm{spin}^- \rho)^{\oplus a}$ , the proposition follows from (9.7). Henceforth assume that  $\mathrm{spin}^+ \rho \not\simeq \eta \otimes \mathrm{spin}^- \rho$ .

We claim that  $\mathrm{spin}^+ \rho(\gamma)_{\mathrm{ss}}$  is regular in  $\mathrm{GL}_{2^{n-1}}$  on a density 1 set of  $\gamma \in \Gamma$ . Define  $X^+$  to be the subset of  $h \in H(\overline{\mathbb{Q}}_\ell)$  such that the semisimple part of  $\overline{\mathrm{spin}^+}(h)$  is non-regular in  $\mathrm{PGL}_{2^{n-1}}$ . Then  $X^+$  is Zariski-closed and conjugation-invariant in  $H$ . To show  $H \neq X^+$ , take  $\tilde{H} \subset \mathrm{GSpin}_{2n}$  to be  $\mathrm{Spin}_{2n}$ ,  $i_{\mathrm{std}}(\mathrm{Spin}_{2n-1})$ , or  $\overline{\mathrm{spin}(\mathrm{Spin}_7)}$  in the three cases, respectively, so that  $\tilde{H}$  surjects onto  $H$ . Then the restriction of  $\mathrm{spin}^+$  via  $\tilde{H} \hookrightarrow \mathrm{GSpin}_{2n}$  is an irreducible representation with distinct weight vectors. (When  $\tilde{H} = i_{\mathrm{std}}(\mathrm{Spin}_{2n-1})$ , the restriction is the spin representation of  $\mathrm{Spin}_{2n-1}$  by Proposition 4.5.) So some element of  $\tilde{H}$  maps to a regular element of  $\mathrm{GL}_{2^{n-1}}$  under  $\mathrm{spin}^+$ . It follows that some element of  $H$  maps to a regular element of  $\mathrm{GL}_{2^{n-1}}$ ; in particular  $H \neq X^+$  and thus  $\dim X^+ < \dim H$ . Therefore the set of  $\gamma$  such that  $\rho^\circ(\gamma) \notin X^+$  has density 1 according to Lemma 1.1, and in this case  $\overline{\mathrm{spin}^+ \rho(\gamma)_{\mathrm{ss}}} = \overline{\mathrm{spin}^+(\rho^\circ(\gamma)_{\mathrm{ss}})}$  is regular. The claim is verified.

Given a square matrix  $g$ , let  $\mathcal{E}\mathcal{V}(g)$  for the multi-set of its eigenvalues. Since  $\mathrm{spin}^+ \rho \not\simeq \eta \otimes \mathrm{spin}^- \rho$ , there exists  $\gamma \in \Gamma$  such that

- $\mathrm{spin}^+ \rho(\gamma)$  has distinct eigenvalues,
- $\mathcal{E}\mathcal{V}(\mathrm{spin}^+ \rho(\gamma)) \neq \mathcal{E}\mathcal{V}(\eta(\gamma) \mathrm{spin}^- \rho(\gamma))$ .

In particular there exists an eigenvalue  $\alpha$  of  $\text{spin}^+\rho(\gamma)$  which is not an eigenvalue of  $\eta(\gamma)\text{spin}^-\rho(\gamma)$ . Then  $\alpha$  appears as an eigenvalue with multiplicity  $a_\pi$  on the right hand side of (10.10). We know from (9.7) that each eigenvalue of  $\rho^{\text{Sh},+}$  and  $\eta \otimes \rho^{\text{Sh},-}$  appears with multiplicity divisible by  $a_\pi$ . Thus  $\alpha$  is an eigenvalue of either  $\rho^{\text{Sh},+}$  or  $\eta \otimes \rho^{\text{Sh},-}$  but not both. This implies, together with (9.7) and the irreducibility of  $\text{spin}^\epsilon \rho$ , that (i)  $(\text{spin}^+\rho)^{\oplus a} \simeq \rho^{\text{Sh},+}$  and  $(\text{spin}^-\rho)^{\oplus a} \simeq \rho^{\text{Sh},-}$ , or (ii)  $(\text{spin}^+\rho)^{\oplus a} \simeq \eta \otimes \rho^{\text{Sh},-}$  and  $(\text{spin}^-\rho)^{\oplus a} \simeq \eta \otimes \rho^{\text{Sh},+}$ . In case (i) we are done. If (ii) occurs, replace  $\rho$  with  $\eta \otimes (\vartheta \rho \vartheta^{-1})$ , where  $\vartheta \in \text{GPin}_{2n}$  is as in (3.7). (Here  $\text{im}(\eta) = \{\pm 1\}$  is viewed as the subgroup of  $\ker(\text{pr}^\circ) = \mathbb{G}_m$ .) Then (10.5) remains valid because the conjugation by  $\vartheta$  is absorbed by  $\overset{\text{out}}{\simeq}$  and because  $\text{pr}^\circ(\eta) = 1$ . We are done in Case 1.

**Case 2.** When  $H^0 = i_{\text{std}}^\circ(G_2)$  and  $n = 4$ . Then  $\rho$  factors through  $(\rho_1, \rho_2) : \Gamma_{E,S} \rightarrow G_2(\overline{\mathbb{Q}}_\ell) \times Z(\overline{\mathbb{Q}}_\ell)$  via  $i_{\text{spin}} : G_2 \hookrightarrow \text{GSpin}_8$ , and  $\rho_1$  has Zariski dense image in  $G_2$ . Thus the 7-dimensional representation  $\text{std}(\rho_1)$  is irreducible. Let  $\omega_\epsilon : Z(\overline{\mathbb{Q}}_\ell) \rightarrow \overline{\mathbb{Q}}_\ell^\times$  denote the central character of  $\text{spin}^\epsilon$ . Both  $\omega_+$  and  $\omega_-$  restrict to the weight 1 character on  $Z^0(\overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell^\times$ , and  $\omega_+/\omega_-$  is the nontrivial quadratic character of  $Z/Z^0 \simeq \{1, z_+\}$ . Since  $\text{spin}^\epsilon$  restricts via  $i_{\text{spin}}$  to  $\text{std} \oplus \mathbf{1}$  on  $G_2$  (observed in §5), we can rewrite (10.10) as (omitting  $\otimes$  for character twists)

$$(10.11) \quad \rho^{\text{Sh},+} \oplus \eta \rho^{\text{Sh},-} \simeq ((\text{std}(\rho_1) \oplus \mathbf{1}) \otimes (\omega_+ \rho_2 \oplus \eta \omega_- \rho_2))^{\oplus a}.$$

Let  $X$  be the subset of  $g \in G_2(\overline{\mathbb{Q}}_\ell)$  such that either of the following fails:

- $\mathcal{E}\mathcal{V}(\text{std}(g))$  is multiplicity-free,
- $-\mathcal{E}\mathcal{V}(\text{std}(g)) \cap \mathcal{E}\mathcal{V}(\text{std}(g))$  is empty.

(Here  $-A$  for a multi-set  $A$  means the new multi-set obtained from  $A$  by taking additive inverses.) Then  $X$  is a proper Zariski-closed subset in  $G_2$  that is conjugation-invariant.<sup>15</sup> By Lemma 1.1, we have a density one set of places  $\mathfrak{p}$  such that  $\rho_1(\text{Frob}_\mathfrak{p}) \notin X$ . Let us divide into two sub-cases.

**Case 2-1.**  $\omega_+ \rho_2 = \eta \omega_- \rho_2$ . Then (10.11) becomes

$$(10.12) \quad \rho^{\text{Sh},+} \oplus \eta \rho^{\text{Sh},-} \simeq ((\text{std}(\rho_1) \otimes \omega_+ \rho_2) \oplus \omega_+ \rho_2)^{\oplus 2a}.$$

Fix a prime  $\mathfrak{p}$  such that  $\rho_1(\text{Frob}_\mathfrak{p}) \notin X$ . We have a multiplicity-free eigenvalue  $\alpha \neq 1$  of  $\text{std}(\rho_1(\text{Frob}_\mathfrak{p}))\omega_+(\text{Frob}_\mathfrak{p})$ . Then  $\alpha$  is an eigenvalue with multiplicity  $2a$  in the right hand side of (10.12), thus also in the left hand side. If  $\alpha$  has multiplicity  $a$  for each of  $\rho^{\text{Sh},+}(\text{Frob}_\mathfrak{p})$  and  $\eta(\text{Frob}_\mathfrak{p})\rho^{\text{Sh},-}(\text{Frob}_\mathfrak{p})$ , then each of  $\rho^{\text{Sh},+}$  and  $\eta \rho^{\text{Sh},-}$  contains  $\text{std}(\rho_1) \otimes \omega_+ \rho_2$  with multiplicity exactly  $a$ . Since  $\dim \rho^{\text{Sh},+} = \dim \eta \rho^{\text{Sh},-}$ , each of the two representations contains  $\omega_+ \rho_2$  with exact multiplicity  $a$  as well. That is,

$$\rho^{\text{Sh},+} \simeq \eta \rho^{\text{Sh},-} \simeq ((\text{std}(\rho_1) \otimes \omega_+ \rho_2) \oplus \omega_+ \rho_2)^{\oplus a},$$

implying (10.8). Now let us exclude the case that the multiplicity of  $\alpha$  is not  $a_\pi$  for either  $\rho^{\text{Sh},+}(\text{Frob}_\mathfrak{p})$  or  $\eta \otimes \rho^{\text{Sh},-}(\text{Frob}_\mathfrak{p})$ . If it were the case, the multiplicity would be  $2a_\pi$  for one and 0 for the other. In particular, one of the two representations would be the direct sum of one-dimensional representations only. However the total dimension of (10.11) is  $16a$ , whereas the total dimension of one-dimensional representations is  $2a$ , contradicting  $\dim \rho^{\text{Sh},+} = \dim \rho^{\text{Sh},-} = 8a$  (with  $n = 4$ ).

**Case 2-2.**  $\omega_+ \rho_2 \neq \eta \omega_- \rho_2$ . Then there exists a place  $\mathfrak{p}$  such that  $\rho_1(\text{Frob}_\mathfrak{p}) \notin X$  and  $\omega_+ \rho_2(\text{Frob}_\mathfrak{p}) \neq \eta \omega_- \rho_2(\text{Frob}_\mathfrak{p})$ . The latter implies that  $\omega_+ \rho_2(\text{Frob}_\mathfrak{p}) = -\eta \omega_- \rho_2(\text{Frob}_\mathfrak{p})$ . Again let  $\alpha \neq 1$  be a multiplicity-free eigenvalue of  $\text{std}(\rho_1(\text{Frob}_\mathfrak{p}))\omega_+(\text{Frob}_\mathfrak{p})$ . The condition  $\rho_1(\text{Frob}_\mathfrak{p}) \notin X$  tells us that  $\alpha$  is not an eigenvalue of  $\text{std}(\rho_1(\text{Frob}_\mathfrak{p}))\eta \omega_- \rho_2(\text{Frob}_\mathfrak{p})$ . Hence  $\alpha$  is an eigenvalue with multiplicity  $a$  in the left hand side of (10.11) evaluated at  $\text{Frob}_\mathfrak{p}$ . Arguing as above, up to replacing  $\rho$  with  $\eta(\vartheta \rho \vartheta^{-1})$ , we have  $\alpha$  appearing as an eigenvalue with multiplicity  $a$  in  $\rho^{\text{Sh},+}(\text{Frob}_\mathfrak{p})$  but not as an eigenvalue of  $\eta \rho^{\text{Sh},-}(\text{Frob}_\mathfrak{p})$ . We deduce that  $\rho^{\text{Sh},+} \supset (\text{std}(\rho_1) \otimes \omega_+ \rho_2)^{\oplus a}$  and

<sup>15</sup>To explicitly see that  $X$  is proper and closed, fix a maximal torus  $T$  of  $G_2$  and an isomorphism  $T \simeq \mathbb{G}_m \times \mathbb{G}_m$  with simple roots  $(b, c) \mapsto b^2c, b^3c^2$ . If the semisimple part of  $g$  is conjugate to  $(b, c) \in T$  then  $\mathcal{E}\mathcal{V}(\text{std}(g)) := \{1, b, b^{-1}, bc, b^{-1}c^{-1}, b^2c, b^{-2}c^{-1}\}$ .

$\eta\rho^{\text{Sh},-} \supset (\text{std}\rho_1 \otimes \eta\omega_-\rho_2)^{\oplus a}$ . In the latter, we can cancel out  $\eta$ . Reading off the central characters of  $\rho^{\text{Sh},\epsilon}$  from (9.7), we can fill in the one-dimensional representations uniquely to obtain

$$\rho^{\text{Sh},\epsilon} \simeq ((\text{std}\rho_1 \otimes \omega_\epsilon\rho_2) \oplus \omega_\epsilon\rho_2)^{\oplus a_\pi}, \quad \epsilon \in \{+, -\}.$$

It follows that (10.8) holds true.

**Case 3.** When  $H^0 = i_{\text{reg}}^\circ(\text{PGL}_2)$ . We see from the first paragraph of §5 that  $i_{\text{reg}}^\circ : \text{PGL}_2 \times Z(\text{SO}_{2n}) \hookrightarrow \text{SO}_{2n}$  pulls back via  $\text{pr}^\circ : \text{GSpin}_{2n} \rightarrow \text{SO}_{2n}$  to  $i_{\text{reg}} : H_1 \times H_2 \hookrightarrow \text{GSpin}_{2n}$ , where

$$H_1 \simeq \begin{cases} \text{GL}_2, & n(n-1)/2 \text{ is odd,} \\ \text{PGL}_2 \times \mathbb{G}_m, & n(n-1)/2 \text{ is even,} \end{cases} \quad H_2 = \{1, z_+\},$$

such that  $H_1$  is the preimage of  $\text{PGL}_2$  via  $\text{pr}^\circ$ . By assumption,  $\rho$  factors through a unique representation

$$(\rho_1, \rho_2) : \Gamma_{E,S} \rightarrow H_1(\overline{\mathbb{Q}}_\ell) \times H_2(\overline{\mathbb{Q}}_\ell).$$

Now  $\rho(\text{Frob}_\mathfrak{p})_{\text{ss}} \in i_{\text{reg}}((H_1 \times H_2)(\overline{\mathbb{Q}}_\ell))$ . By (10.5), conjugating  $\phi_{\pi_\mathfrak{p}}(\text{Frob}_\mathfrak{p})$  if necessary, we may assume that  $\iota \text{pr}^\circ \phi_{\pi_\mathfrak{p}}(\text{Frob}_\mathfrak{p}) \in \text{pr}^\circ(i_{\text{reg}}((H_1 \times H_2)(\overline{\mathbb{Q}}_\ell)))$ . (A priori the latter holds up to the outer automorphism  $\theta$ , but note that  $\theta$  acts as the identity on the image of  $i_{\text{reg}}$ .) Taking the preimage under  $\text{pr}$ , we see that

$$\iota \phi_{\pi_\mathfrak{p}}(\text{Frob}_\mathfrak{p}) \in i_{\text{reg}}(H_1(\overline{\mathbb{Q}}_\ell) \times H_2(\overline{\mathbb{Q}}_\ell)).$$

For  $\mathfrak{p} \notin S$ , define  $h'_{1,\mathfrak{p}}, h''_{1,\mathfrak{p}} \in H_1(\overline{\mathbb{Q}}_\ell)$  and  $h'_{2,\mathfrak{p}}, h''_{2,\mathfrak{p}} \in H_2(\overline{\mathbb{Q}}_\ell)$  by

$$h'_{i,\mathfrak{p}} := \rho_i(\text{Frob}_\mathfrak{p})_{\text{ss}} \text{ for } i = 1, 2, \quad \iota q_\mathfrak{p}^{n(n-1)/4} \phi(\pi_\mathfrak{p})(\text{Frob}_\mathfrak{p}) = i_{\text{reg}}(h'_{1,\mathfrak{p}}, h''_{2,\mathfrak{p}}).$$

(We need not know whether  $h'_{1,\mathfrak{p}}, h''_{2,\mathfrak{p}}$  are uniquely determined by  $\phi_{\pi_\mathfrak{p}}(\text{Frob}_\mathfrak{p})$  up to conjugation. This does not affect the argument below.)

Now we observe that the composition  $\text{spin}^\epsilon \circ i_{\text{reg}} : H_1 \times H_2 \rightarrow \text{GL}_{2n-1}$  for  $\epsilon = +$  (resp.  $\epsilon = -$ ) is  $(\text{spin} \circ j_{\text{reg}}, \mathbf{1})$  (resp.  $(\text{spin} \circ j_{\text{reg}}, \text{sgn})$ ), where  $\text{sgn}$  is the sign character  $H_2(\overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \{\pm 1\}$ . (On the  $H_2$ -factor this is a direct calculation. On  $H_1$ , since  $i_{\text{reg}} : H_1 \times H_2 \rightarrow \text{GSpin}_{2n}$  factors through  $i_{\text{std}} : \text{GSpin}_{2n-1} \hookrightarrow \text{GSpin}_{2n}$ , it follows from the fact that the half-spin representations restrict to the spin representation via  $i_{\text{std}}$ .) To simplify notation, define

$$\eta_\mathfrak{p} := \eta(\text{Frob}_\mathfrak{p}), \quad \rho_\mathfrak{p}^\epsilon := \text{spin}^\epsilon(\rho(\text{Frob}_\mathfrak{p})_{\text{ss}}), \quad \rho_\mathfrak{p}^{\text{Sh},\epsilon} := \rho^{\text{Sh},\epsilon}(\text{Frob}_\mathfrak{p})_{\text{ss}}, \quad \mathfrak{p} \notin S.$$

Recall that  $\rho_\mathfrak{p}^{\text{Sh},\epsilon} \sim i_a \circ i_{\text{reg}}(h'_{1,\mathfrak{p}}, h''_{2,\mathfrak{p}})$ . By the computation of  $\text{spin}^\epsilon \circ i_{\text{reg}}$ , we have

$$(10.13) \quad \begin{cases} \rho_\mathfrak{p}^+ \sim \text{spin}(j_{\text{reg}}(h'_{1,\mathfrak{p}})), \\ \eta_\mathfrak{p} \rho_\mathfrak{p}^- \sim \text{spin}(j_{\text{reg}}(h'_{1,\mathfrak{p}})) \eta_\mathfrak{p} \text{sgn}(h'_{2,\mathfrak{p}}), \end{cases} \quad \begin{cases} \rho_\mathfrak{p}^{\text{Sh},+} \sim i_a \circ \text{spin}(j_{\text{reg}}(h'_{1,\mathfrak{p}})), \\ \eta_\mathfrak{p} \rho_\mathfrak{p}^{\text{Sh},-} \sim i_a \circ \text{spin}(j_{\text{reg}}(h'_{1,\mathfrak{p}})) \eta_\mathfrak{p} \text{sgn}(h'_{2,\mathfrak{p}}). \end{cases}$$

When  $g_1, g_2 \in \text{GL}_{2n-1}$ , write  $g_1 \boxplus g_2 \in \text{GL}_{2n}$  for the image of  $(g_1, g_2)$  under the block diagonal embedding. Comparing this with (10.10) evaluated at  $\text{Frob}_\mathfrak{p}$ , we see that

$$(10.14) \quad \begin{aligned} \rho_\mathfrak{p}^+ \boxplus \eta_\mathfrak{p} \rho_\mathfrak{p}^- &\sim \text{spin}(j_{\text{reg}}(h'_{1,\mathfrak{p}})) \boxplus \text{spin}(j_{\text{reg}}(h'_{1,\mathfrak{p}})) \eta_\mathfrak{p} \text{sgn}(h'_{2,\mathfrak{p}}) \\ &\sim \text{spin}(j_{\text{reg}}(h'_{1,\mathfrak{p}})) \boxplus \text{spin}(j_{\text{reg}}(h'_{1,\mathfrak{p}})) \eta_\mathfrak{p} \text{sgn}(h''_{2,\mathfrak{p}}). \end{aligned}$$

To make a computational argument with eigenvalues, consider the surjection

$$\text{SL}_2 \times \mathbb{G}_m \twoheadrightarrow H_1$$

whose kernel is the diagonally embedded  $\{\pm 1\}$  if  $n(n-1)/2$  is odd, and  $\{(\pm 1, 1)\}$  if  $n(n-1)/2$  is even. Given  $h_1 \in H_1$ , choose a lift  $\tilde{h}_1 \in \text{SL}_2 \times \mathbb{G}_m$  and an element  $(\text{diag}(c, c^{-1}), \lambda) \in \text{SL}_2 \times \mathbb{G}_m$  conjugate to  $\tilde{h}_{1,\text{ss}}$ . Let  $\bar{h}_1 \in \text{PGL}_2$  be the image of  $h_1$ . Suppose that  $\bar{h}_{1,\text{ss}}$  is conjugate to the image as  $\text{diag}(b, 1)$  in  $\text{PGL}_2$ . (Thus  $b$  is well-defined up to taking inverse.) Then we have an explicit description (cf. [Gro00, Prop. 6.1, §7] for the latter)

$$(10.15) \quad \mathcal{E}^\mathcal{V}(\text{std}(i_{\text{reg}}^\circ(\bar{h}_1))) = \{b^{2n-2}, b^{2n-3}, \dots, b, 1, 1\},$$

$$(10.16) \quad \mathcal{E}^\mathcal{V}(\text{spin}(j_{\text{reg}}(h_1))) = \left\{ \lambda \cdot c^{\sum_{i=1}^{n-1} \varepsilon_i i} \mid \varepsilon_i \in \{\pm 1\} \right\}.$$

Let  $X$  be the subset of  $h_1 \in H_1(\overline{\mathbb{Q}}_\ell)$  such that, in terms of  $b, c$  assigned to  $h_{1,ss}$  as above,

- (X1)  $b^m = 1$  for some integer  $m \neq 0$  with  $|m| \leq 2(n+1)$ , or
- (X2)  $c^m = 1$  for some integer  $m \neq 0$  with  $|m| \leq 2^{n+1}n^2$ .

Then  $X$  is a proper Zariski closed subset of  $H_1$  that is invariant under conjugation. Appealing to Lemma 1.1, we see that there is a density one set of places  $\mathcal{D}_1$ , disjoint from  $S$ , such that if  $\mathfrak{p} \in \mathcal{D}_1$  then  $h'_{1,\mathfrak{p}} \notin X$ . Since (X2) is false when  $\mathfrak{p} \in \mathcal{D}_1$ , we see from (10.13) and (10.16) that

$$-\mathcal{E}\mathcal{V}(\rho_{\mathfrak{p}}^+) \cap \mathcal{E}\mathcal{V}(\rho_{\mathfrak{p}}^+) = \emptyset, \quad \mathfrak{p} \in \mathcal{D}_1.$$

(If the intersection were nonempty,  $\lambda c^i = -\lambda c^j$  with  $|i|, |j| \leq n(n-1)/2$ . This would imply  $c^m = 1$  for  $|m| \leq 2n(n-1)$ .) We know from (10.9) that  $i_{\text{reg}}^\circ(\overline{h}'_{1,\mathfrak{p}})\text{sgn}(h'_{2,\mathfrak{p}}) \stackrel{\text{out}}{\sim} i_{\text{reg}}^\circ(\overline{h}''_{1,\mathfrak{p}})\text{sgn}(h''_{2,\mathfrak{p}})$  in  $\text{SO}_{2n}$ . Pick  $b', b'' \in \mathbb{G}_m$  (well-defined up to taking inverses) such that  $\overline{h}'_{1,\mathfrak{p}}, \overline{h}''_{1,\mathfrak{p}}$  are conjugate to the images of  $\text{diag}(b', 1), \text{diag}(b'', 1)$  in  $\text{PGL}_2$ . By (10.15), we have an equality of multi-sets

$$\text{sgn}(h'_{2,\mathfrak{p}}) \cdot \{b'^{2n-2}, b'^{2n-3}, \dots, b', 1, 1\} = \text{sgn}(h''_{2,\mathfrak{p}}) \cdot \{b''^{2n-2}, b''^{2n-3}, \dots, b'', 1, 1\}.$$

For  $\mathfrak{p} \in \mathcal{D}_1$ , neither  $b'$  nor  $b''$  can be an  $m$ -th root of unity for  $1 \leq m \leq 2^{n+1}n^2$  as (X1) is false. Then it is an elementary exercise to verify that  $\text{sgn}(h'_{2,\mathfrak{p}}) = \text{sgn}(h''_{2,\mathfrak{p}})$  and that  $b'' = b'$  or  $b'' = b'^{-1}$ . Thus  $\overline{h}'_{1,\mathfrak{p}} \sim \overline{h}''_{1,\mathfrak{p}}$  in  $\text{PGL}_2$ . We summarize the findings:

$$(10.17) \quad \overline{h}'_{1,\mathfrak{p}} \sim \overline{h}''_{1,\mathfrak{p}} \quad \text{and} \quad h'_{2,\mathfrak{p}} = h''_{2,\mathfrak{p}}, \quad \mathfrak{p} \in \mathcal{D}_1.$$

We divide into two sub-cases.

**Case 3-1.**  $\eta = \text{sgn} \circ \rho_2$ . Then for almost all  $\mathfrak{p}$ ,

$$\eta_{\mathfrak{p}} \text{sgn}(h'_{2,\mathfrak{p}}) = \eta(\text{Frob}_{\mathfrak{p}}) \text{sgn}(\rho_2(\text{Frob}_{\mathfrak{p}})) = \eta(\text{Frob}_{\mathfrak{p}})^2 = 1.$$

By (10.13),  $\text{spin}^+ \rho \simeq \eta \text{spin}^- \rho$ . We also have  $\eta_{\mathfrak{p}} \text{sgn}(h''_{2,\mathfrak{p}}) = 1$  at  $\mathfrak{p} \in \mathcal{D}_1$  since  $h'_{2,\mathfrak{p}} = h''_{2,\mathfrak{p}}$ . Hence  $\rho^{\text{Sh},+} \simeq \eta \rho^{\text{Sh},-}$  by (10.13). Going back to (10.10), it follows that

$$\rho^{\text{Sh},+} \simeq \eta \rho^{\text{Sh},-} \simeq (\text{spin}^+ \rho)^{\oplus a} \simeq (\eta \text{spin}^- \rho)^{\oplus a}.$$

We complete the proof of (10.8) by (10.2).

**Case 3-2.**  $\eta \neq \text{sgn} \circ \rho_2$ . Then  $\eta \cdot (\text{sgn} \circ \rho_2)$  is a nontrivial quadratic character, so there is a density  $1/2$  set of primes  $\mathcal{D}_{1/2}$  such that for  $\mathfrak{p} \notin S$ ,

$$\eta_{\mathfrak{p}} \text{sgn}(h'_{2,\mathfrak{p}}) = \begin{cases} -1, & \mathfrak{p} \in \mathcal{D}_{1/2}, \\ 1, & \mathfrak{p} \notin \mathcal{D}_{1/2}. \end{cases}$$

When  $\mathfrak{p} \in \mathcal{D}_1$  but  $\mathfrak{p} \notin \mathcal{D}_{1/2}$ , we are in a situation similar to Case 3-1. As before, we have

$$(10.18) \quad \rho_{\mathfrak{p}}^{\text{Sh},+} \simeq \eta_{\mathfrak{p}} \rho_{\mathfrak{p}}^{\text{Sh},-} \sim i_a(\rho_{\mathfrak{p}}^+) \sim i_a(\eta_{\mathfrak{p}} \rho_{\mathfrak{p}}^-), \quad \mathfrak{p} \in \mathcal{D}_1, \mathfrak{p} \notin \mathcal{D}_{1/2} \cup S.$$

For  $\mathfrak{p} \in \mathcal{D}_{1/2} \cap \mathcal{D}_1$ , we have  $\eta_{\mathfrak{p}} \text{sgn}(h'_{2,\mathfrak{p}}) = \eta_{\mathfrak{p}} \text{sgn}(h''_{2,\mathfrak{p}}) = -1$ . Applying Lemma 10.4 below to (10.14) (taking  $([b], \lambda), ([c], \delta)$  to be lifts of  $h'_{1,\mathfrak{p}}, h''_{1,\mathfrak{p}}$ ; we have  $[b]^2 \sim [c]^2$  from (10.17)), and comparing with (10.13), we deduce that

- (i)  $\rho_{\mathfrak{p}}^{\text{Sh},+} \sim i_a(\rho_{\mathfrak{p}}^+)$  and  $\eta_{\mathfrak{p}} \rho_{\mathfrak{p}}^{\text{Sh},-} \sim i_a(\eta_{\mathfrak{p}} \rho_{\mathfrak{p}}^-)$ , or
- (ii)  $\eta_{\mathfrak{p}} \rho_{\mathfrak{p}}^{\text{Sh},-} \sim i_a(\rho_{\mathfrak{p}}^+)$  and  $\rho_{\mathfrak{p}}^{\text{Sh},+} \sim i_a(\eta_{\mathfrak{p}} \rho_{\mathfrak{p}}^-)$ .

Only one of the two occurs at each  $\mathfrak{p}$  since no eigenvalue is shared between the  $+$  and  $-$  parts.

The main remaining point is to show that only (i) holds for all  $\mathfrak{p} \in \mathcal{D}_{1/2} \cap \mathcal{D}_1$ , or only (ii) holds. To this end, consider the irreducible representation  $S^{n(n-1)} := (\text{Sym}^{n(n-1)}, \text{id})$  of  $\text{SL}_2 \times \mathbb{G}_m$ . It descends to a representation of  $H_1$  via  $\text{SL}_2 \times \mathbb{G}_m \rightarrow H_1$  (see §5). By the highest weight reason,  $S^{n(n-1)}$  appears in  $\text{spin} \circ j_{\text{reg}}$  with multiplicity one. Hence

$$\dim_{\overline{\mathbb{Q}}_\ell} \text{Hom}_{\Gamma}((S^{n(n-1)}, \mathbf{1}) \circ (\rho_1, \rho_2), \text{spin}^+ \rho) = 1.$$

By (10.10),  $(S^{n(n-1)}, \mathbf{1}) \circ (\rho_1, \rho_2)$  appears in  $\rho^{\text{Sh},+}$  or  $\eta \rho^{\text{Sh},-}$ ; we may assume that this is the case for  $\rho^{\text{Sh},+}$ , replacing  $\rho$  with  $\eta(\vartheta \rho \vartheta^{-1})$  (thus changing  $(\rho_1, \rho_2)$  to  $(\eta \rho_2 \otimes \rho_1, \rho_2)$ ) if necessary. Then  $\text{spin}^+ \rho$  and  $\rho^{\text{Sh},+}$  both contain  $(S^{n(n-1)}, \mathbf{1}) \circ (\rho_1, \rho_2)$ , so  $\rho_{\mathfrak{p}}^+$  and  $\rho_{\mathfrak{p}}^{\text{Sh},+}$  share an eigenvalue.

On the other hand, no eigenvalue is shared between  $\rho_{\mathfrak{p}}^{\text{Sh},+}$  and  $\eta_{\mathfrak{p}}\rho_{\mathfrak{p}}^{\text{Sh},-}$ , and between  $\rho_{\mathfrak{p}}^+$  and  $\eta_{\mathfrak{p}}\rho_{\mathfrak{p}}^-$ . Therefore we must have case (i) at all  $\mathfrak{p} \in \mathcal{D}_{1/2} \cap \mathcal{D}_1$ . Together with (10.18), this implies that  $\rho^{\text{Sh},+} \simeq i_a \circ \text{spin}^+ \rho$  and  $\rho^{\text{Sh},-} \simeq i_a \circ \text{spin}^- \rho$ . Given this, (10.8) follows from (10.2).  $\square$

The following combinatorial lemma was used in the above proof. For  $a \in \overline{\mathbb{Q}}_{\ell}^{\times}$ , write  $[a] := \text{diag}(a, a^{-1}) \in \text{SL}_2(\overline{\mathbb{Q}}_{\ell})$ . Recall we have the maps

$$\text{SL}_2 \xrightarrow{j_{\text{reg}}} \text{GSpin}_{2n-1} \xrightarrow{\text{spin}} \text{GL}_{2n-1}.$$

**Lemma 10.4.** *Let  $([b], \lambda), ([c], \delta) \in \text{SL}_2 \times \mathbb{G}_m$ . Assume that  $[b]^2 \sim [c]^2$  and that  $b^m \neq 1$  for any integer  $m$  with  $|m| \leq 2^{n+1}n^2$ . If moreover<sup>16</sup>*

$$(10.19) \quad \begin{aligned} & -\mathcal{E}\mathcal{V}(\lambda \text{spin}(j_{\text{reg}}([b]))) \cup \mathcal{E}\mathcal{V}(\lambda \text{spin}(j_{\text{reg}}([b]))) \\ & = -\mathcal{E}\mathcal{V}(\delta \text{spin}(j_{\text{reg}}([c]))) \cup \mathcal{E}\mathcal{V}(\delta \text{spin}(j_{\text{reg}}([c]))) \end{aligned}$$

then  $\mathcal{E}\mathcal{V}(\lambda \text{spin}(j_{\text{reg}}([b]))) = e \cdot \mathcal{E}\mathcal{V}(\delta \text{spin}(j_{\text{reg}}([c])))$  with  $e = 1$  or  $e = -1$ .

*Proof.* By scaling  $\lambda, \mu$  and by replacing  $c$  with  $c^{-1}$  if needed, we may assume that  $\delta = 1$  and that  $b = c$  or  $b = -c$ . Multiplying all elements on both sides, we deduce that  $\lambda^{2^n} = 1$ . Recall that  $\mathcal{E}\mathcal{V}(\lambda \text{spin}(j_{\text{reg}}([b])))$  is explicitly described by (10.16).

Assume  $n(n-1)/2$  is odd. Then  $c \in \mathcal{E}\mathcal{V}(\text{spin}(j_{\text{reg}}([c])))$ . By (10.19), there exist  $e \in \{\pm 1\}$  and  $i \in \mathbb{Z}$  with  $|i| \leq n(n-1)/2$  such that  $\lambda b = e \cdot c^i$ . Since  $b^2 = c^2$ , if  $i \neq 1$  then  $b^{2i} = \lambda^{2i} b^2$ , so  $b$  is a  $2^n(i-1)$ -th root of unity, violating the initial assumption. Hence  $\lambda b = e \cdot c$ . By squaring, we obtain  $\lambda b^2 = b^2$ , thus  $\lambda \in \{\pm 1\}$ . Replacing  $([b], \lambda)$  with  $([-b], -\lambda)$  does not change eigenvalues in the lemma, so we may assume  $\lambda = 1$ . It is easy to see from (10.16) that the conclusion of the lemma holds with the same  $e$  as we have chosen.

When  $n(n-1)/2$  is even, we argue similarly using  $c^2 \in \mathcal{E}\mathcal{V}(\text{spin}(j_{\text{reg}}([c])))$  to find  $\lambda b^2 = e \cdot c^2$ , which implies  $\lambda = e$ . Again the conclusion of the lemma holds with this  $e$  as can be seen from (10.16). The proof is finished.  $\square$

**Proposition 10.5.** *We have that*

$$(10.20) \quad \forall \mathfrak{p} \notin S : \quad \rho_{\pi}^C(\text{Frob}_{\mathfrak{p}})_{\text{ss}} \sim \iota q_{\mathfrak{p}}^{n(n-1)/4} \phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}}) \in \text{GSpin}_{2n}(\overline{\mathbb{Q}}_{\ell}).$$

*Proof.* We first establish the claim that  $\chi_{\ell}^{-n(n-1)/2} \iota \omega_{\pi} = \mathcal{N} \rho_{\pi}^C$ , where  $\chi_{\ell}$  is the cyclotomic character and we view  $\omega_{\pi}$  as a Galois character via class field theory. In view of Lemma 5.3, it suffices to check that

$$(10.21) \quad \chi_{\ell}^{-n(n-1)/2} \iota \omega_{\pi} \cdot \text{spin}^{\varepsilon}(\rho_{\pi}^C) \simeq \mathcal{N} \rho_{\pi}^C \cdot \text{spin}^{\varepsilon}(\rho_{\pi}^C), \quad \varepsilon \in \{\pm 1\}.$$

By Lemma 4.2 we have

$$(10.22) \quad \text{spin}^{\varepsilon}(\rho_{\pi}^C) \simeq (\text{spin}^{(-1)^n \varepsilon})^{\vee}(\rho_{\pi}^C) \otimes \mathcal{N} \rho_{\pi}^C.$$

For all  $\mathfrak{p} \notin S$  we apply (10.8) and compute using Lemma 4.2 again (but now locally)

$$(10.23) \quad \begin{aligned} \text{spin}^{\varepsilon}(\rho_{\pi}^C(\text{Frob}_{\mathfrak{p}})_{\text{ss}}) & \simeq \iota \text{spin}^{\varepsilon}(q_{\mathfrak{p}}^{n(n-1)/4} \phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}})) \\ & \simeq \iota q_{\mathfrak{p}}^{n(n-1)/4} (\text{spin}^{(-1)^n \varepsilon})^{\vee}(\phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}})) \otimes \iota \mathcal{N}(\phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}})) \\ & \simeq \iota q_{\mathfrak{p}}^{n(n-1)/2} (\text{spin}^{(-1)^n \varepsilon})^{\vee}(\rho_{\pi}^C(\text{Frob}_{\mathfrak{p}})_{\text{ss}}) \otimes \iota \mathcal{N}(\phi_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}})) \\ & \simeq \chi_{\ell}(\text{Frob}_{\mathfrak{p}})^{-n(n-1)/2} (\text{spin}^{(-1)^n \varepsilon})^{\vee}(\rho_{\pi}^C(\text{Frob}_{\mathfrak{p}})_{\text{ss}}) \iota \omega_{\pi_{\mathfrak{p}}}(\text{Frob}_{\mathfrak{p}}). \end{aligned}$$

In the last isomorphism, we appealed to functoriality of the Satake isomorphism (unramified local Langlands correspondence) with respect to  $\mathbb{G}_m \hookrightarrow \text{GSO}_{2n}$  (dual to  $\mathcal{N} : \text{GSpin}_{2n} \rightarrow \mathbb{G}_m$ ). Therefore  $\text{spin}^{\varepsilon}(\rho_{\pi}^C) \simeq (\text{spin}^{(-1)^n \varepsilon})^{\vee}(\rho_{\pi}^C) \otimes \chi_{\ell}^{-n(n-1)/2} \iota \omega_{\pi}$ . Comparing with (10.22), we obtain (10.21).

<sup>16</sup>Recall we write  $\mathcal{E}\mathcal{V}(g)$  for the set of (generalized) eigenvalues of a matrix  $g \in \text{M}_n$



At this point we have established that for all  $E$ -places  $\mathfrak{p} \notin S$  that

$$\begin{aligned}
 \text{spin}^\varepsilon \rho_\pi^C(\text{Frob}_\mathfrak{p})_{\text{ss}} &\sim \iota q_\mathfrak{p}^{n(n-1)/4} \text{spin}^\varepsilon \phi_{\pi_\mathfrak{p}}(\text{Frob}_\mathfrak{p}) \in \text{GL}_{2n-1}(\overline{\mathbb{Q}}_\ell) && \text{(Prop. 10.3)}, \\
 \text{pr}^\circ \rho_\pi^C(\text{Frob}_\mathfrak{p})_{\text{ss}} &\overset{\text{out}}{\sim} \iota \text{pr}^\circ \phi_{\pi_\mathfrak{p}}(\text{Frob}_\mathfrak{p}) \in \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell) && \text{(Prop. 10.3)}, \\
 \mathcal{N} \rho_\pi^C(\text{Frob}_\mathfrak{p}) &= \iota q_\mathfrak{p}^{n(n-1)/2} \mathcal{N} \phi_{\pi_\mathfrak{p}}(\text{Frob}_\mathfrak{p}) \in \mathbb{G}_m(\overline{\mathbb{Q}}_\ell) && \text{(claim above)}
 \end{aligned}
 \tag{10.24}$$

By [KS16, Lem. 1.1] a semi-simple element  $\gamma$  of  $\text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$  is determined up to conjugacy by the conjugacy classes of  $\text{spin}^+ \gamma, \text{spin}^- \gamma \in \text{GL}_{2n-1}$ ,  $\text{std} \gamma \in \text{GL}_{2n}$  and  $\mathcal{N} \gamma \in \mathbb{G}_m$ . We complete the proof by noting that the two sides of (10.20) become conjugate under  $\text{spin}^+$ ,  $\text{spin}^-$ ,  $\text{std}$ , and  $\mathcal{N}$  by (10.24).  $\square$

## 11. COMPATIBILITY AT UNRAMIFIED PLACES

We continue in the setup of §10 with the same running assumptions. We determined the image of Frobenius under  $\rho_\pi^C$  at each prime away from some finite set  $S$ . Now we compute the image at the finite places  $\mathfrak{p} \nmid \ell$  above  $S \setminus S_{\text{bad}}$ . The argument here follows very closely that of [KS16, Sect. 10]. In fact, it is directly copied in many places. We repeat it for completeness.

**Proposition 11.1.** *Let  $\mathfrak{p}$  be a prime of  $E$  not lying above  $S_{\text{bad}} \cup \{\ell\}$ . Then  $\rho_\pi^C$  is unramified at  $\mathfrak{p}$ . Moreover writing  $\mathfrak{q} := \mathfrak{p} \cap F$ ,*

$$\rho_\pi^C(\text{Frob}_\mathfrak{p})_{\text{ss}} \sim \iota q_\mathfrak{p}^{n(n-1)/4} \phi_{\pi_\mathfrak{q}}(\text{Frob}_\mathfrak{p}) \in \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell).$$

*Proof.* Fix  $\mathfrak{p}$  as in the statement. Let  $p$  denote the prime of  $\mathbb{Q}$  below  $\mathfrak{p}$ . Let  $\pi^\natural$  be a transfer of  $\pi$  to  $G(\mathbb{A}_F)$  from  $G^*(\mathbb{A}_F)$  given by [KS16, Prop 6.3(2)]. Then  $\pi^\natural$  is cuspidal automorphic and

- $\pi_{\mathfrak{q}'}^\natural \simeq \pi_{\mathfrak{q}'}$  at all finite primes  $\mathfrak{q}' \notin S_{\text{bad}}^F \cup \{\mathfrak{q}_{\text{St}}\}$ ,
- $\pi_{\mathfrak{q}_{\text{St}}}^\natural$  is the Steinberg representation up to an unramified twist,
- $\pi_\infty^\natural$  is  $\xi$ -cohomological.

Let  $B(\pi^\natural)$  be the set of cuspidal automorphic representations  $\tau$  of  $G(\mathbb{A}_F)$  such that

- $\tau_{\mathfrak{q}_{\text{St}}}$  and  $\pi_{\mathfrak{q}_{\text{St}}}^\natural$  are isomorphic up to a twist by an unramified character,
- $\tau^{\infty, \mathfrak{q}_{\text{St}}, p}$  and  $\pi_{\mathfrak{q}_{\text{St}}}^{\natural, \infty, \mathfrak{q}_{\text{St}}, p}$  are isomorphic,
- $\tau_p$  is unramified,
- $\tau_\infty$  is  $\xi$ -cohomological

We define an equivalence relation  $\approx$  on the set  $B(\pi^\natural)$  by declaring that  $\tau_1 \approx \tau_2$  if and only if  $\tau_2 \in A(\tau_1)$  (hence,  $\tau_1 \approx \tau_2$  if and only if  $\tau_{1, \mathfrak{q}} \simeq \tau_{2, \mathfrak{q}}$ ). For  $\varepsilon \in \{+, -\}$ , define (true) representations of  $\Gamma_E$  by  $\rho_B^{\text{Sh}, \varepsilon} := \sum_{\tau \in B(\pi^\natural)/\approx} \rho_\tau^{\text{Sh}, \varepsilon}$  (see Theorem 9.7). Put  $b(\pi^\natural) := \sum_{\tau \in B(\pi^\natural)/\approx} a(\tau) \in \mathbb{Z}_{>0}$ . Since  $\text{spin}^\varepsilon \circ \rho_{\pi^\natural}^C$  and  $\text{spin}^\varepsilon \circ \rho_\tau^C$  have the same Frobenius trace at almost all places for  $\tau \in B(\pi^\natural)$ , we deduce that  $\text{spin}^\varepsilon \circ \rho_{\pi^\natural}^C \simeq \text{spin}^\varepsilon \circ \rho_{\pi^\natural}^C$ . Hence

$$(11.1) \quad \rho_B^{\text{Sh}, \varepsilon} \simeq i_{b(\pi^\natural)} \circ \text{spin}^\varepsilon \circ \rho_{\pi^\natural}^C.$$

We adapt the argument of Theorem 9.7. Consider the function  $f$  on  $G(\mathbb{A}_F)$  of the form  $f = f_\infty f_{\mathfrak{q}_{\text{St}}} \mathbf{1}_{K_p} f^{\infty, \mathfrak{q}_{\text{St}}, p}$ , where  $f_\infty$  and  $f_{\mathfrak{q}_{\text{St}}}$  are as in that argument, and  $f^{\infty, \mathfrak{q}_{\text{St}}, p}$  is such that, for all automorphic representations  $\tau$  of  $G(\mathbb{A}_F)$  with  $(\tau^\infty)^K \neq 0$  and  $\text{Tr} \tau_\infty(f_\infty) \neq 0$ , we have:

$$(11.2) \quad \text{Tr} \tau^{\infty, \mathfrak{q}_{\text{St}}, p}(f^{\infty, \mathfrak{q}_{\text{St}}, p}) = \begin{cases} 1 & \text{if } \tau^{\infty, \mathfrak{q}_{\text{St}}, p} \simeq \pi_{\mathfrak{q}_{\text{St}}}^{\natural, \infty, \mathfrak{q}_{\text{St}}, p} \\ 0 & \text{otherwise.} \end{cases}$$

Arguing as in Theorem 9.7 we obtain

$$\iota^{-1} \text{Tr}(\text{Frob}_\mathfrak{p}^j, \rho_B^{\text{Sh}, \varepsilon}) = \sum_{\tau \in B(\pi^\natural)/\approx} a(\tau) \text{Tr} \tau_p(f_j) = \sum_{\tau \in B(\pi^\natural)/\approx} \iota a(\tau) q_\mathfrak{p}^{jn(n-1)/4} \text{Tr}(\text{spin}^\varepsilon(\phi_{\tau_\mathfrak{q}}))(\text{Frob}_\mathfrak{p}^j).$$

Thus the statement follows from Lemma 11.2 below. Indeed the lemma and the last equality imply that

$$\text{Tr} \rho_B^{\text{Sh}, \varepsilon}(\text{Frob}_\mathfrak{p}^j) = b(\pi^\natural) \cdot \iota q_\mathfrak{p}^{jn(n-1)/4} \text{Tr}(\text{spin}^\varepsilon(\phi_{\pi_\mathfrak{q}}))(\text{Frob}_\mathfrak{p}^j), \quad j \gg 1.$$

Comparing this with (11.1), we deduce that

$$\text{spin}^\varepsilon \circ \rho_\pi^C(\text{Frob}_\mathfrak{p})_{\text{ss}} \sim \iota q_\mathfrak{p}^{jn(n-1)/4} \text{spin}^\varepsilon(\phi_{\pi_\mathfrak{q}})(\text{Frob}_\mathfrak{p}).$$

Since we also know the conjugacy relation with  $\text{std}$  and  $\mathcal{N}$  in place of  $\text{spin}^\varepsilon$  from (10.9) and Proposition 10.5 (and the argument at (10.21) in its proof) we conclude as in the proof of Proposition 10.5 that

$$\rho_\pi^C(\text{Frob}_\mathfrak{p})_{\text{ss}} \sim \iota q_\mathfrak{p}^{n(n-1)/4} \phi_{\pi_\mathfrak{p}}(\text{Frob}_\mathfrak{p}) \in \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell).$$

□

**Lemma 11.2.** *With the above notation, if  $\tau \in B(\pi^\natural)$  then  $\tau_\mathfrak{q} \simeq \pi_\mathfrak{q}^\natural$  for all  $\mathfrak{q}$  not above  $S_{\text{bad}} \cup \{\ell\}$ .*

*Proof.* Since Xu's paper [Xu18, Thm. 1.8] is available also for the quasi-split outer forms of  $\text{GSO}_{2n}$ , the argument for [KS16, Lem. 10.1] goes through unchanged, except Corollary 9.8 replaces [KS16, Cor. 8.5], and Lemma 5.3 replaces [KS16, Lem. 5.2]. □

## 12. THE MAIN THEOREM

In this section we prove Theorem A (Theorem 12.5), the main result of this paper. Before doing this we first explain how the  $\Gamma_E$ -representation  $\rho_\pi$  from previous section can be extended to a representation of  $\Gamma$ .

As in Theorem A, let  $\pi$  be a cuspidal automorphic representation of  $G^*(\mathbb{A}_F)$  satisfying (St) and (L-coh). Fix a cuspidal automorphic representation  $\pi^\flat$  of  $\text{SO}_{2n}^{E/F}(\mathbb{A}_F)$  which embeds in  $\pi|_{\text{SO}_{2n}^{E/F}(\mathbb{A}_F)}$  as it is possible by [LS19]. Assume either (std-reg) for  $\pi$  or Hypothesis 6.10 for an  $\text{SO}_{2n}(\mathbb{A}_F)$ -subrepresentation  $\pi^\flat$  of  $\pi$ . Define  $\tilde{\pi} := \pi|_{\text{sim}}^{n(n-1)/4}$ . Then  $\tilde{\pi}$  is  $\xi$ -cohomological and will play the role of  $\pi$  in Sections 10 and 11. Naturally  $\pi^\flat$  is a subrepresentation of  $\tilde{\pi}|_{\text{SO}_{2n}(\mathbb{A}_F)}$  since  $|\text{sim}|$  is trivial when restricted to  $\text{SO}_{2n}(\mathbb{A}_F)$ .

Let  $S^F$  (resp.  $S^E$ ) be the finite set of places of  $F$  (resp.  $E$ ) above  $S := S_{\text{bad}} \cup \{\ell\}$ . From Propositions 10.1 and 11.1, we obtain

$$\rho_\pi^C : \Gamma_{E,S} \rightarrow \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$$

such that for every  $\mathfrak{p} \notin S^E$ , writing  $\mathfrak{q} := \mathfrak{p}|_F$ , we have

$$(12.1) \quad \rho_\pi^C(\text{Frob}_\mathfrak{p})_{\text{ss}} \sim \iota q_\mathfrak{p}^{n(n-1)/4} \phi_{\pi_\mathfrak{q}}(\text{Frob}_\mathfrak{p}) = \iota \phi_{\pi_\mathfrak{q}}(\text{Frob}_\mathfrak{p}).$$

Let us explain the definition of  $\rho_\pi$  on  $\Gamma_{F,S}$ . If  $n$  is even (thus  $E = F$ ) then we simply take  $\rho_\pi := \rho_\pi^C$ . In case  $n$  is odd (so  $[E : F] = 2$ ), fix an infinite place  $y$  of  $F$  and write  $c_y \in \Gamma$  for the corresponding complex conjugation (canonical up to conjugacy). In order to apply Lemma A.1, we check

**Lemma 12.1.** *When  $n$  is odd, we have  $c_y \rho_\pi^C \simeq \theta \circ \rho_\pi^C$ .*

*Proof.* In light of Proposition 5.4, it is enough to check this locally, namely that

$$\rho_\pi^C(c_y \text{Frob}_\mathfrak{p} c_y^{-1})_{\text{ss}} \sim \theta \circ \rho_\pi^C(\text{Frob}_\mathfrak{p})_{\text{ss}} \quad \text{in } \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$$

for almost all primes  $\mathfrak{p}$  of  $E$ . For each  $\mathfrak{p}$ , write  $\mathfrak{q} := \mathfrak{p} \cap F$ . Firstly if  $\mathfrak{q}$  splits in  $E$  as  $\mathfrak{p}c(\mathfrak{p})$  then we use (12.1) to deduce that

$$\rho_\pi^C(c_y \text{Frob}_\mathfrak{p} c_y^{-1})_{\text{ss}} \sim \rho_\pi^C(\text{Frob}_{c(\mathfrak{p})})_{\text{ss}} \sim \iota \phi_{\pi_\mathfrak{q}}(\text{Frob}_{c(\mathfrak{p})}) \sim \iota \theta(\phi_{\pi_\mathfrak{q}}(\text{Frob}_\mathfrak{p})) \sim \theta(\rho_\pi^C(\text{Frob}_\mathfrak{p})).$$

(To see the third conjugacy relation, we argue as follows. From (8.4) we see that an element of  $\text{GSO}_{2n, F_\mathfrak{q}}^{E/F}$  has the form  $(g, \theta(g))$  with  $g \in \text{GSO}_{2n, E_\mathfrak{p}}$  and that  $\text{GSO}_{2n, F_\mathfrak{q}}^{E/F}$  is isomorphic to  $\text{GSO}_{2n, E_\mathfrak{p}}$  and  $\text{GSO}_{2n, E_{c(\mathfrak{p})}}$  by the projection map onto the first and second components, respectively.

Likewise the dual group of  $\text{GSO}_{2n, F_\mathfrak{q}}^{E/F}$  is naturally the subgroup of  $\text{GSpin}_{2n} \times \text{GSpin}_{2n}$  consisting of elements of the form  $(g, \theta(g))$ , the two components corresponding to  $\mathfrak{p}$  and  $c(\mathfrak{p})$ . It follows that  $\phi_{\pi_\mathfrak{q}}(\text{Frob}_{c(\mathfrak{p})}) \sim \theta(\phi_{\pi_\mathfrak{q}}(\text{Frob}_\mathfrak{p}))$ .)

Secondly if  $\mathfrak{q}$  is inert in  $E$  then  $c_y \text{Frob}_{\mathfrak{p}} c_y^{-1} \sim \text{Frob}_{\mathfrak{p}}$ . Thus we need to check that the conjugacy class of  $\rho_{\pi}^C(\text{Frob}_{\mathfrak{p}})_{\text{ss}}$  is  $\theta$ -invariant. Writing  $\theta(\phi_{\pi_{\mathfrak{q}}}(\text{Frob}_{\mathfrak{q}})) = s \rtimes c \in \text{GSpin}_{2n}(\overline{\mathbb{Q}}_{\ell}) \rtimes \Gamma_{E/F}$ ,

$$\theta(\phi_{\pi_{\mathfrak{q}}}(\text{Frob}_{\mathfrak{p}})) \sim \theta(\phi_{\pi_{\mathfrak{q}}}(\text{Frob}_{\mathfrak{q}}^2)) = s\theta(s) \sim \theta(s)s \quad \text{in } \text{GSpin}_{2n}(\overline{\mathbb{Q}}_{\ell}).$$

This implies the desired  $\theta$ -invariance via (12.1). The proof is complete.  $\square$

We are assuming that  $n$  is odd. By Lemmas 12.1 and A.1, we extend  $\rho_{\pi}^C$  to a Galois representation to be denoted  $\rho_{\pi}$ :

$$(12.2) \quad \rho_{\pi} : \Gamma_{F,S} \rightarrow \text{GSpin}_{2n}(\overline{\mathbb{Q}}_{\ell}) \rtimes \Gamma_{E/F}.$$

There are two choices up to conjugacy (Example A.6). We choose one arbitrarily and possibly modify the choice below.

We return to treating both parities of  $n$ . We fixed  $\pi^{\flat}$  above. Theorem 6.3, or Hypothesis 6.10 if (std-reg) is not assumed, supplies us with

$$\rho_{\pi^{\flat}} : \Gamma_{F,S} \rightarrow \text{SO}_{2n}(\overline{\mathbb{Q}}_{\ell}) \rtimes \Gamma_{E/F}$$

such that  $\rho_{\pi^{\flat}}(\text{Frob}_{\mathfrak{q}})_{\text{ss}} \overset{\text{out}}{\sim} \iota \phi_{\pi_{\mathfrak{q}}^{\flat}}(\text{Frob}_{\mathfrak{q}})$  for  $\mathfrak{q}, \mathfrak{p}$  as above. Thanks to (12.1) and the unramified Langlands functoriality with respect to  $\text{SO}_{2n} \rightarrow \text{GSO}_{2n}^{E/F}$  (whose dual morphism is  $\text{pr}^{\circ}$ ),

$$\rho_{\pi^{\flat}}(\text{Frob}_{\mathfrak{p}})_{\text{ss}} \overset{\text{out}}{\sim} \iota \phi_{\pi_{\mathfrak{q}}^{\flat}}(\text{Frob}_{\mathfrak{p}}) \sim \iota \text{pr}^{\circ}(\phi_{\pi_{\mathfrak{q}}}(\text{Frob}_{\mathfrak{p}})) \sim \text{pr}^{\circ}(\rho_{\pi}(\text{Frob}_{\mathfrak{p}})_{\text{ss}}).$$

Thus the conjugacy classes at the left and right ends are  $\text{O}_{2n}(\overline{\mathbb{Q}}_{\ell})$ -conjugate, under the identification  $\text{SO}_{2n}(\overline{\mathbb{Q}}_{\ell}) \rtimes \Gamma_{E/F} = \text{O}_{2n}(\overline{\mathbb{Q}}_{\ell})$ . Since  $\text{O}_{2n}$  is acceptable,  $\rho_{\pi^{\flat}}|_{\Gamma_{E,S}}$  and  $\text{pr}^{\circ} \circ \rho_{\pi}|_{\Gamma_{E,S}}$  are  $\text{O}_{2n}(\overline{\mathbb{Q}}_{\ell})$ -conjugate. Replacing  $\rho_{\pi^{\flat}}$  by an  $\text{O}_{2n}(\overline{\mathbb{Q}}_{\ell})$ -conjugate, we may and will assume that

$$\rho_{\pi^{\flat}}|_{\Gamma_{E,S}} = \text{pr}^{\circ} \circ \rho_{\pi}|_{\Gamma_{E,S}}$$

without disturbing the validity of (SO-i) through (SO-v) in Theorem 6.3. When  $n$  is odd, we take an extra step as follows. Observe that  $\rho_{\pi^{\flat}}$  and  $\text{pr}^{\circ} \circ \rho_{\pi}$  are two  $\text{SO}_{2n}(\overline{\mathbb{Q}}_{\ell}) \rtimes \Gamma_{E/F}$ -valued representations of  $\Gamma_{F,S}$  extending (12.3). If they are not equal then  $\text{pr}^{\circ} \circ \rho_{\pi} = \rho_{\pi^{\flat}} \otimes \chi_{E/F}$  by Example A.5 with  $\chi_{E/F} : \Gamma_F \rightarrow \Gamma_{E/F} \xrightarrow{\sim} \{\pm 1\}$ . Then we go back to (12.2) and replace  $\rho_{\pi}$  with  $\rho_{\pi} \otimes \chi$ , where  $\chi$  is as in Example A.6; this does not affect the discussion between (12.2) and here. Since  $\text{pr}^{\circ} \circ \chi = \chi_{E/F}$ , this ensures that

$$(12.3) \quad \rho_{\pi^{\flat}} = \text{pr}^{\circ} \circ \rho_{\pi}.$$

As in §2, let  $(s_0, s_1, \dots, s_n) \in (\overline{\mathbb{Q}}_{\ell}^{\times})^{n+1}$  denote an element of  $T_{\text{GSpin}}(\overline{\mathbb{Q}}_{\ell}) \subset \text{GSpin}_{2n}(\overline{\mathbb{Q}}_{\ell})$ . This element maps to  $\text{diag}(s_1, \dots, s_n, s_1^{-1}, \dots, s_n^{-1}) \in \text{SO}_{2n}(\overline{\mathbb{Q}}_{\ell})$  under  $\text{pr}^{\circ}$ , and maps to  $s_0^2 s_1 s_2 \cdots s_n$  under the spinor norm  $\mathcal{N}$ .

**Lemma 12.2.** *At every infinite place  $y$  of  $F$ , the following are  $\text{GSpin}_{2n}(\overline{\mathbb{Q}}_{\ell})$ -conjugate:*

$$(12.4) \quad \rho_{\pi}(c_y) \sim \begin{cases} (a, \underbrace{1, \dots, 1}_{n/2}, \underbrace{-1, \dots, -1}_{n/2}), & a \in \{\pm 1\}, \quad n : \text{even}, \\ (1, \underbrace{1, \dots, 1}_{(n-1)/2}, \underbrace{-1, \dots, -1}_{(n-1)/2}, 1) \rtimes c, & n : \text{odd}. \end{cases}$$

*Proof.* In light of (12.3) (which is valid for both odd and even  $n$  as discussed above) and Theorem 6.3 (SO-v) (or Hypothesis 6.10) which describes  $\rho_{\pi^{\flat}}(c_y)$ , the following are  $\text{GSpin}_{2n}(\overline{\mathbb{Q}}_{\ell})$ -conjugate:

$$\text{pr}^{\circ}(\rho_{\pi}(c_y)) \sim \begin{cases} \text{diag}(\underbrace{1, \dots, 1}_{n/2}, \underbrace{-1, \dots, -1}_{n/2}, \underbrace{1, \dots, 1}_{n/2}, \underbrace{-1, \dots, -1}_{n/2}), & n : \text{even}, \\ \text{diag}(\underbrace{1, \dots, 1}_{(n-1)/2}, \underbrace{-1, \dots, -1}_{(n-1)/2}, 1, \underbrace{1, \dots, 1}_{(n-1)/2}, \underbrace{-1, \dots, -1}_{(n-1)/2}, 1) \rtimes \theta, & n : \text{odd}. \end{cases}$$

Therefore  $\rho_{\pi}(c_y)$  is a lift of the right hand side (up to  $\text{GSpin}_{2n}(\overline{\mathbb{Q}}_{\ell})$ -conjugacy) via  $\text{pr}^{\circ}$ . Moreover  $\rho_{\pi}(c_y)^2 = \rho_{\pi}(c_y^2) = 1$ . We claim that these two conditions imply (12.4).

This is straightforward when  $n$  is even. Now suppose that  $n$  is odd. Evidently the right hand side of (12.4) satisfies the two conditions. Any other lift of order 2 can only differ (possibly after conjugation) from the right hand side of (12.4) by scalars  $\{\pm 1\}$ . (Use Lemma 3.1 (ii) and the order two condition.) This implies (12.4) since every  $g \in \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes c$  is conjugate to  $-g$ ; indeed,  $-g = \zeta g \zeta^{-1}$  if  $\zeta \in Z_{\mathrm{Spin}}(\overline{\mathbb{Q}}_\ell)$  is an element of order 4, noting that  $\theta(\zeta) = \zeta^{-1}$ .  $\square$

Let  $\omega_\pi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$  denote the central character of  $\pi$ . By abuse of notation, we still write  $\omega_\pi$  (depending on the choice of  $\iota$ ) for the  $\ell$ -adic character of  $\Gamma_F$  corresponding to  $\omega_\pi$  via class field theory (as in [HT01, pp.20–21]). To make  $\omega_\pi$  explicit, recall that  $\tilde{\pi} = \pi|_{\mathrm{sim}^{n(n-1)/4}}$  is  $\xi$ -cohomological. By condition (cent), the central character of  $\xi$  is  $z \mapsto z^w$  on  $F_y^\times$  at every real place  $y$  of  $F$ , for an integer  $w$  independent of  $y$ . Therefore (recalling  $\mathrm{sim}$  is the squaring map on the center)

$$\omega_{\pi,y}(z) = z^{-w} |z|^{n(n-1)/2} = \mathrm{sgn}(z)^w |z|^{w+n(n-1)/2}, \quad z \in F_y^\times.$$

Then  $\omega_\pi|_{\cdot|_{\mathbb{A}_F^\times}}^{-w-n(n-1)/4}$  is a finite-order Hecke character which is  $\mathrm{sgn}^w$  at every real place.

Hence  $\omega_\pi = \chi_{\mathrm{cyc}}^{-w-n(n-1)/2} \chi_0$ , where  $\chi_{\mathrm{cyc}}$  is the  $\ell$ -adic cyclotomic character, and  $\chi_0$  a finite-order character with  $\chi_0(c_y) = (-1)^w$  at each real place  $y$ . The upshot is that

$$(12.5) \quad \omega_\pi(c_y) = (-1)^{-w-n(n-1)/2} (-1)^w = (-1)^{n(n-1)/2}, \quad y : \text{real place of } F.$$

We are ready to upgrade (12.1) to a compatibility at places of  $F$  for odd  $n$  (thus  $[E : F] = 2$ ).

**Corollary 12.3.** *We have  $\mathcal{N} \circ \rho_\pi = \omega_\pi$ . Moreover, at every finite place  $\mathfrak{q}$  of  $F$  not above  $S_{\mathrm{bad}} \cup \{\ell\}$ ,*

$$\rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}} \sim \iota \phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q}).$$

*Remark 12.4.* The corollary is certainly not automatic from (12.1) since the unramified base change from  $G^*(F_\mathfrak{q})$  to  $G^*(E_\mathfrak{p})$  is not injective when  $\mathfrak{q}$  does not split in  $E$ . Curiously our proof crucially relies on the image of complex conjugation. We have not found a local or global proof only using properties at finite places.

*Proof.* Via the unramified Langlands functoriality with respect to the central embedding  $\mathbb{G}_m \hookrightarrow \mathrm{GSO}_{2n}^{E/F}$ , (12.1) implies that  $\mathcal{N} \circ \rho_\pi|_{\Gamma_E} = \omega_\pi|_{\Gamma_E}$ . If  $n$  is even then  $E = F$  so there is no more to prove as the latter assertion is already true by (12.1).

Henceforth assume that  $n$  is odd (so  $[E : F] = 2$ ). Then either  $\mathcal{N} \circ \rho_\pi = \omega_\pi$  or  $\mathcal{N} \circ \rho_\pi = \omega_\pi \otimes \chi_{E/F}$ , where  $\chi_{E/F} : \Gamma_F \twoheadrightarrow \Gamma_{E/F} \xrightarrow{\sim} \{\pm 1\}$ . To exclude the latter case, let  $y$  be a real place of  $F$ . We have  $\mathcal{N}(\rho_\pi(c_y)) = (-1)^{(n-1)/2}$  from Lemma 12.2, and  $\omega_\pi(c_y) = 1$  from (12.5), but clearly  $\chi_{E/F}(c_y) = -1$ . Then the only possibility is that  $\mathcal{N} \circ \rho_\pi = \omega_\pi$ .

We prove the second assertion. If  $\mathfrak{q}$  splits in  $E$ , this follows immediately from (12.1) for  $\rho_\pi|_{\Gamma_{E,S}}$ . Henceforth assume that  $\mathfrak{q}$  is inert in  $E$ . We have seen that  $\mathrm{pr}^\circ \circ \rho_\pi|_{\Gamma_{E,S}} = \rho_{\pi^\circ}|_{\Gamma_{E,S}}$ . Theorem 6.3 (SO-i) (or Hypothesis 6.10) tells us that

$$\rho_{\pi^\circ}(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}} \sim \iota \phi_{\pi^\circ_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q}) = \iota \mathrm{pr}^\circ(\phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q})).$$

(Note that the outer automorphism ambiguity disappears as it is absorbed by the  $\mathrm{SO}_{2n}$ -conjugacy on the nontrivial coset of  $\mathrm{SO}_{2n} \rtimes \Gamma_{E/F}$ ; since  $\mathfrak{q}$  is inert in  $E$ , the image of  $\mathrm{Frob}_\mathfrak{q}$  in  $\Gamma_{E/F}$  is nontrivial.) Therefore  $\rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}} \sim z \iota \phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q})$  for some  $z \in \overline{\mathbb{Q}}_\ell^\times$ . Taking the spinor norm,

$$\mathcal{N}(z) = (\mathcal{N} \circ \rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}})^{-1} \mathcal{N}(\iota \phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q})) = \omega_\pi(\mathrm{Frob}_\mathfrak{q})^{-1} \omega_\pi(\mathrm{Frob}_\mathfrak{q}) = 1.$$

It follows that  $z \in \{\pm 1\}$ . Since every  $g \in \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes c$  is conjugate to  $-g$  (proof of Lemma 12.2), we conclude that  $\rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}}$  is conjugate to  $\iota \phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q})$ .  $\square$

**Theorem 12.5.** *Theorem A is true.*

*Proof.* Let  $\pi$  be as in the theorem. We fix an automorphic representation  $\pi^b$  of  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$  in  $\pi|_{\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)}$ , take  $\rho_{\pi^b} : \Gamma_F \rightarrow \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$  to be as in Theorem 6.3, or Hypothesis 6.10 if (std-reg) is false, and define

$$(12.6) \quad \rho_\pi : \Gamma_F \rightarrow \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$$

such that  $\rho_{\pi^b} = \mathrm{pr}^\circ \circ \rho_\pi$  as explained at the start of this section. By inflating  $\rho_\pi$  to a representation  $\Gamma_F \rightarrow \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_F$  of Theorem A, but we work with  $\rho_\pi$  in the form of (12.6) as this is harmless for verifying Theorem A.

The equality  $\rho_{\pi^b} = \mathrm{pr}^\circ \circ \rho_\pi$  and Corollary 12.3 imply (A2). Corollary 12.3 exactly gives (A1). Item (A4) is straightforward from Lemma 12.2. To see (A5), note that the image of  $\rho_\pi$  in  $\mathrm{PSO}_{2n}(\overline{\mathbb{Q}}_\ell)$  is the same as the image of  $\rho_{\pi^b}$  in the same group. The Zariski closure of the image is (possibly disconnected and) reductive since  $\rho_{\pi^b}$  is semisimple and contains a regular unipotent element by Corollary 6.7. Hence (A5) is implied by Proposition 5.2. Now  $\rho_\pi$  also contains a regular unipotent in the image, so (A6) and the uniqueness of  $\rho_\pi$  up to conjugacy are consequences of Proposition 5.4.

It remains to verify (A3). We begin with part (b). If  $\pi_{\mathfrak{q}}$  has nonzero invariants under a hyperspecial (resp. Iwahori) subgroup, then  $\pi_{\mathfrak{q}}^b$  and  $\omega_{\pi, \mathfrak{q}}$  enjoy the same property. Therefore (b) follows from (A2) and Theorem 6.3 (SO-iv). To prove part (c), write  $\mathfrak{p}$  for a place of  $E$  above  $\mathfrak{q}$ . Since  $\mathfrak{p}$  is unramified over  $E$ , it suffices to check that  $\rho_\pi|_{\Gamma_E}$  is crystalline at  $\mathfrak{p}$ . Moreover we may assume  $F \neq \mathbb{Q}$  by base change. (If  $F = \mathbb{Q}$  then replace  $F$  with a real quadratic field  $F'$  unramified at  $\ell$ , and  $E$  with  $EF'$ .) Then the Shimura varieties in §9 are proper, and  $\rho_{\pi_{\mathfrak{q}}}^{\mathrm{Sh}, \epsilon}$  is crystalline at all places above  $\ell$  by [Lov17]. Since  $\mathrm{spin} \circ \rho_\pi|_{\Gamma_E}$  embeds in  $\rho_{\pi_{\mathfrak{q}}}^{\mathrm{Sh}, +} \oplus \rho_{\pi_{\mathfrak{q}}}^{\mathrm{Sh}, -}$  (which is isomorphic to the  $a(\pi^b)$ -fold direct sum of  $\mathrm{spin} \circ \rho_\pi$ ), and since  $\mathrm{spin}$  is faithful, we deduce that  $\rho_\pi|_{\Gamma_E}$  is crystalline at  $\mathfrak{p}$  as desired.

Finally we prove (A3), part (a). The new input is the claim that

$$(12.7) \quad \mathrm{spin}^\epsilon(\mu_{\mathrm{HT}}(\rho_{\pi, \mathfrak{q}}, \nu y)) \sim \mathrm{spin}^\epsilon(\nu \mu_{\mathrm{Hodge}}(\xi_y)), \quad \epsilon \in \{\pm\}.$$

Accept this for now. It follows easily from (A2) and Theorem 6.3 (SO-iii) that

$$\mu_{\mathrm{HT}}(\rho_{\pi, \mathfrak{q}}, \nu y) \stackrel{\mathrm{out}}{\sim} \nu \mu_{\mathrm{Hodge}}(\xi_y).$$

If they are  $\mathrm{GSpin}_{2n}$ -conjugate, we are done with (a). Otherwise, we may assume

$$\mu_{\mathrm{HT}}(\rho_{\pi, \mathfrak{q}}, \nu y), \nu \mu_{\mathrm{Hodge}}(\xi_y) \in X_*(T_{\mathrm{GSpin}}), \quad \nu \mu_{\mathrm{Hodge}}(\xi_y) = \theta(\mu_{\mathrm{HT}}(\rho_{\pi, \mathfrak{q}}, \nu y))$$

after conjugation. Then (12.7) would imply that  $\mathrm{spin}^+(\mu_{\mathrm{Hodge}}(\xi_y)) \sim \mathrm{spin}^-(\mu_{\mathrm{Hodge}}(\xi_y))$  but this is impossible in view of (std-reg) by comparing the highest weights described in (2.8). (The highest weights differ only in the  $s_n$ -coordinate, but (std-reg) tells us that the Hodge cocharacter has nontrivial  $s_n$ -coordinate.) To complete the proof of (a), we check the claim (12.7). It follows from Proposition 9.9 that for every  $y : F \hookrightarrow \overline{\mathbb{Q}}_\ell$ ,

$$\mu_{\mathrm{HT}}(\mathrm{spin}^\epsilon \circ \rho_\pi, \nu y) \sim \mathrm{spin}^\epsilon \circ \left( \mu_{\mathrm{Hodge}}(\xi_y) + \frac{n(n-1)}{4} \mathrm{sim} \right),$$

and the left hand side equals  $\mathrm{spin}^\epsilon \circ \mu_{\mathrm{HT}}(\rho_\pi, \nu y)$  by the construction of Hodge–Tate cocharacters. On the other hand, (A2) and (SO-iii) (and the analogue of the latter for the group  $\mathrm{GL}_1$ ) tells us that

$$\begin{aligned} \mathrm{std}^\circ \circ \mu_{\mathrm{HT}}(\rho_\pi, \nu y) &\sim \mathrm{std}^\circ \circ \left( \mu_{\mathrm{Hodge}}(\xi_y) + \frac{n(n-1)}{4} \mathrm{sim} \right), \\ \mathcal{N} \circ \mu_{\mathrm{HT}}(\rho_\pi, \nu y) &\sim \mathcal{N} \circ \left( \mu_{\mathrm{Hodge}}(\xi_y) + \frac{n(n-1)}{4} \mathrm{sim} \right). \end{aligned}$$

Therefore it boils down to the following assertion: if  $\mu_1, \mu_2 \in X_*(T_{\mathrm{GSpin}})$  becomes conjugate after composition with each of  $\mathrm{spin}^+$ ,  $\mathrm{spin}^-$ ,  $\mathrm{std}^\circ$ , and  $\mathcal{N}$ . then  $\mu_1$  and  $\mu_2$  are  $\mathrm{GSpin}_{2n}$ -conjugate. To see this, note that a semi-simple conjugacy class  $\gamma$  in  $\mathrm{GSpin}_{2n}(\mathbb{C})$  is determined by the conjugacy classes  $\mathrm{spin}^\pm(\gamma)$ ,  $\mathcal{N}(\gamma)$  and  $\mathrm{std}(\gamma)$  by the table above Lemma 1.1 and Lemma 1.3(5) of [KS16].

The same statement holds for the cocharacters via the Weyl group-equivariant isomorphism  $X_*(T_{\mathrm{GSpin}}) \otimes_{\mathbb{Z}} \mathbb{C}^\times \rightarrow T_{\mathrm{GSpin}}(\mathbb{C})$ .  $\square$

*Remark 12.6.* Lemma 12.2 tells us that  $\rho_\pi$  is totally odd. Our result also shows that  $\rho_\pi(c_y)$  is as predicted by [BG14, Conj. 3.2.1, 3.2.2] for every infinite place  $y$  of  $F$ . Indeed, as explained in §6 of their paper, their conjectures are compatible with the functoriality. Considering the  $L$ -morphism  ${}^L\mathrm{GSO}_{2n}^{E/F} \rightarrow {}^L\mathrm{SO}_{2n}^{E/F}$  dual to the inclusion  $\mathrm{SO}_{2n}^{E/F} \hookrightarrow \mathrm{GSO}_{2n}^{E/F}$ , we reduce the question to the case of  $\mathrm{SO}_{2n}^{E/F}$  in view of the characterization of  $\rho_\pi(c_y)$  in terms of  $\mathrm{pr}^\circ(\rho_{\pi_\pi}(c_y))$ . The latter is conjugate to  $\rho_{\pi^b}$ , which is as conjectured by *loc. cit.* by Remark 6.4.

*Remark 12.7.* It was easier to determine the Hodge–Tate cocharacter in the  $\mathrm{GSp}$ -case [KS16], thanks to the absence of nontrivial outer automorphisms. In particular we did not need to prove the analogue of Proposition 9.9. Compare with the proof of Theorem 9.1 (iii.a') of *loc. cit.*

### 13. REFINEMENT FOR $\mathrm{SO}_{2n}$ -VALUED GALOIS REPRESENTATIONS

As an application of our results we improve in this section upon Theorem 6.3 by removing the outer-ambiguity of the Frobenius conjugacy classes.

Let  $E/F$  be a quadratic CM extension of  $F$  in case  $n$  is odd, and  $E := F$  for  $n$  even. Let  $\mathrm{SO}_{2n}^{E/F}$  be the corresponding group defined above (6.2). If  $\pi^b$  (resp.  $\pi$ ) is an automorphic representation of  $\mathrm{SO}_{2n}^{E/F}$  we write  $S_{\mathrm{bad}}(\pi^b)$  (resp.  $S_{\mathrm{bad}}(\pi)$ ) for the set of rational prime numbers  $p$ , such that  $p = 2$ ,  $p$  ramifies in  $E$ , or  $\pi_p^b$  (resp.  $\pi_p$ ) is a ramified representation of  $\mathrm{SO}_{2n}^{E/F}$  (resp.  $\mathrm{GSO}_{2n}^{E/F}$ ). For other notation, we refer to Section 1.

In order to be able to extend a given cohomological representation  $\pi^b$  of  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$  to a *cohomological* representation  $\pi$  of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ , the following condition on the central character  $\omega_{\pi^b} =: \mu_2(F) \backslash \mu_2(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$  is necessary in view of condition (cent) of §9. (If  $\pi$  is  $\xi$ -cohomological with  $w \in \mathbb{Z}$  as in (cent) then all  $\omega_{\pi^b, y}$  are trivial, resp. nontrivial, according as  $w$  is even, resp. odd.)

(**cent** $^\circ$ ) The sign character  $\omega_{\pi^b, y} : \mu_2(F_y) = \{\pm 1\} \rightarrow \mathbb{C}^\times$  does not depend on  $y | \infty$ .

**Theorem 13.1.** *Let  $\pi^b$  be a cuspidal automorphic representation of  $\mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)$  satisfying (cent $^\circ$ ), (coh $^\circ$ ), (St $^\circ$ ), and (std-reg $^\circ$ ) of §6. Then there exists a semisimple Galois representation (depending on  $\iota$ )*

$$\rho_{\pi^b} = \rho_{\pi^b, \iota} : \Gamma_F \rightarrow \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$$

satisfying (SO-i)–(SO-v) as in Theorem 6.3 as well as the following.

(SO-i+) For every finite prime  $\mathfrak{q}$  of  $F$  not above  $S_{\mathrm{bad}}(\pi^b) \cup \{\ell\}$ ,

$$\iota \phi_{\pi^b, \mathfrak{q}} \sim \mathrm{WD}(\rho_{\pi^b} |_{\Gamma_{F\mathfrak{q}}})^{\mathrm{F-ss}},$$

as  $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ -parameters.

(SO-iii+) For every  $\mathfrak{q} | \ell$ , the representation  $\rho_{\pi^b, \mathfrak{q}}$  is potentially semistable. For each  $y : F \hookrightarrow \mathbb{C}$  such that  $\iota y$  induces  $\mathfrak{q}$ , we have  $\mu_{\mathrm{HT}}(\rho_{\pi^b, \mathfrak{q}}, \iota y) \sim \iota \mu_{\mathrm{Hodge}}(\xi^b, y)$ .

Condition (SO-i+) characterizes  $\rho_{\pi^b}$  uniquely up to  $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ -conjugation.

*Remark 13.2.* Statement (SO-i+) is stronger than (SO-i) in that the statement is up to  $\mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ -conjugacy, but also weaker as it excludes the places above  $S_{\mathrm{bad}}(\pi^b) \cup \{\ell\}$ . Clearly (SO-iii+) strengthens (SO-iii). If we drop (std-reg $^\circ$ ) from the assumption, then the theorem can be proved by the same argument but conditionally on Hypothesis 6.10.

*Proof of Theorem 13.1.* We have  $\mu_2 = Z(\mathrm{SO}_{2n}^{E/F})$ . We claim that the central character  $\omega_{\pi^b}$  extends (via  $\mu_2(\mathbb{A}_F) \subset \mathbb{A}_F^\times$ ) to a Hecke character

$$\chi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$$

such that  $\chi_y(z) = z^w$  at every infinite place  $y$  for some  $w \in \{0, 1\}$  that is independent of  $y$ . Indeed,  $\omega_{\pi^b}$  easily extends to a quadratic Hecke character  $\chi' : F^\times \backslash \mathbb{A}_F^\times \rightarrow \{\pm 1\}$ . Take  $w = 0$  if  $\omega_{\pi^b}$  is trivial at infinite places and  $w = 1$  otherwise. Denoting by  $|\cdot|$  the absolute value character on  $F^\times \backslash \mathbb{A}_F^\times$ , we see that  $\chi := \chi' |\cdot|^w$  is a desired character.

Consider the multiplication map  $f : \mathrm{GL}_1 \times \mathrm{SO}_{2n}^{E/F} \rightarrow \mathrm{GSO}_{2n}^{E/F}$ . Let  $\xi^b$  be such that  $\pi^b$  is  $\xi^b$ -cohomological. Write  $\varsigma$  for the algebraic character  $z \mapsto z^w$  of  $\mathrm{GL}_1$  over  $F$ . Then  $(\varsigma, \xi^b)$  descends to an algebraic representation  $\xi$  of  $\mathrm{GSO}_{2n}^{E/F}$ .

Let us extend  $\pi^b$  to an irreducible admissible  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$ -representation, by decomposing  $\pi^b = \otimes'_v \pi_v^b$  and taking an irreducible subrepresentation  $\pi_v$  of

$$\mathrm{Ind}_{\mathrm{GL}_1(F_v)\mathrm{SO}_{2n}^{E/F}(F_v)}^{\mathrm{GSO}_{2n}^{E/F}(F_v)} \chi_v \pi_v^b,$$

which is semisimple [Xu16, pp.1832–1833]. Take  $\pi_v$  to be unramified for almost all  $v$ , and define  $\pi := \otimes'_v \pi_v$ . Lemma 5.4 of [Xu18] states that

$$\sum_{\omega \in X/YX(\pi)} m(\pi \otimes \omega) = \sum_{g \in \mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)/\tilde{G}(\pi)\mathrm{GSO}_{2n}^{E/F}(F)} m(\pi^g),$$

where we refer to *loc. cit.* for some undefined notation that is not important for us. Observe that for  $g = 1$  the right hand side is positive. Thus the left hand side is positive, and thus we may (and do) twist  $\pi$  so that it is discrete automorphic.

We now check that  $\pi$  satisfies the conditions of Theorem A. Since  $\pi_\infty$  has the same central and infinitesimal characters as  $\xi^\vee$ , by construction  $\pi_\infty$  is cohomological according to Lemma 7.2. By Lemma 7.1,  $\pi$  satisfies (St) thus also cuspidal. Condition (std-reg) is implied by (std-reg $^\circ$ ) on  $\pi^b$ . Hence we have a Galois representation

$$\rho_\pi : \Gamma \rightarrow \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}.$$

such that for every finite  $F$ -place  $\mathfrak{q}$  not above  $S_{\mathrm{bad}}(\pi) \cup \{\ell\}$ ,

$$(13.1) \quad \rho_\pi(\mathrm{Frob}_\mathfrak{q})_{\mathrm{ss}} \sim \iota \phi_{\pi_\mathfrak{q}}(\mathrm{Frob}_\mathfrak{q}) \in \mathrm{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}.$$

As in the preceding section, we can arrange that  $\rho_{\pi^b} = \mathrm{pr}^\circ \circ \rho_\pi$  (not just up to outer automorphism). The Satake parameter of  $\pi_\mathfrak{q}^b$  is equal to the composition of the Satake parameter of  $\pi_\mathfrak{q}$  with the natural surjection (cf. [Xu18, Lem. 5.2])

$$(\mathrm{pr}^\circ, \mathrm{id}) : \mathrm{GSpin}_{2n}(\mathbb{C}) \rtimes \Gamma \rightarrow \mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma.$$

In particular (SO- $i+$ ) follows from (13.1) for the places not above  $S_{\mathrm{bad}}(\pi) \cup \{\ell\}$ . Similarly (SO- $iii+$ ) follows from Theorem A (A3)(a).

At this point we have not yet completely proved (SO- $i+$ ), as the inclusion  $S_{\mathrm{bad}}(\pi^b) \subset S_{\mathrm{bad}}(\pi)$  is strict in general. Thus it remains to treat  $\mathfrak{q}$  above a prime  $p \in S_{\mathrm{bad}}(\pi) \setminus S_{\mathrm{bad}}(\pi^b)$ . Consider for  $n$  odd (resp. even) the obvious hyperspecial subgroup (recall  $\mathfrak{q} \nmid 2$ )

$$K_\mathfrak{q} := \begin{cases} \{(g, \lambda) \in \mathrm{GL}_{2n}(\mathcal{O}_E \otimes \mathcal{O}_{F_\mathfrak{q}}) \times \mathcal{O}_{F_\mathfrak{q}}^\times \mid \bar{g} = \vartheta^\circ g \vartheta^\circ, g^t \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} g = \lambda \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \det(g) = \lambda^n\} \\ \text{resp.} \\ \{(g, \lambda) \in \mathrm{GL}_{2n}(\mathcal{O}_{F_\mathfrak{q}}) \times \mathcal{O}_{F_\mathfrak{q}}^\times \mid g^t \cdot \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix} \cdot g = \lambda \cdot \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}, \det(g) = \lambda^n\} \end{cases}$$

of  $\mathrm{GSO}_{2n}^{E/F}(F_\mathfrak{q})$ . Define  $K_{0\mathfrak{q}}$  to be the kernel of the similitudes mapping  $K_\mathfrak{q} \rightarrow \mathcal{O}_{F_\mathfrak{q}}^\times$ ,  $(g, \lambda) \mapsto \lambda$ .

Then  $\pi_\mathfrak{q}$  is a ramified representation of  $\mathrm{GSO}_{2n}^{E/F}(F_\mathfrak{q})$ , but has nonzero  $K_{0\mathfrak{q}}$ -fixed vectors, on which  $K_\mathfrak{q}$  acts through nontrivial characters of  $K_\mathfrak{q}/K_{0\mathfrak{q}} \simeq \mathcal{O}_{F_\mathfrak{q}}^\times$ . We fix one such character  $\chi_\mathfrak{q}^0$  of  $K_\mathfrak{q}$ , and do this at every  $\mathfrak{q}$  above  $p$ . Now we globalize  $\{\chi_\mathfrak{q}^0\}_{\mathfrak{q}|p}$  to an algebraic Hecke character  $\chi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$  whose restriction to each  $\mathcal{O}_{F_\mathfrak{q}}^\times$  is  $\chi_\mathfrak{q}^0$ . Define  $\pi' := \pi \otimes \chi^{-1}$ . Then  $\pi'$  also satisfies the conditions of Theorem A. Moreover,  $p \notin S_{\mathrm{bad}}(\pi')$  by construction. Therefore (13.1) is true at each  $\mathfrak{q}|p$ , with  $\pi'$  in place of  $\pi$ . Then (SO- $i+$ ) for  $\mathfrak{q}$  follows as before.  $\square$

## 14. AUTOMORPHIC MULTIPLICITY ONE

Let  $E/F$  be a quadratic CM extension of  $F$  in case  $n$  is odd, and  $E := F$  for  $n$  even. Let  $\mathrm{SO}_{2n}^{E/F}$  and  $\mathrm{GSO}_{2n}^{E/F}$  be the corresponding groups. If  $\pi$  (resp.  $\pi^\flat$ ) is an automorphic representation of  $\mathrm{GSO}_{2n}^{E/F}$  (resp.  $\mathrm{SO}_{2n}^{E/F}$ ), we write  $m(\pi)$  (resp.  $m(\pi^\flat)$ ) for its automorphic multiplicity. In this section we will show that  $m(\pi^\flat)$  and  $m(\pi)$  are 1 for certain classes of automorphic representations of  $\mathrm{SO}_{2n}^{E/F}$  and  $\mathrm{GSO}_{2n}^{E/F}$  (and certain inner forms of those groups). To do this we combine our results with a potential automorphy result, Arthur's result on multiplicities for  $\mathrm{SO}_{2n}^{E/F}$ , and Xu's result on multiplicities for  $\mathrm{GSO}_{2n}^{E/F}$ .

Let  $\pi^\flat$  be a discrete automorphic representation of  $\mathrm{SO}_{2n}^{E/F}$ . Arthur gives in the discussion below [Art13, Thm. 1.5.2] the following result towards the computation of  $m(\pi^\flat)$ . Let  $\psi = \psi_1 \boxplus \cdots \boxplus \psi_r$  be the global (formal) parameter of  $\pi^\flat$  [Art13, 1.4] (cf. Section 6). Technically,  $\psi$  is an automorphic representation  $\pi^\sharp$  of  $\mathrm{GL}_{2n}(\mathbb{A}_F)$ , isomorphic to an isobaric direct sum of cuspidal automorphic representations  $\pi_i^\sharp$  of  $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ , with  $\psi_i$  the formal parameter represented by  $\pi_i^\sharp$ . In terms of these parameters Arthur proves in Theorem 1.5.2 a decomposition of the form

$$L_{\mathrm{disc}}^2(\mathrm{SO}_{2n}^{E/F}(F) \backslash \mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F)) \xrightarrow{\sim} \bigoplus_{\psi \in \tilde{\Psi}_2(\mathrm{SO}_{2n}^{E/F})} \bigoplus_{\pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)} m_\psi \pi$$

as  $\tilde{\mathcal{H}}(\mathrm{SO}_{2n}^{E/F})$ -Hecke module. It takes us too far afield to recall all the notation here, but we emphasize that  $\tilde{\mathcal{H}}(\mathrm{SO}_{2n}^{E/F})$  is the restricted tensor product of the local algebras  $\tilde{\mathcal{H}}(\mathrm{SO}_{2n}^{E/F}(F_v))$  consisting of  $\theta^\circ$ -invariant functions [Art13, before (1.5.3)]. Similarly, the local packet  $\tilde{\Pi}_{\psi_v}(\varepsilon_\psi)$  consists of  $\theta^\circ$ -orbits of representations.

Assume  $\pi^\flat \not\cong \pi^\flat \circ \theta^\circ$ . Both  $\pi^\flat$  and  $\pi^\flat \circ \theta^\circ$  map to the same global parameter  $\psi$ , and are isomorphic as  $\tilde{\mathcal{H}}(\mathrm{SO}_{2n}^{E/F})$ -module. Arthur proves  $m_\psi \leq 2$  for all  $\psi$ . In particular

$$(14.1) \quad m(\pi^\flat) + m(\pi^\flat \circ \theta^\circ) \leq m_\psi \leq 2.$$

On the other hand,  $\theta^\circ$  acts on  $L_{\mathrm{disc}}^2(\mathrm{SO}_{2n}^{E/F}(F) \backslash \mathrm{SO}_{2n}^{E/F}(\mathbb{A}_F))$ , so if  $\pi^\flat$  appears, then  $\pi^\flat \circ \theta^\circ$  also appears. Hence  $m(\pi^\flat), m(\pi^\flat \circ \theta^\circ) \geq 1$ , forcing  $m(\pi^\flat) = 1$  and  $m(\pi^\flat \circ \theta^\circ) = 1$ .

Now assume  $\pi^\flat \simeq \pi^\flat \circ \theta^\circ$ . The global parameters  $\psi$  and  $\psi \circ \theta^\circ$  are locally isomorphic, but not necessarily globally isomorphic. In this case, Arthur's result does not give enough information to compute  $m(\pi^\flat)$ . See also [Wan15, Wan19] where this question is studied in more detail.

**Proposition 14.1.** *Assume  $(\mathrm{coh}^\circ)$ ,  $(\mathrm{St}^\circ)$ , and  $(\mathrm{std}\text{-}\mathrm{reg}^\circ)$ . Let  $\rho_{\pi^\flat}$  be the Galois representation attached to  $\pi^\flat$  (Theorem 6.3). Then*

- (1)  $m_\psi = 2$  if  $\mathrm{std}\rho_{\pi^\flat}$  is irreducible.
- (2) Assume  $\ell \geq 4n$ ,  $\ell \notin S_{\mathrm{bad}}(\pi^\flat)$  and  $\ell$  is in the Fontaine–Lafaille range.<sup>17</sup> Then  $m_\psi = 2$  only if  $\mathrm{std}\rho_{\pi^\flat}$  is irreducible.

In particular  $m_\psi = 2$  if and only if  $\pi \not\cong \pi^\theta$ .

*Remark 14.2.* Condition (disc- $\infty$ ) of §6 is built into the setup of this section, but when (disc- $\infty$ ) fails, we see from Corollary 6.8 and [Art13, Thm. 1.5.2] that  $m(\pi^\flat) = 1$  whenever  $\pi^\flat$  satisfies  $(\mathrm{coh}^\circ)$ ,  $(\mathrm{St}^\circ)$ , and  $(\mathrm{std}\text{-}\mathrm{reg}^\circ)$ .

*Proof.* Arthur proves that  $m_\psi = 2$  if and only if the numbers  $n_i$  are all even ([Art13, ten lines below Thm 1.5.2]). Since  $\pi_{\mathrm{qSt}}^\flat$  is a twist of the Steinberg representation,  $\pi_{\mathrm{qSt}}^\sharp \simeq \mathrm{St}_{2n-1} \boxplus \mathbf{1}$  up to character twist by Proposition 6.1 (Ar2). Consequently we have two possible shapes for the global parameter  $\psi$ : either  $r = 1$ ,  $n_1 = 2n$ , or  $r = 2$ ,  $n_1 = 1$ ,  $n_2 = 2n - 1$  up to swapping the two indices (see argument below (6.3)). Thus  $m_\psi \in \{1, 2\}$  and  $m(\pi_\psi) = 2$  if and only if  $\pi^\sharp$  is cuspidal.

<sup>17</sup>The Fontaine–Lafaille range means the following: for each embedding  $\lambda: F \rightarrow \overline{\mathbb{Q}}_\ell$  and each pair of distinct Hodge–Tate weights  $a, b \in \mathbb{Z}$  of  $\mathrm{std}\rho_\pi^C$  relative  $\lambda$ , we have  $|a - b| \leq \ell - 2$ .



Clearly, if  $\pi^\sharp$  is not cuspidal, then  $\text{std}\rho_{\pi^\flat}$  is reducible. This proves (1). Assume  $\ell$  is as in (2) and (for a contradiction) that  $\pi^\sharp$  is cuspidal but the  $\ell$ -adic representation  $r_\ell := \text{std}\rho_{\pi^\flat}$  is reducible. Then  $r_\ell = r_1 \oplus r_2$  with  $\dim r_1 = 1$  and  $\dim r_2 = 2n - 1$ , and both  $r_1$  and  $r_2$  are self-dual. Twisting  $\pi^\flat$  by a quadratic automorphic character of  $\text{SO}_{2n}^{E/F}(\mathbb{A}_F)$  if needed (cf. first paragraph in the proof of Lemma 7.1 for a local setup), we may assume that  $r_1$  is the trivial representation. We will apply the potential automorphy theorem [BLGGT14, Thm. C] to  $r_2$ . Firstly, they require  $\ell \geq 2(\dim(r_2) + 1) = 4n$ . We check the other hypotheses of that theorem:

- (Unramified almost everywhere). True by (SO-ii) of Theorem 6.3.
- (Odd essential self-duality). This condition is that  $r_2$  factors through  $\text{GO}_{2n-1}(\overline{\mathbb{Q}}_\ell)$  (up to conjugation) and has totally even multiplier character. By Theorem 6.3,  $\rho_{\pi^\flat}$  has image in  $\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell) \rtimes \text{Gal}(E/F) \subset \text{O}_{2n}(\overline{\mathbb{Q}}_\ell)$ . Consequently  $r_2$  has image in  $\text{O}_{2n-1}(\overline{\mathbb{Q}}_\ell) \subset \text{O}_{2n}(\overline{\mathbb{Q}}_\ell)$  up to conjugation. The multiplier character of  $r_2$  is trivial and hence totally even.
- (Potential diagonalizability and regularity). The representation  $r_2|_{\Gamma_{\mathfrak{q}}}$  is crystalline by Theorem 6.3 (SO-iv),  $\ell$  is unramified in  $F/\mathbb{Q}$ , and the Hodge–Tate weights are in the Fontaine–Lafaille range. Lemma 1.4.3 of [BLGGT14] then assures that  $r_2|_{\Gamma_{\mathfrak{q}}}$  is potentially diagonalizable. Theorem 6.3 (SO-iii) and (std-reg) imply that  $r_2$  is regular relative to every embedding  $F \hookrightarrow \overline{\mathbb{Q}}_\ell$ .
- (Irreducibility). We want  $r_2|_{\text{Gal}(\overline{F}/F(\zeta_\ell))}$  to be irreducible (with  $\zeta_\ell$  a primitive  $\ell$ -th root of unity). This is also true, because  $r_2$  has a regular unipotent element in its image, and is semi-simple (irreducible), and therefore strongly irreducible.

Thus the hypotheses are satisfied. By [Thm. C, *loc. cit.*] there exists a finite totally real Galois extension  $F'/F$  such that  $r_2|_{\Gamma_{F'}}$  is automorphic. In particular the  $L$ -function  $L(r_2, s)$  has a meromorphic continuation to the complex plane. In fact, it follows from a Brauer induction argument (see [HSBT10, pf. of Thm. 4.2] and [Tay04, Sect. 5]) and Jacquet–Shalika’s work [JS76] that  $L(r_2, s)$  does not vanish at  $s = 1$ . Since  $r = r_1 \oplus r_2$  with  $r_1$  the trivial representation, we have the factorization

$$(14.2) \quad L(\pi^\sharp, s) = \zeta_F(s) \cdot L(r_2, s),$$

where  $\zeta_F(s)$  is the Dedekind zeta function of  $F$ . Since  $\pi^\sharp$  is cuspidal,  $L(\pi^\sharp, s)$  is an entire function. As  $\zeta_F(s)$  has a pole at  $s = 1$  and  $L(r_2, 1) \neq 0$ , we have a contradiction.  $\square$

**Proposition 14.3.** *Let  $\pi^\flat$  be a cuspidal automorphic representation of  $\text{SO}_{2n}^{E/F}$  satisfying (coh $^\circ$ ), (St $^\circ$ ) and (std-reg $^\circ$ ). Then  $m(\pi^\flat) = 1$ .*

*Proof.* We have  $1 \leq m(\pi^\flat) \leq m_\psi$ , so if  $m_\psi = 1$  we are done. We now show that  $m_\psi = 2$  implies  $\pi^\flat \not\cong \pi^\flat \circ \theta^\circ$ ; then the proposition will follow from the argument at (14.1). Take  $\ell$  sufficiently large so that  $\text{std}\rho_{\pi^\flat}$  is irreducible (Proposition 14.1). The Zariski closure  $I$  of the image of  $\overline{\rho}_{\pi^\flat}$  in  $\text{PSO}_{2n}$ , is either  $\text{PSO}_{2n}$ ,  $\text{SO}_{2n-1}$ ,  $G_2$  or  $\text{PGL}_2$ . If  $I \neq \text{PSO}_{2n}$ , then  $\text{std}\rho_{\pi^\flat}$  is reducible. Hence  $\overline{\rho}_{\pi^\flat}$  has Zariski dense image in  $\text{PSO}_{2n}$ . By the Chebotarev density theorem, there exists an  $F$ -prime  $\mathfrak{q}$  where  $\rho_{\pi^\flat}$  is unramified such that  $\rho_{\pi^\flat}(\text{Frob}_{\mathfrak{q}})_{\text{ss}}$  is not conjugate to  $\theta^\circ \rho_{\pi^\flat}(\text{Frob}_{\mathfrak{q}})_{\text{ss}}$ . By (SO-i) at  $\mathfrak{q}$ , the representation  $\pi_{\mathfrak{q}}^\flat$  is not isomorphic to  $\pi_{\mathfrak{q}}^\flat \circ \theta^\circ$ .  $\square$

Using the work of Xu [Xu18] and our construction of Galois representations, we can prove for  $\text{GSO}_{2n}^{E/F}$  a statement that is very similar to Proposition 14.1:

**Proposition 14.4.** *Let  $\pi$  be a cuspidal automorphic representation of  $\text{GSO}_{2n}^{E/F}$  satisfying (L-coh), (St) and (std-reg). Then  $m(\pi) = 1$ .*

*Proof.* (cf. [KS16, Thm. 12.1]). Let  $\pi^\flat$  be a cuspidal automorphic representation contained in  $\pi$ , such that  $\pi^\flat$  satisfies (coh $^\circ$ ), (St $^\circ$ ) and (std-reg $^\circ$ ). Let  $Y(\pi)$  be the set of characters  $\omega: \text{GSO}_{2n}^{E/F}(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$  which are trivial on  $\text{GSO}_{2n}^{E/F}(F)\mathbb{A}_F^\times \text{SO}_{2n}^{E/F}(\mathbb{A}_F) \subset \text{GSO}_{2n}^{E/F}(\mathbb{A}_F)$  and such that  $\pi \simeq \pi \otimes \omega$ . Bin Xu [Xu18, Prop. 1.7] proves that

$$(14.3) \quad m(\pi) = m_{\tilde{\psi}} |Y(\pi)/\alpha(\mathcal{S}_\phi)|,$$

where  $\tilde{\psi}$  is the global parameter of  $\pi$  as defined in [Xu18, Sect. 3] ( $\tilde{\psi}$  is denoted  $\tilde{\phi}$  there).

We claim that  $Y(\pi) = 1$  in (14.3). Let  $\omega \in Y(\pi)$  and let  $\chi: \Gamma \rightarrow \overline{\mathbb{Q}}_\ell^\times$  be the corresponding character via class field theory. As  $\chi\rho_\pi$  and  $\rho_\pi$  have conjugate Frobenius images at almost all places, we obtain  $\chi\rho_\pi \simeq \rho_\pi$  by Proposition 5.4, and thus  $\chi = 1$  by Lemma 5.3. Hence indeed  $Y(\pi) = 1$ .

Let  $\psi$  denote the Arthur parameter of  $\pi^b$ . (Our  $\psi$  is  $\phi$  in [Xu18].) Then by [loc. cit., Thm. 3.13], the multiplicity  $m_\phi$  considered by Xu is the same as Arthur's multiplicity  $m_\psi$  from [Art13, Thm. 1.5.2]. In [Xu18, Cor. 5.10], Xu states that  $m_{\tilde{\psi}} = m_\psi / \#\Sigma_Y(\pi)$ , where  $\Sigma_Y(\pi) := \Sigma_0 / \Sigma_0(\pi, Y)$ , where  $\Sigma_0$  is the 2-group  $\{1, \theta\}$ , and  $\Sigma_0(\pi, Y)$  is the group of  $\theta' \in \Sigma_0$  such that  $\pi \otimes \omega \simeq \pi^{\theta'}$  for some  $\omega \in Y(\pi)$ . As we have  $Y(\pi) = 1$ , we have  $\#\Sigma_Y(\pi) \in \{1, 2\}$  and it is equal to 2 if and only if  $\pi \not\simeq \pi^\theta$ . Thus  $m_\psi = \#\Sigma_Y(\pi)$  by Proposition 14.1. Now (14.3) simplifies to  $m(\pi) = m_{\tilde{\psi}} = m_\psi / \#\Sigma_Y(\pi) = 1$ , which completes the proof.  $\square$

Let  $G$  be an inner form of the group  $\mathrm{GSO}_{2n}^{E/F}$ , constructed in (8.7) (so  $\mathrm{Res}_{F/\mathbb{Q}}G$  can be promoted to a Shimura datum). We close this section with computing some automorphic multiplicities for these inner forms. In particular we prove that the multiplicities  $a(\cdot)$  appearing in Section 9 are in fact equal to 1.

**Proposition 14.5.** *Let  $\pi$  be a cuspidal automorphic representation of  $G(\mathbb{A}_F)$ , satisfying (coh), (St) and (std-reg). Then  $m(\pi) = 1$ .*

*Proof.* The proof is exactly the same as the argument for [KS16, Thm. 12.2]. The main point is that automorphic representations  $\tau^*$  of  $G^*(\mathbb{A}_F) = \mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$  contributing to the analogue of [loc. cit., Eq. (14.2)] have automorphic multiplicity 1. Notice that [Xu18, Thm. 1.8] may be used again, together with the existence of Galois representations (Theorem A), to prove that for all  $\pi^*$  and  $\tau^*$  contributing to [KS16, Eq. (14.2)] we have  $\tau_{\mathrm{qSt}}^* \simeq \pi_{\mathrm{qSt}}^*$ .  $\square$

## 15. MEROMORPHIC CONTINUATION OF SPIN $L$ -FUNCTIONS

Recall that  $n \in \mathbb{Z}_{\geq 3}$ . Let  $\pi$  be a cuspidal automorphic representation of  $\mathrm{GSO}_{2n}^{E/F}(\mathbb{A}_F)$  unramified away from a finite set of places  $S$ . Throughout this section, we assume (spin-reg) in the introduction, which implies (std-reg). Indeed, if the image of  $(s_0, s_1, \dots, s_n) \in T_{\mathrm{GSpin}}$  under  $\mathrm{spin}^\epsilon$  is regular then  $s_1, \dots, s_n$  must be mutually distinct, as the weights in  $\mathrm{spin}^\epsilon$  are described as the Weyl orbit(s) of (2.7).

**Proposition 15.1.** *Assume that  $\pi$  satisfies (St), (L-coh), and (spin-reg). Let  $n \in \mathbb{Z}_{\geq 3}$ . There exist a number field  $M_\pi$  and a semisimple representation*

$$R_{\pi, \lambda}^\epsilon : \Gamma \rightarrow \mathrm{GL}_{2^n/|\mathfrak{e}|}(\overline{M}_{\pi, \lambda})$$

for each finite place  $\lambda$  of  $M_\pi$  such that the following hold for every  $\epsilon \in \mathfrak{e}$ . (Write  $\ell$  for the rational prime below  $\lambda$ .)

(1) At each place  $\mathfrak{q}$  of  $F$  not above  $S_{\mathrm{bad}} \cup \{\ell\}$ , we have

$$\mathrm{char}(R_{\pi, \lambda}^\epsilon(\mathrm{Frob}_{\mathfrak{q}})) = \mathrm{char}(\mathrm{spin}^\epsilon(\iota\phi_{\pi_{\mathfrak{q}}})) \in M_\pi[X].$$

(2)  $R_{\pi, \lambda}^\epsilon|_{\Gamma_{\mathfrak{q}}}$  is de Rham for every  $\mathfrak{q}|\ell$ . Moreover it is crystalline if  $\pi_v$  is unramified and  $\mathfrak{q} \notin S_{\mathrm{bad}}$ .

(3) For each  $\mathfrak{q}|\ell$  and each  $y : F \hookrightarrow \mathbb{C}$  such that  $\iota y$  induces  $\mathfrak{q}$ , we have  $\mu_{\mathrm{HT}}(R_{\pi, \lambda}^\epsilon|_{\Gamma_{\mathfrak{q}}}, \iota y) = \iota(\mathrm{spin}^\epsilon \circ \mu_{\mathrm{Hodge}}(\phi_{\pi_{\mathfrak{q}}}))$ . In particular  $\mu_{\mathrm{HT}}(R_{\pi, \lambda}^\epsilon|_{\Gamma_{\mathfrak{q}}}, \iota y)$  is a regular cocharacter for each  $y$ .

(4)  $R_{\pi, \lambda}^\epsilon$  is pure.

(5)  $R_{\pi, \lambda}^\epsilon$  maps into  $\mathrm{GSp}_{2^n/|\mathfrak{e}|}(\overline{M}_{\pi, \lambda})$  if  $n \equiv 2, 3 \pmod{4}$  (resp.  $\mathrm{GO}_{2^n/|\mathfrak{e}|}(\overline{M}_{\pi, \lambda})$  if  $n \equiv 0, 1 \pmod{4}$ ) for a nondegenerate alternating (resp. symmetric) pairing on the underlying  $2^n/|\mathfrak{e}|$ -dimensional space over  $\overline{M}_{\pi, \lambda}$ . The multiplier character  $\mu_\lambda^\epsilon : \Gamma \rightarrow \mathrm{GL}_1(\overline{M}_{\pi, \lambda})$  (so that  $R_{\pi, \lambda}^\epsilon \simeq (R_{\pi, \lambda}^\epsilon)^\vee \otimes \mu_\lambda^\epsilon$ ) is totally of sign  $(-1)^{n(n-1)/2}$  and associated with  $\omega_\pi$  via class field theory and  $\iota_\lambda$ .

*Proof.* Let  $M$  be the field of definition of  $\xi$ , which is a finite extension of  $\mathbb{Q}$  in  $\mathbb{C}$ . We can choose  $M_\pi$  to be the field of definition for the  $\pi^\infty$ -isotypic part in the (compact support) Betti cohomology of  $H^\bullet(\mathrm{Sh}^+(\mathbb{C}), \mathcal{L}_\xi) \oplus H^\bullet(\mathrm{Sh}^-(\mathbb{C}), \mathcal{L}_\xi)$  with  $M$ -coefficient. Then  $M_\pi$  is a finite extension of  $M$  in  $\mathbb{C}$ . For each prime  $\ell$  and a finite place  $\lambda$  of  $M_\pi$  above  $\ell$ , extend  $M \hookrightarrow \mathbb{C}$  to an isomorphism  $\overline{M}_{\pi,\lambda} \simeq \mathbb{C}$ . Identifying  $\overline{M}_{\pi,\lambda} \simeq \overline{\mathbb{Q}}_\ell$ , we have  $\iota_\lambda : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ . Take

$$R_{\pi,\lambda}^\epsilon := \mathrm{spin}^\epsilon \circ \rho_{\pi,\iota_\lambda}.$$

Then (1), (2), and (3) follow from (A2) and (A3) of Theorem A, respectively. Part (4) follows from (SO-ii) of Theorem 6.3 via (A2). The first part of (5) holds true since  $\mathrm{spin}^\epsilon : \mathrm{GSpin}_{2n} \rightarrow \mathrm{GL}_{2^{n-1}}$  is an irreducible representation preserving a nondegenerate symplectic (resp. symmetric) pairing up to scalar if  $n$  is 2 (resp. 0) mod 4, and since  $\mathrm{spin} : \mathrm{GPin}_{2n} \rightarrow \mathrm{GL}_{2^n}$  is irreducible and preserves a nondegenerate symplectic (resp. symmetric) pairing up to scalar if  $n$  is 3 (resp. 1) mod 4. Indeed, the irreducibility is standard and the rest follows from Lemma 4.2 (with the pairing given as in the lemma). Lemma 4.2 also tells us that  $\mu_\lambda^\epsilon = \mathcal{N} \circ \rho_{\pi,\iota_\lambda}$ . By (A2),  $\omega_\pi = \mathcal{N} \circ \rho_{\pi,\iota_\lambda}$  so  $\mu_\lambda^\epsilon$  is associated with  $\omega_\pi$ . As in the proof of part 5 of [KS16, Prop. 13.1],  $\omega_\pi \otimes |\cdot|^{n(n-1)/2}$  corresponds to an even Galois character of  $\Gamma$ . (We change  $n(n+1)/2$  in [KS16] to  $n(n-1)/2$  here due to the difference in the definition of (L-coh).) It follows that  $\mu_{\lambda,y}(c_y) = (-1)^{n(n-1)/2}$  for every  $y|\infty$ .  $\square$

Now we apply potential automorphy results to the weakly compatible system of  $R_{\pi,\lambda}^\epsilon$ .

**Theorem 15.2.** *Theorem D is true.*

*Proof.* This follows from [PT15, Thm. A], which can be applied to the weakly compatible system  $\{R_{\pi,\lambda}^\epsilon\}$  thanks to the preceding proposition.  $\square$

*Remark 15.3.* We cannot appeal to the potential automorphy as in [BLGGT14, Thm. A] as  $R_{\pi,\lambda}^\epsilon$  may be reducible. The point of [PT15] is to replace the irreducibility hypothesis with a purity hypothesis (guaranteed by (iv) of Proposition 15.1). We take advantage of this.

## APPENDIX A. EXTENDING A GALOIS REPRESENTATION

Here we investigate the problem of extending a  $\widehat{G}$ -valued Galois representation to an  ${}^L G$ -valued representation over a quadratic extension.

We freely use the notation and terminology of §1. Let  $E$  be a CM quadratic extension over a totally real field  $F$  in an algebraic closure  $\overline{F}$ . Set  $\Gamma = \Gamma_F := \text{Gal}(\overline{F}/F)$ ,  $\Gamma_E := \text{Gal}(\overline{F}/E)$ , and  $\Gamma_{E/F} := \text{Gal}(E/F) = \{1, c\}$ . Let  $G$  be a quasi-split group over  $F$  which splits over  $E$ . Let  $\theta \in \text{Aut}(\widehat{G})$  denote the action of  $c$  on  $\widehat{G}$  (with respect to a pinning over  $F$ ). By  $\widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$ , we mean the  $L$ -group relative to  $E/F$ , namely the semi-direct product such that  $cgc = \theta(g)$  for  $g \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ .

Fix an infinite place  $y$  of  $F$ . Write  $c_y \in \Gamma_F$  for the corresponding complex conjugation (well defined up to conjugacy). Let  $\rho' : \Gamma_E \rightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell)$  be a Galois representation. Define

$${}^{c_y} \rho'(\gamma) := \rho'(c_y \gamma c_y^{-1}).$$

(Of course  $c_y^{-1} = c_y$ .) We will sometimes impose the following hypotheses.

(H1)  $\text{Cent}_{\widehat{G}}(\text{im}(\rho')) = Z(\widehat{G})$ .

(H2) The map  $Z(\widehat{G}) \rightarrow Z(\widehat{G})^\theta$  given by  $z \mapsto z\theta(z)$  is a surjection on  $\overline{\mathbb{Q}}_\ell$ -points.

**Lemma A.1.** *Consider the following statements.*

- (1)  $\rho'$  extends to some  $\rho : \Gamma_F \rightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma_{E/F}$ .
- (2)  ${}^{c_y} \rho' \simeq \theta \circ \rho'$ .
- (3) there exists  $g \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$  such that  $g\theta(g) = 1$  and  $\rho'(c_y \gamma c_y^{-1}) = g\theta(\rho'(\gamma))g^{-1}$  for every  $\gamma \in \Gamma_E$ .

Then (3) $\Leftrightarrow$ (1) $\Rightarrow$ (2). In particular if  $\rho$  is as in (1) then the element  $g$  such that  $\rho(c_y) = g \times c$  enjoys the property of (3). If (H1) and (H2) are satisfied, then we also have (2) $\Rightarrow$ (3), so all three statements are equivalent.

*Remark A.2.* We recommend [BC09, Section A.11] as a useful guide to similar ideas.

*Remark A.3.* Often (2) is the condition to verify to extend a Galois representation, as we did in Lemma 12.1 of this paper.

*Proof.* (3) $\Leftrightarrow$ (1): First we show (3) $\Rightarrow$ (1). Define  $\rho$  by  $\rho|_{\Gamma_E} := \rho'$  and  $\rho(\gamma c_y) := \rho'(\gamma)gc$  ( $\gamma \in \Gamma_E$ ). Then

$$\begin{aligned} \rho(c_y^2) &= gcgc = g\theta(g) = 1, \\ \rho(c_y \gamma c_y^{-1}) &= {}^{c_y} \rho'(\gamma) = g\theta(\rho'(\gamma))g^{-1}, \end{aligned}$$

and using this, one checks that  $\rho$  is a homomorphism on the entire  $\Gamma$ . A similar computation shows (1) $\Rightarrow$ (3) for  $g$  such that  $\rho(c_y) = g \times c$ .

(1) $\Rightarrow$ (2): Write  $\rho(c_y) = gc$  with  $g \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ . For every  $\gamma \in \Gamma_E$ ,

$${}^{c_y} \rho'(\gamma) = \rho(c_y \gamma c_y^{-1}) = gc\rho'(\gamma)c^{-1}g^{-1} = g\theta(\rho'(\gamma))g^{-1}.$$

(2) $\Rightarrow$ (3), assuming (H1) and (H2): There exists  $g \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$  such that

$$(A.1) \quad \rho'(c_y \gamma c_y^{-1}) = g\theta(\rho'(\gamma))g^{-1}, \quad \gamma \in \Gamma_E.$$

Putting  $c_y \gamma c_y^{-1}$  in place of  $\gamma$ , we obtain

$$\rho'(\gamma) = \rho'(c_y^2 \gamma c_y^{-2}) = g\theta(g\theta(\rho'(\gamma))g^{-1})g^{-1} = g\theta(g)\rho'(\gamma)(g\theta(g))^{-1}.$$

Hence  $g\theta(g) \in Z(\widehat{G})$  as  $\text{Cent}_{\widehat{G}}(\rho') = Z(\widehat{G})$  by (H1). As a central element,

$$g\theta(g) = g^{-1}(g\theta(g))g = \theta(g)g = \theta(g\theta(g)),$$

namely  $g\theta(g) \in Z(\widehat{G})^\theta$ . By (H2),  $g\theta(g) = z\theta(z)$  for some  $z \in Z(\widehat{G})$ . Replacing  $g$  with  $gz^{-1}$ , we can arrange that

$$g\theta(g) = 1.$$

This does not affect (A.1) so we are done.  $\square$

**Lemma A.4.** *Assume (H1). Then the set of  $\widehat{G}$ -conjugacy classes of extensions of  $\rho'$  to  $\Gamma$  is an  $H^1(\Gamma_{E/F}, Z(\widehat{G}))$ -torsor if nonempty.*

*Proof.* Fix an extension  $\rho_0$  of  $\rho'$ , which exists by Lemma A.1. If  $\rho$  is another extension of  $\rho'$ , then set  $z := \rho_0(c_y)\rho(c_y)^{-1}$ . Writing  $\rho_0(c_y) = g_0 \rtimes c$  and  $\rho(c_y) = g \rtimes c$ , we have  $zg = g_0$ , and both  $g_0, g$  satisfy the condition of Lemma A.1 (3). It follows that  $z$  centralizes  $\theta(\text{im}(\rho'))$ , hence  $z \in Z(\widehat{G})$ , and also that  $z\theta(z) = 1$ . Thus  $z$  defines a  $Z(\widehat{G})$ -valued 1-cocycle on  $\Gamma_{E/F}$ , and by reversing the process, such a cocycle determines an extension of  $\rho'$ .

Let  $\rho_z$  be the extension given by  $z \in Z(\widehat{G})$  such that  $z\theta(z) = 1$ . It remains to show that  $\rho_z \sim \rho_0$  if and only if  $z = \theta(x)/x$  for some  $x \in Z(\widehat{G})$ . If  $\rho_z \sim \rho_0$  then  $\rho_z = \text{Int}(x)\rho_0$  for some  $x \in \widehat{G}$ . By (H1),  $x \in Z(\widehat{G})$ . Evaluating at  $c_y$ , we obtain  $z^{-1}\rho_0(c_y) = x\rho_0(c_y)x^{-1}$ . Therefore  $z = \theta(x)/x$ . The converse direction is shown similarly by arguing backward.  $\square$

We illustrate assumptions (H1), (H2), and the lemmas in the following examples.

**Example A.5.** Consider  $\widehat{G} = \text{SO}_{2n}$  ( $n \geq 3$ ) with  $\theta$  being the conjugation by  $\vartheta^\circ \in \text{O}_{2n}(\overline{\mathbb{Q}}_\ell) - \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$  as in (2.2). Assume that  $\text{im}(\rho')$  contains a regular element of  $\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ . In this case  $Z(\widehat{G}) = Z(\widehat{G})^\theta = \{\pm 1\}$ . Then (H2) is trivially false but (H1) is true. To see this, by assumption,  $\text{std} \circ \rho'$  is either irreducible or the direct sum of an irreducible  $(2n-1)$ -dimensional representation and a character. In the former case (H1) is clear by Schur's lemma. In the latter case, again by Schur's lemma, a centralizer of  $\text{im}(\rho')$  in  $\text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$  is contained in  $\begin{pmatrix} a^{-1}2^{n-1} & 0 \\ 0 & b \end{pmatrix}$  with  $a, b \in \{\pm 1\}$  up to  $\text{O}_{2n}(\overline{\mathbb{Q}}_\ell)$ -conjugacy. Since the determinant equals 1, we deduce that  $a = b$ , i.e., the centralizer belongs to  $Z(\widehat{G})$ .

We easily compute  $Z^1(\Gamma_{E/F}, Z(\widehat{G})) = H^1(\Gamma_{E/F}, Z(\widehat{G})) \simeq \mathbb{Z}/2\mathbb{Z}$ , the nontrivial element sending  $c$  to  $-1$ . In fact if  $\rho$  extends  $\rho'$  in the setup of the preceding lemmas, the other extension is easily described as  $\rho \otimes \chi_{E/F}$ , where  $\chi_{E/F} : \Gamma \rightarrow \Gamma_{E/F} \xrightarrow{\sim} \{\pm 1\}$ .

**Example A.6.** The main case of interest for us is when

- $\widehat{G} = \text{GSpin}_{2n}$  ( $n \geq 3$ ),
- $\theta$  is the conjugation by an element of  $\text{GPin}_{2n}(\overline{\mathbb{Q}}_\ell) - \text{GSpin}_{2n}(\overline{\mathbb{Q}}_\ell)$ ,
- $\text{im}(\rho')$  contains a regular unipotent.

Since  $Z(\widehat{G})^\theta = \mathbb{G}_m$  (identified with invertible scalars in the Clifford algebra underlying  $\widehat{G}$  as a  $\text{GSpin}$  group; see §3), assumption (H2) is satisfied. (The squaring map  $\mathbb{G}_m \rightarrow \mathbb{G}_m$  is clearly surjective on  $\overline{\mathbb{Q}}_\ell$ -points.) To check (H1),  $\text{Cent}_{\widehat{G}}(\text{im}(\rho'))$  is contained in the preimage of  $\text{Cent}_{\text{SO}_{2n}}(\text{im}(\rho'^\circ))$  via  $\text{pr}^\circ : \text{GSpin}_{2n} \rightarrow \text{SO}_{2n}$ . Since the latter centralizer is  $\{\pm 1\} \subset \text{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ , we see that  $\text{Cent}_{\widehat{G}}(\text{im}(\rho')) \subset \text{pr}'^{-1}(\{\pm 1\}) = Z(\widehat{G})$ .

In the coordinates for  $Z(\widehat{G})$  of Lemma 2.5,  $Z^1(\Gamma_{E/F}, Z(\widehat{G})) = \{(s_0, s_1) : s_1 \in \{\pm 1\}, s_1 = s_0^2\} \simeq \mu_4$ , of which coboundaries are  $\{(\pm 1, 1)\} \simeq \mu_2$ . (The first identification is given by taking the image of  $c$ .) Hence  $H^1(\Gamma_{E/F}, Z(\widehat{G})) \simeq \mathbb{Z}/2\mathbb{Z}$ . Let  $\zeta = (\zeta_4, -1) \in Z(\widehat{G})$ , where  $\zeta_4$  is a primitive fourth root of unity, cf. Lemma 3.7. If  $\rho$  is an extension of  $\rho'$ , then the other extension (up to  $\widehat{G}$ -conjugacy) is described as  $\rho \otimes \chi$ , where  $\chi : \Gamma \rightarrow Z(\widehat{G}) \rtimes \{1, c\}$  is inflated from  $\Gamma_{E/F} \xrightarrow{\sim} \{1, \zeta \rtimes c\}$ . Notice that  $\text{pr}^\circ \circ \chi = \chi_{E/F}$ , for  $\chi_{E/F}$  as in the preceding example.

**Example A.7.** When studying Galois representations arising from automorphic representations on a unitary group  $U_n$  in  $n$  variables, two target groups appear in the literature: the group  $\mathcal{G}_n$  in [CHT08, §2.1] and the C-group of  $U_n$  in [BG14]; the two are isogenous as explained in [BG14, §8.3]. The latter is the  $L$ -group of a  $\mathbb{G}_m$ -extension of  $U_n$ ; it does not satisfy (H2). The former is not an  $L$ -group, but still a semi-direct product  $(\text{GL}_n \times \text{GL}_1) \rtimes \Gamma_{E/F}$ , with  $c(g, \mu) = (\Phi_n g^{-t} \Phi_n^{-1}, \mu)$  for an anti-diagonal matrix  $\Phi_n \in \text{GL}_n$ . As such, the discussion in this appendix goes through for  $\mathcal{G}_n$ . An easy computation shows that  $\mathcal{G}_n$  satisfies (H2) and that  $H^1(\Gamma_{E/F}, Z(\text{GL}_n \times \text{GL}_1)) = \{1\}$  for the given Galois action. Thus provided that  $\rho'$  satisfies (H1) (e.g., if  $\rho'$  is irreducible), an extension of  $\rho'$  exists if and only if  ${}^c \rho' \simeq \theta \circ \rho'$ , and the extension is unique up to conjugacy. Compare this with [CHT08, Lem. 2.1.4] (which allows a general coefficient field of characteristic 0).

APPENDIX B. ON LOCAL  $A$ -PACKETS OF EVEN SPECIAL ORTHOGONAL GROUPS

Let  $F$  be a finite extension of  $\mathbb{Q}_p$ . Suppose that  $E = F$  or that  $E$  is a CM quadratic extension of  $F$ . Let  $\chi_{E/F} : F^\times \rightarrow \{\pm 1\}$  denote the quadratic character associated with  $E/F$  via class field theory. Let  $G := \mathrm{SO}_{2n}^{E/F}$  denote the quasi-split form of the split group  $\mathrm{SO}_{2n}$  over  $F$  twisted by  $\chi_{E/F}$ . Write  $\tilde{\mathrm{Out}}_N(G) := \mathrm{O}_{2n}(\mathbb{C})/\mathrm{SO}_{2n}(\mathbb{C})$  for the outer automorphism group on  $\mathrm{SO}_{2n}(\mathbb{C})$ . Denote by  $\mathbf{1}$  and  $\mathrm{St}$  the trivial and Steinberg representations of  $G(F)$ . We aim to identify local  $A$ -packets containing each of  $\mathbf{1}$  and  $\mathrm{St}$ .

Let  $\mathcal{L}_F := W_F \times \mathrm{SU}(2)$  denote the local Langlands group. Write  $\Psi^+(G)$  for the isomorphism classes of extended  $A$ -parameters, that is, continuous morphisms  $\psi : \mathcal{L}_F \times \mathrm{SU}(2) \rightarrow {}^L G$  such that  $\psi|_{\mathcal{L}_F}$  is an  $L$ -parameter. (Two  $A$ -parameters are considered isomorphic if they are in the same  $\hat{G}$ -orbit.) Write  $\Psi(G)$  for the subset of  $\Psi^+(G)$  consisting of  $\psi \in \Psi^+(G)$  such that the image of  $\psi(\mathcal{L}_F)$  in  $\mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F}$  is bounded. (Such a property is  $\hat{G}$ -invariant.) The set of  $\tilde{\mathrm{Out}}_N(G)$ -orbits in  $\Psi^+(G)$  (resp.  $\Psi(G)$ ) is denoted by  $\tilde{\Psi}^+(G)$  (resp.  $\tilde{\Psi}(G)$ ). The group  $\mathcal{L} \times \mathrm{SU}(2)$  admits the involution permuting the two  $\mathrm{SU}(2)$ -components (acting as the identity on  $W_F$ ). The involution induces an involution on each of  $\tilde{\Psi}^+(G)$  and  $\tilde{\Psi}(G)$ , to be written as  $\psi \mapsto \hat{\psi}$ .

We say  $\psi \in \tilde{\Psi}^+(G)$  is square-integrable if  $\psi$  is stabilized by at most finitely many elements of  $\hat{G}$ . Then  $\psi$  lies in  $\tilde{\Psi}(G)$ . To see this, let  $w \in W_F$  be a lift of (geometric) Frobenius. Then  $\psi(w)^m$  centralizes the image of  $\psi$  for some  $m \in \mathbb{Z}_{\geq 1}$  as in [Del73, proof of Lem. 8.4.3]. Write  $I_F \subset W_F$  for the inertia subgroup. It follows that, replacing  $m$  with a suitable multiple,  $\psi(w)$  has trivial image in  $\mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F}$ . Since  $I_F \times \mathrm{SU}(2) \times \mathrm{SU}(2) \subset \mathcal{L}_F \times \mathrm{SU}(2)$  has already bounded image in  $\mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F}$  under  $\psi$ , we see that  $\psi \in \Psi(G)$ . Denote by  $\Psi_2(G)$  the subset of  $\Psi(G)$  consisting of square-integrable members.

Define  $\psi_{\mathrm{triv}} \in \Psi(G)$  to be the map (up to  $\hat{G}$ -conjugacy) that is trivial on  $\mathcal{L}_F$  and the principal embedding on the  $\mathrm{SU}(2)$ -factor outside  $\mathcal{L}_F$ . Concretely, the latter is the unique embedding (up to isomorphism) whose composition with  $\mathrm{std} : \mathrm{SO}_{2n} \hookrightarrow \mathrm{GL}_{2n}$  is  $\mathrm{Sym}^{2n-2} \oplus \mathbf{1}$ , where  $\mathrm{Sym}^{2n-2}$  (resp.  $\mathbf{1}$ ) denotes the  $(2n-2)$ -th symmetric power (resp. trivial) representations of  $\mathrm{SU}(2)$ . Write  $\psi_{\mathrm{St}} := \hat{\psi}_{\mathrm{triv}}$ . Then  $\psi_{\mathrm{triv}}$  and  $\psi_{\mathrm{St}}$  are  $\tilde{\mathrm{Out}}_N(G)$ -stable.

To every  $\psi \in \tilde{\Psi}(G)$ , Arthur [Art13, Thm. 1.5.1] assigned an  $A$ -packet  $\tilde{\Pi}(\psi)$ , a finite set consisting of  $\tilde{\mathrm{Out}}_N(G)$ -orbits of irreducible unitary representations of  $G(F)$ . Below *loc. cit.* he also defines  $\tilde{\Pi}(\psi)$  for  $\psi \in \tilde{\Psi}^+(G)$ , consisting of  $\tilde{\mathrm{Out}}_N(G)$ -orbits of parabolically induced representations of  $G(F)$  (which need not be irreducible or unitary).

**Proposition B.1.** *Let  $\psi \in \tilde{\Psi}^+(G)$ . The following are true.*

- (1)  $\mathbf{1} \in \tilde{\Pi}(\psi_{\mathrm{triv}})$  and  $\mathrm{St} \in \tilde{\Pi}(\psi_{\mathrm{St}})$ .
- (2) If  $\mathbf{1}$  (resp.  $\mathrm{St}$ ) is a direct summand of a member of  $\tilde{\Pi}(\psi)$  then  $\psi = \psi_{\mathrm{triv}}$  (resp.  $\psi = \psi_{\mathrm{St}}$ ).

*Remark B.2.* We learned the argument for the second part from [MS14, Prop. 8.2], where a similar statement is proved for the symplectic group.

*Proof.* (1) We have a number field  $\dot{F}$ , a finite place  $\mathfrak{q}$ , and a quasi-split form  $\dot{G}$  of the split  $\mathrm{SO}_{2n}$  over  $F$ , such that  $\dot{F}_{\mathfrak{q}} \simeq F$  and  $\dot{G}_{\mathfrak{q}} \simeq G$ . Arthur's global theorem [Art13, Thm. 1.5.2] assigns a global parameter  $\dot{\psi}$  whose packet contains the trivial representation  $\mathbf{1}_{\dot{G}}$  of  $\dot{G}(\mathbb{A}_{\dot{F}})$ . Considering the Satake parameters at almost all places, we identify  $\dot{\psi} = (\mathbf{1} \boxtimes \nu_{2n-1}) \boxplus (\mathbf{1} \boxtimes \nu_1)$  in Arthur's notation [Art13, §1.4], where  $\nu_i$  denotes the  $i$ -dimensional irreducible representation of  $\mathrm{SU}(2)$ . From this, we see that  $\dot{\psi}_{\mathfrak{q}} = \psi_{\mathrm{triv}}$ . It is part of the global theorem that the  $\mathfrak{q}$ -component of  $\mathbf{1}_{\dot{G}}$  belongs to  $\tilde{\Pi}(\dot{\psi}_{\mathfrak{q}})$ . In other words,  $\mathbf{1} \in \tilde{\Pi}(\psi_{\mathrm{triv}})$ . Since the involution  $\psi \mapsto \hat{\psi}$  changes members of  $A$ -packets by the Aubert involution, we deduce that  $\mathrm{St} \in \tilde{\Pi}(\psi_{\mathrm{St}})$ , since the Aubert involution sends  $\mathbf{1}$  to  $\mathrm{St}$ .

(2) It is enough to treat the case of  $\mathrm{St}$ , as the other case follows via the Aubert involution. Assume that  $\mathrm{St}$  is a direct summand of  $\pi \in \tilde{\Pi}^+(\psi)$ . We claim that  $\psi$  is square-integrable (so that  $\psi \in \tilde{\Psi}_2(G)$ ). If false, then  $\psi$  comes from a square integrable parameter on a proper Levi subgroup of  $G$ . The construction of packets in [Art13] (see the proof of Prop. 2.4.3 and the

discussion around (1.5.1) therein) tells us that  $\text{St}$  appears as a direct summand in a parabolic induction, namely

$$(B.1) \quad \text{ind}_{P(F)}^{G(F)}(\sigma) = \text{St} \oplus \pi',$$

where  $P$  is a proper parabolic subgroup of  $G$  with Levi factor  $M$ ,  $\sigma$  a finite-length representation of  $M(F)$ , and  $\pi'$  some representation of  $G(F)$ . Let us show that this is impossible. By devissage, replacing  $\sigma$  with an irreducible subquotient and renaming  $\pi'$  if needed, we may assume that  $\sigma$  is irreducible while maintaining the form of (B.1). Denote by  $J_P$  the normalized Jacquet module relative to  $P$ . By Frobenius reciprocity,

$$0 \neq \text{Hom}_{G(F)}(\text{St}, \text{ind}_{P(F)}^{G(F)}(\sigma)) = \text{Hom}_{M(F)}(\text{St}_M \delta_P^{-1/2}, \sigma),$$

so  $\sigma = \text{St}_M \delta_P^{-1/2}$ . However  $\text{ind}_{P(F)}^{G(F)}(\sigma)$  is reducible and has  $\text{St}$  as a unique quotient, thus contradicting (B.1).

Before we proceed, we recall [Mø09, §4.5] (applicable since Mœglin's  $A$ -packets are compatible with Arthur's by the main results of [Xu17]). Let  $\psi_1, \psi_2 \in \Psi(G)$ . If  $\tilde{\Pi}(\psi_1) \cap \tilde{\Pi}(\psi_2)$  is nonempty, then  $\psi_1 \circ \Delta = \psi_2 \circ \Delta$ , where  $\Delta$  is the diagonal embedding

$$W_F \times \text{SU}(2) \rightarrow \mathcal{L}_F \times \text{SU}(2) = W_F \times \text{SU}(2) \times \text{SU}(2), \quad (w, x) \mapsto (w, x, x).$$

In our case,  $\tilde{\Pi}(\psi) \cap \tilde{\Pi}(\psi_{\text{St}}) \neq \emptyset$ . Hence  $\psi \circ \Delta \simeq \psi_{\text{St}} \circ \Delta$ . In particular,

$$(\text{std} \circ \psi \circ \Delta)|_{\text{SU}(2) \times \text{SU}(2)} \simeq (\text{std} \circ \psi_{\text{St}} \circ \Delta)|_{\text{SU}(2) \times \text{SU}(2)} \simeq \text{Sym}^{2n-2} \oplus \mathbf{1}.$$

It follows via basic representation theory of  $\text{SU}(2)$  that  $\psi$  is trivial on the first  $\text{SU}(2)$  and  $\text{Sym}^{2n-2} \oplus \mathbf{1}$  on the second  $\text{SU}(2)$ , or the other way around. That is,  $\psi \simeq \psi_{\text{St}}$  or  $\psi \simeq \psi_{\text{triv}}$ .

If  $\psi \simeq \psi_{\text{St}}$  then we are done, so suppose that  $\psi \simeq \psi_{\text{triv}}$ . By the initial assumption,  $\text{St} \subset \pi$  for some  $\pi \in \tilde{\Pi}(\psi_{\text{triv}})$ . Now consider  $\dot{F}$ ,  $\dot{G}$ , and  $\dot{\psi} = (\mathbf{1} \boxtimes \nu_{2n-1}) \boxplus (\mathbf{1} \boxtimes \nu_1)$  in the proof of (1) above. Arthur's global packet  $\tilde{\Pi}(\dot{\psi})$  consists of representations  $\dot{\pi}$  of  $G(\mathbb{A}_{\dot{F}})$  such that  $\dot{\psi}_v$  lies in  $\tilde{\Pi}(\dot{\psi}_v)$  at every place  $v$ . Thus there exists  $\dot{\pi} \in \tilde{\Pi}(\dot{\psi})$  such that  $\dot{\pi}_{\mathfrak{q}_{\text{St}}} \simeq \text{St}$ . On the other hand,  $\dot{\pi}_v \simeq \mathbf{1}$  at almost all  $v$  (where  $\dot{\pi}$  is unramified).

For our  $\dot{\psi}$ , the group  $\mathcal{S}_{\dot{\psi}}$  defined in [Art13, §1.4] is trivial. This means that, by [Art13, Thm. 1.5.2], every member of  $\tilde{\Pi}(\dot{\psi})$  is a direct summand in the discrete part of the space of  $L^2$ -automorphic forms on  $G(\mathbb{A}_{\dot{F}})$ . In particular  $\dot{\pi}$  is a  $G(\mathbb{A}_{\dot{F}})$ -subrepresentation in the discrete part. This implies that  $\dim \dot{\pi} = 1$  by an approximation theorem for  $G$  since  $\dim \dot{\pi}_v = 1$  for almost all  $v$ , contradicting  $\dim \dot{\pi}_{\mathfrak{q}_{\text{St}}} = \infty$ .  $\square$

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