Midterm Galois Theory 2019

March 17, 2020

1. Let $K \subset L$ be an algebraic field extension.

1. Prove that for $\alpha, \beta \in L$ we have $[K(\alpha, \beta) : K] \leq [K(\alpha) : K] \cdot [K(\beta) : K]$.
2. Suppose that $[K(\alpha) : K]$ and $[K(\beta) : K]$ are co-prime. Show that in fact equality hold in (1).

2. Correct the following wrong theorem and give a proof of the corrected theorem.

Theorem. *Let $K \subset L$ be a field extension and $\alpha \in L$. Then there is a unique monic and irreducible polynomial $f_K^\alpha \in K[X]$ which has α as a root. Suppose $K(\alpha) \subset L$ is the subfield generated by α. The map*

$$K[X]/(f_K^\alpha) \to K(\alpha), g \mod f_K^\alpha \mapsto g(\alpha)$$

is a field isomorphism and the degree $[K(\alpha) : K]$ is equal to the degree of f_K^α as a polynomial in $K[X]$.

3. Determine the minimal polynomial of $\sqrt{2} + \sqrt{3} \in \mathbb{R}$ over $\mathbb{Q}$.

4. Let $f = X^4 + 1 \in \mathbb{Z}[X]$.

1. Show f is irreducible in $\mathbb{Q}[X]$.
2. Let α be a root of f in an algebraic closure of $\mathbb{Q}$ and write $K = \mathbb{Q}(\alpha)$. Factor the polynomial $X^4 - 4 \in K[X]$ in irreducible factors in $K[X]$.

5. Let K be the field of fractions of the polynomial ring $\mathbb{Q}[x, y, z]$ in three variables x, y and z over $\mathbb{Q}$. Define

$$\alpha = x + y + z, \quad \beta = x^2 + y^2 + z^2, \quad \gamma = x^3 + y^3 + z^3 \in K$$

and let $M = \mathbb{Q}(\alpha, \beta, \gamma) \subset K$ be the subfield generated by these elements.

1. Is the element $x \in K$ algebraic over M. If so, determine the minimal polynomial $f_M^x \in M[X]$.
2. Determine the trancendence degree of M over $\mathbb{Q}$.